HOMOLOGY CYLINDERS AND SKEIN ALGEBRAS

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Abstract. In this paper, we introduce a construction of an invariant for a homology cylinder of a surface Σ. It is an element of the skein algebra of Σ and has two aspects. The first is a quantization of the action of homology cylinders on fundamental groups. In the second aspect, we can extend the Ohtsuki series, one for integral homology spheres, to our invariant. We use the HOMFLY-PT skein algebra in this paper. But the main theorem holds in other skein algebras.

Contents

1. Introduction

In this paper, we introduce a construction of an invariant for a homology cylinder of a surface Σ. It is an element of the skein algebra of Σ and has two aspects.

The first is a quantization of the action of homology cylinders on fundamental groups. Turaev [12] used the word, “quantization”, considering the skein algebras as a refinement of the Goldman Lie algebra. In this paper, we use the word with the same meaning.

In the second aspect, we can extend the Ohtsuki series, an invariant for integral homology spheres, to ours in this paper. In other words, considering them as homology cylinders of the closed disk, the first one equals the second one. The action of the homology cylinders on fundamental groups does not have the information of quantum invariants, but our invariant has.

In this paper, we will explain some essential definitions and main theorems without proof. In our theory, a formula for Dehn twists plays an important role. We use the HOMFLY-PT skein algebras, but the results in this paper hold in other skein algebras.

1.1. **Outline.** In the second section, we define skein modules. Ignoring the h-torsion part, we consider simplified skein modules. Furthermore, we set filtrations of them using "detours".

In the third section, we introduce a formula for the action of Dehn twists on skein modules using the Lie action of a skein algebra. Using it, we construct an embedding from the Torelli group in the completed skein algebras and the Ohtsuki series, which is an invariant for integral homology 3-spheres.

In the fourth section, we explain how to define homology cylinders. Furthermore, we set their action on the completed skein modules. It is an analogy for the one on the completion of the fundmental group.

In the last section, we state our main theorem. We construct an invariant for homology cylinders being an element of the completed skein algebra and explain its properties.

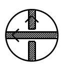
2. Skein modules

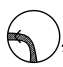
2.1. **Definition of skein modules.** In this subsection, we set the HOMFLY-PT skein modules. Let M be a compact oriented 3-manifolds, β a non-negative integer, and J an embedding $J : \{1, \cdots, 2\beta\} \times I \to \partial M$. Here the symbol I is the unit interval $[0, 1]$. We set $\mathcal{E}(M, J)$ as the set of embeddings $e' : M^{(1)} \times I \to M$ satisfying the following conditions.

- $M^{(1)}$ is an oriented comapct 1-manifold.
- $e'(\partial M^{(1)} \times I) \subset J(\{1, \cdots, 2\beta\} \times I)$.
- The embedding $J^{-1} \circ e'_{|\partial M^{(1)} \times I} : \partial M^{(1)} \times I \to \{1, \cdots, 2\beta\}$ preserves the orientations and indeces the bijective map $\pi_0(M^{(1)}) \to \pi_0(\{1, \cdots, 2\beta\} \times I)$.

We call an isotopy class of an embedding of $\mathcal{E}(M, J)$ a ribbon tangle and denote by $\mathcal{T}(M, J)$ the set $\mathcal{E}(M, J)/\text{isotopy}$.

Let $\mathcal{A}^{\ddagger}(M, J)$ be the quotient of $\mathbb{Q}[\rho][[h]]\mathcal{T}(M, J)$ by the relation

 $-$  $= h$ ,

 $= \exp(\rho h)$ ,

$$\text{(trivial knot)} = \frac{2\sinh(\rho h)}{h}\text{(empty knot)}.$$

Furthermore, in this paper, considering the other one

$$\{y \in \mathcal{A}^{\dagger}(M, J) | \text{there exists m} \in \mathbb{Z}_{\geq 1} \text{ s.t. } h^m y = 0.\},$$

we set the h-torsion-free skein algebra

$$\mathcal{A}(M, J) = \mathcal{A}^{\dagger}(M, J)$$
$$\stackrel{\text{def.}}{=} \mathcal{A}^{\ddagger}(M, J)/\{y \in \mathcal{A}^{\dagger}(M, J) | \text{there exists m} \in \mathbb{Z}_{\geq 1} \text{ s.t. } h^m y = 0.\}.$$

We remark that the $\mathbb{Q}[\rho][[h]]$-module homomorphism map

$$\mathcal{A}(M, J) \to \mathcal{A}(M, J), y \to hy$$

is injective. If β equals 0, we use a simple denote

$$\mathcal{A}(M) \stackrel{\text{def.}}{=} \mathcal{A}(M, J).$$

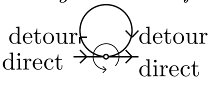
Remark 2.1. *In general, the* $\mathbb{Q}[\rho][[h]]$*-module homomorphism map* $\mathcal{A}^{\ddagger}(M, J) \to \mathcal{A}^{\dagger}(M, J)$ *is not injective. In this paper, we use the simplified skein module* $\mathcal{A}^{\ddagger}(M, J)$ *in this meaning.*

2.2. **Filrtrations and completions.** At first, we set a finite-type filtration of the free $\mathbb{Z}$-module with basis $\mathcal{T}(M, J)$. Using graphs we define them. We prepare the notations of graphs.

- A uni-quad-valent graph is a graph, whose every vertex is either univalent, bivalent, or quadvalent.
- A vertex-ordered uni-bi-quad-valent graph is a uni-bi-quad-valent graph such that, for each vertex of it, we fix a cyclic orientation of edges around the vertex.

Definition 2.2. *A vertex-ordered uni-bi-quad-valent directed detour-graph* Γ *is a vertex-ordered uni-bi-quad-valent directed graph whose edges* $\mathrm{Edge}(\Gamma)$ *are classified as detour edges and direct ones satisfying the conditions.*

- *The indegree of a quad-valent vertex or a bi-valent vertex equals the outdegree of it.*
- *A uni-valent vertex or a bi-valent vertex does not have detour edges.*
- *The neighborhood of a quad-valent is as the figure.*



Let M be a compact oriented 3-manifold, β a non-negative integer, and J an embedding $\{1, \cdots, 2\beta\} \times I \to \partial M$. We set $\mathcal{E}^{Vd}(M, J)$ as the set consisting of all embeddings $e : \Gamma \times I \to M$ satisfying the conditions for a vertex-ordered uni-bi-quad-valent directed detour-graph Γ.

(1) Denoting the set of uni-valent vertex of Γ by $\mathrm{Base}(\Gamma)$, we have

$$e(\mathrm{Base}(\Gamma) \times I) \subset J(\{1, \cdots, 2\beta\} \times I).$$

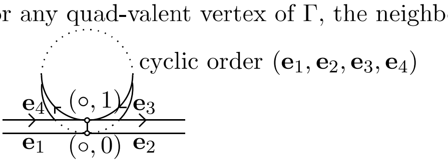
(2) The embedding

$$(\widetilde{J}^{+})^{-1} \circ e_{|\mathrm{Base}(\Gamma) \times I} : \mathrm{Base}(\Gamma) \times I \to \{1, \cdots, 2\beta\} \times I$$

preserves the orientations and induces the bijective map

$$((\widetilde{J})^{-1} \circ e_{|\mathrm{Base}(\Gamma) \times I})_* : \pi_0(\mathrm{Base}(\Gamma) \times I) \to \pi_0(\{1, \cdots, 2\beta\} \times I).$$

(3) For any quad-valent vertex of Γ, the neighborhood of $e(\circ \times I)$ is as the figure.



We denote by $\mathcal{E}^{Vd}_n(M, J) \subset \mathcal{E}^{Vd}(M, J)$ the subset

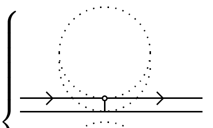
$$\mathcal{E}^{Vd}_n(M, J) \stackrel{\text{def.}}{=} \{e : \Gamma \times I \to M | e \in \mathcal{E}^{Vd}(M, J), \{\text{quad}-\text{valent vertex of } \Gamma\} = n\}.$$

For an embedding $e^{Vd} : \Gamma \times I \to M$ being an element of $\mathcal{E}^{Vd}(M, J)$, we set $\Psi^{\mathcal{T}}_{\mathcal{E}^{Vd}}(e^{Vd}) \in \mathbb{Z}\mathcal{T}(M, J)$ by the following steps.

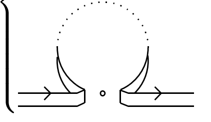
- Let $\circ_1, \cdots, \circ_n$ be the quad-valent vertices of Γ. We denote by $\mathbf{e}_i$ the detour edge attaching $\circ_i$.
- We denote by Cross the set of quad-valent vertices of Γ. For a map $\phi : \mathrm{Cross} \to \{0, +1\}$, we choose the embedding $e_{e^{Vd},\phi} \in \mathcal{E}(M, J)$ satisfying the conditions.
  (1) The image of $e_{e^{Vd},\phi}$ equals the one of the original e^{Vd} as oriented ribbon tangles except for the neighborhood of $\coprod_{j=1}^{n} e^{Vd}(\mathbf{e}_j)$.
  (2) In the neighborhood of $e^{Vd}(\mathbf{e}_j)$, the image of $e_{e^{Vd},\phi}$ is as

$$(\text{the neighborhood of } e^{Vd}(\mathbf{e}_j)) \cap (\text{the image of } e_{e^{Vd},\phi})$$

$=$



if$\phi(\circ_j) = 0$



if$\phi(\circ_j) = 0$.

- The embedding $e_{e^{Vd},\phi}$ represents $T_{e^{Vd},\phi} \in \mathcal{T}(M, J)$.
- We set $\Psi^{\mathcal{T}}_{\mathcal{E}^{Vd}}(e^{Vd}) \in \mathbb{Z}\mathcal{T}(M, \widetilde{J}^{-}, \widetilde{J}^{+})$ as

$$\Psi^{\mathcal{T}}_{\mathcal{E}^{Vd}}(e^{Vd}) \overset{\text{def.}}{=} \sum_{\phi:\mathrm{Cross}\to\{+1,0\}} (-1)^{\sum_{\circ_j \in \mathrm{Cross}(\Gamma)} \phi(\circ_j)} T_{e^{Vd},\phi}.$$

For a non-negative integer n, we denote by $F^n\mathbb{Z}\mathcal{T}(M, J) \subset \mathbb{Z}\mathcal{T}(M, J)$ the submodule generated by

$$\{\Psi^{\mathcal{T}}_{\mathcal{E}^{Vd}}(e^{Vd}) | e^{Vd} \in \mathcal{E}^{Vd}_n(M, J)\}.$$

We set a finite-type filtration $\{F^n\mathcal{A}(M, J)\}_{n\geq 0}$ as the set

$$\cup_{2i+j\geq n}\{h^i X | X \in F^j\mathbb{Z}\mathcal{T}(M, J)\}$$

generates the submodule $F^n\mathcal{A}(M, J)$ as $\mathbb{Q}[\rho][[h]]$-module. Using it, we consider the completion

$$\widehat{\mathcal{A}}(M, J) \overset{\text{def.}}{=} \varprojlim_{i\to\infty} \mathcal{A}(M, J)/F^i\mathcal{A}(M, J)$$

and a finite-type filtration $\{F^n\widehat{\mathcal{A}}(M, J)\}_{n\geq 0}$ of $\widehat{\mathcal{A}}(M, J)$ by

$$F^n\widehat{\mathcal{A}}(M, J) \overset{\text{def.}}{=} \ker(\widehat{\mathcal{A}}(M, J) \to \mathcal{A}(M, J)/F^n\mathcal{A}(M, J)).$$

3. Lie actions and a formula for Dehn twists.

3.1. **Skein algebras and a Lie structure.** Let S be a compact oriented surface, β a non-negative integer, and j be an injective map. We use simple denotes

$$\mathcal{A}(S, j) \overset{\text{def.}}{=} \mathcal{A}(S \times I, j \times \mathrm{id}_I), \quad \widehat{\mathcal{A}}(S, j) \overset{\text{def.}}{=} \widehat{\mathcal{A}}(S \times I, j \times \mathrm{id}_I),$$

$$\mathcal{A}(S) \overset{\text{def.}}{=} \mathcal{A}(S \times I), \quad \widehat{\mathcal{A}}(S) \overset{\text{def.}}{=} \widehat{\mathcal{A}}(S \times I),$$

where the symbol I means the unit interval $[0,1]$. Using the embeddings

$$e_{\mathrm{up}}: S\times I \to S\times I, (p,t)\mapsto (p,\frac{1+t}{2}),$$
$$e_{\mathrm{down}}: S\times I \to S\times I, (p,t)\mapsto (p,\frac{t}{2}),$$

we define the multiple of $\mathcal{A}(S)$ by

$$\mathcal{A}(S)\times\mathcal{A}(S)\to\mathcal{A}(S), L_1\times L_2\mapsto L_1L_2 \overset{\mathrm{def.}}{=} e_{\mathrm{up}}(L_1)\cup e_{\mathrm{down}}(L_2).$$

Furthermore, we set the right and left actions of $\mathcal{A}(S)$ on $\mathcal{A}(S,j)$ by

$$\mathcal{A}(S)\times\mathcal{A}(S,j)\to\mathcal{A}(S,j), L\times T\mapsto LT \overset{\mathrm{def.}}{=} e_{\mathrm{up}}(L)\cup e_{\mathrm{down}}(T),$$
$$\mathcal{A}(S,j)\times\mathcal{A}(S)\to\mathcal{A}(S,j), T\times L\mapsto TL \overset{\mathrm{def.}}{=} e_{\mathrm{up}}(T)\cup e_{\mathrm{down}}(L).$$

Then, for any elements $x,x_1,x_2\in\mathcal{A}(\Sigma)$, $z\in\mathcal{A}(\Sigma,j)$, we have

$$x_1x_2-x_2x_1\in h\mathcal{A}(S),$$
$$xz-zx\in h\mathcal{A}(S,j)$$

by the skein relation. Since the homomorphism map

$$\mathcal{A}(S,j)\to\mathcal{A}(S,j), y\to hy$$

is injective, we can define the bracket and the action by

$$[x_1,x_2] \overset{\mathrm{def.}}{=} \frac{1}{h}(x_1x_2-x_2x_1)\in\mathcal{A}(S),$$
$$\sigma(x)(z) \overset{\mathrm{def.}}{=} \frac{1}{h}(xz-zx)\in\mathcal{A}(S,j).$$

The first makes $(\mathcal{A}(S),[\cdot,\cdot])$ a Lie algebra, and the second $(\mathcal{A}(S,j),\sigma)$ a Lie module of $\mathcal{A}(S)$. The finite type filtrations satisfy the following.

Proposition 3.1 ([11])**.** *For any $n,m\in\mathbb{Z}_{\geq 0}$, we have*

$$F^n\mathcal{A}(S)F^m\mathcal{A}(S)\subset F^{n+m}\mathcal{A}(S)$$
$$F^n\mathcal{A}(S)F^m\mathcal{A}(S,j)\subset F^{n+m}\mathcal{A}(S,j)$$
$$F^n\mathcal{A}(S,j)F^m\mathcal{A}(S)\subset F^{n+m}\mathcal{A}(S,j)$$
$$[F^n\mathcal{A}(S),F^m\mathcal{A}(S)]\subset F^{n+m-2}\mathcal{A}(S)$$
$$\sigma(F^n(\mathcal{A}(S)))(F^m\mathcal{A}(S,j))\subset F^{n+m-2}\mathcal{A}(S,j).$$

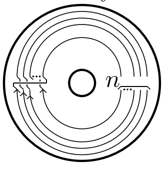
Using this proposition, we can define the above operations in completions such as

$$\widehat{\mathcal{A}}(S)\times\widehat{\mathcal{A}}(S)\to\widehat{\mathcal{A}}(S)$$
$$\widehat{\mathcal{A}}(S)\times\widehat{\mathcal{A}}(S,j)\to\widehat{\mathcal{A}}(S,j)$$
$$\widehat{\mathcal{A}}(S,j)\times\widehat{\mathcal{A}}(S)\to\widehat{\mathcal{A}}(S,j)$$
$$[\cdot,\cdot]:\widehat{\mathcal{A}}(S)\times\widehat{\mathcal{A}}(S)\to\widehat{\mathcal{A}}(S)$$
$$\sigma(\cdot)(\cdot):\widehat{\mathcal{A}}(S)\times\widehat{\mathcal{A}}(S,j)\to\widehat{\mathcal{A}}(S,j).$$

3.2. **A formula for Dehn twists.** In this subsection, we introduce a formula for the action of a Dehn twists using the Lie action. In our theory, this formula plays an important role. There exist other versions of this formula in some skein algebras.

At first, we set a significant element $L_{\mathcal{A}} \in \widehat{\mathcal{A}}(S^1 \times I)$ by the following steps where $S^1 = \mathbb{R}/\mathbb{Z}$.

- For any $n \in \mathbb{Z}_{\geq 1}$, we denote by l_n an element of $\mathcal{A}(S^1 \times I)$ represented by a knot presented by the figure.



- For any $n \in \mathbb{Z}_{\geq 0}$, we set l'_n as

$$l'_n \stackrel{\text{def.}}{=} \begin{cases} \sum_{j=1}^{n} \frac{(-h)^{j-1}}{j} \sum_{i_1+\cdots,i_j=n, i_{j'} \in \mathbb{Z}_{\geq 1}} l_{i_1} \cdots l_{i_j} & (n \in \mathbb{Z}_{\geq 1}) \\ 2\rho & (n = 0). \end{cases}$$

- For any $n \in \mathbb{Z}_{\geq 0}$, we set $l''_n \in \mathcal{A}(S^1 \times I)$ as

$$l''_n \stackrel{\text{def.}}{=} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} (-1)^{n-j} l'_j.$$

  Then we have $l''_n \in F^n \mathcal{A}(S^1 \times I)$.
- Setting a sequence $\{v_n\}_{n \in \mathbb{Z}_{\geq 2}}$ by $\frac{1}{2}(\log(x))^2 = \sum_{n \in \mathbb{Z}_{\geq 2}} v_n (x-1)^n$, we define $L_{\mathcal{A}} \in \widehat{\mathcal{A}}(S^1 \times I)$ as

$$L_{\mathcal{A}} \stackrel{\text{def.}}{=} \sum_{n \in \mathbb{Z}_{\geq 2}} v_n l''_n - \frac{1}{3} \rho^3 h^2.$$

  .

The above equations to define $L_{\mathcal{A}} \in \widehat{\mathcal{A}}(S^1 \times I)$ is complicated, but we can characterize it by the theorem.

Theorem 3.2 ([11]). *Let $t = t_{S^1 \times \{\frac{1}{2}\}}$ be the Dehn twist along the simple closed curve $S^1 \times \{\frac{1}{2}\}$. For $z \in \widehat{\mathcal{A}}(S^1 \times I)$, z satisfies the two conditions*

(1) *We have*

$$\sum_{i \in \mathbb{Z}_{\geq 1}} \frac{(-1)^{i-1}}{i} (t - \mathrm{id})^i (y) = \sigma(z)(y)$$

for any $\beta \in \mathbb{Z}_{\geq 1}$, any $j : \{1, \cdots, 2\beta\} \to \partial(S^1 \times I)$, and any $y \in \widehat{\mathcal{A}}(S^1 \times I, j)$.

(2) *The embedding from the annulus $S^1 \times I$ to the disk D^2 induces a $\mathbb{Q}[\rho][[h]]$-module homomorphism map*

$$e_{\text{trivial knot}} : \widehat{\mathcal{A}}(S^1 \times I) \to \widehat{\mathcal{A}}(D^2) \simeq \mathbb{Q}[\rho][[h]].$$

Then we have

$$e_{\text{trivial knot}}(z) = 0.$$

if and only if $z = L_{\mathcal{A}}$.

The first condition is crucial in our theory. We use the skein relation crucially only to prove the first one. By the second condition, there exists no ambiguity of $L_{\mathcal{A}}$. To construct invariants, we use the second one. Using this theorem, we have the following.

Corollary 3.3. *The orientation preserving diffeomorphism* $i_{\mathrm{rev}} : S^1 \times I \to S^1 \times I, (s,t) \mapsto (-s,-t)$ *induces a* $\mathbb{Q}[\rho][[h]]$*-algebra homomorphism map* $i_{\mathrm{rev}} : \widehat{\mathcal{A}}(S^1 \times I) \to \widehat{\mathcal{A}}(S^1 \times I)$*. Then we have*

$$i_{\mathrm{rev}}(L_{\mathcal{A}}) = L_{\mathcal{A}}.$$

We can prove this corollary by the direct computation but can do it using Theorem 3.2. Since $i_{\mathrm{rev}}(L_{\mathcal{A}})$ also satisfies the two conditions in Theorem 3.2, we have $i_{\mathrm{rev}}(L_{\mathcal{A}}) = L_{\mathcal{A}}$.

We return to the story about a compact oriented surface S. For any simple closed curve $c \subset S$, we choose an embedding $e_c : S^1 \times I \to S$ satisfying

$$e_c(S^1 \times \{\frac{1}{2}\}) = c.$$

Then e_c induces a $\mathbb{Q}[\rho][[h]]$-algebra homomorphism map

$$e_c : \widehat{\mathcal{A}}(S^1 \times I) \to \widehat{\mathcal{A}}(S).$$

We set an element $L_{\mathcal{A}}(c) \in \widehat{\mathcal{A}}(S \times I)$ by

$$L_{\mathcal{A}}(c) \stackrel{\text{def.}}{=} e_c(L_{\mathcal{A}}).$$

By Corollary 3.3, this element $L_{\mathcal{A}}(c)$ does not depend on the choice of the orientation of c. Using this element, we have the following theorem.

Theorem 3.4. *For any* $\beta \in \mathbb{Z}_{\geq 0}$*, any injective map* $j : \{1, \cdots, 2\beta\} \to \partial S$*, and any* $y \in \widehat{\mathcal{A}}(S, j)$*, we have*

$$\sum_{i \in \mathbb{Z}_{\geq 1}} \frac{(-1)^{i-1}}{i} (t_c - \mathrm{id})^i(y) = \sigma(L_{\mathcal{A}})(y).$$

In other words, we have

$$t_c(y) = \exp(\sigma(L_{\mathcal{A}}))(y) \stackrel{\text{def.}}{=} \sum_{i=1}^{\infty} \frac{1}{i!} (\sigma(L_{\mathcal{A}}))^i(y).$$

We prove the theorem using the first condition of Theorem 3.2. We call $L_{\mathcal{A}}(c)$ an element describing a formula for the Dehn twists in the HOMFLY-PT skein algebra. Kawazumi-Kuno [3][4] abd Massuyeau-Turaev [5] discovered one for the Dehn twists in the Goldman Lie algebra. Our formula is an analogy for one of them.

3.3. **Applications of the formula for the Dehn twists.** In this subsection, we introduce some applications of Theorem 3.4. Let $\Sigma_{g,1}$ be a surface of genus g with a connected non-empty boundary. Considering the action of the mapping class group

$$\mathcal{M}(\Sigma_{g,1}) \stackrel{\text{def.}}{=} \pi_0(\mathrm{Diff}\Sigma_{g,1} \text{fixing } \partial\Sigma_{g,1} \text{ pointwise})$$

of $\Sigma_{g,1}$ on the homology groups $H_1(\Sigma, \mathbb{Z})$, we call its kernel

$$\ker(\mathcal{M}(\Sigma_{g,1}) \to \mathrm{Aut}(H_1(\Sigma_{g,1}, \mathbb{Z}))$$

the Torelli group and denote it by $\mathcal{I}(\Sigma_{g,1})$. It is well-known that the set

$$\{t_{c_1}t_{c_2}^{-1}|(c_1,c_2) : \text{bounding pair}\}$$

generates $\mathcal{I}(\Sigma_{g,1})$, where a bounding pair is a pair of simple closed curves bounding a compact subsurface.

Theorem 3.5. *We can consider the set $F^3\widehat{\mathcal{A}}(\Sigma_{g,1})$ as a group whose multiple is the Baker-Campbell-Hausdorff series* bch. *We set a group homomorphism $\zeta : \mathcal{I}(\Sigma_{g,1}) \to (F^3\widehat{\mathcal{A}}(\Sigma_{g,1}), \mathrm{bch})$ by*

$$\zeta(t_{c_1}t_{c_2}^{-1}) = L_{\mathcal{A}}(c_1) - L_{\mathcal{A}}(c_2).$$

Then it is well-defined and injective.

We prove the theorem by Theorem 3.4 and Putman's relation [6] of the Torelli group. It is needless to say that the first condition in Theorem 3.4 is essential, but the second one is also essential.

Using this embedding, we construct an invariant for integral homology 3-spheres. We obtain them by the following steps.

- We take a standard embedding $e_{H_g} : H_g \to S^3$ from a handle body H_g in the sphere S^3, which means the closure $\overline{S^3\backslash e_{H_g}(H_g)}$ is also a handle body. We remark that we do not need the assumption that e_{H_g} is standard.
- We take a diffeomorphism representing an element $\xi \in \mathcal{I}(\Sigma_{g,1})$ and denote it by the same symbol ξ.
- We fix a closed disk D in the boundary ∂H_g of the handle body H_g and consider the closure $\overline{\partial H_g\backslash D}$ as $S_{g,1}$. We obtain a 3-manifold

$$S^3(e_{H_g},\xi) \stackrel{\text{def.}}{=} H_g \cup_{e_{H_g|\partial H_g}\circ(\xi\cup \mathrm{id}_D)} \overline{H_g\backslash e_{H_g}(H_g)}$$

  gluing the two handle bodies H_g and $\overline{H_g\backslash e_{H_g}(H_g)}$ by a new map $e_{H_g|\partial H_g}\circ(\xi\cup \mathrm{id}_D)$. Then $S^3(e_{H_g},\xi)$ is an integral homology 3-sphere. Conversely, it is well-known that we can get any integral homology 3-sphere in this way.

About the integral homology 3-sphere $S^3(e_{H_g},\xi)$ obtained in this way, we set a series defined by the following.

- Considering $\Sigma_{g,1}$ as an embedded surface in the sphere S^3, the tubular neighborhood

$$e_{\Sigma_{g,1}} : \Sigma_{g,1}\times I \to S^3$$

  of the embedding induces a $\mathbb{Q}[\rho][[h]]$-module homomorphism map

$$e_{\Sigma_{g,1}} : \widehat{\mathcal{A}}(\Sigma_{g,1}) \to \widehat{\mathcal{A}}(S^3) \simeq \mathbb{Q}[\rho][[h]].$$

- Using the injective map ζ in Theorem 3.5, we set a series $z_{\mathcal{A}}(S^3(e_{H_g},\xi))$ as

$$z_{\mathcal{A}}(S^3(e_{H_g},\xi)) \stackrel{\text{def.}}{=} \sum_{i=1}^{\infty}\frac{1}{i!h^i}e_{\Sigma_{g,1}}((\zeta(\xi))^i) \in \mathbb{Q}[\rho][[h]].$$

Then we have the following theorem.

Theorem 3.6. *The above map*

$$z_{\mathcal{A}} : \{\text{diffeomorphism types of integral homology cylinders}\} \to \mathbb{Q}[\rho][[h]]$$

is well-defined. In other words, for any integral homology 3-sphere M, the series $z_{\mathcal{A}}(M)$ is an invariant.

Using the Reidemeister-Singer stabilizer, we prove the theorem. Furthermore, in our recent work, this invariant has the following property.

Theorem 3.7. *For any integral homology 3-sphere M, changing variables*

$$\rho \mapsto \frac{N \log q}{q - q^{-1}}, \quad h \mapsto -q + q^{-1},$$

we obtain a series $(z_{\mathcal{A}}(M))_{(N))}$ from $z_{\mathcal{A}}(M)$. Then $(z_{\mathcal{A}}(M))_{(N))}$ equals the $\mathrm{sl}(N)$*-quantum invariant, the* $\mathrm{sl}(N)$*-Ohtsuki series.*

The theorem says that the invariant

$$z_{\mathcal{A}} : \{\text{diffeomorphism types of integral homology cylinders}\} \to \mathbb{Q}[\rho][[h]]$$

has no new information but all information of $\mathrm{sl}(N)$-quantum invariant.

Remark 3.8. *The above results hold in Kauffman bracket skein modules. For details, you can see our paper* [8][9][10].

4. Homolocy cylinders

In this section, we recall the definition of homology cylinders and introduce an action of homology cylinders on skein modules.

Let Σ be a compact, connected, and oriented surface with a non-empty boundary. A homology cylinder of Σ is a pair (M, α) of a 3-manifold and a diffeomorphism $\alpha : \partial M \to \partial(\Sigma \times I)$ satisfying the two conditions. The first is that M is compact, connected, and oriented. The second is that α has the property

$$\begin{aligned}&\ker(\alpha : H_1(\partial(\Sigma \times I), \mathbb{Z}) \to H_1(M, \mathbb{Z}))\\ &= \ker(H_1(\partial(\Sigma \times I), \mathbb{Z}) \to H_1(\partial(\Sigma \times I), \mathbb{Z})\text{induced by the natural embedding}).\end{aligned}$$

For two homology cylinders (M^1, α^1) and (M^2, α^2), if a diffeomorphism $\chi : M^1 \to M^2$ satisfies $\alpha^1 = \alpha^2 \circ \chi_{|\partial M^1}$, we call they are isomorphic. We denote by $\mathcal{H}(\Sigma)$ the set of isomorphic classes of homology cylinders of Σ.

We can define the composition of $\mathcal{H}(\Sigma)$. We fix two homology cylinders (M^1, α^1) and (M^2, α^2) of Σ. We set a new 3-manifold $M^1 \circ M^2$ as the quotient of $M^1 \coprod M^2$ by the relation

$$\alpha^2(p, 1) \sim \alpha^1(p, 0)$$

and a new diffeomorphism $(\alpha^1 \sqcup \alpha^2)_{M^1 \circ M^2} : \partial(\Sigma \times I) \to \partial(M^1 \circ M^2)$ as

$$(\alpha^1 \sqcup \alpha^2)_{M^1 \circ M^2}(p, t) = \begin{cases} \alpha_1(p, 1) & \text{if } t = 1 \\ \alpha_1(p, 2t - 1) & \text{if } t \in [\frac{1}{2}, 1] \\ \alpha_2(p, 2t) & \text{if } t \in [0, \frac{1}{2}] \\ \alpha_2(p, 0) & \text{if } t = 0. \end{cases}$$

Then the pair $(M^1 \circ M^2, (\alpha^1 \sqcup \alpha^2)_{M^1 \circ M^2})$ is also a homology cylinder. The composition

$$\begin{aligned}(\cdot) \circ (\cdot) :&\mathcal{H}(\Sigma) \times \mathcal{H}(\Sigma) \to \mathcal{H}(\Sigma),\\ &((M^1, \alpha^1), (M^2, \alpha^2)) \mapsto (M^1 \circ M^2, (\alpha^1 \sqcup \alpha^2)_{M^1 \circ M^2})\end{aligned}$$

makes $\mathcal{H}(\Sigma)$ a monoid.

We fix a homology cylinder (M, α) and an injective map $j : \{1, \cdots, 2\beta\} \to \partial\Sigma$. We consider the following three embeddings.

- Let $\widetilde{\alpha} : \partial(\Sigma \times I) \times I \hookrightarrow M$ be the tubular neighborhood of the embedding surface $\partial M = \alpha(\partial(\Sigma \times I))$.
- Let $\iota_0 : \Sigma \times I \to \partial(\Sigma \times I) \times I$ be the tubular neighborhood of the embedding surface $\Sigma \times \{0\} \times \{1\}$ satisfying the conditions.
  - For any $p \in \Sigma$, we have $\iota_0(p, 0) = (p, 0, 1)$.
  - There exist a positive number $\epsilon > 0$ such that $\iota_0(p, t) = (p, \epsilon t, 1)$ for any $p \in \partial\Sigma$ and $t \in [0, 1]$.
- Let $\iota_1 : \Sigma \times I \to \partial(\Sigma \times I) \times I$ be the tubular neighborhood of the embedding surface $\Sigma \times \{1\} \times \{1\}$ satisfying the conditions.
  - For any $p \in \Sigma$, we have $\iota_1(p, 1) = (p, 1, 1)$.
  - There exist a positive number $\epsilon > 0$ such that $\iota_1(p, 1 - t) = (p, 1 - \epsilon t, 1)$ for any $p \in \partial\Sigma$ and $t \in [0, 1]$.

The compositions $\widetilde{\alpha} \circ \iota_0$ and $\widetilde{\alpha} \circ \iota_1$ of them induce the $\mathbb{Q}[\rho][[h]]$-module homomorphism maps

$$\begin{aligned}(\widetilde{\alpha} \circ \iota_0)_* &: \widehat{\mathcal{A}}(\Sigma, j) \to \widehat{\mathcal{A}}(M, \alpha \circ (j \times \mathrm{id}_I)),\\ (\widetilde{\alpha} \circ \iota_1)_* &: \widehat{\mathcal{A}}(\Sigma, j) \to \widehat{\mathcal{A}}(M, \alpha \circ (j \times \mathrm{id}_I)).\end{aligned}$$

Then we have the following.

Theorem 4.1. *Using the above notation, the* $\mathbb{Q}[\rho][[h]]$*-module homomorphism maps*

$$\begin{aligned}(\widetilde{\alpha} \circ \iota_0)_* &: \widehat{\mathcal{A}}(\Sigma, j) \to \widehat{\mathcal{A}}(M, \alpha \circ (j \times \mathrm{id}_I)),\\ (\widetilde{\alpha} \circ \iota_1)_* &: \widehat{\mathcal{A}}(\Sigma, j) \to \widehat{\mathcal{A}}(M, \alpha \circ (j \times \mathrm{id}_I))\end{aligned}$$

are isomorphisms.

It is easy to check that the maps $(\widetilde{\alpha} \circ \iota_0)_*$ and $(\widetilde{\alpha} \circ \iota_1)_*$ are surjective. We can prove that $(\widetilde{\alpha} \circ \iota_0)_*$ and $(\widetilde{\alpha} \circ \iota_1)_*$ are injective using the formula for Dehn twists. We remark that we need the skein modules to be h-torsion free in our proof.

Remark 4.2. *By the above theorem, the structure of the skein algebra* $\widehat{\mathcal{A}}(M, j)$ *simplified in two ways becomes clear. The first is to simplify it like Stallings's theorem* [7] *in a group. The second is to ignore the h-torsion part of* $\mathcal{A}^{\ddagger}(M, j)$.

Using the above theorem, we set a monoid homomorphism map as

$$\Psi_{\mathcal{H}}^{\mathrm{Aut}\mathcal{A}} : \mathcal{H}(\Sigma) \to \mathrm{Aut}(\widehat{\mathcal{A}}(\Sigma, j)), (M, \alpha) \to (\widetilde{\alpha} \circ \iota_1)_*^{-1} \circ (\widetilde{\alpha} \circ \iota_0)_*.$$

We call the map $\Psi_{\mathcal{H}}^{\mathrm{Aut}\mathcal{A}}$ an action of $\mathcal{H}(\Sigma)$ on the completed skein algebra $\widehat{\mathcal{A}}(\Sigma, j)$. This monoid homomorphism map is related closely to an invariant defined in the next section.

5. Main theorems

We can construct an invariant

$$\widetilde{\zeta}_{\mathcal{A}}:\mathcal{H}(\Sigma)\to\widehat{\mathcal{A}}(\Sigma)$$

for homology cylinders in a similar way in Theorem 3.6. We do it in the following. First, we obtain a homology cylinder in the steps.

- We take an embedding $e_{H_g}:H_g\to\Sigma\times I$ from a handle body H_g in the 3-manifold $\Sigma\times I$.
- We take a diffeomorphism of the surface $\Sigma_{g,1}$ representing an element $\xi\in\mathcal{I}(\Sigma_{g,1})$ and denote it by the same symbol ξ.
- Taking a closed disk D in the boundary ∂H_g, we consider the closure $\overline{\partial H_g\backslash D}$ as the surface $\Sigma_{g,1}$. We get a 3-manifold $(\Sigma\times I)(e_{H_g},\xi)$, the quotient of the disjoint sum $H_g\coprod\overline{\Sigma\times I\backslash e_{H_g}(H_g)}$ by the relation

  $$x\in\partial H_g\sim e_{H_g|\partial H_g}\circ(\xi\cup\mathrm{id}_D)(x)\in\overline{\Sigma\times I\backslash e_{H_g}(H_g)}.$$

  Then the pair $(\Sigma\times I)(e_{H_g},\xi)=((\Sigma\times I)(e_{H_g},\xi),\mathrm{id}_{\partial(\Sigma\times I)})$ is a homology cylinder. Conversely, Habegger [1] proved that we obtain any homology cylinder in this way.

Next, we construct an invariant $\widetilde{\zeta}_{\mathcal{A}}((\Sigma\times I)(e_{H_g},\xi))\in\widehat{\mathcal{A}}(\Sigma)$ of $(\Sigma\times I)(e_{H_g},\xi)$ in the following steps.

- Considering the surface $\Sigma_{g,1}$ is an embedded one in $\Sigma\times I$, the tubular neighborhood

  $$e_{\Sigma_{g,1}}:\Sigma_{g,1}\times I\to\Sigma\times I$$

  induces a homomorphism map

  $$e_{\Sigma_{g,1}}:\widehat{\mathcal{A}}(\Sigma_{g,1})\to\widehat{\mathcal{A}}(\Sigma).$$

- Using the embedding $\zeta_{\mathcal{A}}$ in Theorem 3.5, we set an element $\widetilde{\zeta}_{\mathcal{A}}((\Sigma\times I)(e_{H_g},\xi))$ of the completion $\widehat{\mathcal{A}}(\Sigma)$ as

  $$\widetilde{\zeta}_{\mathcal{A}}((\Sigma\times I)(e_{H_g},\xi))\stackrel{\text{def.}}{=}h\log(e_{\Sigma_{g,1}}(\exp\frac{\zeta_{\mathcal{A}}(\xi)}{h})).$$

Then the theorem holds.

Theorem 5.1. *The above map*

$$\widetilde{\zeta}_{\mathcal{A}}:\mathcal{H}(\Sigma)\to F^3\widehat{\mathcal{A}}(\Sigma)$$

is well-defined. In other words, for an isomorphic class $\Xi\in\mathcal{H}(\Sigma)$ of a homology cylinder, $\widetilde{\zeta}_{\mathcal{A}}(\Xi)$ is an invariant. Furthermore, we have

$$\Psi_{\mathcal{H}}^{\mathrm{Aut}\mathcal{A}}(\Xi)(y)=\exp(\sigma(\widetilde{\zeta}_{\mathcal{A}}(\Xi)))(y)$$

for any $\beta\in\mathbb{Z}_{\geq0}$, any injective map $j:\{1,\cdots,2\beta\}\to\partial\Sigma$, and any $y\in\widehat{\mathcal{A}}(\Sigma,j)$.

In this way, we obtain an invariant $\widetilde{\zeta}_{\mathcal{A}}:\mathcal{H}(\Sigma)\to F^3\widehat{\mathcal{A}}(\Sigma)$ for it describing the action $\Psi_{\mathcal{H}}^{\mathrm{Aut}\mathcal{A}}$ on the completed skein module $\mathcal{A}(\Sigma,j)$. Furthermore, $\widetilde{\zeta}_{\mathcal{A}}$ has the following property.

Theorem 5.2. *We fix an embedding $e : \Sigma \times I \to S^3$, which induces the two maps*

$$e_\sharp : \mathcal{H}(\Sigma) \to \{\text{diffeomorphism types of integral homology cylinders}\},$$
$$e_* : \widehat{\mathcal{A}}(\Sigma) \to \widehat{\mathcal{A}}(S^3).$$

Here we consider $\widehat{\mathcal{A}}(S^3)$ as $\mathbb{Q}[\rho][[h]]$. Then we have

$$z_{\mathcal{A}}(e_\sharp(\Xi)) = e_*(\exp(\frac{\widetilde{\zeta}(\Xi)}{h}))$$

for any $\Xi \in \mathcal{H}(\Sigma)$.

Remark 5.3. *It is unclear whether the element*

$$\widetilde{\zeta}_{\mathcal{A}}((\Sigma \times I)(e_{H_g}, \xi)) \stackrel{\text{def.}}{=} h \log(e_{\Sigma_{g,1}}(\exp \frac{\zeta_{\mathcal{A}}(\xi)}{h}))$$

belongs to $F^3\widehat{\mathcal{A}}(\Sigma)$. Using another algebraic definition not introduced in the paper, we can check $\widetilde{\zeta}_{\mathcal{A}}((\Sigma \times I)(e_{H_g}, \xi)) \in F^3\widehat{\mathcal{A}}(\Sigma)$.

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