PURE BRAID GROUPS IN MAPPING CLASS GROUPS OF SURFACES

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ABSTRACT. This paper summarizes the author's talk at RIMS workshop entitled "Geometry of discrete groups and hyperbolic spaces" in 2021. We give a necessary and sufficient condition for embedding pure braid groups into the mapping class groups of surfaces. This is joint work with Erika Kuno.

1. Background and main results

Let $S^b_{g,p}$ be a connected orientable surface of genus g with p punctures and b boundary components. In the case where $b=0$ or $p=0$, we drop the suffix that denotes 0, excepting g, from $S^b_{g,p}$. For example, the 2-sphere $S^0_{0,0}$ is simply denoted as S_0. The *mapping class group* $\mathrm{Mod}(S^b_{g,p})$ of $S^b_{g,p}$ is the group of orientation-preserving homeomorphisms of $S^b_{g,p}$, fixing the boundary pointwise, up to isotopy relative to the boundary. We write B_n for the *braid group* on n strands, which is identified with $\mathrm{Mod}(S^1_{0,n})$. The *pure braid group* $\mathrm{PMod}(S^1_{0,n})$ is denoted by PB_n. We define the Euler charcteristic of $S^b_{g,p}$ as

$$\chi^b_{g,p} := 2-2g-p-b.$$

Theorem 1.1. *$PB_n \hookrightarrow \mathrm{Mod}(S_{g,p})$ if and only if*

$$n \leq \begin{cases} 1 & ((g,p)\in\{(0,0),(0,1),(0,2),(0,3)\}) \\ 2 & ((g,p)\in\{(0,4),(1,0),(1,1)\}) \\ -\chi_{g,p} & (g=0,\ p\geq 5) \\ 2-\chi_{g,p} & (g\geq 2,\ p=0) \\ 1-\chi_{g,p} & (otherwise). \end{cases}$$

Theorem 1.2. *Suppose $b\geq 1$. Then $PB_n \hookrightarrow \mathrm{Mod}(S^b_{g,p})$ if and only if*

$$n \leq \begin{cases} 2-\chi^b_{g,p} & (g\geq 1,\ p+b\leq 2) \\ 1-\chi^b_{g,p} & (otherwise). \end{cases}$$

The study of injective homomorphisms between the mapping class groups of surfaces has been extensively developed by various researchers. Aramayona–Leininger–Souto [1], Birman–Hilden [3] and Paris–Rolfsen [13] gave injective homomorphisms induced by inclusion maps and (possibly branched) coverings of surfaces. Aramayona–Souto [2] proved that $\mathrm{PMod}(S^b_{g,p})$ cannot be embedded in $\mathrm{PMod}(S^{b'}_{g',p'})$ if $g\geq 6$ and either $g'\leq 2g-2$ or $g'=2g-1$ and $p'+b'\geq 1$. Ivanov–McCarthy [8] proved

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that every injective homomorphism between the mapping class groups of surfaces with the almost same topological complexity is necessarily an isomorphism. Castel [5] characterized (injective) homomorphisms from the braid groups to the mapping class groups of surfaces without punctures by defining transvections of monodromy representations of the braid groups. Castel's result implies neither Theorem 1.1 nor Theorem 1.2, because there is an injective homomorphism from a pure braid group into the mapping class group of a surface, which cannot be extended to the braid group (see Proposition 2.7). The topological complexity [4] and virtual cohomological dimension [7] are obstructions to the existence of an injective homomorphism between finite index subgroups of mapping class groups. Theoretically, right-angled Artin groups in mapping class groups can also be such obstructions. What can say about injective homomorphisms between finite index subgroups of mapping class groups by using right-angled Artin groups? Theorems 1.1 and 1.2 give an answer to this question.

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2. Embedding pure braid groups

In this section, we explain how to obtain injective homomorphisms from (pure) braid groups into the mapping class groups of surfaces.

We first review homomorphisms induced by embeddings of surfaces. Let $i\colon S \to F$ be an inclusion map between connected orientable surfaces of finite type. In this paper we call i an *extension* of S for convenience sake. The extension i is said to be *admissible* if $i(S)$ is a closed subset of F and every component of $i(\partial S)$ is not parallel to a component of ∂F. Additionally, we say that an admissible extension i is *annular* if $F \setminus \mathrm{Int}(i(S))$ is a disjoint union of annuli.

Proposition 2.1 ([13])**.** *Suppose that $i\colon S \to F$ is an admissible extension.*

(1) *If the extension i is annular, then the kernel of the induced homomorphism $\mathrm{Mod}(S) \to \mathrm{Mod}(F)$ is a free abelian group generated by*

$$\mathcal{A} = \{[T_{c_1}][T_{c_2}]^{-1} \mid c_1, c_2 \text{ are the boundary components of an outer annulus}\}.$$

(2) *If $F \setminus \mathrm{Int}(i(S))$ is a disjoint union of annuli and once-punctured disks, then the kernel of the induced homomorphism $\mathrm{Mod}(S) \to \mathrm{Mod}(F)$ is a free abelian group generated by $\mathcal{A}$ and*

$$\{[T_c] \mid c \text{ is the boundary component of an outer once-punctured disk}\}.$$

Here, T_c is a Dehn twist about a curve c.

When each component of the exterior of an admissible extension has negative Euler characteristic, then the extension induces an injective homomorphism.

Proposition 2.2 ([13])**.** *Suppose that $F \setminus i(S)$ is a disjoint union of surfaces with negative Euler characteristics. Then $\mathrm{Mod}(S) \hookrightarrow \mathrm{Mod}(F)$.*

We say that an admissible extension i is *hyperbolic* if i satisfies the assumption in Proposition 2.2. The embedding given in the following corollary is induced by a hyperbolic extension $S^1_{0,n} \to S^1_{0,m}$.

Corollary 2.3. *If $n \leq m$, then $B_n \hookrightarrow B_m$.*

We next recall injective homomorphisms from braid groups induced by branched coverings of surfaces. Consider the double branched covering $S^2_{g-1,0} \to S^1_{0,2g}$ induced by the hyperelliptic involution of $S^2_{g-1,0}$. According to the Birman–Hilden theory, it follows that B_{2g} is embedded in $\mathrm{Mod}(S^2_{g-1})$ as the symmetric mapping class subgroup with respect to the hyperelliptic involution. By gluing S^2_{g-1} and $S^{b+2}_{0,p}$ along their boundaries, we obtain an admissible extension $S^2_{g-1} \to S^b_{g,p}$. This extension is either annular or hyperbolic, thereby inducing a homomorphism $\mathrm{Mod}(S^2_{g-1}) \to \mathrm{Mod}(S^b_{g,p})$, whose restriction to the subgroup B_{2g} is injective. Therefore, B_{2g} is embedded in $\mathrm{Mod}(S^b_{g,p})$.

Similarly, the double branched covering $S^1_g \to S^1_{0,2g+1}$ induces an embedding $B_{2g+1} \hookrightarrow \mathrm{Mod}(S^1_g)$. If $b + p \geq 2$, then an extension $S^1_g \to S^b_{g,p}$ is hyperbolic and induces an embedding $B_{2g+1} \hookrightarrow \mathrm{Mod}(S^b_{g,p})$. For more information about the Birman–Hilden theory, see [3] and [12].

We also use the following embedding to establish our main theorems.

Proposition 2.4. $B_{2g+2} \hookrightarrow \mathrm{Mod}(S^1_{g,1})$.

Definition 2.5. Let S be a surface with punctures and F a surface. By $\overline{S}$, we denote the compactification of S that has circles at infinity (if $S \cong S^b_{g,p}$, then $\overline{S} \cong S^{b+p}_g$). An extension $i\colon S \to F$ is said to be *pseudo-annular* if i is obtained from a disjoint union of $\overline{S}$ and copies of an annulus and once-punctured disk by gluing each annulus A (resp. once-punctured disk D) to $\overline{S}$ along their boundaries so that a boundary component of A (resp. the boundary of D) is identified with a component of $\partial\overline{S} - S$, and that another boundary component of A is either identified with a component of $\partial\overline{S}$ or unattached. Note that any pseudo-annular extension is not admissible, because the image is not a closed subset.

The pure mapping class group of a sphere is naturally embedded in the mapping class group of its compactification as follows.

Lemma 2.6 ([6])**.** $\mathrm{PMod}(S^b_{0,p}) \times \mathbb{Z}^b \cong \mathrm{PMod}(S^{p+b}_0)$.

Every pseudo-annular extension of a sphere induces an injective homomorphism between the pure mapping class groups.

Proposition 2.7. *Let $S \to F$ be a pseudo-annular extension. If S is a sphere, then* $\mathrm{PMod}(S)$ *is embedded in* $\mathrm{PMod}(F)$.

3. Right-angled Artin subgroups in mapping class groups

For a finite simple graph Γ, we define the *right-angled Artin group* $A(\Gamma)$ to be the group with the following finite presentation:

$$A(\Gamma) := \langle V(\Gamma) \mid v_i v_j = v_j v_i \text{ if } \{v_i, v_j\} \in E(\Gamma)\rangle.$$

Here, $V(\Gamma)$ is the vertex set of Γ and $E(\Gamma)$ is the edge set of Γ. The following theorem implies that every right-angled Artin group is a subgroup of the mapping class group of some surface.

Theorem 3.1 (Koberda's embedding theorem [11]). *Let S be a surface with negative Euler characteristic, and Γ a finite graph. If $\Gamma \leq \mathcal{C}(S)$, then $A(\Gamma) \hookrightarrow \mathrm{Mod}(S)$.*

Here, $\mathcal{C}(S)$ is the 1-skeleton of the Harvey's curve complex of a surface S.
Let C_m^c denote the complement graph of the cyclic graph on m vertices.

Theorem 3.2. *$A(C_m^c)$ is embedded in $\mathrm{Mod}(S_{g,p})$ if and only if m satisfies*

$$m \leq \begin{cases} 0 & ((g,p) \in \{(0,0),(0,1),(0,2),(0,3)\}) \\ 3 & ((g,p) \in \{(0,4),(1,0),(1,1)\}) \\ 5 & ((g,p) = (1,2),(0,5)) \\ 2g+2 & (g \geq 2,\ p = 0) \\ 2g+p+1 & (g \geq 2,\ 1 \leq p \leq 2) \\ 2g+p & (\textit{otherwise}). \end{cases}$$

Theorem 3.3. *$A(C_m^c) \times \mathbb{Z}$ is embedded in $\mathrm{Mod}(S_{g,p})$ if and only if m satisfies*

$$m \leq \begin{cases} 0 & (g,p) \in \{(0,4),(1,1)\} \\ 3 & (g,p) \in \{(0,5),(1,2)\} \\ p-1 & (g = 0,\ p \geq 6) \\ p+2 & (g = 1,\ p \geq 3) \\ 2g+1 & (g \geq 2,\ p = 0) \\ 2g+p & (g \geq 2,\ p \geq 1). \end{cases}$$

Theorem 3.4 ([9]). *Suppose $n \geq 3$. $A(C_{n+1}^c) \times \mathbb{Z} \hookrightarrow PB_n$.*

The "if parts" of the above theorems can be proved by applying Koberda's embedding theorem. In order to prove the "only if parts", we use the normal form theorem, due to Kim–Koberda[10], for injective homomorphisms from right-angled Artin groups into the mapping class groups of surfaces.

Theorems 1.1 and 1.2 are consequences of the results in Sections 2 and 3.

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