# Attainments of the Bayesian information bounds for the escort distribution

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Abstract

We show some necessary and sufficient conditions for the attainment of the Borovkov-Sakhanenko and van Trees Bayesian information bounds when the underlying distribution is the escort distribution of an exponential family and the conjugate or the Jeffreys prior. In this paper we consider the case where the order of the escort distribution is a constant and the parameter of exponential family is a random variable. Some examples attaining the bounds are also given.

## 1 Introduction

Let $X$ be a random variable according to the probability density function (pdf) $g(x|\theta)(\theta \in \Theta)$ with respect to a $\sigma$-finite measure $\mu$, where $\Theta = [a, b](-\infty \leq a < b \leq \infty)$ is a parameter space of $\theta$. Let $\lambda(\theta)$ be a prior pdf of $\theta$ with respect to the Lebesgue measure. We denote the support of a function $k(\theta)$ of $\theta$ as $\mathrm{supp}(k)$. In this paper, we assume the smoothness of $g(x|\theta)$ and $\lambda(\theta)$ with respect to the parameter $\theta$. Consider the Bayesian estimation for a differentiable function $\psi(\theta)$ of $\theta$ under quadratic loss $L(\theta, a) = (a - \psi(\theta))^2$. Denote the Fisher information number of $\theta$ as

$$I(\theta) = E_\theta\left[\left\{\frac{\partial}{\partial\theta}\log g(x|\theta)\right\}^2\right] = \int \frac{\{\frac{\partial}{\partial\theta}g(x|\theta)\}^2}{g(x|\theta)}d\mu.$$

Hereafter, we will often omit the variables of the functions and denote the derivative with respect to $\theta$ of a function $k(\theta)$ by $k'(\theta)$ or $k'$.

Let $\hat{\psi} = \hat{\psi}(X)$ be an estimator of $\psi(\theta)$. Let $h$ be a differentiable function satisfying $\mathrm{supp}(h) \subset \mathrm{supp}(\lambda)$, and $h(\theta)g(x|\theta) = \psi(\theta)h(\theta)g(x|\theta) = 0$ at $\theta = a$ and $b$ for almost $x$. Borovkov [5] shows that, under the regularity assumptions, it holds

$$E[(\hat{\psi} - \psi)^2] \geq \frac{\left\{E\left[\frac{\psi' h}{\lambda}\right]\right\}^2}{E\left[\frac{h^2 I}{\lambda^2}\right] + E\left[\left(\frac{h'}{\lambda}\right)^2\right]}. \tag{1}$$

If we adopt $h = \lambda$ in (1), we have the *Van Trees inequality* (van Trees [12])

$$E[(\hat{\psi} - \psi)^2] \geq \frac{\{E[\psi']\}^2}{E[I] + \hat{I}(\lambda)}$$

with $\hat{I}(\lambda) = E[\{(\log\lambda)'\}^2]$. Moreover, if we adopt $h = \psi'\lambda/I$ in (1), we have the *Borovkov-Sakhanenko* *inequality* (Borovkov and Sakhanenko [6])

$$E[(\hat{\psi}-\psi)^2] \geq \frac{\left\{E\left[\frac{(\psi')^2}{I}\right]\right\}^2}{E\left[\frac{(\psi')^2}{I}\right]+E\left[\left\{\frac{(\psi'\lambda/I)'}{\lambda}\right\}^2\right]}.$$

Recently, Abu-Shanab and Veretennikov [2] and Koike [8] showed that the Borovkov-Sakhanenko bound is asymptotically better than the van Trees bound, and is asymptotically optimal. However, the superiority between the Borovkov-Sakhanenko and the van Trees bounds in non-asymptotical sense has been unknown so far. Thus, it is worth examining the attainments for these bounds.

Targhetta [11] obtained the necessary and sufficient condition for the attainment of the van Trees bound when the underlying distribution is an exponential family with natural parameter. Koike [9] extended the result of Targhetta [11] to the Borovkov-Sakhanenko and the van Trees bounds when underlying distribution is an exponential family and the conjugate prior, and estimand is a function of the parameter. Also, the attainments under the Jeffreys prior were also considered in [9].

The notion of escort distribution was introduced by Beck and Schlögl [6]. It is known that, to a given probability distribution, the escort distribution has the ability to scan the structure of the original probability distribution. Also, the expectation under the escort distribution, or the $q$-expectation value, plays a crucial role in the formulation of nonextensive statistical mechanics. Abe [1] showed quantitatively that it is inappropriate to use the original distribution instead of the escort distribution for calculating the expectation values of physical quantities in nonextensive statistical mechanics. Tanaka [10] discussed the meaning of the escort distribution using the $q$-Gaussian distribution.

Banno and Koike [3] considered the case where the order of the escort distribution is a random variable and the parameter of exponential family is a constant. We deal with the reverse case of [3] in this paper. We consider the necessary and sufficient conditions for the attainments of the van Trees bound (van Trees [12]) and the Borovkov-Sakhanenko bound (Borovkov and Sakhanenko [6]) when the underlying distribution is the escort distribution for an exponential family and the conjugate prior, and estimand is a function of the parameter of the exponential family. Moreover, we also consider the attainments under the Jeffreys prior. Obtained results are different from [3] since the Fisher information in these cases are different from the ones in [3].

## 2 Setup

### Exponential family and the conjugate family

Let $f(x|\theta)$ be the pdf of exponential family given by

$$f(x|\theta) = \exp\{\phi(\theta)t(x) + \beta(\theta)\} \quad (\theta \in \Theta \subset \mathbb{R})$$

with respect to a $\sigma$-finite measure $\mu$, where $\phi(\theta)$ and $\beta(\theta)$ are smooth. Then, its *escort distribution* $g(x|\theta, q)$ of order $q \neq 0$ is defined by

$$g(x|\theta,q) = \frac{f(x|\theta)^q}{\int f(x|\theta)^q d\mu} = \exp[q\phi(\theta)t(x) + \{q\beta(\theta) - m(\theta,q)\}], \tag{2}$$

where $\exp\{m(\theta, q)\} = \int f(x|\theta)^q d\mu$. We assume without loss of generality that $\phi$ and $q\beta - m$ are linearly independent. Suppose that a random variable $X$ is distributed according to the escort distribution $g(x|\theta, q)$. Hereafter, we assume that the parameter $(q, \theta)$ is hybrid, that is, we treat $\theta$ as random and $q$ as non-random. The Fisher information of $\theta$ in (2) is given by

$$I(\theta) = -E_\theta \left[ \frac{\partial^2}{\partial\theta^2} \log g \right] = -\phi'(\theta) \left( \frac{q\beta'(\theta) - m'(\theta)}{\phi'(\theta)} \right)'. \tag{3}$$

Suppose that $\theta$ is distributed according to a pdf $\lambda(\theta)$ with respect to the Lebesgue measure. Then, the posterior pdf $j(\theta|x, q)$ of $\theta$ is

$$j(\theta|x, q) \propto g(x|\theta, q)\lambda(\theta).$$

In this paper, we adopt two prior distributions as below.

i. *Conjugate prior distribution.* The prior pdf is given by

$$\lambda(\theta) \propto \exp[c_1\phi(\theta) + c_2\{q\beta(\theta) - m(\theta, q)\}], \tag{4}$$

where $c_1$ and $c_2$ are constants independent of $\theta$.

ii. *Jeffreys prior distribution.* The prior pdf is given by

$$\lambda(\theta) \propto \{I(\theta)\}^{1/2} = \exp\{\alpha(\theta)\}. \tag{5}$$

Although the Jeffreys prior can often be improper, in this paper, we assume that $\lambda(\theta)$ is proper, that is,

$$\int \lambda(\theta) d\theta = 1.$$

## 3 Attainment of the Borovkov-Sakhanenko bound

Koike [9] considered the attainment condition of the Borovkov-Sakhanenko bound when the underlying distribution is an exponential family with random parameter $\theta$. Banno and Koike [3] considered the attainment condition of the Borovkov-Sakhanenko bound when the underlying distribution is the escort distribution of an exponential family with non-random parameter $\theta$ and random parameter $q$. In this section, we consider the attainments for the Borovkov-Sakhanenko bound under (2) with random parameter $\theta$ and non-random parameter $q$.

The attainment of the Borovkov-Sakhanenko is equivalent to

$$\frac{\frac{\partial}{\partial\theta}\{g(x|\theta, q)\lambda(\theta)\psi'(\theta)/I(\theta)\}}{g(x|\theta, q)\lambda(\theta)} = A(\hat{\psi}(x) - \psi(\theta))$$

from the condition of the attainment of the equality in the Caucy-Schwarz inequality (Borovkov and Sakhanenko [6], Koike [7]), where $A$ is a constant independent of $x$ and $\theta$. This means

$$\log(g\lambda) \propto A\left(\hat{\psi} \int \frac{I}{\psi'} d\theta - \int \frac{\psi}{\psi'} I d\theta\right) + \log\left|\frac{I}{\psi'}\right|,$$

and hence, the posterior pdf is

$$j(\theta|x, q) \propto \exp\left[A\left(\hat{\psi} \int \frac{I}{\psi'} d\theta - \int \frac{\psi}{\psi'} I d\theta\right) + \log\left|\frac{I}{\psi'}\right|\right]. \tag{6}$$

### 3.1 Conjugate prior distribution

In this subsection, we consider the attainment condition under (2) and (4). On the other hand, from the definition of $g(x|\theta, q)$ and $\lambda(\theta)$, we have

$$j(\theta|x,q) \propto \exp[\phi(\theta)\{qt(x)+c_1\}+(c_2+1)\{q\beta(\theta)-m(\theta,q)\}]. \tag{7}$$

From (6) and (7), we obtain

$$\hat{\psi}=t(x), \quad A\int \frac{I}{\psi'}d\theta = q\phi(\theta), \tag{8}$$

and

$$-A\int \frac{\psi}{\psi'}Id\theta + \log\left|\frac{I}{\psi'}\right| = c_1\phi + (c_2+1)(q\beta - m) + \text{const.} \tag{9}$$

Differentiating the right equation of (8) with respect to $\theta$, we have

$$A\frac{I}{\psi'} = q\phi'. \tag{10}$$

By substituting (3) to this, we have

$$-A\left(\frac{q\beta'-m'}{\phi'}\right)' = q\psi',$$

so that

$$\psi = -\frac{A}{q}\frac{q\beta'-m'}{\phi'} + l_1, \tag{11}$$

where $l_1$ is a constant independent of $\theta$. we have, by substituting (10) and (11) to (9),

$$-l_1 q\phi + A(q\beta - m) + \log|\phi'| = c_1\phi + (c_2+1)(q\beta - m) + \text{const.} \tag{12}$$

In the next step, we deal with (12) separately, according to whether $\phi$, $q\beta - m$, and $\log|\phi'|$ are linearly independent or not.

(i) The case where $\phi$, $q\beta - m$, and $\log|\phi'|$ are linearly independent, or $\log|\phi'|$ is constant.

If $\phi'$, $q\beta' - m'$, and $\phi''/\phi'$ are linearly independent, then $A = c_2+1, l_1 = -c_1/q, \phi'' = 0$ from (12). Hence, we can express $\phi = n_1\theta + n_2$, where $n_1, n_2$ are constants. Here, without loss of generality, we may assume $\phi = \theta$. In this case, we have

$$\psi = -\frac{c_2+1}{q}(q\beta' - m') - \frac{c_1}{q}$$

from (11). Now, the Bayes estimator $\hat{\psi}_B$ of $\psi$ is given by

$$\hat{\psi}_B = E[\psi|x] = -\frac{c_2+1}{q}E[q\beta' - m'|x] - \frac{c_1}{q}.$$

By differentiating (7) with respect to $\theta$, we have

$$\frac{\partial}{\partial\theta}j(\theta|x,q) = \{(qt+c_1)+(c_2+1)(q\beta'-m')\}j(\theta|x,q).$$

Hence,

$$(c_2+1)E[q\beta'-m'|x] = [j(\theta|x,q)]_a^b - (qt+c_1)$$

and

$$\hat{\psi}_B = t - \frac{[j(\theta|x,q)]_a^b}{q}.$$

Since the attainment can be deduced only from $\hat{\psi}_B = \hat{\psi} = t$, the necessary and sufficient condition for the attainment of the Borovkov-Sakhanenko bound is

$$\hat{\psi} = t, \ \ \phi(\theta) = \theta, \ \ \psi = -\frac{c_2+1}{q}(q\beta' - m') - \frac{c_1}{q}, \ \ [j(\theta|x,q)]_a^b = 0.$$

Thus, we have the following theorem.

**Theorem 1.** Assume (2) and (4). If $\phi$, $q\beta - m$, and $\log|\phi'|$ are linearly independent, or $\log|\phi'|$ is constant, then the necessary and sufficient condition for the attainment of the Borovkov-Sakhanenko bound is

$$\phi(\theta) = \theta, \ \ \psi = -\frac{c_2+1}{q}(q\beta' - m') - \frac{c_1}{q}, \ \ \hat{\psi} = t, \ \ [j(\theta|x,q)]_a^b = 0.$$

(ii) The case when $q\beta - m, \phi$, and $\log|\phi'|$ are linearly dependent. Since $q\beta - m$ and $\phi$ are linearly independent, and $q\beta - m, \phi$ and $\log|\phi'|$ are linearly dependent, there exist $a_1$ and $a_2$, not all zero, such that $\log|\phi'| = a_1(q\beta - m) + a_2\phi$. Then, from (12), we have $A = -a_1 + c_2 + 1, l_1 = (a_2 - c_1)/q$. Hence,

$$\psi = \frac{a_1 - c_2 - 1}{q}\frac{q\beta' - m'}{\phi'} + \frac{a_2 - c_1}{q}$$

from (11). Then, we may assume, without loss of generality, $a_1 - c_2 - 1 \neq 0$. The Bayes estimator $\hat{\psi}_B$ of $\psi$ is

$$\hat{\psi}_B = E[\psi|x] = \frac{a_1 - c_2 - 1}{q}E\left[\frac{q\beta' - m'}{\phi'}\middle| x\right] + \frac{a_2 - c_1}{q}.$$

By differentiating (7) with respect to $\theta$, we have

$$\frac{\partial}{\partial\theta}j(\theta|x,q) = \{\phi'(qt + c_1) + (c_2+1)(q\beta' - m')\}j(\theta|x,q).$$

Hence,

$$(c_2+1)E\left[\frac{q\beta' - m'}{\phi'}\middle| x\right] = \int_a^b \frac{1}{\phi'}\frac{\partial j(\theta|x,q)}{\partial\theta}d\theta - (qt + c_1). \tag{13}$$

Integration by parts gives

$$\int_a^b \frac{1}{\phi'}\frac{\partial j(\theta|x,q)}{\partial\theta}d\theta = \left[\frac{j(\theta|x,q)}{\phi'}\right]_a^b + \int_a^b \frac{\phi''}{(\phi')^2}j(\theta|x,q)d\theta.$$

Since $\log|\phi'| = a_1(q\beta - m) + a_2\phi$,

$$(\log|\phi'|)' = \phi''/\phi' = a_1(q\beta' - m') + a_2\phi'.$$

Hence,

$$\frac{\phi''}{(\phi')^2} = a_1\frac{q\beta' - m'}{\phi'} + a_2$$

and

$$\int_a^b \frac{\phi''}{(\phi')^2}j(\theta|x,q)d\theta = a_1E\left[\frac{q\beta' - m'}{\phi'}\middle| x\right] + a_2.$$

So, we have

$$\int_a^b \frac{1}{\phi'}\frac{\partial j(\theta|x,q)}{\partial\theta}d\theta = \left[\frac{j(\theta|x,q)}{\phi'}\right]_a^b + a_1 E\left[\left.\frac{q\beta' - m'}{\phi'}\right| x\right] + a_2.$$

Substituting this into (13), we have

$$E\left[\left.\frac{q\beta' - m'}{\phi'}\right| x\right] = \frac{1}{c_2+1}\left[\frac{j(\theta|x,q)}{\phi'}\right]_a^b + \frac{a_1}{c_2+1}E\left[\left.\frac{q\beta' - m'}{\phi'}\right| x\right] + \frac{a_2}{c_2+1} - \frac{qt+c_1}{c_2+1}.$$

Rearranging this with respect to $E\left[\left.\frac{q\beta'-m'}{\phi'}\right| x\right]$ gives

$$E\left[\left.\frac{q\beta' - m'}{\phi'}\right| x\right] = \frac{1}{-a_1+c_2+1}\left\{\left[\frac{j(\theta|x,q)}{\phi'}\right]_a^b + (a_2 - qt - c_1)\right\}.$$

Hence,

$$\hat{\psi}_B = t - \frac{[j(\theta|x,q)]_a^b}{q}.$$

**Theorem 2.** Assume (2) and (4). If $q\beta - m$, $\phi$, and $\log|\phi'|$ are linearly dependent, then the necessary and sufficient condition for the attainment of the Borovkov-Sakhanenko bound is

$$\psi = -\frac{c_2+1}{q}(q\beta' - m') - \frac{c_1}{q}, \quad \hat{\psi} = t, \quad [j(\theta|x,q)]_a^b = 0.$$

## 3.2 Jeffreys prior distribution

In this subsection, we consider the attainment condition under (2) and (5). Then, the posterior density is given by

$$j(\theta|x,q) \propto \exp\{q\phi(\theta)t(x) + q\beta(\theta) - m(\theta,q) + \alpha(\theta)\}. \tag{14}$$

Then, by (6) and (14), we have

$$\hat{\psi}(x) = t(x), \qquad A\int \frac{I}{\psi'}d\theta = q\phi(\theta), \tag{15}$$

and

$$-A\int \frac{\psi}{\psi'}I d\theta + \log\left|\frac{I}{\psi'}\right| = q\beta(\theta) - m(\theta,q) + \alpha(\theta) + \text{const.} \tag{16}$$

Differentiating the right equation of (15) with respect to $\theta$, yields to

$$A\frac{I}{\psi'} = q\phi'. \tag{17}$$

This can be deformed to

$$\psi = -\frac{A}{q}\frac{q\beta' - m'}{\phi'} + l_1 \tag{18}$$

by (3), where $l_1$ is a constant independent of $\theta$. By substituting (17) and (18) to (16), we obtain

$$(A-1)(q\beta - m) - l_1 q\phi + \log|\phi'| - \alpha = \text{const.} \tag{19}$$

In the next step, we deal with (19) separately, according to whether $\phi, q\beta - m, \log|\phi'|$ and $\alpha$ are linearly independent or not.

(i)The case when $\phi, q\beta - m, \log|\phi'|$ and $\alpha$ are linearly independent. If $\phi', q\beta' - m', \phi''/\phi'$ and $\alpha'$ are linearly independent, then $A = 1, l_1 = 0$ from (19). Hence, we have

$$\psi = -\frac{1}{q}\frac{q\beta' - m'}{\phi'}$$

from (18). Now, the Bayes estimator $\hat{\psi}_B$ of $\psi$ is given by

$$\hat{\psi}_B = E[\psi|x] = -\frac{1}{q}E\left[\frac{q\beta' - m'}{\phi'}\middle| x\right].$$

In order to attain the bound, it is necessary to have $\hat{\psi}_B = \hat{\psi} = t$. By differentiating (14) with respect to $\theta$, we have

$$\frac{\partial}{\partial\theta}j(\theta|x, q) = (q\phi' t + q\beta' - m' + \alpha')j(\theta|x, q).$$

Thus, we obtain

$$E\left[\frac{q\beta' - m'}{\phi'}\middle| x\right] = \int_a^b \frac{1}{\phi'}\frac{\partial}{\partial\theta}j(\theta|x, q)d\theta - \int_a^b \frac{\alpha'}{\phi'}j(\theta|x, q)d\theta - qt(x).$$

Integration by parts gives

$$\int_a^b \frac{1}{\phi'}\frac{\partial}{\partial\theta}j(\theta|x, q)d\theta = \left[\frac{j(\theta|x, q)}{\phi'}\right]_a^b + \int_a^b \frac{\phi''}{(\phi')^2}j(\theta|x, q)d\theta.$$

Since $(\log|\phi'|)' = \phi''/\phi' = \alpha'$,

$$\int_a^b \frac{\phi''}{(\phi')^2}j(\theta|x, q)d\theta = \int_a^b \frac{\alpha'}{\phi'}j(\theta|x, q)d\theta$$

and

$$E\left[\frac{q\beta' - m'}{\phi'}\middle| x\right] = \left[\frac{j(\theta|x, q)}{\phi'}\right]_a^b - qt(x).$$

Hence,

$$\hat{\psi}_B = t(x) - \frac{1}{q}\left[\frac{j(\theta|x, q)}{\phi'}\right]_a^b.$$

**Theorem 3.** Assume (2) and (5). If$\phi, q\beta - m, \log|\phi'|$ and $\alpha$ are linearly independent, then the necessary and sufficient condition for the attainment of the Borovkov-Sakhanenko bound is

$$\psi = -\frac{1}{q}\frac{q\beta' - m'}{\phi'}, \quad \hat{\psi} = t, \quad [j(\theta|x, q)]_a^b = 0.$$

(ii) The case when $q\beta - m$, $\phi$ and $\log|\phi'|$ are linearly independent and $q\beta - m$, $\phi$, $\log|\phi'|$ and $\alpha$ are linearly dependent. Then, there exist $a_1, a_2$ and $a_3$, not all zero, such that $\alpha = a_1(q\beta - m) + a_2\phi + a_3\log|\phi'|$. Then, from (19), we have $A = a_1 + 1, l_1 = -a_2/q, a_3 = 1$. Hence,

$$\psi = -\frac{a_1 + 1}{q}\frac{q\beta' - m'}{\phi'} - \frac{a_2}{q}.$$

The Bayes estimator $\hat{\psi}_B$ of $\psi$ is

$$\hat{\psi}_B = E[\psi|x] = -\frac{a_1 + 1}{q}E\left[\frac{q\beta' - m'}{\phi'}\middle| x\right] - \frac{a_2}{q}.$$

Since $(\log|\phi'|)' = \phi''/\phi' = \alpha' - a_1(q\beta' - m') - a_2\phi'$,

$$E\left[\frac{\phi''}{(\phi')^2}\middle|x\right] = E\left[\frac{\alpha'}{\phi'}\middle|x\right] - a_1E\left[\frac{q\beta'-m'}{\phi'}\middle|x\right] - a_2.$$

Hence,

$$E\left[\frac{q\beta'-m'}{\phi'}\middle|x\right] = \left[\frac{j(\theta|x,q)}{\phi'}\right]_a^b - a_1E\left[\frac{q\beta'-m'}{\phi'}\middle|x\right] - a_2 - qt(x)$$

and

$$(a_1+1)E\left[\frac{q\beta'-m'}{\phi'}\middle|x\right] = \left[\frac{j(\theta|x,q)}{\phi'}\right]_a^b - a_2 - qt(x).$$

Hence,

$$\hat{\psi}_B = t(x) - \frac{1}{q}\left[\frac{j(\theta|x,q)}{\phi'}\right]_a^b.$$

**Theorem 4.** Assume (2) and (5). If $q\beta - m, \phi$ and $\log|\phi'|$ are linearly independent and $q\beta - m, \phi, \log|\phi'|$ and $\alpha$ are linearly dependent, then the necessary and sufficient condition for the attainment of the Borovkov-Sakhanenko bound is

$$\psi = -\frac{a_1+1}{q}\frac{q\beta'-m'}{\phi'} - \frac{a_2}{q}, \quad \hat{\psi} = t, \quad [j(\theta|x,q)]_a^b = 0.$$

(iii) The case when $q\beta - m, \phi$ and $\log|\phi'|$ are linearly dependent and $q\beta - m, \phi$ and $\alpha$ are linearly dependent. Then, there exist $a_1, a_2, b_1$ and $b_2$, not all zero, such that $\log|\phi'| = a_1(q\beta - m) + a_2\phi$ and $\alpha = b_1(q\beta - m) + b_2\phi$. Then, from (19), we have $A = 1 - a_1 + b_1, \ l_1 = (a_2 - b_2)/q$. Hence,

$$\psi = -\frac{1-a_1+b_1}{q}\frac{q\beta'-m'}{\phi'} + \frac{a_2-b_2}{q}$$

from (19). The Bayes estimator $\hat{\psi}_B$ of $\psi$ is

$$\hat{\psi}_B = E[\psi|x] = -\frac{1-a_1+b_1}{q}E\left[\frac{q\beta'-m'}{\phi'}\middle|x\right] + \frac{a_2-b_2}{q}.$$

Since $(\log|\phi'|)' = a_1(q\beta' - m') + a_2\phi'$ and $\alpha' = b_1(q\beta' - m') + b_2\phi'$,

$$E\left[\frac{\phi''}{(\phi')^2}\middle|x\right] = a_1E\left[\frac{q\beta'-m'}{\phi'}\middle|x\right] + a_2$$

and

$$E\left[\frac{\alpha'}{\phi'}\middle|x\right] = b_1E\left[\frac{q\beta'-m'}{\phi'}\middle|x\right] + b_2.$$

Hence,

$$E\left[\frac{q\beta'-m'}{\phi'}\middle|x\right] = \left[\frac{j(\theta|x,q)}{\phi'}\right]_a^b + a_1E\left[\frac{q\beta'-m'}{\phi'}\middle|x\right] + a_2 - b_1E\left[\frac{q\beta'-m'}{\phi'}\middle|x\right] - b_2 - qt(x).$$

Rearranging this with respect to $E\left[\frac{q\beta'-m'}{\phi'}\middle|x\right]$ gives

$$E\left[\frac{q\beta'-m'}{\phi'}\middle|x\right] = \frac{1}{(1-a_1+b_1)}\left\{\left[\frac{j(\theta|x,q)}{\phi'}\right]_a^b + a_2 - b_2 - qt(x)\right\}.$$

Thus,

$$\hat{\psi}_B = t(x) - \frac{1}{q}\left[\frac{j(\theta|x,q)}{\phi'}\right]_a^b.$$

**Theorem 5.** Assume (2) and (5). If $q\beta - m, \phi$ and $\log|\phi'|$ are linearly dependent and $q\beta - m, \phi$ and $\alpha$ are linearly independent, then the necessary and sufficient condition for the attainment of the Borovkov-Sakhanenko bound is

$$\psi = -\frac{1 - a_1 + b_1}{q}\frac{q\beta' - m'}{\phi'} + \frac{a_2 - b_2}{q}, \quad \hat{\psi} = t, \quad [j(\theta|x, q)]_a^b = 0.$$

## 4 Attainment of the van Trees bound

In this section, we consider the attainment of the van Trees bound.

To attain the van Trees bound is equivalent to

$$\frac{\partial}{\partial\theta}[\log\{j(\theta|x, q)\}] = A\{\hat{\psi}(x) - \psi(\theta)\},$$

where A is a constant independent of $x$ and $\theta$. This means

$$j(\theta|x, q) \propto \exp\left(A\hat{\psi}\theta - A\int\psi d\theta\right). \tag{20}$$

### 4.1 Conjugate prior distribution

At first, we consider the attainment condition under (2) and (4). Then, the posterior density of $\theta$ is (7) and (20). From (7) and (20), we have

$$\hat{\psi} = t(x), \quad A\theta = q\phi(\theta),$$

and

$$-A\int\psi d\theta = \phi c_1 + (q\beta - m)(c_2 + 1) + \text{const.}$$

we get $A = q, \phi = \theta$. Then, differentiating (4.2) with respect to $\theta$ leads to

$$\psi = -\frac{c_2 + 1}{q}(q\beta' - m') - \frac{c_1}{q}.$$

In a similar way to Subsection 3.1, it is verified that the Bayes estimator $\hat{\psi}_B = t$ can attain the van Trees bound.

**Theorem 6.** Under (2) and (4), the necessary and sufficient condition for the attainment of the van Trees bound is

$$\psi = -\frac{c_2 + 1}{q}(q\beta' - m') - \frac{c_1}{q}, \quad \hat{\psi} = t, \quad [j(\theta|x, q)]_a^b = 0.$$

### 4.2 Jeffreys prior distribution

Secondly, we consider the attainment condition under (2) and (5). Then, the posterior density of $\theta$ is (14) and (20). From (14) and (20), we have

$$\hat{\psi}(x) = t(x), \quad A\theta = q\phi,$$

$$-A\int\psi d\theta = q\beta - m + \alpha + \text{const.} \tag{21}$$

Without loss of generality, we may assume $A = q$. Then, differentiating (21) with regard to $\theta$ leads to

$$-q\psi = q\beta' - m' + \alpha'.$$

Then, since $\phi = \theta$ and $I = -(q\beta - m)''$,

$$\alpha = \frac{\log\{-(q\beta - m)''\}}{2}.$$

Thus, we get

$$\psi = -\frac{1}{q}(q\beta - m)' - \frac{1}{2q}\frac{(q\beta - m)'''}{(q\beta - m)''}.$$

Moreover, for the posterior density (14), since

$$\frac{\partial}{\partial\theta} j(\theta|x, q) = \{qt + (q\beta - m)' + \alpha'\} j(\theta|x, q),$$

the Bayes estimator $\hat{\psi}_B$ of $\psi$ is

$$\hat{\psi}_B = E[\psi|x] = -\frac{E[q\beta' - m'|x]}{q} - \frac{E[\alpha'|x]}{q} = t - \frac{[j(\theta|x, q)]_a^b}{q}.$$

To attain the bound, it is necessary to have $\hat{\psi}_B = \hat{\psi} = t$. Therefore, we have the following theorem.

**Theorem 7.** Under (2) and (5), the necessary and sufficient condition for the attainment of the van Trees bound is

$$\psi = \frac{1}{q}(q\beta - m)' - \frac{1}{2q}\frac{(q\beta - m)'''}{(q\beta - m)''}, \quad \hat{\psi} = t, \quad [j(\theta|x, q)]_a^b = 0.$$

## 5 Examples

In this section, we show two examples attaining the bounds.

**Example 1.** (Exponential distribution) Suppose that a random variable $X$ given $\theta$ is distributed according to the Exponential distribution Exp($\theta$). Then, the pdf of $X$ given $\theta$ is

$$f(x|\theta) = \theta\exp\{-\theta x\} = \exp\{-\theta x + \log\theta\}. \quad (\theta > 0)$$

Hence, $\phi(\theta) = \theta,\ t(x) = -x,\ \beta(\theta) = \log\theta$ and the pdf of the escort distribution is

$$g(x|q, \theta) = (q\theta)\exp\{-(q\theta)x\} = \exp\{-q\theta x + \log q\theta\}. \quad (q > 0)$$

So, $m(q, \theta) = (q - 1)\log\theta - \log q$, $q\beta - m = \log\theta + \log q$ and $(q\beta - m)'' = -1/q^2$. If the prior distribution of $\theta$ is the conjugate one (4), then the pdf of $q$ is

$$\lambda(\theta) \propto \exp\{c_1\theta + c_2(\log\theta + \log q)\} \propto \exp\{-b\theta + (a - 1)\log\theta\}$$

with $a = c_2 + 1$, $b = -c_1$. Note that the assumption $a > 2$ and $b > 0$ is necessary so that the prior is proper and the second order moment of $\theta$ exists. Since $\phi$, $q\beta - m$ and $\log|\phi'|$ are linearly independent and $[j(\theta|q, x)]_a^b = 0$, the necessary and sufficient condition for the attainments of the Borovkov-Sakhanenko and the van Trees bound is

$$\psi = -\frac{a}{q}\frac{1}{\theta} + \frac{b}{q}, \quad \hat{\psi} = -x$$

from Theorems 1 and 3. In fact, by simple calculation, it can be verified

$$E[(\hat{\psi}-\psi)^2] = \frac{1}{q^2}\frac{ab^2}{(a-1)(a-2)},$$

which is equal to the Borovkov-Sakhanenko and the van Trees bounds. In this model, since the Jeffreys prior is improper, we do not deal with it here.

**Example 2.** (Bernoulli trial, logit model) Suppose that a random variable $X$ given $p$ is distributed according to the Bernoulli distribution Ber($p$). Then, the pdf of $X$ given $p$ is

$$f_0(x|p) = p^x(1-p)^{1-x} = \exp\left\{x\log\frac{p}{1-p} + \log(1-p)\right\}$$
$$(0 < p < 1,\ p \neq 1/2 \text{ and } x = 0, 1). \tag{22}$$

In this example, we consider the logit parameter, that is, $\theta = \log\{p/(1-p)\}$ as the natural parameter. In this model, we have $p = e^{\theta}/(1+e^{\theta})$, and (22) can be expressed as

$$f(x|\theta) = \exp\{x\theta - \log(1+e^{\theta})\}.$$

Hence, $\phi(\theta) = \theta, t(x) = x, \beta(\theta) = -\log(1+e^{\theta})$ and its escort distribution is

$$g(x|q,\theta) = \exp\{(q\theta)x - \log(1+e^{q\theta})\}\ \ (-\infty < q < \infty).$$

So, $m(q,\theta) = \log(1+e^{q\theta}) - q\log(1+e^{\theta})$, $q\beta - m = -\log(1+e^{q\theta})$ and $(q\beta - m)'' = -q^2e^{q\theta}/(1+e^{q\theta})^2$. If the prior density of $\theta$ is the conjugate one (4), then

$$\lambda(\theta) \propto \exp\left[c_1\theta + c_2\{-\log(1+e^{q\theta})\}\right] \propto \frac{e^{aq\theta}}{(1+e^{q\theta})^{a+b}}$$

with $a = c_1/q > 0$ and $b = c_2 - a > 0$. Then, the necessary and sufficient condition for the attainment of the Borovkov-Sakhanenko and the van Trees bound is

$$\psi = (a+b+1)\frac{e^{q\theta}}{1+e^{q\theta}} - a,\ \ \hat{\psi} = x$$

from Theorems 1 and 3. Thus, it can be calculated

$$E\left[(\hat{\psi}-\psi)^2\right] = \frac{ab}{a+b},$$

which is equal to the Borovkov-Sakhanenko and the van Trees bounds.

On the other hand, the Jeffreys prior density equals

$$\lambda(\theta) = \frac{|q|}{\pi}\frac{e^{\frac{1}{2}q\theta}}{1+e^{q\theta}},$$

which is proper. Then,

$$\begin{aligned}\exp\{\alpha(\theta)\} &\propto \exp\left\{-\log(1+e^{q\theta}) + \frac{1}{2}q\theta\right\}\\ &= \exp\left[a_1\left\{-\log\left(1+e^{q\theta}\right)\right\} + a_2\theta\right],\end{aligned}$$

that is, $a_1 = 1, a_2 = q/2$. Hence, the necessary and sufficient condition for the attainment of the Borovkov-Sakhanenko and the van Trees bounds is

$$\psi = \frac{2e^{q\theta}}{1+e^{q\theta}} - \frac{1}{2}, \quad \hat{\psi} = x$$

from Theorems 4 and 7. We can verify

$$E\left[\left(\hat{\psi} - \psi\right)^2\right] = \frac{1}{4},$$

which is equal to the Borovkov-Sakhanenko and the van Trees bounds.

## Acknowledgement

This research was supported by JSPS KAKENHI, grant numbers JP20K11702. This work was also supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

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