Construction of p-energy and associated energy measures on the Sierpiński carpet

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Abstract. We establish the existence of a scaling limit $\mathcal{E}_p$ of discrete p-energies on the graphs approximating the planar Sierpiński carpet for $p > \dim_{\mathrm{ARC}}(\mathsf{SC})$, where $\dim_{\mathrm{ARC}}(\mathsf{SC})$ is the Ahlfors regular conformal dimension of the Sierpiński carpet. Furthermore, the function space $\mathcal{F}_p$ defined as the collection of functions with finite p-energies is shown to be a reflexive and separable Banach space that is dense in the set of continuous functions with respect to the supremum norm. In particular, $(\mathcal{E}_2, \mathcal{F}_2)$ recovers the canonical regular Dirichlet form constructed by Barlow and Bass [5] or Kusuoka and Zhou [47]. We also provide $\mathcal{E}_p$-energy measures associated with the constructed p-energy and investigate its basic properties like self-similarity and chain rule.

Contents

1 Introduction

On Euclidean spaces, the *nonlinear potential theory* is built on the theory of the $(1, p)$-Sobolev spaces $W^{1,p}$ and the p-energy $\int |\nabla f|^p \, dx$. The main aim of this paper is to construct and study p-energies on the planar Sierpinski carpet as a prototype of nonlinear potential theory on complicated metric spaces like "fractals" (see also [35, Problem 7.6]). There has been significant progress on "analysis and probability" on complicated spaces beyond Euclidean spaces over the last several decades. The earlier works are the constructions of diffusion processes, which is called the *Brownian motions*, on self-similar sets in 1980s and 1990s. (For details and precise history of "analysis on fractals", see the ICM survey of Kumagai [45] for example.) In particular, the planar Sierpiński carpet (see Figure 1), SC for short, is one of the successful examples. The Brownian motions on the SC was constructed by Barlow and Bass in [5], where they obtained the Brownian motion as a scaling limit of Brownian motions on Euclidean regions approximating the SC. From an analytic viewpoint, the result of Barlow and Bass gives 2-energy $\mathcal{E}_2$ and the associated $(1, 2)$-"Sobolev" space $\mathcal{F}_2$, namely *regular Dirichlet form* on the SC. Recall that a tuple of 2-energy $\int |\nabla f|^2 \, dx$ (on $L^2(\mathbb{R}^N, dx)$) and $(1, 2)$-Sobolev space $W^{1,2}$ is a typical example of regular Dirichlet forms, which corresponds to the classical Brownian motion on $\mathbb{R}^N$. Although it is difficult to define the gradient ∇f on the SC, we can say that a suitable 2-energy "$\int |\nabla f|^2 \, dx$" exists on the SC. Later, Kusuoka and Zhou [47] gave an alternative construction of a regular Dirichlet form as a scaling limit of discrete 2-energies on a series of graphs approximating the SC as shown in Figure 2. Our work gives a "canonical" construction of p-energy $\mathcal{E}_p$ and the associated $(1, p)$-"Sobolev" space $\mathcal{F}_p$ on the SC, which play the same roles as the pair of $\int |\nabla f|^p \, dx$ and the Sobolev space $W^{1,p}$, by extending and simplifying the method of Kusuoka and Zhou.

Let us describe briefly our strategy to construct $(\mathcal{E}_p, \mathcal{F}_p)$ on the SC. We write (K, d, μ) to denote the SC as a metric measure space, that is, K is the Sierpiński carpet, d is the Euclidean metric of $\mathbb{R}^2$ and μ is the $\dim_{\mathrm{H}}(K, d)$-dimensional Hausdorff measure on (K, d), where $\dim_{\mathrm{H}}(K, d) = \log 8/\log 3$ is the Hausdorff dimension of (K, d). Let $\{G_n\}_{n\geq 1}$ be a series of finite graphs approximating the SC whose edge set is denoted by E_n (see Figure 2

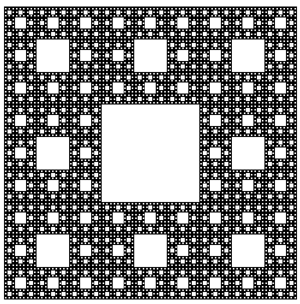

Figure 1: The planar Sierpiński carpet

Figure 2: Graphical approximation $\{G_n\}_{n\geq 1}$ of the SC (This figure draws G_1 and G_2 in blue)

and Definition 2.9). Then discrete p-energy $\mathcal{E}_p^{G_n}$ on G_n is

$$\mathcal{E}_p^{G_n}(f) = \frac{1}{2}\sum_{(x,y)\in E_n} |M_n f(x) - M_n f(y)|^p ,$$

where M_n is a discretization of $f \in L^p(K,\mu)$ to $\mathbb{R}^{G_n}$ (see Section 2 for its definition). To obtain an appropriate non-trivial limit of discrete p-energies, some renormalization is necessary (see [4] for example). We will see that the behavior of $\mathcal{R}_p^{(n)}$ defined as

$$\mathcal{R}_p^{(n)} := \left(\inf\left\{\mathcal{E}_p^{G_n}(M_n f) \;\middle|\; \begin{array}{c} f \in L^p(K,\mu) \text{ with } M_n f \equiv 0 \text{ on the left side of } G_n \\ \text{and } M_n f \equiv 1 \text{ on the right side} \end{array}\right\}\right)^{-1}$$

gives us the proper renormalization constant of discrete p-energies $\mathcal{E}_p^{G_n}$. In fact, for $p = 2$, Barlow and Bass [6] have proved that there exist constants $\rho_2 > 0$ (the so-called *resistance scaling factor*) and $C \geq 1$ such that

$$C^{-1}\rho_2^n \leq \mathcal{R}_2^{(n)} \leq C\rho_2^n, \quad n \in \mathbb{N}. \tag{1.1}$$

What Kusuoka and Zhou have shown is that, roughly speaking, the Dirichlet form $(\mathcal{E}_2, \mathcal{F}_2)$ on the SC is obtained as

$$\mathcal{F}_2 = \left\{ f \in L^2(K,\mu) \;\middle|\; \sup_{n\geq 1} \rho_2^n \mathcal{E}_2^{G_n}(M_n f) < \infty \right\}$$

and $\mathcal{E}_2(f) = \lim_{k\to\infty} \mathcal{E}_2^{G_{n_k}}(M_{n_k} f)$ for some subsequence $\{n_k\}_{k\geq 1}$.

By using p-combinatorial modulus, which is one of fundamental tools in "quasiconformal geometry", Bourdon and Kleiner [14] have generalized (1.1), i.e. they have ensured the existence of a constant $\rho_p > 0$ such that

$$C^{-1}\rho_p^n \leq \mathcal{R}_p^{(n)} \leq C\rho_p^n, \quad n \in \mathbb{N}. \tag{1.2}$$

Then our $(1,p)$-"Sobolev" space $\mathcal{F}_p$ equipped with the norm $\|\cdot\|_{\mathcal{F}_p}$ is defined by

$$\mathcal{F}_p = \left\{ f \in L^p(K,\mu) \;\middle|\; \sup_{n\geq 1} \rho_p^n \mathcal{E}_p^{G_n}(M_n f) < \infty \right\},$$

and

$$\|f\|_{\mathcal{F}_p} = \|f\|_{L^p} + \left(\sup_{n\geq 1} \rho_p^n \mathcal{E}_p^{G_n}(M_n f)\right)^{1/p}.$$

Under the following assumption (see Assumption 4.2):

$$p \text{ satisfies } p > 1 \text{ and } \rho_p > 1, \tag{1.3}$$

we will prove that $\mathcal{F}_p$ is continuously embedded in the Hölder space:

$$C^{0,\theta_p} = \left\{ f \colon K \to \mathbb{R} \;\middle|\; \sup_{x\neq y\in K} \frac{|f(x) - f(y)|}{d(x,y)^{\theta_p}} < \infty \right\},$$

where $\theta_p \coloneqq \log \rho_p / p \log 3$ (Theorem 5.1). This embedding result is very powerful. Indeed, we will deduce the *closedness*, i.e. $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ is a Banach space, and the *regularity*, i.e. $\mathcal{F}_p$ is dense in $C(K) = \{f \colon K \to \mathbb{R} \mid f \text{ is continuous}\}$ with the sup norm, from this embedding (see Theorems 5.2 and 5.5).

Moreover, the *separability* of $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ will be deduced from the *reflexivity* of $\mathcal{F}_p$ (Theorems 5.9 and 5.10). Thanks to the separability, one easily sees that, by the diagonal procedure, a subsequential limit $\lim_{k\to\infty} \rho_p^{n_k} \mathcal{E}_p^{G_{n_k}}(M_{n_k} f)$ exists for all $f \in \mathcal{F}_p$. Our final object $\mathcal{E}_p$ called the p-energy on the SC will be constructed through these subsequential limits[1].

The assumption (1.3) is essential for the continuous embedding of $\mathcal{F}_p$ in the Hölder space C^{0,θ_p} and has a close connection with the *Ahlfors regular conformal dimension* $\dim_{\mathrm{ARC}}(K,d)$ which is defined by

$$\dim_{\mathrm{ARC}}(K,d) = \inf\left\{ \alpha \;\middle|\; \begin{array}{c} \text{there exists a metric } \rho \text{ on } K \text{ which is} \\ \text{quasisymmetric to } d \text{ and } \alpha\text{-Ahlfors regular} \end{array} \right\}. \tag{1.4}$$

(For the precise definitions of *Ahlfors regularity* and being *quasisymmetric*, see (2.1) and Definition 4.1.) Indeed, by results of Carrasco Piaggio [18] and Kigami [41], the condition (1.3) is equivalent to

$$p > \dim_{\mathrm{ARC}}(K,d). \tag{1.5}$$

We expect that this condition (1.5) represents a "low-dimensional" phase. More precisely, we regard the Hölder embedding $\mathcal{F}_p \hookrightarrow C^{0,\theta_p}$ as a generalization of the classical Sobolev embedding (a consequence of Morrey's inequality). For this reason, we naturally arrive at the following conjecture.

Conjecture 1.1. $\dim_{\mathrm{ARC}}(K,d) = \inf\{p \mid \mathcal{F}_p$ *is embedded in a subset of* $C(K).\}$.

[1] To construct "canonical" p-energy $\mathcal{E}_p$ on the SC, we need to follow some additional procedures as shown in the work of Kusuoka and Zhou. In particular, we introduce new graphs $\{\mathbb{G}_n\}_{n\geq 1}$ and consider discrete p-energies on them. These procedures are described in Section 6. See Theorem 2.20 for the meaning of canonical p-energy.

To show this conjecture, what we need is the regularity of $\mathcal{F}_p$, i.e. the density of $\mathcal{F}_p \cap C(K)$ in $C(K)$ with the sup norm, for $p \leq \dim_{\mathrm{ARC}}(K, d)$ (see [7] for $p = 2$). This is a big open problem for future work.

Besides our "Sobolev spaces" $\mathcal{F}_p$, there has already been an established theory of "Sobolev spaces on metric measure spaces" based on the notion of *upper gradients*, which is a counter part of $|\nabla f|$ introduced by Heinonen and Koskela in [28]. We refer to [27, 29] for details. From the viewpoint of this theory, our $(1, p)$-"Sobolev" space $\mathcal{F}_p$ can be seen as a *fractional Korevaar–Schoen* Sobolev space. Indeed, we will give the following representation of $\mathcal{F}_p$ (Theorem 2.18):

$$\mathcal{F}_p = \left\{ f \in L^p(K, \mu) \,\middle|\, \varlimsup_{r \downarrow 0} \int_K \fint_{B_d(x,r)} \frac{|f(x) - f(y)|^p}{r^{\beta_p}} \, d\mu(y) d\mu(x) < \infty \right\}, \tag{1.6}$$

where $\beta_p = \log(8\rho_p)/\log 3$. When $p = 2$, this result is well-known (see [24, 46] for example) and the parameter β_2 is called the *walk dimension*. For detailed expositions of the walk dimension, see [24, 45, 46] for example. If $\beta_p = p$, then the expression (1.6) coincides with (a slight modification of) the Korevaar–Schoen $(1, p)$-Sobolev space [43, 44]. However, it is well-known that a strict inequality $\beta_2 > 2$ holds on the SC (see [7, Proposition 5.1] or [33]). This phenomenon suggests that the existing theory of "Sobolev spaces on metric measure spaces" do not give any non-trivial $(1, p)$-Sobolev spaces on the SC[2]. This is one of the reasons why we try to provide an alternative theory of $(1, p)$-"Sobolev" space and p-energy on the SC.

Another major objective of this paper is the $\mathcal{E}_p$-energy measures associated with p-energy $\mathcal{E}_p$. In terms of a Dirichlet form $(\mathcal{E}_2, \mathcal{F}_2)$, $\mathcal{E}_2$-energy measure of a function $f \in \mathcal{F}_2$ is defined as the unique Borel measure $\mu^2_{\langle f \rangle}$ on K such that

$$\int_K g \, d\mu^2_{\langle f \rangle} = \mathcal{E}_2(f, fg) - \frac{1}{2}\mathcal{E}_2(f^2, g), \quad g \in \mathcal{F}_2. \tag{1.7}$$

(Note that we can define the form $\mathcal{E}_2(f, g)$ by the polarization: $\mathcal{E}_2(f, g) := \frac{1}{4}\bigl(\mathcal{E}_2(f + g) - \mathcal{E}_2(f - g)\bigr)$.) This measure plays the role of $|\nabla f(x)|^2 \, dx$ if the underlying space is Euclidean. On the other hand, for any $f \in \mathcal{F}_2$ with $\mathcal{E}_2(f) \neq 0$, the $\mathcal{E}_2$-energy measure $\mu^2_{\langle f \rangle}$ and the $\log_3 8$-dimensional Hausdorff measure μ on the SC are mutually singular due to the fact that $\beta_2 > 2$ by a result of Hino [31]. See [36] for an extension of this fact to general metric measure Dirichlet spaces. This phenomenon is also far different from "smooth" settings and motivates the study of $\mathcal{E}_2$-energy measures on fractals.

For general p, due to the lack of a counterpart of the expression in the right-hand side of (1.7), we will choose to generalize Hino's alternative method of the construction of $\mathcal{E}_2$-energy measure. Namely, for any $f \in \mathcal{F}_p$, we first construct a measure $\mathfrak{m}^p_{\langle f \rangle}$ on the shift space $\{1, \dots, 8\}^{\mathbb{N}}$ associated with the SC and define our $\mathcal{E}_p$-energy measure $\mu^p_{\langle f \rangle}$ as the pushforward measure of $\mathfrak{m}^p_{\langle f \rangle}$ under the natural quotient map $\pi \colon \{1, \dots, 8\}^{\mathbb{N}} \to K$ (see

[2] It is also well-known that the *Newtonian* $(1, p)$*-Sobolev space* on the SC becomes $L^p(K, \mu)$ due to the lack of plenty rectifiable curves in the SC. See [48, Proposition 4.3.3] and [29, Proposition 7.1.33] for example.

Proposition 2.3 for a description of π), i.e. $\mu^p_{\langle f\rangle}(A) = \mathfrak{m}^p_{\langle f\rangle}(\pi^{-1}(A))$ for any Borel set A of K. Then our $\mathcal{E}_p$-energy measure $\mu^p_{\langle f\rangle}$ is associated with $\mathcal{E}_p$ in the sense that $\mu^p_{\langle f\rangle}(K) = \mathcal{E}_p(f)$ (for more details on relations between $\mu^p_{\langle f\rangle}$ and $\mathcal{E}_p$, see Theorem 2.22-(c)).

Furthermore, we will show the *chain rule*: for any $\Phi \in C^1(\mathbb{R})$ with $\Phi(0) = 0$,

$$\mu^p_{\langle \Phi(f)\rangle}(dx) = |\Phi'(x)|^p\, \mu^p_{\langle f\rangle}(dx). \tag{1.8}$$

When $p = 2$, the chain rule (1.8) is proved by using integral expressions of $\mathcal{E}_2$ (see [23, (3.2.12)] for example), but such representations take full advantage of the fact that $p = 2$. Alternatively, we prove (1.8) by introducing a new series of graphs $\{\mathbb{G}_n\}_{n\geq 1}$ (see the beginning of subsection 6.1), which is embedded in the SC, and analyzing discrete p-energies $\{\mathcal{E}_p^{\mathbb{G}_n}(\Phi(f))\}_{n\geq 1}$. This approach is actually valid since our p-energies are based on subsequential limits of $\{\rho_p^n \mathcal{E}_p^{\mathbb{G}_n}\}_{n\geq 1}$.

In the very recent paper [38], Kigami extends the construction of p-energy $\mathcal{E}_p$ in this paper to *p-conductively homogeneous* compact metric spaces (see the introduction of [38]), which is based on the theory of weighted partition (see [41, Definition 2.2.1]) and includes new construction results even if $p = 2$. However, the construction of $\mathcal{E}_p$-energy measures associated with the p-energy $\mathcal{E}_p$ is not treated in his general framework.

Outline. This paper is organized as follows. In Section 2, we prepare basic frameworks in this paper and state the main results. Sections 3 and 4 are devoted to extending results of Kusuoka and Zhou to fit our purpose. Section 3 is a collection of basic estimates of (p, p)-Poincaré constants and $\mathcal{R}_p^{(n)}$. In Section 4, we prove powerful results concerning (p, p)-Poincaré constants (uniform Hölder estimates and a condition called p-Knight Move (KM_p) for example) under Assumption 4.2 and finish all preparations to construct p-energy $\mathcal{E}_p$ and $(1, p)$-"Sobolev" space $\mathcal{F}_p$. Section 5 is devoted to investigating detailed properties of $\mathcal{F}_p$. Then, in Section 6, we introduce another graphical approximation $\{\mathbb{G}_n\}_{n\geq 1}$ and construct a canonical p-energy $\mathcal{E}_p$ (see Theorem 2.17 for the precise meaning of canonical). Section 7 is devoted to discussions on $\mathcal{E}_p$-energy measures. Finally, the appendix contains proofs of some elementary lemmas.

Notation. In this paper, we use the following notation and conventions.

(1) $\mathbb{N} := \{n \in \mathbb{Z} \mid n > 0\}$ and $\mathbb{Z}_{\geq 0} := \mathbb{N} \cup \{0\}$.

(2) We set $a \vee b := \max\{a, b\}$, $a \wedge b := \min\{a, b\}$ for $a, b \in [-\infty, \infty]$.

(3) For any countable set V, we define $\mathbb{R}^V := \{f \mid f\colon V \to \mathbb{R}\}$.

(4) For $f\colon \mathbb{R} \to \mathbb{R}$, define $\mathrm{Lip}(f) := \sup_{x\neq y\in\mathbb{R}} \frac{|f(x)-f(y)|}{|x-y|}$.

(5) Let X be a compact topological space. We set $C(X) := \{f\colon X \to \mathbb{R} \mid f \text{ is continuous}\}$ and write its sup norm by $\|f\|_{C(X)} := \sup_{x\in X} |f(x)|$.

(6) Let X be a topological space and let A be a subset of X. The topological boundary of A is denoted by ∂A, that is $\partial A := \overline{A}^X \setminus \mathrm{int}_X A$.

(7) Let (X, d) be a metric space. The open ball with center $x \in X$ and radius $r > 0$ is denoted by $B_d(x, r)$, that is,

$$B_d(x, r) := \{y \in X \mid d(x, y) < r\}.$$

If the metric d is clear in context, then we write $B(x, r)$ for short.

(8) Let K be a compact metrizable space and let $\mathcal{B}(K)$ denote the Borel σ-algebra of K. Let μ be a Borel (regular) measure on K. For any $A \in \mathcal{B}(K)$ with $\mu(A) > 0$ and $f \in L^1(K, \mu)$, we define

$$⨍_A f \, d\mu := \frac{1}{\mu(A)} \int_A f \, d\mu.$$

2 Preliminary and results

2.1 Sierpiński carpet and graphical approximations

In this paper, our target space is always the standard planar Sierpiński carpet (Figure 1). We start with its definition and standard notions. The reader is referred to [40] for further background and more general framework, namely, self-similar structure.

Definition 2.1 (The planar Sierpiński carpet)**.** Let $a = 3$ and let $N_* = 8$. Let $S := \{1, \dots, N_*\}$ and define $p_i \in \mathbb{R}^2$ by setting

$$\begin{aligned} &p_1 = (-1/2, -1/2), && p_2 = (0, -1/2), && p_3 = (1/2, -1/2), && p_4 = (1/2, 0), \\ &p_5 = (1/2, 1/2), && p_6 = (0, 1/2), && p_7 = (-1/2, 1/2), && p_8 = (-1/2, 0), \end{aligned}$$

and define $f_i \colon \mathbb{R}^2 \to \mathbb{R}^2$ by $f_i(x) := a^{-1}(x - p_i) + p_i$ for each $i \in S$. Note that f_i is an a^{-1}-similitude and p_i is the fixed point of f_i. (See Figure 3.) Let K be the self-similar set associated with $\{f_i\}_{i \in S}$, that is, the unique non-empty compact subset of $\mathbb{R}^2$ such that $K = \bigcup_{i \in S} f_i(K)$. We set $F_i := f_i|_K$ and use d to denote the Euclidean metric restricted to K. Then the triple $(K, S, \{F_i\}_{i \in S})$ is called the *(standard) planar Sierpiński carpet*.

Throughout this paper, $(K, S, \{F_i\}_{i \in S})$ is the planar Sierpiński carpet.

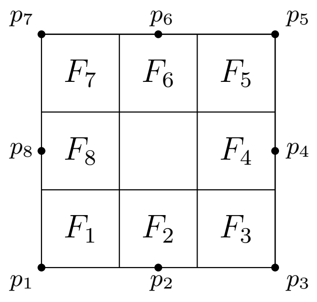

Figure 3: The similitudes $\{F_i\}_{i \in S}$ and fixed points $\{p_i\}_{i \in S}$

Definition 2.2 (Word and shift spaces).

(1) We set $W_m := S^m = \{w_1 \cdots w_m \mid w_i \in S \text{ for } i \in \{1, \dots, m\}\}$ for $m \in \mathbb{N}$ and $W_\# := \bigcup_{m=1}^{\infty} W_m$. For $w = w_1 \cdots w_m \in W_\#$, the unique $m \in \mathbb{N}$ with $w \in W_m$ is denoted by $|w|$ and set $F_w := F_{w_1} \circ \cdots \circ F_{w_m}$, $K_w := F_w(K)$, $O_w := K_w \setminus \bigcup_{v \in W_m; v \neq w}(K_w \cap K_v)$, and $[w]_n := w_1 \cdots w_n$ for $n \in \{1, \dots, m\}$. We define $W_0 := \{\emptyset\}$ and $[w]_0 := \emptyset$, where $\emptyset$ is an empty word. We also set $i^m := i \cdots i \in W_m$ for each $i \in S$. For $n \in \mathbb{N}$ and non-empty subset A of W_n, we define $A \cdot W_m$ by setting

$$A \cdot W_m = \{v_1 \cdots v_n w_1 \cdots w_m \mid v_1 \cdots v_n \in A, w_1 \cdots w_m \in W_m\}.$$

When $A = \{v\}$ for some $v \in W_m$, we write $v \cdot W_m$ to denote $\{v\} \cdot W_m$ for simplicity.

(2) The collection of one-sided infinite sequences of symbols S is denoted by Σ, that is,

$$\Sigma = \{\omega = \omega_1\omega_2\omega_3 \cdots \mid \omega_i \in S \text{ for any } i \in \mathbb{N}\},$$

which is called the *one-sided shift space* of symbols S. We define the *shift map* $\sigma : \Sigma \to \Sigma$ by $\sigma(\omega_1\omega_2\cdots) = \omega_2\omega_3\cdots$ for each $\omega_1\omega_2\cdots \in \Sigma$. The branches of σ are denoted by σ_i $(i \in S)$, namely $\sigma_i : \Sigma \to \Sigma$ is defined as $\sigma_i(\omega_1\omega_2\cdots) = i\omega_1\omega_2\cdots$ for each $i \in S$ and $\omega_1\omega_2\cdots \in \Sigma$. For $w = w_1 \cdots w_m \in W_\#$, we write $\sigma_w = \sigma_{w_1} \circ \cdots \circ \sigma_{w_m}$ and $\Sigma_w := \sigma_w(\Sigma)$. For $\omega = \omega_1\omega_2\cdots \in \Sigma$ and $m \in \mathbb{Z}_{\geq 0}$, we define $[\omega]_m = \omega_1 \cdots \omega_m \in W_m$.

We consider Σ as a topological space equipped with the product topology of $S^{\mathbb{N}}$. Then the following fact is elemental (see [40, Theorem 1.2.3]).

Proposition 2.3. *For any $\omega = \omega_1\omega_2\cdots \in \Sigma$, the set $\bigcap_{m\geq 1} K_{[\omega]_m}$ contains only one point. If we define $\pi\colon \Sigma \to K$ by $\{\pi(\omega)\} = \bigcap_{m\geq 1} K_{[\omega]_m}$, then π is a continuous surjective map. Furthermore, it holds that $\pi \circ \sigma_i = F_i \circ \pi$ for each $i \in S$.*

Let μ be the self-similar probability measure on K with weight $(1/N_*, \dots, 1/N_*)$, namely μ is the unique Borel probability measure on K such that $\mu = N_*(\mu \circ F_i)$ for any $i \in S$. It is known that μ is a constant multiple of the α-dimensional Hausdorff measure, where $\alpha := \log N_* / \log a$ is the Hausdorff dimension of K with respect to the metric d (see [40, Theorem 1.5.7] for example). Furthermore, d is *α-Ahlfors regular*, that is, there exists a constant $C_{\mathrm{AR}} \geq 1$ such that

$$C_{\mathrm{AR}}^{-1} r^{\alpha} \leq \mu(B(x, r)) \leq C_{\mathrm{AR}} r^{\alpha}, \tag{2.1}$$

for any $x \in K$ and $r \in (0, \operatorname{diam} K)$. The following two lemmas on self-similar measure μ are standard (see [33, Lemmas 3.2 and 3.3] for example).

Lemma 2.4. *Let $v, w \in W_\#$ satisfy $|v| = |w|$ and $v \neq w$. Then $\mu(K_v \cap K_w) = 0$ and $\mu(K_v) = N_*^{-|v|}$.*

Lemma 2.5. *Let $w \in W_\#$ and let $f\colon K \to [-\infty, \infty]$ be Borel measurable. Then*

$$\int_K |f \circ F_w| \, d\mu = N_*^{|w|} \int_{K_w} |f| \, d\mu \quad \text{and} \quad \int_{K_w} \left|f \circ F_w^{-1}\right| d\mu = N_*^{-|w|} \int_K |f| \, d\mu.$$

Now, we define some operators that are frequently used in this paper.

Definition 2.6. Let $p \in [1, \infty)$. For $w \in W_{\#}$, we define F_w^*, $(F_w)_*\colon L^p(K, \mu) \to L^p(K, \mu)$ by setting

$$F_w^* f \coloneqq f \circ F_w, \qquad (F_w)_* f \coloneqq \begin{cases} f \circ F_w^{-1} & \text{on } K_w, \\ 0 & \text{on } K \setminus K_w, \end{cases}$$

for each $f \in L^p(K, \mu)$. For $n \in \mathbb{N}$, define $M_n\colon L^p(K, \mu) \to \mathbb{R}^{W_n}$ by setting

$$M_n f(w) \coloneqq \fint_{K_w} f \, d\mu = N_*^n \int_{K_w} f \, d\mu, \quad w \in W_n,$$

for each $f \in L^p(K, \mu)$.

Note that, from Lemma 2.5, $M_n f(w) = \int_K F_w^* f \, d\mu$ for any $f \in L^p(K, \mu)$, which implies that, for $m \in \mathbb{N}$ and $v \in W_m$,

$$M_n(F_v^* f)(w) = \int_K F_w^*(F_v^* f) \, d\mu = \int_K F_{vw}^* f \, d\mu = M_{n+m} f(vw). \tag{2.2}$$

Next, we observe important geometric properties of the Sierpiński carpet. Define

$$n(x, y) \coloneqq \max \left\{ m \in \mathbb{Z}_{\geq 0} \,\middle|\, \begin{array}{c} \text{there exist } v, w \in W_m \text{ such that} \\ x \in K_v,\, y \in K_w \text{ and } K_v \cap K_w \neq \emptyset \end{array} \right\},$$

where we set $K_\emptyset \coloneqq K$. Then the following lemma says that the Sierpiński carpet equipped with the Euclidean metric is 1-*adapted* (see [41, Definition 2.4.1]). We refer to [41, Example 2.4.2] for its proof.

Lemma 2.7. *There exists a constant* $C_{\mathrm{AD}} \geq 1$ *such that*

$$C_{\mathrm{AD}}^{-1} \, a^{-n(x,y)} \leq d(x, y) \leq C_{\mathrm{AD}} \, a^{-n(x,y)}, \quad x, y \in K.$$

The following set $\mathrm{Sym}(K)$ describes the symmetries of the Sierpiński carpet.

Definition 2.8. We define $\mathrm{Sym}(K)$ by setting

$$\mathrm{Sym}(K) = \{I, -I, T_{\mathrm{v}}, -T_{\mathrm{v}}, T_+, -T_+, R_+, -R_+\},$$

where I is the identity matrix and

$$T_{\mathrm{v}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T_+ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad R_+ = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We also set $T_{\mathrm{h}} \coloneqq -T_{\mathrm{v}}$, $T_- \coloneqq -T_+$ and $R_- \coloneqq -R_+$. Note that $T(K) = K$ for any $T \in \mathrm{Sym}(K)$.

Finally, we introduce graphical approximations of K and related notations.

Definition 2.9. We define $G_n := (W_n, E_n)$ by setting

$$E_n := \{(v, w) \mid v, w \in W_n \text{ with } v \neq w \text{ and } K_v \cap K_w \neq \emptyset\}.$$

This series of graphs is called the horizontal network in [41]. We also define

$$\widetilde{E}_n := \{(v, w) \in E_n \mid K_v \cap K_w \text{ is a line segment}\}.$$

We use d_{G_n} to denote the graph distance of G_n. For $n, m, k \in \mathbb{N}$ and $w \in W_m$, we define a subset $\mathcal{B}_n(w, k)$ of W_{n+m} by setting

$$\mathcal{B}_n(w, k) := \bigcup_{v \in W_m; d_{G_m}(v,w) \leq k} v \cdot W_n.$$

A boundary $\partial_* G_n$ of the graph G_n is the set of words that the associated n-cells intersect with the topological boundary of $[-1/2, 1/2]^2$, that is,

$$\partial_* G_n := \{w \in W_n \mid K_w \cap \partial[-1/2, 1/2]^2 \neq \emptyset\}.$$

We conclude this subsection by observing an important fact on the degree of a series of graphs $\{G_n\}_{n \geq 1}$. Define

$$D_* := \sup_{w \in W_\#} \#\{v \in W_{|w|} \mid (v, w) \in E_{|w|}\}. \tag{2.3}$$

Then it is immediate that $D_* < \infty$ ($D_* = 7$ for the planar Sierpiński carpet).

2.2 p-energies and Poincaré constants on finite graphs

In this subsection, we review some basic results and definitions in discrete nonlinear potential theory and introduce (p, p)-Poincaré constants that will play essential roles in this paper.

Let $G = (V, E)$ be a non-directed, connected, simple finite graph, and let $p > 1$.

Definition 2.10. For $f : V \to \mathbb{R}$, we define its *p-energy* $\mathcal{E}_p^G(f)$ by setting

$$\mathcal{E}_p^G(f) := \frac{1}{2} \sum_{(x,y) \in E} |f(x) - f(y)|^p.$$

Definition 2.11. For disjoint subsets A, B of V, we define their *p-conductance* $C_p^G(A, B)$ by setting

$$C_p^G(A, B) := \inf\left\{\mathcal{E}_p^G(f) \;\middle|\; f|_A \equiv 1, f|_B \equiv 0\right\}.$$

For a given subset A of V, define

$$E^A := \{(x, y) \in E \mid x, y \in A\},$$

and

$$\mathcal{E}_p^A(f) := \frac{1}{2} \sum_{(x,y) \in E^A} |f(x) - f(y)|^p.$$

We also set

$$\overline{A} := \{x \in V \mid x \in A \text{ or } (x, y) \in E \text{ for some } y \in A\}.$$

Then the following monotonicity of p-conductance is immediate (see [50, Proposition 3.7-(2)] for example).

Proposition 2.12. *Let $A, B, A', B' \subseteq V$ with $A \subseteq A'$ and $B \subseteq B'$. Then $C_p^G(A,B) \leq C_p^G(A',B')$.*

The following property states the Markov property of discrete p-energy. (This naming is borrowed from the case $p = 2$.) This is also immediate from the definition.

Proposition 2.13. *Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ with $\mathrm{Lip}(\varphi) \leq 1$. Then $\mathcal{E}_p^G(\varphi \circ f) \leq \mathcal{E}_p^G(f)$ for any $f\colon V \to \mathbb{R}$. In particular, if we define $f^{\#} := (f \vee 0) \wedge 1$, then $\mathcal{E}_p^G(f^{\#}) \leq \mathcal{E}_p^G(f)$.*

Next we define some types of (p,p)-Poincaré constants. Let ν be a non-negative measure on V, and let $\partial_* G \subsetneq V$ be a given non-empty subset.

Definition 2.14. For a non-empty subset A of V and $f\colon A \to \mathbb{R}$, define its mean $\langle f \rangle_{A,\nu}$ by setting

$$\langle f \rangle_{A,\nu} := \frac{1}{\sum_{x \in A} \nu(x)} \sum_{x \in A} f(x)\nu(x).$$

We define $\lambda_p^{(G,\nu)}$ on (G,ν) by setting

$$\lambda_p^{(G,\nu)} := \sup\left\{ \frac{\sum_{x \in V} \left| f(x) - \langle f \rangle_{V,\nu} \right|^p \nu(x)}{\mathcal{E}_p^G(f)} \;\middle|\; f\colon V \to \mathbb{R} \text{ with } \mathcal{E}_p^G(f) \neq 0 \right\}.$$

We consider its Dirichlet boundary conditioned version $\lambda_{*,p}^{(G,\nu)}(\partial_* G)$ defined as

$$\lambda_{*,p}^{(G,\nu)}(\partial_* G) := \sup\left\{ \frac{\left|\langle f \rangle_{V,\nu}\right|^p}{\mathcal{E}_p^G(f)} \;\middle|\; f\colon V \to \mathbb{R} \text{ with } \mathcal{E}_p^G(f) \neq 0 \text{ and } f|_{\partial_* G} \equiv 0 \right\}.$$

For disjoint subsets A, B of V, we also define

$$\sigma_p^{(G,\nu)}(A,B) := \sup\left\{ \frac{\left|\langle f \rangle_{A,\nu} - \langle f \rangle_{B,\nu}\right|^p}{\mathcal{E}_p^{A \cup B}(f)} \;\middle|\; f\colon A \cup B \to \mathbb{R} \text{ with } \mathcal{E}_p^{A \cup B}(f) \neq 0 \right\}.$$

By standard arguments in calculus of variations (see [50, proof of Lemma 3.3] for example), one can easily prove the following proposition.

Proposition 2.15. *Let A, B be non-empty disjoint subsets of V, and let $\partial_* G \subsetneq V$ be non-empty.*

(1) *There exists a unique $f \in \mathbb{R}^V$ such that $f|_A \equiv 1$, $f|_B \equiv 0$ and $\mathcal{E}_p^G(f) = C_p^G(A,B)$.*

(2) *There exists a unique $f \in \mathbb{R}^V$ such that $f|_{\partial_* G} \equiv 0$, $\langle f \rangle_{V,\nu} = 1$ and $\mathcal{E}_p^G(f)^{-1} = \lambda_{*,p}^{(G,\nu)}(\partial_* G)$.*

(3) *If both A and B are connected, then there exists a unique $f \in \mathbb{R}^V$ such that $\langle f \rangle_{A,\nu} = 1$, $\langle f \rangle_{B,\nu} = 0$ and $\mathcal{E}_p^{A \cup B}(f)^{-1} = \sigma_p^{(G,\nu)}(A,B)$.*

We conclude this subsection by introducing notations of these quantities in specific settings. We mainly consider p-conductance and (p, p)-Poincaré constants on approximating graphs G_m introduced in subsection 2.1. Note that, by the self-similarity of the Sierpiński carpet, each subgraph $(w \cdot W_m, E^{w \cdot W_m})$ is a copy of G_m for any $w \in W_\#$ and $m \in \mathbb{N}$. Recall that μ denotes the self-similar probability measure on K with weight $(1/N_*, \dots, 1/N_*)$. We consider that μ is also a measure on W_n by setting $\mu(w) := \mu(K_w) = N_*^{-n}$ for each $w \in W_n$. Then, for any subset A of W_n and $f : A \to \mathbb{R}$,

$$\langle f \rangle_{A,\mu} = \frac{1}{\# A} \sum_{w \in A} f(w),$$

and thus we write $\langle f \rangle_A$ to denote $\langle f \rangle_{A,\mu}$ for simplicity. For $w \in W_\#$ and $k \in \mathbb{N}$, we define

$$C_p^{(n)} := \sup_{w \in W_\#} C_p^{G_{n+|w|}}\bigl(w \cdot W_n, W_{n+|w|} \setminus \mathcal{B}_n(w, 1)\bigr),$$

and $\mathcal{R}_p^{(n)} := \bigl(C_p^{(n)}\bigr)^{-1}$. We also set $\lambda_p^{(n)} := \lambda_p^{(G_n,\mu)}$ and $\lambda_{*,p}^{(n)} := \lambda_{*,p}^{(G_n,\mu)}(\partial_* G_n)$. Finally, for $v, w \in W_\#$ with $|v| = |w|$, define

$$\sigma_p^{(n)}(v, w) := \sigma_p^{(G_{|v|+n},\mu)}(v \cdot W_n, w \cdot W_n),$$

and

$$\sigma_p^{(n)} := \sup_{m \ge 1} \max_{(v,w) \in \widetilde{E}_m} \sigma_p^{(n)}(v, w).$$

Remark 2.16. Our definitions of Poincaré constants are slightly changed from the original definitions adopted in [47]. Indeed, $N_*^{-n}\lambda_2^{(n)}$ in our notation is the same as λ_n in [47]. The situations are the same for other Poincaré constants $\sigma_2^{(n)}, \lambda_{*,2}^{(n)}$.

2.3 Main results

Now, we are ready to state the main results of this paper. The following two theorems state detailed properties of our $(1, p)$-"Sobolev" space $\mathcal{F}_p$ on the Sierpiński carpet.

Theorem 2.17. *Let ρ_p be the constant appearing in* (1.2). *Assume that $p > \dim_{\mathrm{ARC}}(K, d)$. Then a function space $\mathcal{F}_p$ defined as*

$$\mathcal{F}_p := \left\{ f \in L^p(K, \mu) \;\middle|\; \sup_{n \ge 1} \rho_p^n \, \mathcal{E}_p^{G_n}(M_n f) < \infty \right\}$$

is a reflexive and separable Banach space equipped with a norm $\|\cdot\|_{\mathcal{F}_p}$ defined by

$$\|f\|_{\mathcal{F}_p} := \|f\|_{L^p} + \left(\sup_{n \ge 1} \rho_p^n \, \mathcal{E}_p^{G_n}(M_n f) \right)^{1/p}.$$

Moreover, $\mathcal{F}_p$ is continuously embedded in a Hölder space $C^{0,(\beta_p - \alpha)/p}$ on K, where $\beta_p := \log(N_* \rho_p)/\log a$ and

$$C^{0,(\beta_p - \alpha)/p} := \left\{ f \in C(K) \;\middle|\; \sup_{x \ne y \in K} \frac{|f(x) - f(y)|}{d(x, y)^{(\beta_p - \alpha)/p}} < \infty \right\}.$$

Furthermore, $\mathcal{F}_p$ is dense in $C(K)$ with respect to the sup norm.

Theorem 2.18 (Theorem 5.15)**.** *Assume that $p > \dim_{\mathrm{ARC}}(K, d)$. Let β_p be the same constant as in Theorem 2.17. Then $\mathcal{F}_p$ has the following expression:*

$$\mathcal{F}_p = \left\{ f \in L^p(K, \mu) \,\middle|\, \varlimsup_{r \downarrow 0} \int_K \fint_{B(x,r)} \frac{|f(x) - f(y)|^p}{r^{\beta_p}} \, d\mu(y) d\mu(x) < \infty \right\}. \tag{2.4}$$

Note that $\beta_p \geq p$ for any $p > 1$ (see Proposition 3.6). Moreover, a strict inequality $\beta_p > p$ holds (see Section 6):

Theorem 2.19. *It holds that $\rho_p > N_*^{-1} a^p$. In particular, $\beta_p > p$ for any $p > 1$.*

Next, in Section 6, we construct a "canonical" p-energy $\mathcal{E}_p$ on the Sierpiński carpet, which satisfies the following properties. For the definition of Clarkson's inequality, see Definition 5.6.

Theorem 2.20. *Assume that $p > \dim_{\mathrm{ARC}}(K, d)$ and let $\rho_p > 1$ be the same constant as in Theorem 2.17. Then there exists a functional $\mathcal{E}_p \colon \mathcal{F}_p \to [0, \infty)$ such that $\mathcal{E}_p(\,\cdot\,)^{1/p}$ is a semi-norm satisfying Clarkson's inequality and $\|\cdot\|_{\mathcal{E}_p} := \|\cdot\|_{L^p} + \mathcal{E}_p(\,\cdot\,)^{1/p}$ is equivalent to $\|\cdot\|_{\mathcal{F}_p}$. Furthermore, $(\mathcal{E}_p, \mathcal{F}_p)$ satisfies the following conditions:*

1. *$\mathbb{1}_K \in \mathcal{F}_p$, and, for $f \in \mathcal{F}_p$, $\mathcal{E}_p(f) = 0$ if and only if f is constant. Furthermore, $\mathcal{E}_p(f + a\mathbb{1}_K) = \mathcal{E}_p(f)$ for any $f \in \mathcal{F}_p$ and $a \in \mathbb{R}$;*
2. (Regularity) *$\mathcal{F}_p$ is dense in $C(K)$ with respect to the sup norm;*
3. (Markov property) *if $f \in \mathcal{F}_p$ and $\varphi \colon \mathbb{R} \to \mathbb{R}$ with $\mathrm{Lip}(\varphi) \leq 1$, then $\varphi \circ f \in \mathcal{F}_p$ and $\mathcal{E}_p(\varphi \circ f) \leq \mathcal{E}_p(f)$;*
4. (Symmetry) *if $f \in \mathcal{F}_p$ and $T \in \mathrm{Sym}(K)$, then $f \circ T \in \mathcal{F}_p$ and $\mathcal{E}_p(f \circ T) = \mathcal{E}_p(f)$;*
5. (Self-similarity) *it holds that*

$$\mathcal{F}_p = \{ f \in C(K) \mid F_i^* f \in \mathcal{F}_p \text{ for all } i \in S \} \tag{2.5}$$

and, for every $f \in \mathcal{F}_p$,

$$\mathcal{E}_p(f) = \rho_p \sum_{i \in S} \mathcal{E}_p(F_i^* f). \tag{2.6}$$

Remark 2.21. When $p = 2$, there exists a unique Dirichlet form (up to constant multiples) satisfying all conditions (1)-(5) by [8, Theorem 1.2], [32, Proposition 5.1] and [34, Proposition 5.9] [3]. This is the reason why we say that a p-energy $\mathcal{E}_p$ satisfying these conditions (1)-(5) is canonical. However, we do not know whether or not such uniqueness also holds for p-energy.

[3] To be precise, the uniqueness was proved in [8] in an alternative formulation of (1)-(5). In particular, there is no proof of the self-similarity condition (5) in [8]. The identity (2.5) was proved in [32, Proposition 5.1] and an explicit proof of (2.6) was given in [34, Proposition 5.9].

Lastly, we will introduce $\mathcal{E}_p$-energy measure $\mu^p_{\langle f\rangle}$ for $f \in \mathcal{F}_p$ and establish a few properties of it in Section 7 (Theorems 7.3, 7.4 and 7.6).

Theorem 2.22. *Assume that $p > \dim_{\mathrm{ARC}}(K, d)$. For any $f \in \mathcal{F}_p$, there exists a Borel finite measure $\mu^p_{\langle f\rangle}$ on K with $\mu^p_{\langle f\rangle}(K) = \mathcal{E}_p(f)$ satisfying the following conditions:*
(1) *if $f, g \in \mathcal{F}_p$ and $A \in \mathcal{B}(K)$ satisfy $(f - g)|_A \equiv$ const., then $\mu^p_{\langle f\rangle}(A) = \mu^p_{\langle g\rangle}(A)$;*
(2) (Chain rule) *for any $\Phi \in C^1(\mathbb{R})$, it holds that $\mu^p_{\langle \Phi(f)\rangle}(dx) = |\Phi'(x)|^p \, \mu^p_{\langle f\rangle}(dx)$;*
(3) (Self-similarity) *for any $n \in \mathbb{N}$, it holds that*

$$\mu^p_{\langle f\rangle}(dx) = \rho_p^n \sum_{w\in W_n} (F_w)_*\mu^p_{\langle F_w^* f\rangle}(dx),$$

where $(F_w)_*\mu^p_{\langle F_w^* f\rangle}(A) \coloneqq \mu^p_{\langle F_w^* f\rangle}(F_w^{-1}(A))$ for any $A \in \mathcal{B}(K)$.

As mentioned in the introduction, this measure $\mu^p_{\langle f\rangle}$ plays the role of $|\nabla f(x)|^p \; dx$ in the case of Euclidean spaces. To treat $\mathcal{E}_2$-energy measures, there are established frameworks in terms of Dirichlet forms. For further development of $\mathcal{E}_p$-energy measures, the lack of p-energy form "$\mathcal{E}_p(f, g)$" (formally written as $(f, g) \mapsto \int \langle |\nabla f|^{p-2} \nabla f, \nabla g\rangle \, dx$) is a big obstacle. This paper contains no results in this direction.

Remark 2.23. For the Sierpiński gasket, Herman, Peirone and Strichartz [30] have constructed p-energy $\mathcal{E}^{\mathrm{HPS}}_p(f)$ and Strichartz and Wong [51] have suggested an approach to interpret $\mathcal{E}^{\mathrm{HPS}}_p(f, g)$ as subderivatives of $t \mapsto \frac{1}{p}\mathcal{E}^{\mathrm{HPS}}_p(f + tg)$ at $t = 0$. The notion of p-harmonicity and p-Laplacian based on this form $\mathcal{E}^{\mathrm{HPS}}_p(f, g)$ are also considered in [51].

3 Estimates of Poincaré constants and conductances

In this section and Section 4, we investigate relations among (p, p)-Poincaré constants $\lambda^{(n)}_p$, $\lambda^{(n)}_{*,p}$, $\sigma^{(n)}_p$ and p-conductances $\mathcal{C}^{(n)}_p$ (and its reciprocal $\mathcal{R}^{(n)}_p$). Almost all parts of this section are p-energy analogs of [47, Section 2]. The ultimate goal is to show that $\lambda^{(n)}_p$, $\sigma^{(n)}_p$ and $\mathcal{R}^{(n)}_p$ are comparable without depending on the level n. In particular, the estimates $\sigma^{(n)}_p, \lambda^{(n)}_p \le C\mathcal{R}^{(n)}_p$ will be needed in later sections (especially Corollary 4.16 and 4.17). However, we need some hard preparations to this end. In the case $p = 2$, this was done in [47, Theorem 7.16] under two assumptions: [47, (B-1) and (B-2)]. The following conditions are generalizations of these assumptions to fit our p-energy context.

(B_p) There exist $k_* \in \mathbb{Z}_{\ge 0}$ and a positive constant C_* (that depends only on p and N_*) such that $\sigma^{(n)}_p \le C_*\lambda^{(n+k_*)}_{*,p}$ for every $n \in \mathbb{N}$.

(KM_p) There exists a positive constant C_{KM} such that $\lambda^{(n)}_p \le C_{\mathrm{KM}}\,\mathcal{R}^{(n)}_p$ for every $n \in \mathbb{N}$.

A proof of (B_2) for the Sierpiński carpet is given in [47, Proposition 8.1], and we also prove (B_p) for all p by a similar method to theirs. The condition (KM_p) is essential for our goals. We prove (KM_p) and show that $\lambda^{(n)}_p$, $\sigma^{(n)}_p$ and $\mathcal{R}^{(n)}_p$ are comparable in the next section (see Theorem 4.13). This section is devoted to a part of preparations toward Theorem 4.13.

Remark 3.1. Kusuoka and Zhou prove (KM_2) using the result of Barlow and Bass [5] that is called the *Knight Move argument* (see [47, Theorem 7.16]). The original Knight Move condition [47, condition (KM)] is a uniform estimate for discrete harmonic functions with some boundary conditions. We can check that a p-harmonic analog of [47, (KM)] is equivalent to (KM_p), and so we call the condition (KM_p) p-Knight Move instead. (See [17] for the case $p = 2$. We can also show this equivalence in our context, but we will not use this fact in this paper.) A recent study by Kigami reveals new important aspects of the condition (KM_p), and he names this condition the p-conductive homogeneity (see [38, Theorem 1.1 and 1.3]).

Let us start by preparing some basic facts. The following proposition is easily derived from the definition of (p, p)-Poincaré constants (for $p = 2$, see [47, Proposition 1.5]).

Proposition 3.2. *Let $n, m \in \mathbb{N}$, $w \in W_m$ and $f\colon W_{n+m} \to \mathbb{R}$.*

(1) *It holds that*

$$\sum_{v \in w \cdot W_n} \left| f(v) - \langle f \rangle_{w \cdot W_n} \right|^p \leq N_*^n \lambda_p^{(n)} \mathcal{E}_p^{w \cdot W_n}(f).$$

In particular,

$$\sum_{v \in W_n} \left| f(v) - \langle f \rangle_{W_n} \right|^p \leq N_*^n \lambda_p^{(n)} \mathcal{E}_p^{G_n}(f). \tag{3.1}$$

(2) *It holds that*

$$\left| \langle f \rangle_{w \cdot W_n} - \langle f \rangle_{W_{n+m}} \right|^p \leq N_*^m \lambda_p^{(n+m)} \mathcal{E}_p^{G_{n+m}}(f). \tag{3.2}$$

Moreover, for $w \in W_n$, $k \in \{1, \dots, n\}$ and $f\colon W_n \to \mathbb{R}$,

$$\left| \langle f \rangle_{[w]_{n-k} \cdot W_k} - \langle f \rangle_{[w]_{n-k+1} \cdot W_{k-1}} \right|^p \leq N_* \lambda_p^{(k)} \mathcal{E}_p^{[w]_{n-k} \cdot W_k}(f). \tag{3.3}$$

(3) *For any $n, m \in \mathbb{N}$, $(v, w) \in \widetilde{E}_m$ and $f\colon W_{n+m} \to \mathbb{R}$,*

$$\left| \langle f \rangle_{v \cdot W_n} - \langle f \rangle_{w \cdot W_n} \right|^p \leq \sigma_p^{(n)} \mathcal{E}_p^{\{v,w\} \cdot W_n}(f). \tag{3.4}$$

Proof. (1) This is immediate from the definition.

(2) Note that a simple computation yields that

$$\langle f \rangle_{w \cdot W_n} - \langle f \rangle_{W_{n+m}} = N_*^{-n} \sum_{v \in w \cdot W_n} (f(v) - \langle f \rangle_{W_{n+m}}).$$

Applying Hölder's inequality, we have that

$$\begin{aligned} \left| \langle f \rangle_{w \cdot W_n} - \langle f \rangle_{W_{n+m}} \right|^p &\leq N_*^{-pn} \cdot N_*^{(p-1)n} \sum_{v \in w \cdot W_n} \left| f(v) - \langle f \rangle_{W_{n+m}} \right|^p \\ &\leq N_*^m \cdot N_*^{-(n+m)} \sum_{v \in W_{n+m}} \left| f(v) - \langle f \rangle_{W_{n+m}} \right|^p \leq N_*^m \lambda_p^{(n+m)} \mathcal{E}_p^{G_{n+m}}(f), \end{aligned}$$

which proves (3.2). Lastly, by viewing $[w]_{n-k} \cdot W_k$ as a copy of W_k, we see that the estimate (3.2) becomes (3.3).

(3) It is obvious from the definition. □

For $n, m \in \mathbb{N}$, we define $P_{n+m,n} \colon \mathbb{R}^{W_{n+m}} \to \mathbb{R}^{W_n}$ by setting

$$P_{n+m,n}f(w) \coloneqq \langle f \rangle_{w \cdot W_m}, \quad w \in W_n.$$

When m is clear in the context, we abbreviate P_n to denote $P_{n+m,n}$. Note that $\langle P_{n+m,m}f \rangle_{W_n} = \langle f \rangle_{W_{n+m}}$ by a simple calculation.

While the following lemma is also immediate from the definition of $\sigma_p^{(n)}$ (for $p = 2$, see [47, Lemma 2.12]), this lemma will derive some important properties later. In particular, the weak monotonicity (Corollary 4.17) comes from this lemma.

Lemma 3.3. *For every $n, m \in \mathbb{N}$, $f \colon W_{n+m} \to \mathbb{R}$ and a subset A of W_n,*

$$\mathcal{E}_p^A(P_{n+m,n}f) \le 2^{p-1} D_* \sigma_p^{(m)} \mathcal{E}_p^{A \cdot W_m}(f). \tag{3.5}$$

Proof. If f is a constant function on W_{n+m}, then we have nothing to be proved. Let $f \colon W_{n+m} \to \mathbb{R}$ be a function that is not constant. For each $v, w \in W_n$, define a function $\tilde{f}[v,w]$ on W_{n+m} by setting $\tilde{f}[v,w] \coloneqq \mathcal{E}_p^{\{v,w\} \cdot W_m}(f)^{-1/p} \cdot f$. Now, it is a simple computation that

$$\begin{aligned}
\mathcal{E}_p^A(P_n f) &= \sum_{(v,w) \in E_n^A} \left| \langle f \rangle_{v \cdot W_m} - \langle f \rangle_{w \cdot W_m} \right|^p \\
&= \sum_{(v,w) \in E_n^A} \left| \left\langle \tilde{f}[v,w] \right\rangle_{v \cdot W_m} - \left\langle \tilde{f}[v,w] \right\rangle_{w \cdot W_m} \right|^p \mathcal{E}_p^{\{v,w\} \cdot W_m}(f) \\
&\le 2^{p-1} \sigma_p^{(m)} \sum_{(v,w) \in \widetilde{E}_n^A} \mathcal{E}_p^{\{v,w\} \cdot W_m}(f) \\
&\le 2^{p-1} D_* \sigma_p^{(m)} \mathcal{E}_p^{A \cdot W_m}(f).
\end{aligned}$$

□

The following theorem by Bourdon and Kleiner [14] describes a crucial behavior of $C_p^{(n)}$.

Theorem 3.4 ([14, Proposition 3.6 and Lemma 4.4])**.** *There exists a constant $C_{\mathrm{Mult}} \ge 1$ (depending only on p and the data of the Sierpiński carpet) such that*

$$C_{\mathrm{Mult}}^{-1} C_p^{(n)} C_p^{(m)} \le C_p^{(n+m)} \le C_{\mathrm{Mult}} C_p^{(n)} C_p^{(m)}, \tag{3.6}$$

for every $n, m \in \mathbb{N}$. In particular, the limit $\lim_{n\to\infty} \bigl(C_p^{(n)}\bigr)^{1/n} =: \rho_p^{-1} > 0$ *exists and*

$$C_{\mathrm{Mult}}^{-1} \rho_p^{-n} \le C_p^{(n)} \le C_{\mathrm{Mult}} \rho_p^{-n}, \quad \textit{for any } n \in \mathbb{N}.$$

The inequality $C_p^{(n+m)} \le C_{\mathrm{Mult}} C_p^{(n)} C_p^{(m)}$ in the above is called the *submultiplicative inequality* (first proof was essentially given in [6] for the Sierpiński carpet when $p = 2$, and [14, Proposition 3.6], [18, Lemma 3.7], [41, Lemma 4.9.3] proved for all p in some general frameworks using p-combinatorial modulus). The converse inequality is called the *supermultiplicative inequality*.

Remark 3.5. More precisely, Bourdon and Kleiner [14] have proven the multiplicative inequality for another p-conductance $\mathcal{C}_p^{(n)}(\mathrm{L} \leftrightarrow \mathrm{R})$ (its definition will be given in Section 4) by using p-combinatorial modulus. They also have shown that $\mathcal{C}_p^{(n)}$ and $\mathcal{C}_p^{(n)}(\mathrm{L} \leftrightarrow \mathrm{R})$ have the same behavior (we will prove this fact in Lemma 4.12 by a simple combinatorial argument), and thus Theorem 3.4 holds.

The constant ρ_p in the above theorem will play indispensable roles in this paper. The following proposition is an extension of [47, Proposition 2.7] and gives an estimate of ρ_p.

Proposition 3.6. *There exists a positive constant $C_{3.6}$ depending only on p, D_* and C_{AD} such that*

$$\mathcal{C}_p^{(n)} \leq C_{3.6}(N_* a^{-p})^n,$$

for every $n \in \mathbb{N}$. In particular, it holds that $\rho_p \geq N_*^{-1} a^p$.

Proof. Let $z \in W_m$ and set $A := z \cdot W_n$, $B := W_{n+m} \setminus \mathcal{B}_n(z,1)$, $K_A := \bigcup_{w \in A} K_w$, and $K_B := \bigcup_{w \in B} K_w$. Then, by Lemma 2.7, we have that $\operatorname{dist}(K_A, K_B) := \inf\{d(x,y) \mid x \in K_A, y \in K_B\} \geq C_{\mathrm{AD}}^{-1} a^{-m}$. Define a continuous function $f \colon K \to \mathbb{R}$ by setting

$$f(x) := \frac{\operatorname{dist}(x, K_A)}{\operatorname{dist}(K_A, K_B)} \wedge 1$$

for each $x \in K$, where $\operatorname{dist}(x, F) := \inf_{y \in F} d(x,y)$ for any subset F of K. Then it is immediate that $f|_{K_A} \equiv 0$ and $f|_{K_B} \equiv 1$, and thus $M_{n+m} f|_A \equiv 0$ and $M_{n+m} f|_B \equiv 1$. This yields that $\mathcal{C}_p^{G_{n+m}}(A,B) \leq \mathcal{E}_p^{G_{n+m}}(M_{n+m} f)$.

Next, we will estimate the p-energy of $M_{n+m} f$ by estimating distances. For $(v,w) \in E_{n+m}$, by the triangle inequality, we have

$$|\operatorname{dist}(F_v(x), K_A) - \operatorname{dist}(F_w(x), K_A)| \leq d(F_v(x), F_w(x)) \leq c_1 a^{-(n+m)},$$

where $c_1 := 2\operatorname{diam}(K)$. By Lemma 2.5,

$$\begin{aligned}
&|M_{n+m} f(v) - M_{n+m} f(w)| \\
&\quad = \left| \int_K (F_v^*) f \, d\mu - \int_K (F_w^*) f \, d\mu \right| \\
&\quad \leq \frac{1}{\operatorname{dist}(K_A, K_B)} \int_K |\operatorname{dist}(F_v(x), K_A) - \operatorname{dist}(F_w(x), K_A)| \, d\mu(x) \\
&\quad \leq c_1 C_{\mathrm{AD}} \, a^{-n}.
\end{aligned}$$

Consequently, we conclude that

$$\begin{aligned}
\mathcal{C}_p^{G_{n+m}}(A,B) &\leq \mathcal{E}_p^{G_{n+m}}(M_{n+m} f) \\
&\leq \sum_{w \in \mathcal{B}_n(z,1)} \sum_{v \in W_{n+m}} |M_{n+m} f(v) - M_{n+m} f(w)|^p \, \mathbb{1}_{E_{n+m}}(v,w) \\
&\leq (c_1 C_{\mathrm{AD}})^p D_* \big(\#\mathcal{B}_n(z,1)\big) a^{-pn} \\
&\leq (c_1 C_{\mathrm{AD}})^p D_* (D_* + 1) N_*^n a^{-pn}.
\end{aligned}$$

□

In the rest of this subsection, we extend estimates in [47] that hold without assuming (KM$_p$). First, we will see some relations with $\lambda_{*,p}^{(n)}$ and the other (p, p)-Poincaré constants. Since "gluing" maximizers of $\lambda_{*,p}^{(n)}$ does not increase energies, the next proposition follows (for $p = 2$, see [47, Proposition 2.11]).

Proposition 3.7. *For every $n \geq 1$, it holds that $\lambda_{*,p}^{(n)} \leq N_* \lambda_p^{(n+1)}$.*

Proof. Let $f\colon W_n \to \mathbb{R}$ satisfy $f|_{\partial_* G_n} \equiv 0$, $\mathcal{E}_p^{G_n}(f) = 1$, and $\langle f \rangle_{W_n} = \bigl(\lambda_{*,p}^{(n)}\bigr)^{1/p}$ (see Proposition 2.15-(2)). Fix $i \neq j \in S$ and define $f_*\colon W_{n+1} \to \mathbb{R}$ by

$$f_*(z) \coloneqq \begin{cases} f(w) & \text{if } z = iw \text{ for some } w \in W_n, \\ -f(w) & \text{if } z = jw \text{ for some } w \in W_n, \\ 0 & \text{otherwise.} \end{cases}$$

Since $f|_{\partial_* G_n} = 0$, we easily see that $\langle f_* \rangle_{W_{n+1}} = 0$ and $\mathcal{E}_p^{G_{n+1}}(f_*) = 2$. By Hölder's inequality,

$$\begin{aligned} N_*^{-n} \sum_{z \in W_{n+1}} \bigl|f_*(z) - \langle f_* \rangle_{W_{n+1}}\bigr|^p &= 2(\#W_n)^{-1} \sum_{w \in W_n} |f(w)|^p \\ &\geq 2(\#W_n)^{-p} \left|\sum_{w \in W_n} f(w)\right|^p = 2\bigl|\langle f \rangle_{W_n}\bigr|^p = 2\lambda_{*,p}^{(n)}. \end{aligned}$$

Moreover, from Proposition 3.2-(1), the term $N_*^{-n} \sum_{z \in W_{n+1}} \bigl|f_*(z) - \langle f_* \rangle_{W_{n+1}}\bigr|^p$ is bounded above by $N_* \lambda_p^{(n+1)} \mathcal{E}_p^{G_{n+1}}(f_*) = 2N_* \lambda_p^{(n+1)}$, and so we finish the proof. □

The condition (B$_p$) plays a converse role of the above proposition. We heavily use the symmetries of the Sierpiński carpet to prove (B$_p$).

Proposition 3.8. *There exists a positive constant $C_{3.8}$ depending only on p and N_* such that $\sigma_p^{(n)} \leq C_{3.8} \lambda_{*,p}^{(n+2)}$ for any $n \in \mathbb{N}$. In particular, (B$_p$) holds (with $k_* = 2$).*

Proof. The proof is a straightforward modification of [47, Proposition 8.1]. Let $n \in \mathbb{N}$. By the self-similarity and symmetries of the Sierpiński carpet, one easily sees that $\sigma_p^{(n)} = \sigma_p^{G_{n+1}}(1 \cdot W_n, 8 \cdot W_n)$. We define an affine transformation $T_{1,8}$ of $\mathbb{R}^2$ by setting

$$T_{1,8}\bigl((x, y)\bigr) \coloneqq \Bigl(x, -\frac{1}{3} - y\Bigr),$$

for each $(x, y) \in \mathbb{R}^2$. Note that $T_{1,8}(K_i) = K_j$ if $(i, j) = (1, 8), (8, 1)$. Now, by the symmetry, there exist $\iota_{1,\mathrm{v}} : 1 \cdot W_n \to 1 \cdot W_n$, $\iota_{1\to 8}\colon 1 \cdot W_n \to 8 \cdot W_n$, $\iota_{8,\mathrm{v}} : 8 \cdot W_n \to 8 \cdot W_n$, $\iota_{8\to 1}\colon 8 \cdot W_n \to 1 \cdot W_n$, $\iota_+\colon W_n \to W_n$, $\iota_-\colon W_n \to W_n$ such that

$$\begin{aligned} K_{\iota_{1,\mathrm{v}}(w)} &= F_1(T_{\mathrm{v}}(F_1^{-1}(K_w))) \quad \text{and} \quad K_{\iota_{1\to 8}(w)} = T_{1,8}(K_w) && \text{for any } w \in 1 \cdot W_n, \\ K_{\iota_{8,\mathrm{v}}(w)} &= F_8(T_{\mathrm{v}}(F_8^{-1}(K_w))) \quad \text{and} \quad K_{\iota_{8\to 1}(w)} = T_{8,1}(K_w) && \text{for any } w \in 8 \cdot W_n, \\ K_{\iota_+(w)} &= T_+(K_w) \quad \text{and} \quad K_{\iota_-(w)} = T_-(K_w) && \text{for any } w \in W_n. \end{aligned}$$

From Proposition 2.15, there exists $f\colon \{1,8\}\cdot W_n \to \mathbb{R}$ with $\mathcal{E}_p^{\{1,8\}\cdot W_n}(f) = 1$ such that $\left|\langle f\rangle_{1\cdot W_n} - \langle f\rangle_{8\cdot W_n}\right|^p = \sigma_p^{(n)}$. Furthermore, again from Proposition 2.15, we have that $f(\iota_{i,\mathrm{h}}(w)) = f(w)$ for each $w \in i\cdot W_n$ $(i = 1,8)$ and $f(\iota_{i\to j}(w)) = -f(w)$ for each $(i,j) \in \{(1,8),(8,1)\}$, $w \in i\cdot W_n$. It will suffice to consider the case $f|_{8\cdot W_n} \geq 0$. Now, define $g_0, g_\pm\colon W_n \to \mathbb{R}$ by setting $g_0(w) \coloneqq f(8\cdot w) \geq 0$,

$$g_+(w) \coloneqq \begin{cases} g_0(T_+(w)) & \text{if } w \in W_n \text{ satisfies } K_w \cap \{(x,y)\in\mathbb{R}^2 \mid y \geq x\} \neq \emptyset, \\ g_0(w) & \text{otherwise,} \end{cases}$$

and

$$g_-(w) \coloneqq \begin{cases} g_0(T_-(w)) & \text{if } w \in W_n \text{ satisfies } K_w \cap \{(x,y)\in\mathbb{R}^2 \mid y \geq -x\} \neq \emptyset, \\ g_0(w) & \text{otherwise,} \end{cases}$$

for each $w \in W_n$. Then it is immediate that $\mathcal{E}_p^{G_n}(g_0) \leq \mathcal{E}_p^{\{1,8\}\cdot W_n}(f) = 1$ and $\langle g_0\rangle_{W_n} = \langle f\rangle_{8\cdot W_n} = \frac{1}{2}\bigl(\sigma_p^{(n)}\bigr)^{1/p}$. It is also obvious that $\mathcal{E}_p^{G_n}(g_\pm) \leq 2\mathcal{E}_p^{G_n}(g_0)$. Next, define $g\colon W_{n+2} \to \mathbb{R}$ by setting

$$g(v) \coloneqq \begin{cases} g_-(w), & \text{if } v = 25w \in 25\cdot W_n \text{ for some } w \in W_n, \\ g_0(w), & \text{if } v = 26w \in 26\cdot W_n \text{ for some } w \in W_n, \\ g_+(w), & \text{if } v = 27w \in 27\cdot W_n \text{ for some } w \in W_n, \\ 0, & \text{otherwise,} \end{cases}$$

for each $v \in W_{n+2}$. This construction yields that $g|_{\partial_* G_{n+2}} \equiv 0$. Furthermore, we see that

$$\mathcal{E}_p^{\{26,2j\}\cdot W_n}(g) = \mathcal{E}_p^{26\cdot W_n}(g) + \mathcal{E}_p^{2j\cdot W_n}(g) \quad \text{for each } j = 5,7,$$

and

$$\mathcal{E}_p^{\{24,25,37\}\cdot W_n}(g) \vee \mathcal{E}_p^{\{15,27,28\}\cdot W_n}(g) \leq 3\mathcal{E}_p^{\{1,8\}\cdot W_n}(f \vee 0).$$

Therefore, it follows that

$$\begin{aligned} \langle g\rangle_{W_{n+2}} &= N_*^{-(n+2)} \sum_{i\in\{0,+,-\}} \sum_{w\in W_n} g_i(w) = N_*^{-2} \sum_{i\in\{0,+,-\}} \langle g_i\rangle_{W_n} \\ &\geq N_*^{-2}\,\langle g_0\rangle_{W_n} = \frac{1}{2}N_*^{-2}\bigl(\sigma_p^{(n)}\bigr)^{1/p}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}_p^{G_{n+2}}(g) &\leq \mathcal{E}_p^{26\cdot W_n}(g) + \mathcal{E}_p^{\{24,25,37\}\cdot W_n}(g) + \mathcal{E}_p^{\{15,27,28\}\cdot W_n}(g) + 6 \\ &\leq 7 + 6\mathcal{E}_p^{\{1,8\}\cdot W_n}(f\vee 0) \leq 13. \end{aligned}$$

Hence, by putting $g_* \coloneqq \mathcal{E}_p^{G_{n+2}}(g)^{-1}\cdot g$, we conclude that

$$\sigma_p^{(n)} \leq (2N_*^2)^p \mathcal{E}_p^{G_{n+2}}(g)\bigl(\langle g_*\rangle_{W_{n+2}}\bigr)^p \leq 13\cdot(2N_*^2)^p \lambda_{*,p}^{(n+2)},$$

which shows (B_p), where $k_* = 2$ and $C_{3.8} = 13\cdot(2N_*^2)^p$. □

Next, we see relations between two (p, p)-Poincaré constants $\lambda_p^{(n)}$ and $\sigma_p^{(n)}$. The following proposition states that the submultiplicative inequality of $\sigma_p^{(n)}$ holds (for $p = 2$, see [47, a part of Proposition 2.13]).

Proposition 3.9. (1) *For any $n, m \in \mathbb{N}$, we have*

$$\lambda_p^{(n+m)} \leq 2^{p-1} D_*(\lambda_p^{(m)} N_*^{-n} + \lambda_p^{(n)} \sigma_p^{(m)}).$$

(2) *For any $n, m \in \mathbb{N}$, we have $\sigma_p^{(n+m)} \leq 2^{p-1} D_* \sigma_p^{(n)} \sigma_p^{(m)}$.*

Proof. (1) Let $f \colon W_{n+m} \to \mathbb{R}$ with $\mathcal{E}_p^{G_{n+m}}(f) = 1$. Then we see from Proposition 3.2-(1) and Lemma 3.3 that

$$\begin{aligned}
&N_*^{-(n+m)} \sum_{v \in W_n} \sum_{w \in v \cdot W_m} \left| f(w) - \langle f \rangle_{W_{n+m}} \right|^p \\
&\leq 2^{p-1} N_*^{-(n+m)} \sum_{v \in W_n} \sum_{w \in v \cdot W_m} \left(\left| f(w) - \langle f \rangle_{v \cdot W_m} \right|^p + \left| \langle f \rangle_{v \cdot W_m} - \left\langle P_{n+m,n} f \right\rangle_{W_n} \right|^p \right) \\
&\leq 2^{p-1} \lambda_p^{(m)} N_*^{-n} + 2^{p-1} N_*^{-n} \sum_{v \in W_n} \left| P_{n+m,n} f(v) - \left\langle P_{n+m,n} f \right\rangle_{W_n} \right|^p \\
&\leq 2^{p-1} D_* \big(\lambda_p^{(m)} N_*^{-n} + \lambda_p^{(n)} \sigma_p^{(m)} \big),
\end{aligned}$$

where we used the following estimate in the second inequality:

$$N_*^{-(n+m)} \sum_{v \in W_n} \sum_{w \in v \cdot W_m} \left| f(w) - \langle f \rangle_{v \cdot W_m} \right|^p \leq \lambda_p^{(m)} N_*^{-n} \sum_{v \in W_n} \mathcal{E}_p^{v \cdot W_m}(f) \leq \lambda_p^{(m)} N_*^{-n}.$$

Since f with $\mathcal{E}_p^{G_{n+m}}(f) = 1$ is arbitrary, we obtain the desired estimate.

(2) Let $k \in \mathbb{N}$, let $(v, w) \in \widetilde{E}_k$ and let $f \in \mathbb{R}^{W_{n+m+k}}$ satisfy $\mathcal{E}_p^{\{v,w\} \cdot W_{n+m}}(f) = 1$. Note that $\langle f \rangle_{v \cdot W_{n+m}} = \langle P_{n+k} f \rangle_{v \cdot W_n}$, where $P_{n+k} = P_{n+m+k, n+k}$. Indeed,

$$\begin{aligned}
\langle f \rangle_{v \cdot W_{n+m}} &= N_*^{-(n+m)} \sum_{v' \in W_n} \sum_{z \in vv' \cdot W_m} f(z) = N_*^{-n} \sum_{v' \in W_n} \langle f \rangle_{vv' \cdot W_m} \\
&= N_*^{-n} \sum_{v' \in W_n} P_{n+k} f(vv') = \langle P_{n+k} f \rangle_{v \cdot W_n}.
\end{aligned}$$

Similar computation yields that $\langle f \rangle_{v \cdot W_{n+m}} = \langle P_{m+k} f \rangle_{v \cdot W_m}$. From Lemma 3.3, we have that

$$\begin{aligned}
\left| \langle f \rangle_{v \cdot W_{n+m}} - \langle f \rangle_{w \cdot W_{n+m}} \right|^p &= \left| \langle P_{n+k} f \rangle_{v \cdot W_n} - \langle P_{n+k} f \rangle_{w \cdot W_n} \right|^p \\
&\leq \sigma_p^{(n)} \mathcal{E}_p^{\{v,w\} \cdot W_n}(P_{n+k} f) \\
&\leq 2^{p-1} D_* \sigma_p^{(n)} \sigma_p^{(m)}.
\end{aligned}$$

The desired result is immediate from this estimate. □

In the rest of this subsection, we prove the following relation with (p, p)-Poincaré constant $\lambda_p^{(n)}$ and p-conductance $C_p^{(n)}$ (see [47, Proposition 2.10] for $p = 2$).

Proposition 3.10. *For every $n, m, k \in \mathbb{N}$ with $W_k \setminus \partial_* G_k \neq \emptyset$,*

$$\frac{\#(W_k \setminus \partial_* G_k)}{N_*^k \#(\partial_* G_k)} \mathcal{R}_p^{(m)} \lambda_p^{(n)} \leq C_{3.10} \lambda_p^{(n+m+k)},$$

where $C_{3.10}$ is a positive constant depending only on p and D_*.

Remark 3.11. For the Sierpiński carpet graph, it holds that $W_k \setminus \partial_* G_k \neq \emptyset$ whenever $k \geq 2$. We write $k_0 := 2$ to denote the minimum k such that $W_k \setminus \partial_* G_k \neq \emptyset$.

Similar ideas of its proof appear in many contexts (see [6, proof of Theorem 3.3], [14, proof of Proposition 3.6] for example). In the following two lemmas, we prepare estimates for "partition of unity" (for $p = 2$, see [47, Lemmas 2.8 and 2.9]).

Lemma 3.12. *Let $n, m \in \mathbb{N}$, and let $\{\varphi_w^{(m)}\}_{w \in W_n}$ be a family of $[0, 1]$-valued functions on W_{n+m} such that $\sum_{w \in W_n} \varphi_w^{(m)} \equiv 1$ and $\varphi_w^{(m)}|_{W_{n+m} \setminus \mathcal{B}_m(w,1)} \equiv 0$ for each $w \in W_n$. If $f : W_n \to \mathbb{R}$, then*

$$\mathcal{E}_p^{G_{n+m}}(f_*) \leq C_{3.12} \mathcal{E}_p^{G_n}(f) \max_{w \in W_n} \mathcal{E}_p^{G_{n+m}}(\varphi_w^{(m)})$$

where $C_{3.12}$ is a positive constant depending only on p and D_* and $f_* : W_{n+m} \to \mathbb{R}$ is defined as

$$f_*(z) := \sum_{w \in W_n} f(w) \varphi_w^{(m)}(z), \quad z \in W_{n+m}.$$

Proof. For each $z, z' \in W_{n+m}$, we set

$$A(z, z') := \left\{ w \in W_n \,\middle|\, \varphi_w^{(m)}(z) \vee \varphi_w^{(m)}(z') > 0 \right\}.$$

Since $\mathrm{supp}[\varphi_w^{(m)}] \subseteq \mathcal{B}_m(w, 1)$, we can verify that there exists $M \in \mathbb{N}$ depending only on D_* such that $\#A(z, z') \leq M$ for any $n, m \in \mathbb{N}$ and $z, z' \in W_{n+m}$. Furthermore, we see that

$$f_*(z) - f_*(z') = \sum_{w \in A(z,z')} f(w)(\varphi_w^{(m)}(z) - \varphi_w^{(m)}(z'))$$

and $\sum_{w \in A(z,z')} (\varphi_w^{(m)}(z) - \varphi_w^{(m)}(z')) = 0$. From these identities, we have that

$$\begin{aligned} \text{(3.7)} \qquad \mathcal{E}_p^{(n+m)}(f_*) &= \frac{1}{2} \sum_{w \in W_n} \sum_{z \in w \cdot W_m} \sum_{\substack{z' \in W_{n+m}; \\ (z,z') \in E_{n+m}}} |f_*(z) - f_*(z')|^p \\ &= \frac{1}{2} \sum_{w \in W_n} \sum_{z \in w \cdot W_m} \sum_{\substack{z' \in W_{n+m}; \\ (z,z') \in E_{n+m}}} \left| \sum_{v \in A(z,z')} (f(v) - f(w))(\varphi_v^{(m)}(z) - \varphi_v^{(m)}(z')) \right|^p. \end{aligned}$$

By Hölder's inequality we obtain

$$\text{(3.8)} \qquad \left| \sum_{v \in A(z,z')} (f(v) - f(w))(\varphi_v^{(m)}(z) - \varphi_v^{(m)}(z')) \right|^p$$

$$\leq \left(\sum_{v \in A(z,z')} |f(v) - f(w)|^p \right) \left(\sum_{v \in A(z,z')} \left| \varphi_v^{(m)}(z) - \varphi_v^{(m)}(z') \right|^{p/(p-1)} \right)^{p-1}.$$

To bound the term $\sum_{v \in A(z,z')} |f(v) - f(w)|^p$, for each $v \in A(z,z')$, $w \in W_n$ with $z \in w \cdot W_m$, and $(z,z') \in E_{n+m}$, we find a path $[w^1, \dots, w^l]$ in G_n from v to w with $l \leq 3$, that is, $(w^i, w^{i+1}) \in E_n$ or $w^i = w^{i+1}$ for each $i = 1, \dots, l-1$, and $w^1 = v$, $w^l = w$. Define

$$\Gamma_{\leq 3}^{(n)}(w) := \left\{ [w^1, \dots, w^l] \;\middle|\; \begin{array}{c} l \leq 3,\, w^i \in W_n,\, w^l = w, \text{ and} \\ (w^i, w^{i+1}) \in E_n \text{ for each } i = 1, \dots, l-1 \end{array} \right\} \neq \emptyset.$$

Then, for any $(z,z') \in E_{n+m}$, we see that

$$\begin{aligned}
&\sum_{v \in A(z,z')} |f(v) - f(w)|^p \\
&\leq C_1 \sum_{v \in A(z,z')} \sum_{i=1}^{l-1} \left| f(w^i) - f(w^{i+1}) \right|^p \\
&\leq C_1 \sum_{[w^1,\dots,w^l] \in \Gamma_{\leq 3}^{(n)}(w)} \sum_{i=1}^{l-1} \left| f(w^i) - f(w^{i+1}) \right|^p =: S_f(w),
\end{aligned} \tag{3.9}$$

where C_1 is a constant depending only on p, l. Note that the number $\#\Gamma_{\leq 3}^{(n)}(w)$ is bounded above by a constant depending only on D_*. Thus we conclude that there exists a constant C_2 depending only on C_1 and D_* such that $\sum_{w \in W_n} S_f(w) \leq C_2 \mathcal{E}_p^{G_n}(f)$ for any $n \in \mathbb{N}$ and $f \colon W_n \to \mathbb{R}$. Combining these estimates (3.7), (3.8) and (3.9), we obtain

$$\begin{aligned}
&\mathcal{E}_p^{G_{n+m}}(f_*) \\
&\leq \frac{1}{2} \sum_{w \in W_n} S_f(w) \sum_{z \in w \cdot W_m} \sum_{\substack{z' \in W_{n+m}; \\ (z,z') \in E_{n+m}}} \left(\sum_{v \in A(z,z')} \left| \varphi_v^{(m)}(z) - \varphi_v^{(m)}(z') \right|^{p'} \right)^{p-1} \\
&\leq M^{p-1} \sum_{w \in W_n} S_f(w) \sum_{z \in w \cdot W_m} \sum_{\substack{z' \in W_{n+m}; \\ (z,z') \in E_{n+m}}} \max_{v \in W_n} \left| \varphi_v^{(m)}(z) - \varphi_v^{(m)}(z') \right|^p \\
&\leq M^{p-1} \sum_{w \in W_n} S_f(w) \max_{v \in W_n} \mathcal{E}_p^{\overline{w \cdot W_m}}(\varphi_v^{(m)}) \\
&\leq C_2 M^{p-1} \mathcal{E}_p^{G_n}(f) \max_{v \in W_n} \mathcal{E}_p^{G_{n+m}}(\varphi_v^{(m)}).
\end{aligned}$$

□

Lemma 3.13. *Let $n, m, k \in \mathbb{N}$ with $W_k \setminus \partial_* G_k \neq \emptyset$. If $f \colon W_n \to \mathbb{R}$, then there exists a function $f_* \colon W_{n+m+k} \to \mathbb{R}$ satisfying*

$$f_*(v) = f(w) \quad \text{if } w \in W_n \text{ and } v \in ww' \cdot W_m \text{ for some } w' \in W_k \setminus \partial_* G_k, \tag{3.10}$$

and

$$\mathcal{E}_p^{G_{n+m+k}}(f_*) \leq C_{3.13} \#(\partial_* G_k) C_p^{(m)} \mathcal{E}_p^{G_n}(f), \tag{3.11}$$

where $C_{3.13}$ is a positive constant depending only on p and D_*.

Proof. For each $w \in W_{n+k}$, let $h_w^{(m)} \colon W_{n+m+k} \to \mathbb{R}$ satisfy $h_w^{(m)}|_{w\cdot W_m} \equiv 1$, $h_w^{(m)}|_{W_{n+m+k}\setminus \mathcal{B}_m(w,1)} \equiv 0$, and

$$\mathcal{E}_p^{G_{n+m+k}}\bigl(h_w^{(m)}\bigr) = C_p^{G_{n+m+k}}\bigl(w \cdot W_m, W_{n+m+k} \setminus \mathcal{B}_m(w,1)\bigr).$$

Define $\Psi := \sum_{w\in W_{n+k}} h_w^{(m)}$. Then it is obvious that $\Psi \geq 1$, and so a family $\bigl\{\varphi_w^{(m)}\bigr\}_{w\in W_{n+k}}$ given by $\varphi_w^{(m)} := \Psi^{-1} h_w^{(m)}$ satisfies the conditions in Lemma 3.12. For each $f \colon W_n \to \mathbb{R}$, define a function $f_* \colon W_{n+m+k} \to \mathbb{R}$ by setting

$$f_*(v) := \sum_{z\in W_{n+k}} f([z]_n)\varphi_z^{(m)}(v), \quad v \in W_{n+m+k}.$$

We will prove that f_* is the required function.

First, we will check (3.10). Define $f_{n+k} \colon W_{n+k} \to \mathbb{R}$ by

$$f_{n+k}(w) := f([w]_n), \quad w \in W_{n+k}.$$

Since $\operatorname{supp}[\varphi_w^{(m)}] \subseteq \mathcal{B}_m(w,1)$, we can write

$$f_*(v) = \sum_{\substack{z \,\in\, W_{n+k}: \\ v \,\in\, \mathcal{B}_m(z,1)}} f_{n+k}(z)\varphi_z^{(m)}(v), \quad v \in W_{n+m+k}.$$

Let $v \in W_{n+m+k}$ and $w \in W_n$ such that $v \in ww' \cdot W_m$ for some $w' \in W_k \setminus \partial_* G_k$. From $w' \notin \partial_* G_k$, it follows that $\mathcal{B}_m(ww',1) \cap (w \cdot W_{m+k})^c = \emptyset$, and thus, for any $z \in W_{n+k}$ with $v \in \mathcal{B}_m(z,1)$, we obtain $[z]_n = w$. From this observation, it holds that $f_{n+k}(z) = f(w)$, and thus we obtain

$$f_*(v) = \sum_{\substack{z \,\in\, W_{n+k}; \\ v \,\in\, \mathcal{B}_m(z,1)}} f([z]_n)\varphi_z^{(m)}(v) = \sum_{\substack{z \,\in\, W_{n+k}; \\ v \,\in\, \mathcal{B}_m(z,1)}} f(w)\varphi_z^{(m)}(v) = f(w),$$

which proves (3.10).

To prove (3.11), it will suffice to show the bound

$$\max_{w\in W_{n+k}} \mathcal{E}_p^{G_{n+m+k}}\bigl(\varphi_w^{(m)}\bigr) \leq c_1 \mathcal{C}_p^{(m)}, \tag{3.12}$$

where c_1 is a positive constant depending only on p and D_*. Indeed, by Lemma 3.12, we have

$$\begin{aligned}
\mathcal{E}_p^{G_{n+m+k}}(f_*) &\leq C_{3.12}\mathcal{E}_p^{G_{n+k}}(f_{n+k}) \max_{w\in W_{n+k}} \mathcal{E}_p^{G_{n+m+k}}(\varphi_w^{(m)}) \\
&\leq 2C_{3.12}\#(\partial_* G_k)\mathcal{E}_p^{G_n}(f) \max_{w\in W_{n+k}} \mathcal{E}_p^{G_{n+m+k}}(\varphi_w^{(m)}).
\end{aligned}$$

A combination of this estimate and (3.12) yields (3.11). Towards proving (3.12), we start by observing that $\#A_m(w) \leq M$ for some constant M depending only on D_*, where

$$A_m(w) := \bigl\{z \in W_{n+k} \bigm| \mathcal{B}_m(w,1) \cap \overline{\mathcal{B}_m(z,1)} \neq \emptyset\bigr\},$$

for each $w \in W_{n+k}$. Indeed, we have $z \in A_m(w)$ if $z \in W_{n+k}$ satisfies $h_w^{(m)}(v) \wedge h_z^{(m)}(v) \neq 0$ for some $v \in W_{n+m+k}$, and thus we obtain $\#A_m(w) \leq M$ for any $n, m, k \in \mathbb{N}$ and $w \in W_{n+k}$ by a similar reason to the bound of $A(z, z')$ in the proof of Lemma 3.12. Now, it is a simple computation that, for any $v, v' \in W_{n+m+k}$,

$$\varphi_w^{(m)}(v) - \varphi_w^{(m)}(v') = \frac{1}{\Psi(v)\Psi(v')}\Big(\Psi(v)\big(h_w^{(m)}(v) - h_w^{(m)}(v')\big) - h_w^{(m)}(v)\big(\Psi(v) - \Psi(v')\big)\Big).$$

From this identity, we have that

$$\begin{aligned}
&\mathcal{E}_p^{G_{n+m+k}}\big(\varphi_w^{(m)}\big) \\
&\leq 2^{p-1}\sum_{(v,v')}\left(\frac{1}{|\Psi(v')|^p}\left|h_w^{(m)}(v) - h_w^{(m)}(v')\right|^p + \frac{\left|h_w^{(m)}(v)\right|^p}{|\Psi(v)\Psi(v')|^p}|\Psi(v) - \Psi(v')|^p\right) \\
&\leq 2^{p-1}\left(\mathcal{E}_p^{G_{n+m+k}}(h_w^{(m)}) + \sum_{(v,v')}\left|h_w^{(m)}(v)\right|^p\left|\sum_{w' \in A_m(w)}(h_{w'}^{(m)}(v) - h_{w'}^{(m)}(v'))\right|^p\right) \\
&\underset{(*)}{\leq} (2M)^{p-1}\left(\mathcal{E}_p^{G_{n+m+k}}(h_w^{(m)}) + \sum_{w' \in A_m(w)}\mathcal{E}_p^{G_{n+m+k}}(h_{w'}^{(m)})\right) \\
&\leq (2M)^{p-1}(M+1)\max_{w \in W_{n+k}}\mathcal{E}_p^{G_{n+m+k}}(h_w^{(m)}) \leq (2M)^{p-1}(M+1)C_p^{(m)},
\end{aligned}$$

where the symbol $\sum_{(v,v')}$ denotes the summation over $(v, v') \in E_{n+m+k}$ and we used Hölder's inequality and $h_w^{(m)} \leq 1$ in $(*)$. This shows (3.12). □

With these preparations in place, we are ready to prove Proposition 3.10.

Proof of Proposition 3.10. Let $f\colon W_n \to \mathbb{R}$ with $\mathcal{E}_p^{G_n}(f) = 1$, and let $f_* \in \mathbb{R}^{W_{n+m+k}}$ be a function obtained by applying Lemma 3.13 to f. From Lemma 3.13 and $\mathcal{E}_p^{G_n}(f) = 1$, we have $\mathcal{E}_p^{G_{n+m+k}}(f_*) \leq C_{3.13}\#(\partial_* G_k)C_p^{(m)}$. On the one hand, Proposition 3.2-(1) yields that

$$N_*^{-(n+m+k)}\sum_{w \in W_{n+m+k}}\left|f_*(w) - \langle f_* \rangle_{W_{n+m+k}}\right|^p \leq C_{3.13}\lambda_p^{(n+m+k)}\#(\partial_* G_k)C_p^{(m)}.$$

On the other hand, from the property (3.10), we have that

$$\begin{aligned}
&N_*^{-(n+m+k)}\sum_{w \in W_{n+m+k}}\left|f_*(w) - \langle f_* \rangle_{W_{n+m+k}}\right|^q \\
&\geq N_*^{-(n+m+k)}\sum_{w \in W_n}\sum_{w' \in W_k \setminus \partial_* G_k}\sum_{z \in ww' \cdot W_m}\left|f(w) - \langle f_* \rangle_{W_{n+m+k}}\right|^p \\
&\geq \#(W_k \setminus \partial_* G_k)N_*^{-(n+k)}\sum_{w \in W_n}\left|f(w) - \langle f_* \rangle_{W_{n+m+k}}\right|^p \\
&\geq 2^{-p}\frac{\#(W_k \setminus \partial_* G_k)}{\#W_k}N_*^{-n}\sum_{w \in W_n}\left|f(w) - \langle f \rangle_{W_n}\right|^p,
\end{aligned}$$

where we used $\sum_{w \in W_n}|f(w) - c|^p \geq 2^{-p}\sum_{w \in W_n}\left|f(w) - \langle f \rangle_{W_n}\right|^p$ for any $c \in \mathbb{R}$ (see [10, Lemma 4.17] for example) in the last line. Since f with $\mathcal{E}_p^{G_n}(f) = 1$ is arbitrary, we obtain the desired estimate. □

We conclude this section by proving the submultiplicative inequality of $\lambda_p^{(n)}$ (see [47, Theorem 2.1] for $p = 2$). Moreover, we have the following theorem (see [47, Proposition 2.13 and 4.1] in the case of $p = 2$).

Theorem 3.14. *There exists a positive constant $C_{3.14}$ depending only on p, D_*, N_* and k_0 such that the following statements hold:*

(1) *for every $n, m \in \mathbb{N}$,*

$$\lambda_p^{(n)} \le C_{3.14} N_*^m \lambda_p^{(m)} \sigma_p^{(n)}; \tag{3.13}$$

(2) *for every $n \in \mathbb{N}$,*

$$C_{3.14}^{-1} \sigma_p^{(n)} \le \lambda_p^{(n)} \le C_{3.14} \sigma_p^{(n)}; \tag{3.14}$$

(3) *for every $n, m \in \mathbb{N}$,*

$$\lambda_p^{(n+m)} \le C_{3.14} \lambda_p^{(n)} \lambda_p^{(m)}; \tag{3.15}$$

(4) *for every $n \in \mathbb{N}$,*

$$C_{3.14}^{-1} \lambda_p^{(n)} \le \lambda_{*,p}^{(n)} \le C_{3.14} \lambda_p^{(n)}; \tag{3.16}$$

(5) *for every $n \in \mathbb{N}$,*

$$\mathcal{R}_p^{(n)} \le C_{3.14} \lambda_p^{(n)}. \tag{3.17}$$

Proof. (1) By Proposition 3.10, we have that $\mathcal{R}_p^{(m-k_0)} \lambda_p^{(n)} \le c_1 \lambda_p^{(n+m)}$ for all $n \in \mathbb{N}$ and $m \ge k_0$, where c_1 depends only on k_0, N_* and $C_{3.10}$. Combining this estimate with Proposition 3.9-(1), we obtain

$$\mathcal{R}_p^{(m-k_0)} \lambda_p^{(n)} \le c_2 (N_*^{-m} \lambda_p^{(n)} + \lambda_p^{(m)} \sigma_p^{(n)}),$$

where $c_2 := 2^{p-1} c_1 D_*$. Since $N_*^m \mathcal{R}_p^{(m-k_0)} \to \infty$ as $m \to \infty$ by Proposition 3.6, there exists $M_0 \in \mathbb{N}$ such that

$$\inf_{m \ge M_0} N_*^m \mathcal{R}_p^{(m-k_0)} \ge c_2 + 1.$$

From these estimates, we have that $(c_2 + 1) N_*^{-m} \lambda_p^{(n)} \le c_2 (N_*^{-m} \lambda_p^{(n)} + \lambda_p^{(m)} \sigma_p^{(n)})$ for all $n \in \mathbb{N}$ and $m \ge M_0$. Hence, we conclude that

$$\lambda_p^{(n)} \le c_2 N_*^m \lambda_p^{(m)} \sigma_p^{(n)}, \tag{3.18}$$

for all $n \in \mathbb{N}$ and $m \ge M_0$, which implies (3.13).

(2) Applying Propositions 3.9-(2), 3.8 and 3.7, we obtain $\sigma_p^{(n)} \le C_{3.8} D_* \sigma_p^{(3)} \lambda_p^{(n)}$ for all $n \ge 4$, which implies that $\sigma_p^{(n)} \le c_3 \lambda_p^{(n)}$ for any $n \in \mathbb{N}$, where c_3 is a positive constant

depending only on p, N_*, D_*. By (3.18), we have $\lambda_p^{(n)} \le c_2 N_*^{M_0} \lambda_p^{(M_0)} \sigma_p^{(n)}$ for any $n \in \mathbb{N}$. Hence we get (3.14).

(3) The submultiplicative inequality (3.15) is immediate from Proposition 3.9-(2) and (3.14).

(4) Applying Proposition 3.9-(2) and 3.8, we have that $\sigma_p^{(n)} \le C_{3.8} D_* \sigma_p^{(2)} \lambda_{*,p}^{(n)}$ for all $n \ge 3$, which together with (3.14) implies that

$$\lambda_p^{(n)} \le c_4 \lambda_{*,p}^{(n)} \quad \text{for any } n \in \mathbb{N}, \tag{3.19}$$

where c_4 is a positive constant depending only on p, N_*, D_* and k_0. The converse inequality of (3.19) is immediate from Proposition 3.7 and (3.15).

(5) We immediately get (3.17) from Proposition 3.10 and (3.15). □

4 Uniform Hölder estimate and Knight Move

This section gives p-energy analogs of [47, Lemma 3.9, Proposition 3.10, Theorem 7.2 and (B-2)]. In particular, we prove a uniform Hölder estimate without depending on levels of graphical approximation (Theorem 4.5) and, by using this Hölder estimate, we check (KM$_p$). To obtain useful Hölder type estimates, a "low-dimensional" condition: Assumption 4.2, which is written by using the *Ahlfors regular conformal dimension*, will be essential. The notion of Ahlfors regular conformal dimension was implicitly introduced by Bourdon and Pajot [15]. The exact definition of this dimension is as follows.

Definition 4.1. Let X be a metrizable space (without isolated points) and let d_i $(i = 1, 2)$ be compatible metrics on X. We say that d_1 and d_2 are *quasisymmetric* to each other if there exists a homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$ such that

$$\frac{d_2(x,a)}{d_2(x,b)} \le \eta\left(\frac{d_1(x,a)}{d_1(x,b)}\right),$$

for every triple $x, a, b \in X$ with $x \ne b$. (It is easy to show that being quasisymmetric gives an equivalence relation among metrics.) The *Ahlfors regular conformal gauge* $\mathcal{J}_{\mathrm{AR}}(X, d_1)$ of (X, d_1) is defined as

$$\mathcal{J}_{\mathrm{AR}}(X, d_1) := \left\{ d_2 \;\middle|\; \begin{array}{c} d_2 \text{ is a metric on } X,\ d_2 \text{ is quasisymmetric to } d_1, \\ \text{and } d_2 \text{ is } \alpha'\text{-Ahlfors regular for some } \alpha' > 0. \end{array} \right\}.$$

(For the definition of Ahlfors regularity, recall (2.1). Note that $\dim_{\mathrm{H}}(X, d_2) = \alpha'$ if d_2 is α'-Ahlfors regular.) Then the *Ahlfors regular conformal dimension* (ARC-dimension for short) of (X, d_1) is

$$\dim_{\mathrm{ARC}}(X, d_1) := \inf_{d_2 \in \mathcal{J}_{\mathrm{AR}}(X, d_1)} \dim_{\mathrm{H}}(X, d_2). \tag{4.1}$$

The notion of quasisymmetric and the exact definition of ARC-dimension are not essential in this paper. We refer the reader to a monograph [48] and surveys [12, 42] for details of the ARC-dimension and related subjects.

The following assumption describes our "low-dimensional" setting (see [47, the condition (R)] in the context of probability theory).

Assumption 4.2. A real number p satisfies $p > \dim_{\mathrm{ARC}}(K, d)$.

Remark 4.3. The following bound concerning the Ahlfors regular conformal dimension of the Sierpiński carpet is known:

$$1 + \frac{\log 2}{\log 3} \leq \dim_{\mathrm{ARC}}(K, d) < \alpha = \frac{\log N_*}{\log a}.$$

The lower bound follows from a general result due to Tyson [52]. The strict inequality in the upper bound is proved by Keith and Laakso [37]. Note that Assumption 4.2 implies that $p > 1$.

To promote understanding Assumption 4.2, we recall characterization results by Carrasco Piaggio [18] and Kigami [41] in our setting. Recall $\lim_{n\to\infty}\bigl(\mathcal{R}_p^{(n)}\bigr)^{1/n} = \rho_p$ (Theorem 3.4).

Theorem 4.4 ([18, Theorem 1.3], [41, Theorem 4.6.9 and 4.7.6])**.** *It holds that*

$$\dim_{\mathrm{ARC}}(K, d) = \inf\bigl\{p \bigm| \lim_{n\to\infty} \mathcal{R}_p^{(n)} = \infty\bigr\}.$$

Moreover, Assumption 4.2 is equivalent to $\rho_p > 1$.

We define

$$\beta_p := \frac{\log N_* \rho_p}{\log a}. \tag{4.2}$$

Then note that Assumption 4.2 is also equivalent to $\beta_p - \alpha > 0$.

4.1 Uniform Hölder estimate

In this subsection, we prove the following Hölder type estimate.

Theorem 4.5. *Suppose Assumption 4.2 holds. Then there exists a constant $\widetilde{C}_{\mathrm{UH}} > 0$ (depending only on $p, D_*, N_*, a, \rho_p, C_{\mathrm{Mult}}$ and k_0) such that, for any $n, m \in \mathbb{N}$, $z \in W_m$, $v, w \in \mathcal{B}_n(z, 1)$ and $f\colon W_{n+m} \to \mathbb{R}$,*

$$|f(v) - f(w)|^p \leq \widetilde{C}_{\mathrm{UH}} \lambda_p^{(n+m)} \mathcal{E}_p^{G_{n+m}}(f) a^{-(\beta_p - \alpha)m}. \tag{4.3}$$

This theorem is proved by iterating Proposition 3.2-(2). Kusuoka and Zhou [47] prepared a general estimate using signed measures ([47, Lemma 3.9]) to show Hölder type estimates, but we need only the case of Dirac measures for our purpose. Thus we give a simplified extension of [47, Lemma 3.9 and Proposition 3.10] as follows.

Lemma 4.6. *Let $p > 1$. Let $n, m \in \mathbb{N}$, let $v \in W_m$ and let $f\colon W_{n+m} \to \mathbb{R}$. Then, for any $w \in v \cdot W_n$,*

$$\left|f(w) - \langle f \rangle_{v\cdot W_n}\right| \le N_*^{1/p} \mathcal{E}_p^{v\cdot W_n}(f)^{1/p} \sum_{k=1}^{n} \left(\lambda_p^{(k)}\right)^{1/p}. \tag{4.4}$$

Proof. Let $w \in v \cdot W_n$ and set $w_l := [w]_l$ for each $l = m, \dots, n+m$. Note that $w_m = v$ and $w_{n+m} = w$. From Proposition 3.2-(2), we see that

$$\begin{aligned}
\left|f(w) - \langle f \rangle_{v\cdot W_n}\right| &\le \sum_{l=m}^{n+m-1} \left|\langle f \rangle_{w_l \cdot W_{n+m-l}} - \langle f \rangle_{w_{l+1}\cdot W_{n+m-l-1}}\right| \\
&\le N_*^{1/p} \mathcal{E}_p^{v\cdot W_n}(f)^{1/p} \sum_{l=m}^{n+m-1} \big(\lambda_p^{(n+m-l)}\big)^{1/p} \\
&= N_*^{1/p} \mathcal{E}_p^{v\cdot W_n}(f)^{1/p} \sum_{k=1}^{n} \big(\lambda_p^{(k)}\big)^{1/p}.
\end{aligned}$$

□

Then we can prove Theorem 4.5 by simple computations.

Proof of Theorem 4.5. Set $\theta_p := \beta_p - \alpha = \log \rho_p / \log a$ for simplicity. Then, by the supermultiplicative inequality of $C_p^{(n)}$ (Theorem 3.4), we have that

$$C_p^{(n)} \le C_{\mathrm{Mult}} a^{-n\theta_p} \quad \text{for every } n \in \mathbb{N}. \tag{4.5}$$

From (4.5), Proposition 3.10 and Theorem 3.14-(3), we have $\lambda_p^{(n)} \le c_1 \lambda_p^{(n+m)} a^{-m\theta_p}$ for every $n, m \in \mathbb{N}$, where $c_1 > 0$ depends only on $C_{\mathrm{Mult}}, C_{3.10}, C_{3.14}, k_0, N_*, \lambda_p^{(k_0)}$. In particular,

$$\lambda_p^{(k)} \le c_1 \lambda_p^{(n+m)} a^{-(n+m-k)\theta_p} \quad \text{for every } n, m, k \in \mathbb{N} \text{ with } k \le n. \tag{4.6}$$

By Lemma 4.6, for any $z \in W_m$, $v \in z \cdot W_n$ and $w \in \mathcal{B}_n(z, 1)$,

$$\begin{aligned}
&|f(v) - f(w)| \\
&\le \left|f(v) - \langle f \rangle_{z\cdot W_n}\right| + \left|\langle f \rangle_{z\cdot W_n} - \langle f \rangle_{[w]_m \cdot W_n}\right| + \left|f(w) - \langle f \rangle_{[w]_m\cdot W_n}\right| \\
&\le \sum_{i=1,2} \left|\langle f \rangle_{z^i\cdot W_n} - \langle f \rangle_{z^{i+1}\cdot W_n}\right| + 2N_*^{1/p} \mathcal{E}_p^{G_{n+m}}(f)^{1/p} \sum_{k=1}^{n} \big(\lambda_p^{(k)}\big)^{1/p},
\end{aligned}$$

where $z^i \in W_m$ ($i = 1, 2, 3$) with $z^1 = z$, $z^3 = [w]_m$ satisfy $(z^i, z^{i+1}) \in \widetilde{E}_m$ for $i = 1, 2$ or $z^1 = z^2$. Thanks to Proposition 3.2-(3), Theorem 3.14-(2) and (4.6), we see that

$$\begin{aligned}
|f(v) - f(w)| &\le 2\mathcal{E}_p^{G_{n+m}}(f)^{1/p} \left(\big(\sigma_p^{(n)}\big)^{1/p} + N_*^{1/p} \sum_{k=1}^{n} \big(\lambda_p^{(k)}\big)^{1/p} \right) \\
&\le c_2 \mathcal{E}_p^{G_{n+m}}(f)^{1/p} \sum_{k=1}^{n} \big(\lambda_p^{(k)}\big)^{1/p} \\
&\le c_1^{1/p} c_2 \big(\lambda_p^{(n+m)} \mathcal{E}_p^{G_{n+m}}(f)\big)^{1/p} \sum_{k=1}^{n} a^{-(n+m-k)\theta_p/p},
\end{aligned}$$

where c_2 is a positive constant depending only on $p, N_*, C_{3.14}$. Since $\theta_p > 0$ and

$$\begin{aligned}\sum_{k=1}^{n} a^{-(n+m-k)\theta_p/p} &= a^{-m\theta_p/p} \sum_{k=0}^{n-1} a^{-k\theta_p/p} \\ &= a^{-m\theta_p/p} \cdot \frac{1 - a^{-n\theta_p/p}}{1 - a^{-\theta_p/p}} \le \frac{1}{1 - a^{-\theta_p/p}} \cdot a^{-m\theta_p/p},\end{aligned}$$

we have the desired estimate for $v \in z \cdot W_n$ and $w \in \mathcal{B}_n(z, 1)$. By chaining this, we complete the proof. □

Remark 4.7. We used the supermultiplicative inequality of $C_p^{(n)}$ to derive (4.5) in the above proof, but we can show similar Hölder estimates without using the supermultiplicative inequality. (Note that the limit $\rho_p = \lim_{n\to\infty} \big(\mathcal{R}_p^{(n)}\big)^{1/n}$ exists by the submultiplicative inequality of $C_p^{(n)}$.) Indeed, for any $\theta \in (0, \theta_p)$, we have that $C_p^{(n)} \le c_1 a^{-n\theta}$ for every $n \in \mathbb{N}$. If one uses this estimate instead of (4.5), then one obtains the following Hölder estimate:

$$|f(v) - f(w)|^p \le \widetilde{C}_{\mathrm{UH}} \lambda_p^{(n)} \mathcal{E}_p^{G_n}(f) a^{-\theta m}. \tag{4.7}$$

In subsection 4.3, we will derive the supermultiplicative inequality of $C_p^{(n)}$ as a consequence of (KM$_p$). Since the continuity (4.7) is enough to prove (KM$_p$) (see subsection 4.2), the use of supermultiplicative inequality of $C_p^{(n)}$ is not essential to obtain (4.3).

4.2 Proof of (KM$_p$)

The aim of this subsection is to prove (KM$_p$) under Assumption 4.2. Our strategy for proving (KM$_p$) comes from a recent study by Cao and Qiu [17], where they give an "analytic" proof of (KM$_2$) using estimates of Poincaré constants in [47]. Although our proof of (KM$_p$) is similar to the argument in [17, Section 4], we give a complete proof of (KM$_p$) for the reader's convenience. Our argument will depend heavily on the uniform Hölder estimate (Theorem 4.5) and on behaviors of "chain" type p-conductance $C_p^{(n,M)}$ (its definition will be given later).

Let us start by introducing a new graph $G_{n,M}$, which is a "horizontal chain" consisting of M copies of G_n. The exact definition of $G_{n,M}$ is as follows. Let $n, M \in \mathbb{N}$ with $M \ge 2$ and pick $m \in \mathbb{N}$ such that $3^m \ge M$. Then there exists a simple path $[w^1, \dots, w^M]$ in G_m such that

$$F_{w^i}\big(K^{\mathrm{R}}\big) = F_{w^{i+1}}\big(K^{\mathrm{L}}\big) \quad \text{for each } i = 1, \dots, M-1,$$

where

$$K^{\mathrm{L}} := K \cap (\{-1/2\} \times [-1/2, 1/2]), \quad K^{\mathrm{R}} := K \cap (\{1/2\} \times [-1/2, 1/2]). \tag{4.8}$$

(K^{L} denotes the left line segment of the Sierpiński carpet. K^{R} is the right line segment.) Then we define $G_{n,M} = (V_{n,M}, E_{n,M})$ as a subgraph of G_{n+m} (recall Definition 2.11) given by

$$V_{n,M} := \bigcup_{i=1}^{M} w^i \cdot W_n \quad \text{and} \quad E_{n,M} := E_{n+m}^{V_{n,M}}.$$

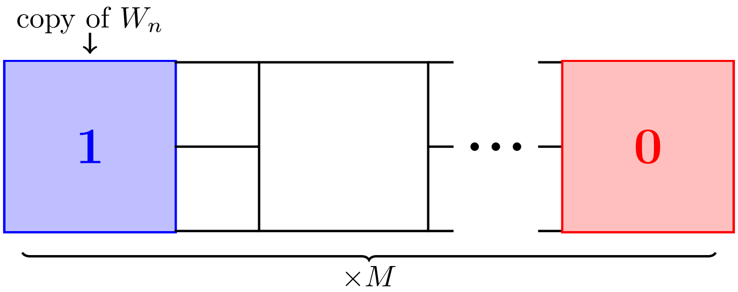

Figure 4: The conductance $C_p^{(n,M)}$

We also set

$$V^{\mathrm{L}}_{n,M} := w^1 \cdot W_n \quad \text{and} \quad V^{\mathrm{R}}_{n,M} := w^M \cdot W_n.$$

Now, we define $C_p^{(n,M)}$ (see Figure 4) by setting

$$C_p^{(n,M)} := C_p^{G_{n,M}}(V^{\mathrm{L}}_{n,M}, V^{\mathrm{R}}_{n,M}).$$

We easily see that this definition does not depend on choices of large $m \in \mathbb{N}$ and horizontal chain $[w^1, \dots, w^M]$.

The following lemma describes a key behavior of $C_p^{(n,M)}$.

Lemma 4.8. *Let $p > 1$. For every $M \geq 3$ there exists a constant $C(M) \geq 1$ depending only on p, D_*, N_*, M such that*

(4.9) $$C(M)^{-1} C_p^{(n)} \leq C_p^{(n,M)} \leq C(M) C_p^{(n)} \quad \textit{for any } n \in \mathbb{N}.$$

Its proof will be a straightforward modification of [17, Lemma 4.7]. Towards proving Lemma 4.8, let us start by providing preparations from *asymptotic geometry*. The following definition extends the notion of *rough isometry among graphs* to that among sequences of graphs.

Definition 4.9. For each $i = 1, 2$, let $\{G_n^i = (V_n^i, E_n^i)\}_{n \geq 1}$ be a series of finite graphs with

(4.10) $$D_*^i := \sup_{n \in \mathbb{N}} \max_{x \in V_n^i} \#\{y \in V_n^i \mid (x, y) \in E_n^i\} < \infty.$$

We say that a family of maps $\{\varphi_n\}_{n \geq 1}$, where $\varphi_n \colon V_n^1 \to V_n^2$, is a *uniform rough isometry* from $\{G_n^1\}_{n \geq 1}$ to $\{G_n^2\}_{n \geq 1}$ if:

(1) there exist constants C_1, C_2 such that, for every $n \in \mathbb{N}$ and $x, y \in V_n^1$,

$$C_1^{-1} d_{G_n^1}(x, y) - C_2 \leq d_{G_n^2}(\varphi_n(x), \varphi_n(y)) \leq C_1 d_{G_n^1}(x, y) + C_2;$$

(2) there exists a constant C_3 such that, for every $n \in \mathbb{N}$,

$$\bigcup_{x \in V_n^1} B_{d_{G_n^2}}(\varphi_n(x), C_3) = V_n^2;$$

(3) there exists a constant C_4 such that, for every $n \in \mathbb{N}$ and $x \in V_n^1$,

$$C_4^{-1} \leq \frac{\#\{y' \in V_n^2 \mid (\varphi_n(x), y') \in E_n^2\}}{\#\{y \in V_n^1 \mid (x, y) \in E_n^1\}} \leq C_4.$$

Remark 4.10. Since each φ_n is a rough isometry from G_n^1 to G_n^2, there exists a rough isometry $\widetilde{\varphi}_n$ from G_n^2 to G_n^1. Moreover, we can choose $\widetilde{\varphi}_n$ so that $\{\widetilde{\varphi}_n\}_{n\geq 1}$ is a uniform rough isometry from $\{G_n^2\}_{n\geq 1}$ to $\{G_n^1\}_{n\geq 1}$. Consequently, being uniform rough isometry gives an equivalence relation among series of finite graphs satisfying (4.10).

Then the following stability result holds. Its proof is a straightforward modification of [50, proof of Lemma 8.5], and so we omit it here (see Appendix A.1 for a proof).

Lemma 4.11. *Let $\{G_n^i = (V_n^i, E_n^i)\}_{n\geq 1}$ be a series of finite graphs with*

$$D_*^i := \sup_{n\in\mathbb{N}} \max_{x\in V_n^i} \#\{y \in V_n^i \mid (x, y) \in E_n^i\} < \infty,$$

for each $i = 1, 2$, and let $\varphi_n \colon V_n^1 \to V_n^2$ be a uniform rough isometry from $\{G_n^1\}_{n\geq 1}$ to $\{G_n^2\}_{n\geq 1}$. Then there exists a positive constant C_{URI} (depending only on C_1, C_2 in Definition 4.9, D_*^1 and p) such that

$$\mathcal{E}_p^{G_n^1}(f \circ \varphi_n) \leq C_{\mathrm{URI}}\, \mathcal{E}_p^{G_n^2}(f), \tag{4.11}$$

for every $n \in \mathbb{N}$ and $f \colon V_n^2 \to \mathbb{R}$. In particular,

$$C_p^{G_n^1}(\varphi_n^{-1}(A_n), \varphi_n^{-1}(B_n)) \leq C_{\mathrm{URI}} C_p^{G_n^2}(A_n, B_n)$$

for every $n \in \mathbb{N}$, where A_n, B_n are disjoint subsets of V_n^2.

Next, let us observe that $C_p^{(n)}$ behaves similarly to the conductance that appeared in the work of Barlow and Bass (see the quantity R_n^{-1} in [6]). To state rigorously, we define

$$W_n^{\mathrm{L}} := \{w \in W_n \mid K_w \cap \{-1/2\} \times [-1/2, 1/2] \neq \emptyset\},$$

$$W_n^{\mathrm{R}} := \{w \in W_n \mid K_w \cap \{1/2\} \times [-1/2, 1/2] \neq \emptyset\},$$

and

$$C_p^{(n)}(\mathrm{L} \leftrightarrow \mathrm{R}) := C_p^{G_n}(W_n^{\mathrm{L}}, W_n^{\mathrm{R}}).$$

The next lemma is proved in [14, proof of Lemma 4.4] using p-combinatorial modulus instead of p-conductance. We give a simple proof for the reader's convenience.

Lemma 4.12. *Let $p > 1$. There exists a constant $C_{4.12} \geq 1$ depending only on p, D_*, N_* such that*

$$C_{4.12}^{-1} C_p^{(n)} \leq C_p^{(n)}(\mathrm{L} \leftrightarrow \mathrm{R}) \leq C_{4.12} C_p^{(n)} \quad \textit{for any } n \in \mathbb{N}.$$

Proof. By the symmetries and self-similarity of the Sierpiński carpet, there exist $m \in \mathbb{N}$ and $w \in W_m$ such that $\mathcal{C}_p^{(n)} = \mathcal{C}_p^{G_{n+m}}\bigl(w \cdot W_n, W_{n+m} \setminus \mathcal{B}_n(w,1)\bigr)$. Define $A_n := \overline{w \cdot W_n}$ and $B_n := \overline{W_{n+m} \setminus \mathcal{B}_n(w,1)}$. It is immediate from Proposition 2.12 that $\mathcal{C}_p^{G_{n+m}}(A_n, B_n) \geq \mathcal{C}_p^{(n)}$. For large $n \in \mathbb{N}$, one can easily construct a rough isometry $\varphi_n\colon W_{n+m} \to W_{n+m}$ (with $C_1 = 1$ and $C_2 = 4$ in Definition 4.9) such that

$$\varphi_n(A_n \setminus w \cdot W_n) \subseteq w \cdot W_n \quad \text{and} \quad \varphi_n\bigl(B_n \setminus (W_{n+m} \setminus \mathcal{B}_n(w,1))\bigr) \subseteq W_{n+m} \setminus \mathcal{B}_n(w,1).$$

Note that $\mathcal{C}_p^{G_{n+m}}(A_n \setminus w \cdot W_n, B_n \setminus (W_{n+m} \setminus \mathcal{B}_n(w,1))) = \mathcal{C}_p^{G_{n+m}}(A_n, B_n)$. Applying Lemma 4.11 and Proposition 2.12, we deduce that there exists $c_1 > 0$ (depending only on p, D_*) such that, for any $n \in \mathbb{N}$,

$$\mathcal{C}_p^{G_{n+m}}(A_n, B_n) \leq c_1 \mathcal{C}_p^{(n)}.$$

Let $f\colon W_{n+m} \to \mathbb{R}$ satisfy $f|_{A_n} \equiv 1$, $f|_{B_n} \equiv 0$ and $\mathcal{E}_p^{G_{n+m}}(f) = \mathcal{C}_p^{G_{n+m}}(A_n, B_n)$. If $v \in W_m$ satisfies $(v, w) \in \widetilde{E}_m$, then we have that

$$\mathcal{C}_p^{G_{n+m}}(A_n, B_n) \geq \mathcal{E}_p^{v \cdot W_n}(f) \geq \mathcal{C}_p^{(n)}(\mathrm{L} \leftrightarrow \mathrm{R}),$$

and thus we conclude that $\mathcal{C}_p^{(n)}(\mathrm{L} \leftrightarrow \mathrm{R}) \leq c_1 \mathcal{C}_p^{(n)}$.

To prove the converse, we set

$$C_n := \{w \in W_n \mid K_w \cap [-1/6, 1/6]^2 \neq \emptyset\}.$$

By the cutting law of p-conductances (see [50, Proposition 3.18] for example), it follows that

$$\mathcal{C}_p^{G_{n+m}}(A_n, B_n) \leq \mathcal{C}_p^{G_{n+1}}\bigl(C_{n+1}, \partial_* G_{n+1}\bigr).$$

Recall the definition of $\mathrm{Sym}(K)$ (see Definition 2.8). For each $T \in \mathrm{Sym}(K)$, let $\iota_T\colon W_n \to W_n$ such that $T(K_w) = K_{\iota_T(w)}$ for any $w \in W_n$. Let $f_n\colon W_n \to \mathbb{R}$ satisfy

$$f_n|_{W_n^{\mathrm{L}}} \equiv 1, \quad f_n|_{W_n^{\mathrm{R}}} \equiv 0, \quad \text{and} \quad \mathcal{E}_p^{G_n}(f_n) = \mathcal{C}_p^{(n)}(\mathrm{L} \leftrightarrow \mathrm{R}).$$

Furthermore, we set

$$K_{\mathrm{I}} := \{(x,y) \in K \mid y \leq -x \text{ and } y \leq x\},\ K_{\mathrm{II}} := \{(x,y) \in K \mid y \geq -x \text{ and } y \leq x\},$$
$$K_{\mathrm{III}} := \{(x,y) \in K \mid y \geq -x \text{ and } y \geq x\},\ K_{\mathrm{IV}} := \{(x,y) \in K \mid y \leq -x \text{ and } y \geq x\}.$$

For any $w \in W_{n+1}$, we define $[w]_{-1} \in W_n$ as $w = i[w]_{-1}$ for some $i \in S$. Then we define $g_{n+1}\colon W_{n+1} \to \mathbb{R}$ by setting

$$g_{n+1}(w) := \begin{cases} f_n\bigl(\iota_{T_-}([w]_{-1})\bigr) & \text{if } K_w \cap K_{\mathrm{I}} \neq \emptyset, \\ f_n([w]_{-1}) & \text{if } K_w \cap K_{\mathrm{II}} \neq \emptyset, \\ f_n\bigl(\iota_{T_+}([w]_{-1})\bigr) & \text{if } K_w \cap K_{\mathrm{III}} \neq \emptyset, \\ f_n\bigl(\iota_{T_+}(\iota_{T_-}([w]_{-1}))\bigr) & \text{if } K_w \cap K_{\mathrm{IV}} \neq \emptyset. \end{cases}$$

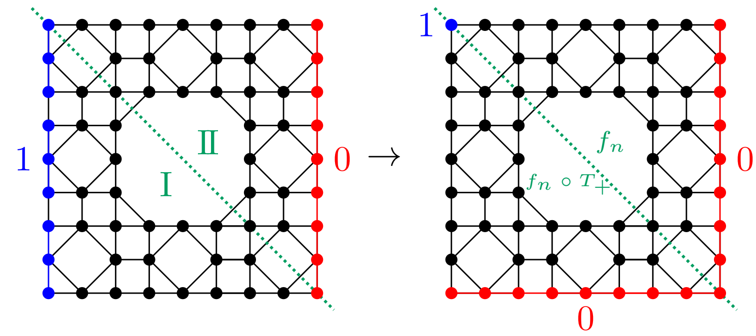

Figure 5: The definition of g_{n+1} on $3 \cdot W_n$

(See Figure 5.) The symmetry of f_n yields that $\mathcal{E}_p^{\{i,j\}\cdot W_n}(g_{n+1}) = 2\mathcal{E}_p^{G_n}(f_n)$ for any $i \neq j \in S$. Hence,

$$C_p^{G_{n+1}}\bigl(C_{n+1}, \partial_* G_{n+1}\bigr) \leq \mathcal{E}_p^{G_{n+1}}(g_{n+1}) = N_* \mathcal{E}_p^{G_n}(f_n) = N_* C_p^{(n)}(\mathrm{L} \leftrightarrow \mathrm{R}),$$

which together with $C_p^{G_{n+1}}\bigl(C_{n+1}, \partial_* G_{n+1}\bigr) \geq C_p^{(n)}$ proves the lemma. □

Now we are ready to prove Lemma 4.8.

Proof of Lemma 4.8. Thanks to Lemma 4.12, it will suffice to compare $C_p^{(n)}(\mathrm{L} \leftrightarrow \mathrm{R})$ and $C_p^{(n,M)}$. Suppose that $G_{n,M}$ is realized as a subgraph of G_{n+m} using a chain $[w^1, \dots, w^M]$ in G_m, that is, $V_{n,M} = \bigcup_{i=1}^{M} w^i \cdot W_n$. First, we consider the case $M = 3$. By the monotonicity of p-conductance (Proposition 2.12), we immediately have that $C_p^{(n,3)} \leq C_p^{(n)}(\mathrm{L} \leftrightarrow \mathrm{R})$. To prove the converse, we will use a stability result: Lemma 4.11. Let us define a subgraph $\widetilde{G}_{n,3} = (\widetilde{V}_{n,3}, \widetilde{E}_{n,3})$ of $G_{n,3}$ by

$$\widetilde{V}_{n,3} \coloneqq \{w^1 v \mid v \in W_n^{\mathrm{R}}\} \cup w^2 \cdot W_n \cup \{w^3 v \mid v \in W_n^{\mathrm{L}}\},$$

$$\widetilde{E}_{n,3} \coloneqq \{(v, w) \in E_{n,3} \mid v, w \in \widetilde{V}_{n,3}\}.$$

Then we easily see that

$$C_p^{(n,3)} = C^{\widetilde{G}_{n,3}}\bigl(\{w^1 v \mid v \in W_n^{\mathrm{R}}\}, \{w^3 v \mid v \in W_n^{\mathrm{L}}\}\bigr).$$

Define $\varphi_n \colon W_n \to \widetilde{V}_{n,3}$ by

$$\varphi_n(w) \coloneqq \begin{cases} w^1 \widehat{w} & \text{if } w \in W_n^{\mathrm{L}}, \\ w^2 w & \text{if } w \notin W_n^{\mathrm{L}} \cup W_n^{\mathrm{R}}, \\ w^3 \widehat{w} & \text{if } w \in W_n^{\mathrm{R}}, \end{cases}$$

where $\widehat{w}$ denotes a unique element such that $T_{\mathrm{v}}(K_w) = K_{\widehat{w}}$. Then $\{\varphi_n\}_{n \geq 1}$ is a uniform rough isometry between $\{G_n\}_{n \geq 1}$ and $\{\widetilde{G}_{n,3}\}_{n \geq 1}$ (with $C_1 = 1, C_2 = 2$ in Definition 4.9). Applying Lemma 4.11, we get $C_p^{(n)}(\mathrm{L} \leftrightarrow \mathrm{R}) \leq c_1 C_p^{(n,3)}$, where $c_1 > 0$ depends only on p and D_*.

Next, let us consider the case $M_k \coloneqq 2^k + 2$ for $k \in \mathbb{N}$. Let $f_{n,k} \colon V_{n,M_k} \to \mathbb{R}$ satisfy

$$f_{n,k}|_{V_{n,M_k}^{\mathrm{L}}} \equiv 1, \quad f_{n,k}|_{V_{n,M_k}^{\mathrm{R}}} \equiv 0, \quad \text{and} \quad \mathcal{E}_p^{G_{n,M_k}}(f_{n,k}) = C_p^{(n,M_k)}.$$

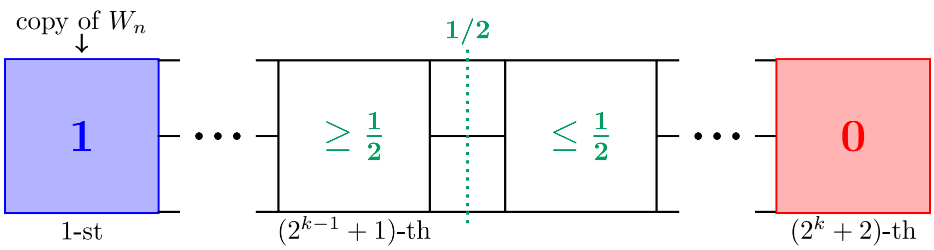

Figure 6: Values of $f_{n,k}$

Then one can prove that

$$\min\{f_{n,k}(v) \mid v \in w^i \cdot W_n, 1 \leq i \leq 2^{k-1}+1\} \geq \frac{1}{2}. \tag{4.12}$$

(See Figure 6.) Indeed, one easily sees from the symmetry of V_{n,M_k} and the uniqueness of $f_{n,k}$ (Proposition 2.15) that a function $\widehat{f_{n,k}}$ given by

$$\widehat{f_{n,k}}(v) := \begin{cases} f_{n,k}(v) \vee (1 - f_{n,k}(v)) & \text{if } v \in \bigcup_{i=1}^{2^{k-1}+1} w^i \cdot W_n, \\ f_{n,k}(v) \wedge (1 - f_{n,k}(v)) & \text{if } v \in \bigcup_{i=2^k}^{2^k+2} w^i \cdot W_n, \end{cases}$$

satisfies $\widehat{f_{n,k}}|_{V^{\mathrm{L}}_{n,M_k}} \equiv 1$, $\widehat{f_{n,k}}|_{V^{\mathrm{R}}_{n,M_k}} \equiv 0$ and $\mathcal{E}_p\big(\widehat{f_{n,k}}\big) \leq \mathcal{E}_p(f_{n,k})$. Again applying the uniqueness, we have that $\widehat{f_{n,k}} = f_{n,k}$. Hence (4.12) holds. As an immediate consequence of (4.12) and the symmetry, we also have

$$\max\{f_{n,k}(v) \mid v \in w^i \cdot W_n, 2^k \leq i \leq 2^k+2\} \leq \frac{1}{2}. \tag{4.13}$$

By (4.12), (4.13) and the Markov property of p-energies on graphs (Proposition 2.13),

$$\begin{aligned} C_p^{(n,M_k)} &= \mathcal{E}_p^{V_{n,M_k}}(f_{n,k}) \\ &\geq \frac{1}{2}\left[\mathcal{E}_p^{V_{n,M_k}}\left(f_{n,k} \vee \frac{1}{2}\right) + \mathcal{E}_p^{V_{n,M_k}}\left(f_{n,k} \wedge \frac{1}{2}\right)\right] = 2^{-p} C_p^{(n,M_{k-1})}. \end{aligned}$$

Iterating this estimate, we conclude that $C_p^{(n,M_k)} \geq 2^{-pk} C_p^{(n,3)}$ for any $k \in \mathbb{N}$. Since $C_p^{(n,M)} \geq C_p^{(n,M')}$ for $M \leq M'$, we obtain the desired estimate for general M. $\square$

Finally, we prove (KM$_p$). We mainly follow the method in [17, Lemma 4.8].

Theorem 4.13. *Suppose Assumption 4.2 holds. Then* (KM$_p$) *holds.*

Proof. Let $f_n \colon W_n \to \mathbb{R}$ satisfy

$$\mathcal{E}_p^{G_n}(f_n) = 1, \quad f_n|_{\partial_* G_n} \equiv 0, \quad \text{and} \quad \langle f_n \rangle_{W_n} = \big(\lambda_{*,p}^{(n)}\big)^{1/p}.$$

Note that f_n is non-negative. Pick $w^* \in W_n$ such that $f_n(w^*) = \max_{w \in W_n} f_n(w)$. Then we easily see that $f_n(w^*) \geq \big(\lambda_{*,p}^{(n)}\big)^{1/p}$. Since $\beta_p - \alpha > 0$ (Assumption 4.2), we can choose $l \geq 1$ such that

$$\big(\widetilde{C}_{\mathrm{UH}} a^{-l(\beta_p - \alpha)}\big)^{1/p} \leq \frac{1}{16 N_*}, \tag{4.14}$$

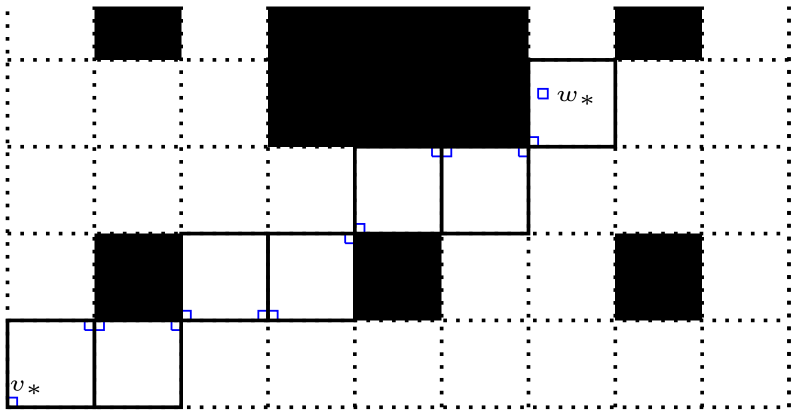

Figure 7: A pick of corners (each of blue square describes $i^*i_{k+1}w^{2k}$ or $i^*i_{k+1}w^{2k+1}$)

where $\widetilde{C}_{\mathrm{UH}}$ is a constant appeared in Theorem 4.5. As a consequence of this choice and Theorem 4.5, for any $n \in \mathbb{N}$,

$$\max\left\{|f_n(v) - f_n(w)| \,\middle|\, k \leq n - l, z \in W_{l+k}, v, w \in z \cdot W_{n-l-k}\right\} \leq \frac{1}{16N_*}\left(\lambda_{*,p}^{(n)}\right)^{1/p}. \tag{4.15}$$

Now, we consider a chain of $(n-2)$-cells from $\partial_* G_n$ to w^* in the $(n-1)$-cell containing w^*. To state explicitly, define $i^* := [w^*]_1 \in S$ and $S_{\mathsf{Corner}} := \{1, 3, 5, 7\}$. Fix $j \in S_{\mathsf{Corner}}$ such that $v^* := i^* j^{n-1} \in \partial_* G_n$ and $w^* \notin i^* j \cdot W_{n-2}$. Then there exist $L \in \{2, \dots, 5\}$ and $i_1, \dots, i_L \in S$ such that $i_1 = j$ and $(i_k, i_{k+1}) \in E_1$ for each $k = 1, \dots, L-1$. Furthermore, we can pick "corners" $w^1, w^2, \dots, w^{2L-1}, w^{2L} \in W_{n-2}(S_{\mathsf{Corner}})$ in the chain of cells $[i^* i_1 \cdot W_{n-2}, \dots, i^* i_L \cdot W_{n-2}]$ satisfying

$$i^* i_1 w^1 = v^*, \quad i^* i_L w^{2L} = w^*, \quad \text{and} \quad (i^* i_k w^{2k}, i^* i_{k+1} w^{2k+1}) \in E_n,$$

for each $k = 1, \dots, L-1$ (see Figure 7). Since $f_n(v^*) = 0$ and $\mathcal{E}_p^{G_n}(f_n) = 1$,

$$\begin{aligned}
&\left(\lambda_{*,p}^{(n)}\right)^{1/p} \\
&\leq |f_n(v^*) - f_n(w^*)| \\
&\leq \sum_{k=1}^{L-1}\left(\left|f_n(i^* i_k w^{2k-1}) - f_n(i^* i_k w^{2k})\right| + \left|f_n(i^* i_k w^{2k}) - f_n(i^* i_{k+1} w^{2k+1})\right|\right) \\
&\leq L - 1 + \sum_{k=1}^{L-1}\left|f_n(i^* i_k w^{2k-1}) - f_n(i^* i_k w^{2k})\right|.
\end{aligned}$$

Now, we have $\lambda_{*,p}^{(n)} \to \infty$ as $n \to \infty$ by Assumption 4.2 and Theorem 3.14. Hence we may assume that

$$\sum_{k=1}^{L-1}\left|f_n(i^* i_k w^{2k-1}) - f_n(i^* i_k w^{2k})\right| \geq \frac{1}{2}\left(\lambda_{*,p}^{(n)}\right)^{1/p},$$

for all large n. Then there exists $k \in \{1, \dots, L-1\}$ such that

$$\left|f_n(i^* i_k w^{2k-1}) - f_n(i^* i_k w^{2k})\right| \geq \frac{1}{2(L-1)}\left(\lambda_{*,p}^{(n)}\right)^{1/p}.$$

Moreover, there exists $(j_0, j_1) \in \{(1,3),(1,7),(3,5),(5,7)\}$ such that

$$\left|f_n(i^* i_k j_0^{n-2}) - f_n(i^* i_k j_1^{n-2})\right| \geq \frac{1}{4(L-1)}\bigl(\lambda_{*,p}^{(n)}\bigr)^{1/p}. \tag{4.16}$$

We may assume that $f_n(i^* i_k j_1^{n-2}) \geq f_n(i^* i_k j_0^{n-2})$. Then we see from (4.15) and (4.16) that

$$\begin{aligned}\delta_n &:= \min_{i^* i_k j_1^l \cdot W_{n-l-2}} f_n - \max_{i^* i_k j_0^l \cdot W_{n-l-2}} f_n \\ &\geq \left[\frac{1}{4(L-1)} - 2\cdot\frac{1}{16N_*}\right]\bigl(\lambda_{*,p}^{(n)}\bigr)^{1/p} \geq \frac{1}{8(L-1)}\bigl(\lambda_{*,p}^{(n)}\bigr)^{1/p},\end{aligned}$$

where we used a bound $L-1 \leq N_*$ in the last inequality. Let us consider the "horizontal chain of $(n-l-2)$-cell" from $i^* i_k j_1^l \cdot W_{n-l-2}$ to $i^* i_k j_0^l \cdot W_{n-l-2}$. Then, by putting $g_n := \bigl((\delta_n^{-1} f_n + c_n) \vee 0\bigr) \wedge 1$, where c_n is a constant such that $g_n|_{i^* i_k j_\varepsilon^l \cdot W_{n-l-2}} \equiv \varepsilon$ for $\varepsilon = 0, 1$, we have that

$$C_p^{(n-l-2,3^l)} \leq \mathcal{E}_p^{G_n}(g_n) \leq \delta_n^{-p}\mathcal{E}_p^{G_n}(f_n) \leq \bigl(8(L-1)\bigr)^p \bigl(\lambda_{*,p}^{(n)}\bigr)^{-1}.$$

From Lemma 4.8, Theorem 3.4 (especially the submultiplicative inequality of $C_p^{(n)}$) and 3.14-(4), we conclude that

$$\lambda_p^{(n)} \leq \bigl(8(L-1)\bigr)^p C_{3.14} C_{\mathrm{Mult}} C(l) \mathcal{R}_p^{(l+2)} \cdot \mathcal{R}_p^{(n)}.$$

This proves (KM$_p$), where $C_{\mathrm{KM}} = \bigl(8(L-1)\bigr)^p C_{3.14} C_{\mathrm{Mult}} C(l) \mathcal{R}_p^{(l+2)}$ that depends only on $p, D_*, a, N_*, k_0, \widetilde{C}_{\mathrm{UH}}$, and C_{Mult}. □

4.3 Consequences of (KM$_p$)

In this subsection, we see three important consequences of (KM$_p$). Throughout this subsection, we always suppose Assumption 4.2 holds.

First, we derive the supermultiplicative inequalities of (p,p)-Poincaré constants.

Theorem 4.14. *There exists a positive constant $C_{4.14}$ (depending only on p, D_*, N_*, k_0 and C_{KM}) such that*

$$\lambda_p^{(n)}\lambda_p^{(m)} \leq C_{4.14}\lambda_p^{(n+m)} \quad \textit{for any } n, m \in \mathbb{N}. \tag{4.17}$$

Proof. From Proposition 3.10 and Theorem 3.14-(3), we have that $\lambda_p^{(n)}\mathcal{R}_p^{(m)} \leq c_1\lambda_p^{(n+m)}$ for any $n, m \in \mathbb{N}$, where c_1 depends only on p, D_*, N_*, k_0. By (KM$_p$), we deduce that

$$C_{\mathrm{KM}}^{-1}\lambda_p^{(n)}\lambda_p^{(m)} \leq \lambda_p^{(n)}\mathcal{R}_p^{(m)} \leq c_1\lambda_p^{(n+m)}. \qquad \square$$

Remark 4.15. Since $\sigma_p^{(n)}$ and $\lambda_p^{(n)}$ are comparable by Theorem 3.14-(2), we also have the supermultiplicative inequality of $\sigma_p^{(n)}$. Moreover, by Theorem 3.14-(5) and (KM$_p$), $\mathcal{R}_p^{(n)}$ and $\lambda_p^{(n)}$ are also comparable, and thus we deduce the multiplicative inequality of $\mathcal{R}_p^{(n)}$:

$$c^{-1}\mathcal{R}_p^{(n)}\mathcal{R}_p^{(m)} \leq \mathcal{R}_p^{(n+m)} \leq c\mathcal{R}_p^{(n)}\mathcal{R}_p^{(m)}, \quad \text{for every } n, m \in \mathbb{N},$$

where c is a positive constant depending only on $p, D_*, N_*, k_0, C_{\mathrm{KM}}$. Recall that we have only used the submultiplicative inequality of $\mathcal{R}_p^{(n)}$ to obtain (4.5) in the proof of Theorem 4.5. One can avoid this use as in Remark 4.7 (see also [47, Theorem 7.16] in the case $p = 2$). Therefore, our arguments to prove Theorem 4.14 give an alternative proof of submultiplicative inequality of $\mathcal{R}_p^{(n)}$ under Assumption 4.2.

From these supermultiplicative inequalities of both $\sigma_p^{(n)}$ and $\lambda_p^{(n)}$, we deduce the following behaviors of (p, p)-Poincaré constants:

$$\lambda_p^{(n)} \leq c_*\rho_p^n \quad \text{and} \quad \sigma_p^{(n)} \leq c_*\rho_p^n \quad \text{for every } n \in \mathbb{N}, \tag{4.18}$$

where $\rho_p = \lim_{n\to\infty}\bigl(\mathcal{R}_p^{(n)}\bigr)^{1/n}$ (see Theorem 3.4) and c_* depends only on p, D_*, N_*, k_0 and C_{KM}. Now, we define the rescaled discrete p-energy $\widetilde{\mathcal{E}}_p^{G_n}$ by setting

$$\widetilde{\mathcal{E}}_p^{G_n}(f) := \rho_p^n \mathcal{E}_p^{G_n}(f),$$

for each $n \in \mathbb{N}$ and $f\colon W_n \to \mathbb{R}$. Then, by Theorem 4.5, the following estimate is obvious.

Corollary 4.16. *For every $n, m \in \mathbb{N}$, $z \in W_m$, $v, w \in \mathcal{B}_n(z, 1)$ and $f\colon W_{n+m} \to \mathbb{R}$,*

$$|f(v) - f(w)|^p \leq C_{\mathrm{UH}}\,\widetilde{\mathcal{E}}_p^{G_{n+m}}(f)a^{-(\beta_p-\alpha)m}, \tag{4.19}$$

where $C_{\mathrm{UH}} := \widetilde{C}_{\mathrm{UH}}c_*$ depending only on $p, D_*, N_*, k_0, a, \rho_p, C_{\mathrm{Mult}}, C_{\mathrm{KM}}$.

Lastly, we observe a monotonicity result (the so-called *weak monotonicity* in [25]). This is proved in [47, Proposition 5.2] for $p = 2$.

Corollary 4.17. *For every $n, m \in \mathbb{N}$ and $f \in L^p(K, \mu)$,*

$$\widetilde{\mathcal{E}}_p^{G_n}(M_n f) \leq C_{\mathrm{WM}}\,\widetilde{\mathcal{E}}_p^{G_{n+m}}(M_{n+m} f), \tag{4.20}$$

where $C_{\mathrm{WM}} := c_* D_*$ (that depends only on p, D_*, N_*, k_0, C_{KM}). In particular,

$$\sup_{n\in\mathbb{N}} \widetilde{\mathcal{E}}_p^{G_n}(M_n f) \leq C_{\mathrm{WM}} \varliminf_{n\to\infty} \widetilde{\mathcal{E}}_p^{G_n}(M_n f). \tag{4.21}$$

Proof. Note that by Lemma 2.4, it follows that for any $n, m \in \mathbb{N}$, $w \in W_n$ and $f \in L^p(K, \mu)$,

$$P_{n+m,n}(M_{n+m} f)(w) = N_*^n \sum_{v\in w\cdot W_m} \int_{K_v} f\, d\mu = M_n f(w).$$

Thus we have $P_{n+m,n}(M_{n+m} f) = M_n f$. Then, by Lemma 3.3 and (4.18), we get (4.20). □

Remark 4.18. One can derive a uniform Harnack type estimate for discrete p-harmonic functions as an application of (KM$_p$). For $p = 2$, this was done by [5, Theorem 3.1] or [47, Lemma 7.8]. We expect that such type estimate will be important for future work, but we omit this since its proof does not fit the purpose of this paper.

5 The domain of p-energy

This section aims to prove a part of our main results: Theorems 2.17 and 2.18.

In view of Corollary 4.17, the following quantity:

$$|f|_{\mathcal{F}_p} \coloneqq \sup_{n\in\mathbb{N}} \widetilde{\mathcal{E}}_p^{G_n}(M_n f)^{1/p}.$$

describes the limit behavior of rescaled p-energy $\widetilde{\mathcal{E}}_p^{G_n}(M_n f)$. Then we easily see that $|\cdot|_{\mathcal{F}_p}$ defines a ($[0,\infty]$-valued) semi-norm. We also define a function space $\mathcal{F}_p$ and its norm $\|\cdot\|_{\mathcal{F}_p}$ by setting

$$\mathcal{F}_p \coloneqq \{f \in L^p(K,\mu) \mid |f|_{\mathcal{F}_p} < \infty\} \quad \text{and} \quad \|f\|_{\mathcal{F}_p} \coloneqq \|f\|_{L^p} + |f|_{\mathcal{F}_p}.$$

Ideally, $\mathcal{F}_p$ plays the same role as the Sobolev space $W^{1,p}$ in smooth settings like Euclidean spaces. As stated in [35, Section 7], this $(1,p)$-"Sobolev" space $\mathcal{F}_p$ should be *closable* and have *regularity*, that is,

- any Cauchy sequence $\{f_n\}_{n\geq 1}$ in $|\cdot|_{\mathcal{F}_p}$ with $f_n \to 0$ in L^p converges to 0 in $\mathcal{F}_p$;
- $\mathcal{F}_p \cap C(K)$ is dense in $C(K)$ with respect to the sup norm.

We prove these properties in subsection 5.1. In addition, the *separability* of $\mathcal{F}_p$ is proved in subsection 5.2. The separability will be essential to follow our construction of p-energy in section 6. We also see in subsection 5.3 that $\mathcal{F}_p$ has a Besov-like representation, which is an extension of results for $\mathcal{F}_2$ in [24].

Throughout this section, we suppose Assumption 4.2 holds.

5.1 Closability and regularity

First, we derive the following Hölder estimate from the uniform Hölder estimates on graphical approximations (Corollary 4.16) in the same way as [38, Lemmas 6.10 and 6.13].

Theorem 5.1. *There exists a positive constant $C_{\mathrm{H\ddot{o}l}}$ (depending only on C_{UH}, C_{AD}, p, ρ_p and a) such that every $f \in \mathcal{F}_p$ has a continuous modification $f_* \in C(K)$ with*

$$|f_*(x) - f_*(y)|^p \leq C_{\mathrm{H\ddot{o}l}}\, |f|_{\mathcal{F}_p}^p\, d(x,y)^{\beta_p - \alpha},$$

for every $x, y \in K$. Moreover, the inclusion map $\mathcal{F}_p \ni f \mapsto f_* \in C(K)$ is injective. In particular, $\mathcal{F}_p$ is continuously embedded in the Hölder space $C^{0,(\beta_p-\alpha)/p}$.

Proof. Let $f \in \mathcal{F}_p$. By Lemmas 2.4 and 2.5, for each $n \geq 1$, we have that $\int_K f\, d\mu = N_*^{-n} \sum_{w\in W_n} M_n f(w)$. From this identity, there exists $w(n) \in W_n$ for each $n \in \mathbb{N}$ such that $M_n\, |f|\, (w(n)) \leq \int_K |f|\, d\mu$. Then, by Corollary 4.16, for any $n \in \mathbb{N}$ and $v \in W_n$,

$$|M_n f(v)|^p \leq 2^{p-1}\, |M_n f(v) - M_n f(w(n))|^p + 2^{p-1}\, |M_n f(w(n))|^p$$

$$\leq 2^{p-1} C_{\mathrm{UH}} |f|_{\mathcal{F}_p}^p + 2^{p-1} \left(\int_K |f|^p \, d\mu \right)^p$$
$$\leq 2^{p-1} C_{\mathrm{UH}} |f|_{\mathcal{F}_p}^p + 2^{p-1} \|f\|_{L^p}^p ,$$

and hence we obtain the following uniform bound of $f \in \mathcal{F}_p$:

$$\sup_{n \geq 1} \max_{v \in W_n} |M_n f(v)| \leq c_1 \|f\|_{\mathcal{F}_p} < \infty, \tag{5.1}$$

where $c_1 > 0$ depends only on p and C_{UH}.

For each $n \in \mathbb{N}$, enumerate the elements W_n as $W_n = \{w_n(1), \ldots, w_n(N_*^n)\}$ and inductively define $\{\widehat{K}_{w_n(i)}\}_{i=1}^{N_*^n}$ as follows: $\widehat{K}_{w_n(1)} \coloneqq K_{w_n(1)}$ and

$$\widehat{K}_{w_n(i+1)} \coloneqq K_{w_n(i+1)} \setminus \bigcup_{j \leq i} \widehat{K}_{w_n(j)}.$$

Note that each $\widehat{K}_{w_n(i)}$ is a Borel set of K, $\widehat{K}_{w_n(i)}$ $(i = 1, \ldots, N_*^n)$ are disjoint, and $K = \bigcup_{i=1}^{N_*^n} \widehat{K}_{w_n(i)}$. Also, by Lemma 2.4, we have $\mu\bigl(K_w \setminus \widehat{K}_w\bigr) = 0$ for any $w \in W_n$. Next, define a Borel measurable function $f_n \colon K \to \mathbb{R}$ by setting

$$f_n \coloneqq \sum_{w \in W_n} M_n f(w) \mathbb{1}_{\widehat{K}_w}.$$

Then (5.1) yields that

$$\sup_{n \geq 1} \sup_{x \in K} |f_n(x)| \leq c_1 \|f\|_{\mathcal{F}_p}. \tag{5.2}$$

Let $n \in \mathbb{N}$, $x \neq y \in K$ and set $n_* \coloneqq n(x, y) \in \mathbb{Z}_{\geq 0}$. In case when $n > n_*$, then there exist $v, w \in W_{n_*}$ such that $x \in K_v$, $y \in K_w$ and $K_v \cap K_w \neq \emptyset$. We can find $v', w' \in W_n$ such that $x \in \widehat{K}_{v'}$ and $y \in \widehat{K}_{w'}$. Then $v' \in \mathcal{B}_{n-n_*}(v, 1)$, $w' \in \mathcal{B}_{n-n_*}(w, 1)$ and $\mathcal{B}_{n-n_*}(v, 1) \cap \mathcal{B}_{n-n_*}(w, 1) \neq \emptyset$. Fix $z' \in \mathcal{B}_{n-n_*}(v, 1) \cap \mathcal{B}_{n-n_*}(w, 1)$. Applying Corollary 4.16, we have that

$$\begin{aligned} |f_n(x) - f_n(y)|^p &\leq 2^{p-1} \bigl(|M_n f(v') - M_n f(z')|^p + |M_n f(z') - M_n f(w')|^p \bigr) \\ &\leq 2^p C_{\mathrm{UH}} \widetilde{\mathcal{E}}_p^{G_n}(M_n f) a^{-(\beta_p - \alpha) n_*} \\ &\leq 2^p C_{\mathrm{UH}} C_{\mathrm{AD}} |f|_{\mathcal{F}_p}^p \, d(x, y)^{\beta_p - \alpha}, \end{aligned}$$

where we used Lemma 2.7 in the last line. In the case where $n \leq n_*$, then there exist $v, w \in W_n$ such that $x \in K_v$, $y \in K_w$ and $K_v \cap K_w \neq \emptyset$. Let $v', w' \in W_n$ with $x \in \widehat{K}_{v'}$ and $y \in \widehat{K}_{w'}$. Then $[v', v, w, w']$ is a path in G_n, and hence we have that

$$\begin{aligned} &|f_n(x) - f_n(y)|^p \\ &\leq 3^{p-1} \bigl(|M_n f(v') - M_n f(v)|^p + |M_n f(v) - M_n f(w)|^p + |M_n f(w) - M_n f(w')|^p \bigr) \\ &\leq 3^p \mathcal{E}_p^{G_n}(M_n f) \leq 3^p \, |f|_{\mathcal{F}_p}^p \, \rho_p^{-n}. \end{aligned}$$

As a result of this observation, we conclude that

$$|f_n(x) - f_n(y)|^p \leq c_2 \, |f|_{\mathcal{F}_p}^p \left(d(x, y)^{\beta_p - \alpha} + \rho_p^{-n} \right), \quad f \in \mathcal{F}_p, n \in \mathbb{N}, x, y \in K, \tag{5.3}$$

where c_2 is a positive constant depending only on $p, C_{\mathrm{UH}}, C_{\mathrm{AD}}$.

Thanks to (5.2) and (5.3), we can apply an Arzelá–Ascoli type argument for $\{f_n\}_{n\geq 1}$ (see [38, Lemma D.1]). For reader's convenience, we provide a complete proof. Set $A_n := \{F_w(p_i)\}_{i\in S, w\in W_n}$ and $A := \bigcup_{n\geq 1} A_n$. Then A is a countable dense subset of K. Since $\{f_n(x)\}_{n\geq 1}$ is bounded for each $x \in A$ by (5.2), by a diagonal argument, we obtain a subsequence $\{n_k\}_{k\geq 1}$ such that $\{f_{n_k}(x)\}_{k\geq 1}$ converges as $k \to \infty$ for any $x \in A$. Define $f_*(x) := \lim_{k\to\infty} f_{n_k}(x)$ for any $x \in A$. From (5.3) and Assumption 4.2, we see that

$$|f_*(x) - f_*(y)|^p \leq c_2 \, |f|_{\mathcal{F}_p}^p \, d(x,y)^{\beta_p - \alpha} \quad \text{for any } x, y \in A.$$

Since A is dense in K, f_* is extended to a continuous function on K, which is again denoted by $f_* \in C(K)$, and it follows that

$$|f_*(x) - f_*(y)|^p \leq c_2 \, |f|_{\mathcal{F}_p}^p \, d(x,y)^{\beta_p - \alpha} \quad \text{for any } x, y \in K.$$

(We can also show that $\sup_{x\in K} \left|f_*(x) - f_{n_k}(x)\right| \to 0$ as $k \to \infty$. For a proof, see [38, Lemma D.1].) Then, for any $m \in \mathbb{N}$ and $w \in W_m$, we have $\int_{K_w} f_{n_k} \, d\mu \to \int_{K_w} f_* \, d\mu \quad$ as $k \to \infty$. By Lemma 2.4 and $\mu\bigl(K_w \setminus \widehat{K}_w\bigr) = 0$,

$$\begin{aligned}
\int_{K_w} f_{n_k} \, d\mu &= \int_{K_w} \sum_{z\in W_{n_k}} M_{n_k} f(z) \mathbb{1}_{\widehat{K}_z} \, d\mu = \int_{K_w} \sum_{z\in w\cdot W_{n_k - m}} M_{n_k} f(z) \mathbb{1}_{K_z} \, d\mu \\
&= \sum_{z\in w\cdot W_{n_k-m}} \frac{1}{\mu(K_z)} \left(\int_{K_z} f \, d\mu\right) \int_{K_w} \mathbb{1}_{K_z} \, d\mu = \int_{K_w} f \, d\mu,
\end{aligned}$$

whenever $w \in W_m$ and $n_k > m$. Letting $k \to \infty$, we obtain $\int_{K_w} f_* \, d\mu = \int_{K_w} f \, d\mu$ for all $w \in W_\#$. By Lemma 2.4 and Dynkin's π-λ theorem, we conclude that f_* is a continuous modification of f. The injectivity of $f \mapsto f_*$ is obvious. We complete the proof. □

Next, we prove the closability by proving that $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ is complete. See also [38, Lemmas 6.15 and 6.16].

Theorem 5.2. *$(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ is a Banach space.*

Proof. Let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence in $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$. Then $\{f_n\}_{n\geq 1}$ converges to some $f \in L^p(K, \mu)$ in L^p. Fix $x_0 \in K$ and set $g_n := f_n - f_n(x_0)$. Then, by the Hölder estimate in Theorem 5.1, for all $n, m \geq 1$ and $x \in K$,

$$\begin{aligned}
|g_n(x) - g_m(x)|^p &\leq C_{\mathrm{H\ddot{o}l}} \, |f_n - f_m|_{\mathcal{F}_p}^p \, d(x, x_0)^{\beta_p - \alpha} \\
&\leq C_{\mathrm{H\ddot{o}l}} (\operatorname{diam} K)^{\beta_p - \alpha} \, |f_n - f_m|_{\mathcal{F}_p}^p,
\end{aligned}$$

and hence we obtain $\|f_n - f_m\|_{C(K)} \leq c \, |f_n - f_m|_{\mathcal{F}_p}$, where $c := C_{\mathrm{H\ddot{o}l}} (\operatorname{diam} K)^{\beta_p - \alpha}$. This estimate implies that $\{g_n\}_{n\geq 1}$ is a Cauchy sequence in $C(K)$. Since $C(K)$ is complete, $\{g_n\}_{n\geq 1}$ converges to some $g \in C(K)$ in the sup norm.

It is immediate that $\{f_n - g_n\}_{n\geq 1}$ converges to $f - g$ in L^p, and thus we can pick a subsequence $\{n_k\}_{k\geq 1}$ so that $f_{n_k} - g_{n_k} \to f - g$ for μ-a.e. as $k \to \infty$. On the other hand, the definition of g_n implies that $f_n - g_n \equiv f_n(x_0)$. Hence the limit $\lim_{k\to\infty} f_{n_k}(x_0) =: A$ exists and $f - g = A$ for μ-a.e. In particular, f admits a continuous modification. We again write f to denote this continuous version. Then f is the limit of f_{n_k}. Indeed, we have

$$\left\|f - f_{n_k}\right\|_{C(K)} \leq \left\|g - g_{n_k}\right\|_{C(K)} + \left|f_{n_k}(x_0) - A\right| \to 0 \quad \text{as } k \to \infty.$$

Since $\{f_n\}_{n\geq 1}$ is a Cauchy sequence in $\mathcal{F}_p$, for any $\varepsilon > 0$ there exists $N(\varepsilon) \geq 1$ such that

$$\sup_{i\wedge j\geq N(\varepsilon)} \sup_{k\geq 1} \widetilde{\mathcal{E}}_p^{G_k}(M_k f_{n_i} - M_k f_{n_j}) \leq \varepsilon,$$

which implies that

$$\sup_{i\geq N(\varepsilon)} \sup_{k\geq 1} \widetilde{\mathcal{E}}_p^{G_k}(M_k f_{n_i} - M_k f) \leq \varepsilon. \tag{5.4}$$

Therefore, we have that, for large $i \geq 1$ with $n_i \geq N(\varepsilon)$,

$$\widetilde{\mathcal{E}}_p^{G_k}(M_k f)^{1/p} \leq \widetilde{\mathcal{E}}_p^{G_k}(M_k f_{n_i} - M_k f)^{1/p} + \widetilde{\mathcal{E}}_p^{G_k}(M_k f_{n_i})^{1/p} \leq \varepsilon + \sup_{n\geq 1} |f_n|_{\mathcal{F}_p},$$

which implies that $f \in \mathcal{F}_p$. In addition, (5.4) yields that $\left\|f - f_{n_i}\right\|_{\mathcal{F}_p} \to 0$ as $i \to \infty$.

The convergence $\|f - f_n\|_{\mathcal{F}_p} \to 0$ is easily derived by applying the above arguments for any subsequence of $\{f_n\}_{n\geq 1}$. We complete the proof. □

Moreover, we can show that $\mathcal{F}_p$ is compactly embedded in $L^p(K, \mu)$.

Proposition 5.3. *The inclusion map from $\mathcal{F}_p$ to $L^p(K, \mu)$ is a compact operator.*

Proof. Let $\{f_n\}_{n\geq 1}$ be a bounded sequence in $\mathcal{F}_p$. Since the embedding of $\mathcal{F}_p$ in $C^{0,(\beta_p-\alpha)/p}$ is continuous, we obtain a subsequence $\{f_{n_k}\}_{k\geq 1}$ and $f \in C(K)$ such that f_{n_k} converges to f in the sup norm by applying the Arzelá–Ascoli theorem. This proves our assertion. □

Towards the regularity of $\mathcal{F}_p$, the following lemma gives a "partition of unity" in $\mathcal{F}_p$. See also [38, Lemma 6.18].

Lemma 5.4. *There exists a family $\{\varphi_w\}_{w\in W_\#}$ in $\mathcal{F}_p$ such that*

(a) *for any $w \in W_\#$, $0 \leq \varphi_w \leq 1$;*

(b) *for any $n \in \mathbb{N}$, $\sum_{w\in W_n} \varphi_w \equiv 1$;*

(c) *for any $n \in \mathbb{N}$ and $w \in W_n$, $\mathrm{supp}[\varphi_w] \subseteq U_1^{(n)}(w)$, where $U_1^{(n)}(w)$ is defined as*

$$U_1^{(n)}(w) := \bigcup_{v\in W_n; d_{G_n}(v,w)\leq 1} K_v;$$

(d) *there exists a constant $C_{5.4} > 0$ (depending only on $p, D_*, C_{\mathrm{Mult}}, C_{\mathrm{WM}}$) such that*

$$|\varphi_w|_{\mathcal{F}_p}^p \le C_{5.4}\rho_p^n \quad \text{for any } n \in \mathbb{N} \text{ and } w \in W_n.$$

Proof. For each $n, m \in \mathbb{N}$ and $w \in W_m$, let $\psi_w^{(n)} \colon W_{n+m} \to [0,1]$ satisfy $\psi_w^{(n)}|_{w\cdot W_n} \equiv 1$, $\psi_w^{(n)}|_{W_{n+m}\setminus\mathcal{B}_n(w,1)} \equiv 0$ and $\mathcal{E}_p^{G_{n+m}}(\psi_w^{(n)}) = C_p^{G_{n+m}}(w\cdot W_m, W_{n+m}\setminus\mathcal{B}_n(w,1))$. Define $\Psi_m^{(n)} := \left(\sum_{v\in W_m}\psi_v^{(n)}\right)^{-1}$ and $\varphi_w^{(n)} := \Psi_m^{(n)}\psi_w^{(n)}$. Note that $\varphi_w^{(n)}$ coincides with the function denoted by the same symbol in the proof of Lemma 3.13. We also set $\widetilde{\varphi}_w^{(n)} \colon K \to \mathbb{R}$ by setting

$$\widetilde{\varphi}_w^{(n)} := \sum_{z\in W_{n+m}} \varphi_w^{(n)}(z)\mathbb{1}_{\widehat{K}_z},$$

where $\{\widehat{K}_z\}_{z\in W_\#}$ is the same as in the proof of Theorem 5.1. Then $M_{n+m}\widetilde{\varphi}_w^{(n)} = \varphi_w^{(n)}$ and, from the estimate (3.12) in the proof of Lemma 3.13, we have that $\widetilde{\mathcal{E}}_p^{G_{n+m}}(\varphi_w^{(n)}) \le c_1\rho_p^m$ for all $w \in W_m$ and $n \in \mathbb{N}$, where $c_1 > 0$ depends only on $p, D_*, C_{\mathrm{Mult}}$. In particular, by Theorem 5.1, we obtain

$$\left|\widetilde{\varphi}_w^{(n)}(x) - \widetilde{\varphi}_w^{(n)}(y)\right|^p \le c_1 C_{\mathrm{H\ddot{o}l}}\,\rho_p^m d(x,y)^{\beta_p-\alpha},$$

for $x, y \in K$ with $n(x,y) < n+m$. Similarly to the Arzelá–Ascoli type argument in the proof of Theorem 5.1, we can find a subsequence $\{n_k\}_{k\ge1}$ and a continuous function $\varphi_w \in C(K)$ such that $\lim_{k\to\infty}\widetilde{\varphi}_w^{(n_k)}(x) = \varphi_w(x)$ for any $x \in K$ and

$$|\varphi_w(x) - \varphi_w(y)|^p \le c_1 C_{\mathrm{H\ddot{o}l}}\,\rho_p^{|w|} d(x,y)^{\beta_p-\alpha} \quad \text{for any } x, y \in K.$$

Then the properties (a), (b) and (c) are immediate from this convergence and the associated properties of $\widetilde{\varphi}_w^{(n)}$, so it will suffice to show (d). By the weak monotonicity (Corollary 4.17),

$$\begin{aligned}
\widetilde{\mathcal{E}}_p^{G_l}(M_l\widetilde{\varphi}_w^{(n_k)}) &\le C_{\mathrm{WM}}\widetilde{\mathcal{E}}_p^{G_{n_k+m}}(M_{n_k+m}\widetilde{\varphi}_w^{(n_k)}) \\
&\le C_{\mathrm{WM}}\sup_{n\ge1}\widetilde{\mathcal{E}}_p^{G_{n+m}}(\varphi_w^{(n)}) \\
&\le c_1 C_{\mathrm{WM}}\rho_p^m,
\end{aligned}$$

whenever $l \le n_k + m$. Taking $k \to \infty$ and supremum over $l \in \mathbb{N}$ in this estimate, we conclude that $|\varphi_w|_{\mathcal{F}_p}^p \le c_1 C_{\mathrm{WM}}\rho_p^m$ for all $m \ge 1$ and $w \in W_m$. This completes the proof. □

Now, define a subspace $\mathcal{H}_p^\star$ of $\mathcal{F}_p$ by setting

$$\mathcal{H}_p^\star := \left\{ \sum_{w\in A} a_w\varphi_w \;\middle|\; A \text{ is a finite subset of } W_\#,\ a_w \in \mathbb{R} \text{ for each } w \in A \right\}, \tag{5.5}$$

where $\{\varphi_w\}_{w\in W_\#}$ is a family of functions in $\mathcal{F}_p$ appeared in Lemma 5.4. Then we achieve the regularity of $\mathcal{F}_p$ (see also [38, Lemma 6.19]).

Theorem 5.5. *The space $\mathcal{H}_p^\star$ is dense in $C(K)$ with respect to the sup norm. In particular, $\mathcal{F}_p$ is dense in $C(K)$.*

Proof. Let $f \in C(K)$ and define f_n by setting $f_n := \sum_{w \in W_n} M_n f(w) \varphi_w \in \mathcal{H}_p^\star$. Then

$$\begin{aligned} |f(x) - f_n(x)| &\leq D_*^2 \max_{w \in W_n; x \in \operatorname{supp}[\varphi_w]} |f(x) - M_n f(w)| \\ &\leq D_*^2 \max_{w \in W_n} \sup_{x \in U_1(w), y \in K_w} |f(x) - f(y)| . \end{aligned}$$

Since $\max_{w \in W_n} \operatorname{diam} U_1^{(n)}(w) \to 0$ and f is uniformly continuous, $\|f - f_n\|_{C(K)} \to 0$. □

5.2 Separability

In this subsection, we prove that $\mathcal{F}_p$ is separable with respect to $\|\cdot\|_{\mathcal{F}_p}$. In the case $p = 2$, this is done by applying easy functional analytic arguments since the polarization formula of $\mathcal{E}_2$ yields a non-negative definite closed quadratic form. For example, by Proposition 5.3, the inclusion map from $\mathcal{F}_2$ to $L^2(K, \mu)$ is a compact operator, and thus there exists a countable complete orthonormal system of $\mathcal{F}_2$ (see [21, Exercise 4.2 and Corollary 4.2.3] for example). One can also give a short proof of the separability of $\mathcal{F}_2$ using resolvents (see [23, proof of Theorem 1.4.2-(iii)] for example). However, it is hopeless to execute similar arguments for general p.

To overcome this difficulty, we directly show that the space $\mathcal{H}_p^\star$ defined in (5.5) is dense in $\mathcal{F}_p$ and hence $\mathbb{Q}$-hull of $\{\varphi_w\}_{w \in W_\#}$ is also dense. Our strategy is standard in calculus of variations, namely, we extract a strong convergent approximation from $\mathcal{H}_p^\star$ by using Mazur's lemma (see [16, Corollary 3.8] for example). To this end, it will be a key ingredient to ensure the reflexivity of $\mathcal{F}_p$, which is deduced from a combination of *Clarkson's inequality* and the Milman–Pettis theorem (see [16, Theorem 3.31] for example). We will derive Clarkson's inequality by using Γ-convergence to find a norm of $\mathcal{F}_p$ having the required properties.

We start by recalling Clarkson's inequality.

Definition 5.6 (Clarkson's inequality)**.** Let $(\mathcal{X}, \|\cdot\|)$ be a norm space. We say that $(\mathcal{X}, \|\cdot\|)$ satisfies Clarkson's inequality if one of the following holds:

(1) Let $p \in (1, 2]$. For every $x, y \in \mathcal{X}$, it holds that

$$\|x + y\|^{\frac{p}{p-1}} + \|x - y\|^{\frac{p}{p-1}} \leq 2\big(\|x\|^p + \|y\|^p\big)^{\frac{1}{p-1}};$$

(2) Let $p \in [2, \infty)$. For every $x, y \in \mathcal{X}$, it holds that

$$\|x + y\|^p + \|x - y\|^p \leq 2^{p-1}\big(\|x\|^p + \|y\|^p\big).$$

It is well-known that L^p-norm on a measurable space satisfies Clarkson's inequality, and that a norm space satisfying Clarkson's inequality is *uniformly convex*.

Next, let us recall the definition of Γ-convergence and its basic properties. The reader is referred to [20] for details on Γ-convergence.

Definition 5.7 (Γ-convergence)**.** Let $\{\Phi_n\}_{n\geq 1}$ be a sequence of $[-\infty, \infty]$-valued functional on $L^p(K, \mu)$. We say that a functional $\Phi\colon L^p(K, \mu) \to [-\infty, \infty]$ is a Γ-limit of $\{\Phi_n\}_{n\geq 1}$ as $n \to \infty$ if the following two inequalities hold;

(1) (liminf inequality) If $f_n \to f$ in L^p, then $\Phi(f) \leq \varliminf_{n\to\infty} \Phi_n(f_n)$.

(2) (limsup inequality) For any $f \in L^p(K, \mu)$, there exists a sequence $\{f_n\}_{n\geq 1}$ such that

$$f_n \to f \text{ in } L^p \text{ and } \varlimsup_{n\to\infty} \Phi_n(f_n) \leq \Phi(f). \tag{5.6}$$

A sequence $\{f_n\}_{n\geq 1}$ satisfying (5.6) is called a *recovery sequence* of f.

Since the Sierpiński carpet K is separable, the following fact holds.

Theorem 5.8 ([20, Theorem 8.5])**.** *Let $\{\Phi_n\}_{n\geq 1}$ be a sequence of functionals on $L^p(K, \mu)$. Then there exists a subsequence $\{n_k\}_{k\geq 1}$ and a functional Φ on $L^p(K, \mu)$ such that Φ is a Γ-limit of $\{\Phi_{n_k}\}_{k\geq 1}$.*

Now, we regard $\widetilde{\mathcal{E}}_p^{G_n}(\,\cdot\,)$ as a $[0, \infty]$-valued functional on $L^p(K, \mu)$ defined by $f \mapsto \widetilde{\mathcal{E}}_p^{G_n}(M_n f)$. Then, by Theorem 5.8, there exists a Γ-convergent subsequence $\{\widetilde{\mathcal{E}}_p^{G_{n_k}}(\,\cdot\,)\}_{k\geq 1}$ and we write $\mathsf{E}_p(\,\cdot\,)$ to denote its Γ-limit. We define $|||\cdot|||_{\mathcal{F}_p} := \left(\|\cdot\|_{L^p}^p + \mathsf{E}_p(\,\cdot\,)\right)^{1/p}$. This new "norm" $|||\cdot|||_{\mathcal{F}_p}$ establishes the reflexivity. (We also need to show that $|||\cdot|||_{\mathcal{F}_p}$ is a norm.)

Theorem 5.9. *The norm $|||\cdot|||_{\mathcal{F}_p}$ is equivalent to $\|\cdot\|_{\mathcal{F}_p}$ and satisfies Clarkson's inequality. In particular, the Banach space $\mathcal{F}_p$ is reflexive.*

Proof. Let $f, g \in L^p(K, \mu)$ and let $\{f_n\}_{n\geq 1}$, $\{g_n\}_{n\geq 1}$ be their recovery sequences throughout the proof. To verify the triangle inequality of $|||\cdot|||_{\mathcal{F}_p}$, define

$$\|f\|_{p,n} := \left(\|f\|_{L^p}^p + \widetilde{\mathcal{E}}_p^{G_n}(M_n f)\right)^{1/p}.$$

Note that the Γ-limit of $\{\|\cdot\|_{p,n_k}\}_{k\geq 1}$ coincides with $|||\cdot|||_{\mathcal{F}_p}$ and that the norm $\|\cdot\|_{p,n}$ can be regarded as a L^p-norm on $K \sqcup W_n$. Using the triangle inequality of $\|\cdot\|_{p,n}$, we see that

$$\begin{aligned} |||f+g|||_{\mathcal{F}_p} \leq \varliminf_{k\to\infty} \left\|f_{n_k} + g_{n_k}\right\|_{p,n_k} &\leq \varlimsup_{k\to\infty} \left\|f_{n_k}\right\|_{p,n_k} + \varlimsup_{k\to\infty} \left\|g_{n_k}\right\|_{p,n_k} \\ &\leq |||f|||_{\mathcal{F}_p} + |||g|||_{\mathcal{F}_p}, \end{aligned}$$

and thus $|||\cdot|||_{\mathcal{F}_p}$ is an extended norm on $L^p(K, \mu)$ (we admit $|||f|||_{\mathcal{F}_p} = \infty$).

Next, we prove $C_{\mathrm{WM}}^{-1} |f|_{\mathcal{F}_p}^p \leq \mathsf{E}_p(f) \leq |f|_{\mathcal{F}_p}^p$ for every $f \in L^p(K, \mu)$ to conclude that $|||\cdot|||_{\mathcal{F}_p}$ and $\|\cdot\|_{\mathcal{F}_p}$ are equivalent. From the liminf inequality, we immediately have that $\mathsf{E}_p(f) \leq |f|_{\mathcal{F}_p}^p$ for $f \in L^p(K, \mu)$. To prove the converse, note that $M_n f_{n_k}(w) \to M_n f(w)$ for any $w \in W_n$ as $k \to \infty$ by the dominated convergence theorem. By the weak monotonicity (Corollary 4.17), we obtain

$$\widetilde{\mathcal{E}}_p^{G_n}(M_n f) \leq C_{\mathrm{WM}} \varliminf_{k\to\infty} \widetilde{\mathcal{E}}_p^{G_{n_k}}(M_{n_k} f_{n_k}) \leq C_{\mathrm{WM}} \mathsf{E}_p(f)$$

for all $n \geq 1$. We thus conclude that $|f|_{\mathcal{F}_p}^p \leq C_{\mathrm{WM}} \mathsf{E}_p(f)$.

The rest of the proof is mainly devoted to Clarkson's inequalities. First, we consider the case $p \leq 2$. By Clarkson's inequality for $\|\cdot\|_{p,n}$, we have

$$\|f+g\|_{p,n}^{\frac{p}{p-1}} + \|f-g\|_{p,n}^{\frac{p}{p-1}} \leq 2\bigl(\|f\|_{p,n}^p + \|g\|_{p,n}^p\bigr)^{\frac{1}{p-1}}.$$

Thus, we see that

$$\begin{aligned}
|||f+g|||_{\mathcal{F}_p}^{\frac{p}{p-1}} + |||f-g|||_{\mathcal{F}_p}^{\frac{p}{p-1}} &\leq \varliminf_{k\to\infty} \left\|f_{n_k}+g_{n_k}\right\|_{p,n_k}^{\frac{p}{p-1}} + \varliminf_{k\to\infty} \left\|f_{n_k}-g_{n_k}\right\|_{p,n_k}^{\frac{p}{p-1}} \\
&\leq 2 \varlimsup_{k\to\infty} \bigl(\left\|f_{n_k}\right\|_{p,n_k}^p + \left\|g_{n_k}\right\|_{p,n_k}^p\bigr)^{\frac{1}{p-1}} \\
&\leq 2\bigl(\varlimsup_{k\to\infty} \left\|f_{n_k}\right\|_{p,n_k}^p + \varlimsup_{k\to\infty} \left\|g_{n_k}\right\|_{p,n_k}^p\bigr)^{\frac{1}{p-1}} \\
&\leq 2\bigl(|||f|||_{\mathcal{F}_p}^p + |||g|||_{\mathcal{F}_p}^p\bigr)^{\frac{1}{p-1}},
\end{aligned}$$

which is Clarkson's inequality of $|||\cdot|||_{\mathcal{F}_p}$ when $p \leq 2$. Similarly, we get Clarkson's inequality for $p \geq 2$.

Consequently, we get a new norm $|||\cdot|||_{\mathcal{F}_p}$ of $\mathcal{F}_p$ satisfying Clarkson's inequality. Thus, we see that the Banach space $(\mathcal{F}_p, |||\cdot|||_{\mathcal{F}_p})$ is uniformly convex (see [16, proof of Theorem 4.10 and its remark] for example). Therefore, we finish the proof by the Milman–Pettis theorem (see [16, Theorem 3.31] for example). □

Theorem 5.10. *The space $\mathcal{H}_p^\star$ defined in* (5.5) *is dense in $\mathcal{F}_p$. Furthermore, $\mathcal{F}_p$ is separable.*

Proof. Recall the definition of $\widetilde{\varphi}_w^{(n)}$ in the construction of φ_w (see the proof of Lemma 5.4). By the diagonal procedure, we can pick a subsequence $\{n_k\}_{k\geq 1}$ such that $\{\widetilde{\varphi}_w^{(n_k)}\}_{k\geq 1}$ converges to φ_w with respect to the sup norm for all $w \in W_\#$. Next, for $f \in \mathcal{F}_p$, we define f_n and $f_n^{(k)}$ by setting

$$f_n := \sum_{w\in W_n} M_n f(w)\varphi_w, \quad f_n^{(k)} := \sum_{w\in W_n} M_n f(w)\widetilde{\varphi}_w^{(n_k)}.$$

Similarly to the proof of Theorem 5.5, we see that $\{f_n\}_{n\geq 1}$ converges to f with respect to the sup norm. Also, by Lemma 3.12, we obtain $\widetilde{\mathcal{E}}_p^{G_{n+n_k}}(M_{n+n_k} f_n^{(k)}) \leq c_1 |f|_{\mathcal{F}_p}^p$, where $c_1 > 0$ depends only on $p, D_*, C_{\mathrm{Mult}}$. Thus, by the weak monotonicity (Corollary 4.17), it holds that $\widetilde{\mathcal{E}}_p^{G_l}(M_l f_n^{(k)}) \leq c_1 C_{\mathrm{WM}} |f|_{\mathcal{F}_p}^p$ whenever $l \leq n + n_k$. Letting $k \to \infty$, we see that $\{f_n\}_{n\geq 1}$ is bounded in $\mathcal{F}_p$. Since $\mathcal{F}_p$ is reflexive, we may assume that a subsequence $\{f_{n_k}\}_{k\geq 1}$ converges to f weakly in $\mathcal{F}_p$. By Mazur's lemma (see [16, Corollary 3.8] for example), we can find a sequence $\{g_l\}_{l\geq 1}$ such that each g_l is a convex combination of $\{f_{n_k}\}_{k\geq 1}$ and $g_l \to f$ in $\mathcal{F}_p$ as $l \to \infty$. In particular, we obtain $\overline{\mathcal{H}_p^\star}^{\|\cdot\|_{\mathcal{F}_p}} = \mathcal{F}_p$. Clearly, the $\mathbb{Q}$-hull of $\{\varphi_w\}_{w\in W_\#}$ also gives this approximation, that is,

$$\overline{\left\{\sum_{w\in A} a_w\varphi_w \,\middle|\, A \text{ is a finite subset of } W_\# \text{ and } a_w \in \mathbb{Q}\ (w \in A)\right\}}^{\|\cdot\|_{\mathcal{F}_p}} = \overline{\mathcal{H}_p^\star}^{\|\cdot\|_{\mathcal{F}_p}}.$$

Therefore, $\mathcal{F}_p$ is separable. □

5.3 Lipschitz–Besov type expression

This subsection is devoted to proving Theorem 2.18. Let us start by introducing the definition of Lipschitz–Besov spaces on the Sierpiński carpet (K, d, μ) in some specific cases (see [11] for example). Recall that $a = 3$ denotes the inverse of the contraction ratio of F_i (see Definition 2.1).

Definition 5.11. For $s \in (0, \infty)$, $p \in [1, \infty)$, the *Lipschitz–Besov space* $\Lambda^s_{p,\infty}$ is defined as

$$\Lambda^s_{p,\infty} := \{ f \in L^p(K, \mu) \mid |f|_{\Lambda^s_{p,\infty}} < \infty \},$$

where

$$|f|_{\Lambda^s_{p,\infty}} := \sup_{n \in \mathbb{N}} \left(\int_K \fint_{B(x,a^{-n})} \frac{|f(x) - f(y)|^p}{a^{-nsp}} \, d\mu(y) d\mu(x) \right)^{1/p}.$$

We also define its norm $\|\cdot\|_{\Lambda^s_{p,\infty}}$ by setting $\|f\|_{\Lambda^s_{p,\infty}} := \|f\|_{L^p} + |f|_{\Lambda^s_{p,\infty}}$.

Then $(\Lambda^s_{p,\infty}, \|\cdot\|_{\Lambda^s_{p,\infty}})$ is a Banach space. Furthermore, for any $c \in [1, \infty)$ there exists a positive constant $C_{\mathrm{LB}}(c)$, which depends only on $c, a, \alpha, C_{\mathrm{AR}}$, such that, for any $f \in L^p(K, \mu)$,

$$|f|^p_{\Lambda^s_{p,\infty}} \le C_{\mathrm{LB}}(c) \sup_{n \in \mathbb{N}} \int_K \fint_{B(x,ca^{-n})} \frac{|f(x) - f(y)|^p}{a^{-nsp}} \, d\mu(y) d\mu(x), \tag{5.7}$$

and

$$\begin{aligned} C_{\mathrm{LB}}(c)^{-1} \varlimsup_{n \to \infty} & \int_K \fint_{B(x,ca^{-n})} \frac{|f(x) - f(y)|^p}{a^{-nsp}} \, d\mu(y) d\mu(x) \\ &\le \varlimsup_{r \downarrow 0} \int_K \fint_{B(x,r)} \frac{|f(x) - f(y)|^p}{r^{sp}} \, d\mu(y) d\mu(x) \\ &\le C_{\mathrm{LB}}(c) \varlimsup_{n \to \infty} \int_K \fint_{B(x,ca^{-n})} \frac{|f(x) - f(y)|^p}{a^{-nsp}} \, d\mu(y) d\mu(x). \end{aligned} \tag{5.8}$$

First, we prove a (p, p)-Poincaré inequality in the sense of Kumagai and Sturm (see [46, pp. 315]). Recall that $\alpha = \log N_* / \log a$ denotes the Hausdorff dimension of K, and that $\beta_p = \log(N_* \rho_p) / \log a$, where ρ_p is the resistance scaling factor in Theorem 3.4.

Lemma 5.12. *There exists a positive constant $C_{\mathrm{PI\text{-}KS}}$ (depending only on p, ρ_p, a, $C_{\mathrm{H\ddot{o}l}}$, C_{WM} and $\mathrm{diam}(K)$) such that*

$$a^{\beta_p n} \sum_{w \in A} \int_{K_w} |f(x) - M_n f(w)|^p \, d\mu(x) \le C_{\mathrm{PI\text{-}KS}} \varliminf_{l \to \infty} \widetilde{\mathcal{E}}^{A \cdot W_l}_p (M_{l+n} f), \tag{5.9}$$

for every $n \in \mathbb{N}$, $f \in \mathcal{F}_p$ and every subset $A \subseteq W_n$. In particular, it holds that

$$a^{\beta_p n} \sum_{w \in W_n} \int_{K_w} |f(x) - M_n f(w)|^p \, d\mu(x) \le C_{\mathrm{PI\text{-}KS}} \, |f|^p_{\mathcal{F}_p}.$$

Proof. Let $f \in \mathcal{F}_p$ be the continuous version. Then, by the mean value theorem, for any $n \in \mathbb{N}$ and $w \in W_n$ there exists $x_w \in K_w$ such that $f(x_w) = M_n f(w)$. From the Hölder estimate (Theorem 5.1), we have that, for any $x \in K_w$,

$$\begin{aligned}|f(x) - M_n f(w)|^p &= \left|F_w^* f(F_w^{-1}(x)) - F_w^* f(F_w^{-1}(x_w))\right|^p \\ &\le C_{\text{Höl}} \left|F_w^* f\right|_{\mathcal{F}_p}^p d(F_w^{-1}(x), F_w^{-1}(x_w))^{\beta_p - \alpha} \\ &\le C_{\text{Höl}} \operatorname{diam}(K)^{\beta_p - \alpha} \left|F_w^* f\right|_{\mathcal{F}_p}^p .\end{aligned}$$

Consequently, we obtain

$$\int_{K_w} |f(x) - M_n f(w)|^p \, d\mu(x) \le C_{\text{Höl}} \operatorname{diam}(K)^{\beta_p - \alpha} a^{-\alpha n} \left|F_w^* f\right|_{\mathcal{F}_p}^p .$$

Summing over $w \in W_n$, we conclude that

$$\begin{aligned}&\sum_{w \in A} \int_{K_w} |f(x) - M_n f(w)|^p \, d\mu(x) \\ &\le \left(C_{\text{Höl}} \operatorname{diam}(K)^{\beta_p - \alpha}\right) a^{-\alpha n} \sum_{w \in A} \left|F_w^* f\right|_{\mathcal{F}_p}^p \\ &\le \left(C_{\text{Höl}} C_{\text{WM}} \operatorname{diam}(K)^{\beta_p - \alpha}\right) a^{-\alpha n} \sum_{w \in A} \varliminf_{l \to \infty} \widetilde{\mathcal{E}}_p^{G_l}\left(M_l(F_w^* f)\right) \\ &\le \left(C_{\text{Höl}} C_{\text{WM}} \operatorname{diam}(K)^{\beta_p - \alpha}\right) a^{-\alpha n} \varliminf_{l \to \infty} \sum_{w \in A} \widetilde{\mathcal{E}}_p^{G_l}\left(M_l(F_w^* f)\right),\end{aligned}$$

where we used the weak monotonicity (Corollary 4.17) in the second line. From (2.2), we see that $\sum_{w \in A} \widetilde{\mathcal{E}}_p^{G_l}\left(M_l(F_w^* f)\right) \le \rho_p^{-n} \widetilde{\mathcal{E}}_p^{A \cdot W_l}(M_{l+n} f)$. In particular, we obtain

$$a^{-\alpha n} \varliminf_{l \to \infty} \sum_{w \in A} \widetilde{\mathcal{E}}_p^{G_l}\left(M_l(F_w^* f)\right) \le a^{-\beta_p n} \varliminf_{l \to \infty} \widetilde{\mathcal{E}}_p^{A \cdot W_l}(M_{l+n} f),$$

which proves (5.9). □

Next, we give an extension of [24, Theorem 4.11-(iii)]. This is essentially proved in [2, Theorem 5.1], so its proof is omitted here. We give a direct proof without using heat kernels in Appendix A.2 for the reader's convenience.

Lemma 5.13. *Let $\beta > \alpha$ and $p > 1$. Then there exists a positive constant $C_{5.13}$ (depending only on $p, \beta, \alpha, C_{\text{AR}}$) such that*

$$\begin{aligned}&|f(x) - f(y)|^p \\ &\quad \le C_{5.13} d(x,y)^{\beta - \alpha} \sup_{r \in (0, 3d(x,y)]} r^{-\beta} \int_K \fint_{B(z,r)} |f(z) - f(z')|^p \, d\mu(z') d\mu(z),\end{aligned}$$

for every $f \in \Lambda_{p,\infty}^{\beta/p}$ and μ-a.e. $x, y \in K$. In particular, for any $g \in C(K)$ and $x, y \in K$,

$$|g(x) - g(y)|^p \le C_{5.13} \left|g\right|_{\Lambda_{p,\infty}^{\beta/p}}^p d(x,y)^{\beta - \alpha}.$$

An important consequence of the above lemma is the following type "(p,p)-Poincaré inequality".

Lemma 5.14. *Let $\beta > \alpha$ and $p > 1$. Then there exists a positive constant $C_{\text{PI-LB}}$ (depending only on $p, \beta, \alpha, a, \operatorname{diam}(K), C_{\text{AR}}$) such that*

$$\tag{5.10} \begin{aligned} &a^{\beta n} \sum_{w\in W_n} \int_{K_w} |f(x) - M_n f(w)|^p \, d\mu(x) \\ &\leq C_{\text{PI-LB}} \sup_{r\in(0,3\operatorname{diam}(K)a^{-n}]} r^{-\beta} \int_K \fint_{B(x,r)} |f(x) - f(y)|^p \, d\mu(y) d\mu(x), \end{aligned}$$

for every $n \in \mathbb{N}$ and $f \in C(K)$.

Proof. We adopt a method in [25, proof of Theorem 3.5] and generalize it to fit our context. Let $f \in \Lambda^{\beta/p}_{p,\infty}$, let $w = w_1 \cdots w_n \in W_n$ and fix $k \in \mathbb{N}$ which we choose later. Then, by the mean value theorem, there exists $x_w \in K_w$ such that $M_n f(w) = f(x_w)$. Let $\omega \in \pi^{-1}(\{x_w\})$ such that $[\omega]_n = w$. For each $m \in \mathbb{Z}_{\geq 0}$, we define $w(m) := [\omega]_{n+km} \in W_{n+km}$. Then, for $z_m \in K_{w(m)}$ $(m = 0, \ldots, n)$,

$$\tag{5.11} \begin{aligned} &|f(x_w) - f(z_0)|^p \\ &\leq 2^{p-1} |f(x_w) - f(z_n)|^p + 2^{p-1} \sum_{i=0}^{n-1} 2^{i(p-1)} |f(z_i) - f(z_{i+1})|^p . \end{aligned}$$

Integrating (5.11), we obtain

$$\tag{5.12} \begin{aligned} &\fint_{K_w} |f(z) - M_n f(w)|^p \, d\mu(z) \\ &\leq 2^{p-1} \fint_{K_{w(n)}} |f(x_w) - f(z_n)|^p \, d\mu(z_n) \\ &\quad + 2^{2(p-1)} \sum_{i=0}^{n-1} 2^{i(p-1)} \fint_{K_{w(i)}} \fint_{K_{w(i+1)}} |f(z_i) - f(z_{i+1})|^p \, d\mu(z_{i+1}) d\mu(z_i). \end{aligned}$$

Set $c := 3\operatorname{diam}(K)$ and define

$$\tag{5.13} S^{(n)}_{p,\beta}(f) := \sup_{r\in(0,ca^{-n}]} r^{-\beta} \int_K \fint_{B(z,r)} |f(z) - f(z')|^p \, d\mu(z') d\mu(z).$$

By Lemma 5.13, the first term of the right-hand side of (5.12) has a bound as follows:

$$\tag{5.14} \begin{aligned} \fint_{K_{w(n)}} |f(x_w) - f(z_n)|^p \, d\mu(z_n) &\leq C_{5.13} S^{(n)}_{p,\beta}(f) \fint_{K_{w(n)}} d(x_w, z_n)^{\beta-\alpha} \, d\mu(z_n) \\ &\leq \left(C_{5.13} \operatorname{diam}(K)^{\beta-\alpha}\right) a^{-(n+kn)(\beta-\alpha)} S^{(n)}_{p,\beta}(f). \end{aligned}$$

For the second term of (5.12), we see that

$$\tag{5.15} 2^{i(p-1)} \fint_{K_{w(i)}} \fint_{K_{w(i+1)}} |f(z_i) - f(z_{i+1})|^p \, d\mu(z_{i+1}) d\mu(z_i)$$

$$\leq c_1\, 2^{i(p-1)} a^{\alpha k} a^{2\alpha(n+ki)} \int_{K_{w(i)}} \int_{B(z_i, ca^{-(n+ki)})} |f(z_i) - f(z_{i+1})|^p \, d\mu(z_{i+1}) d\mu(z_i),$$

where $c_1 > 0$ depends only on c and C_{AR}.

Now, we consider $k \in \mathbb{N}$ large enough so that

$$k(\beta - \alpha) \geq \alpha \quad \text{and} \quad N_* a^{-(\beta-\alpha)k} \vee 2^{p-1} a^{-(\beta-\alpha)k} < 1. \tag{5.16}$$

Then, by summing (5.14) and (5.15) over $w \in W_n$, we have from (5.16) that

$$\begin{aligned}
&\sum_{w \in W_n} \fint_{K_{w(n)}} |f(x_w) - f(z_n)|^p \, d\mu(z_n) \\
&\leq \left(C_{5.13} \operatorname{diam}(K)^{\beta-\alpha}\right) S^{(n)}_{p,\beta}(f) N_*^n a^{-(n+kn)(\beta-\alpha)} \\
&\leq \left(C_{5.13} \operatorname{diam}(K)^{\beta-\alpha}\right) S^{(n)}_{p,\beta}(f)\, a^{-(\beta-\alpha)n},
\end{aligned}$$

and from Lemma 2.4 and (5.16) that

$$\begin{aligned}
&\sum_{i=0}^{n-1} \sum_{w \in W_n} 2^{i(p-1)} \fint_{K_{w(i)}} \fint_{K_{w(i+1)}} |f(z_i) - f(z_{i+1})|^p \, d\mu(z_{i+1}) d\mu(z_i) \\
&\leq c_1 a^{\alpha k} \sum_{i=0}^{n-1} 2^{i(p-1)} a^{2\alpha(n+ki)} \int_K \int_{B(x, ca^{-(n+ki)})} |f(x) - f(y)|^p \, d\mu(y) d\mu(x) \\
&\leq c_2\, a^{\alpha k} S^{(n)}_{p,\beta}(f) \left(\sum_{i=0}^{n-1} 2^{i(p-1)} a^{-(\beta-\alpha)(n+ki)} \right) \\
&\leq c_2\, a^{\alpha k} S^{(n)}_{p,\beta}(f) \left(\sum_{i=0}^{\infty} \left[2^{p-1} a^{-(\beta-\alpha)k}\right]^i \right) a^{-(\beta-\alpha)n} =: c_3 S^{(n)}_{p,\beta}(f)\, a^{-(\beta-\alpha)n},
\end{aligned}$$

where $c_2, c_3 > 0$ depend only on $C_{\mathrm{AR}}, c, \beta, \alpha, p$. From these estimates and (5.12), we finish the proof. □

Now we are ready to finish the proof of Theorem 2.18.

Theorem 5.15.

$$\mathcal{F}_p = \Lambda^{\beta_p/p}_{p,\infty} = \left\{ f \in L^p(K,\mu) \,\middle|\, \varlimsup_{r \downarrow 0} \int_K \fint_{B(x,r)} \frac{|f(x) - f(y)|^p}{r^{\beta_p}} \, d\mu(y) d\mu(x) < \infty \right\}.$$

Proof. Let $c > 0$ such that $\max_{(v,w) \in E_n} \sup_{x \in K_v, y \in K_w} d(x,y) < ca^{-n}$. We can choose such c depending only on C_{AD} by Lemma 2.7 and we may assume that $c \geq 3 \operatorname{diam}(K)$ without loss of generality. Then $y \in B(x, ca^{-n})$ whenever $(v,w) \in E_n$ and $x \in K_v$, $y \in K_w$. Let $\beta > \alpha$ and set

$$A^{(n)}_{p,\beta}(f) := a^{\beta n} \int_K \fint_{B(x, ca^{-n})} |f(x) - f(y)|^p \, d\mu(y) d\mu(x),$$

and define $S^{(n)}_{p,\beta}(f)$ as in (5.13) for each $n \in \mathbb{N}$ and $f \in L^p(K,\mu)$. Then, thanks to Lemma 5.12, it will suffice to show the following two estimates:

$$A^{(n)}_{p,\beta_p}(f) \leq c_{5.17} \left(\widetilde{\mathcal{E}}^{G_n}_p(M_n f) + a^{\beta_p n} \sum_{w \in W_n} \int_{K_w} |f(x) - M_n f(w)|^p \, d\mu(x) \right) \tag{5.17}$$

for any $f \in L^p(K,\mu)$ and

$$a^{(\beta-\alpha)n} \cdot \mathcal{E}_p^{G_n}(M_n f) \le c_{5.18} S_{p,\beta}^{(n)}(f) \tag{5.18}$$

for any $f \in C(K)$, where $c_{5.17}, c_{5.18}$ are positive constants without depending on f and n. Indeed, by (5.7), (5.17) and Lemma 5.12, we immediately see that

$$C_{\mathrm{LB}}(c)^{-1} \, |f|^p_{\Lambda_{p,\infty}^{\beta_p/p}} \le \sup_{n\in\mathbb{N}} A_{p,\beta_p}^{(n)}(f) \le c_{5.17}(1 + C_{\text{PI-KS}}) \, |f|^p_{\mathcal{F}_p} . \tag{5.19}$$

Additionally, by the weak monotonicity (Corollary 4.17), (5.18) and (5.8), we have that

$$\begin{aligned} |f|^p_{\mathcal{F}_p} &\le c_{5.18} C_{\mathrm{WM}} \varlimsup_{r\downarrow 0} r^{-\beta_p} \int_K \fint_{B(x,r)} |f(x) - f(y)|^p \, d\mu(y) d\mu(x) \\ &\le C \varlimsup_{n\to\infty} a^{\beta_p n} \int_K \fint_{B(x,a^{-n})} |f(x) - f(y)|^p \, d\mu(y) d\mu(x), \end{aligned} \tag{5.20}$$

where $C = c_{5.18} C_{\mathrm{WM}} C_{\mathrm{LB}}(c) C_{\mathrm{LB}}(1)$. Our assertion follows from (5.19) and (5.20).

The rest of the proof is devoted to proving (5.17) and (5.18). First, we will prove (5.17). Let $x, y \in K$ with $d(x,y) < ca^{-n}$. Then, by the metric doubling property of K (see [27, pp. 81] for example), there exists a constant $L \ge 2$ depending only on C_{AR} such that, for any $w \in W_n$ with $x \in K_w$, we can choose $v \in W_n$ satisfying $d_{G_n}(v,w) \le L$ and $y \in K_v$. From this observation, we have that

$$A_{p,\beta_p}^{(n)}(f) \le a^{\beta_p n} \sum_{\substack{v,w\in W_n; \\ d_{G_n}(v,w)\le L}} \int_{K_w} \frac{1}{\mu(B(x, ca^{-n}))} \int_{K_v} |f(x) - f(y)|^p \, d\mu(y) d\mu(x). \tag{5.21}$$

To estimate the integral in (5.21), let $v, w \in W_n$ with $d_{G_n}(v,w) \le L$. Then we can pick a path $[w^0, w^1, \dots, w^L]$ in G_n from w to v, that is, w^i $(i = 0, \dots, L)$ satisfy $w^0 = w$, $w^L = v$ and

$$w^{i-1} = w^i \text{ or } (w^{i-1}, w^i) \in E_n \text{ for each } i = 1, \dots, L.$$

Let $x_i \in K_{w^i}$ for each $i = 0, \dots, L$. Then Hölder's inequality implies that

$$|f(x_0) - f(x_L)|^p \le L^{p-1} \sum_{i=1}^{L} |f(x_{i-1}) - f(x_i)|^p .$$

Now, by integrating this, we deduce that

$$\begin{aligned} &\left(\prod_{i=1}^{L-1} \mu(K_{w^i})\right) \int_{K_w} \int_{K_v} |f(x) - f(y)|^p \, d\mu(x) d\mu(y) \\ &\le L^{p-1} \sum_{i=1}^{L} \frac{\prod_{j=0}^{L} \mu(K_{w^j})}{\mu(K_{w^{i-1}})\mu(K_{w^i})} \int_{K_{w^{i-1}}} \int_{K_{w^i}} |f(x_i) - f(x_{i-1})|^p \, d\mu(x_i) d\mu(x_{i-1}). \end{aligned}$$

Since μ is the self-similar measure with weights $(N_*^{-1}, \dots, N_*^{-1})$, it is a simple computation that

$$\frac{\prod_{j=0}^{L} \mu(K_{w^j})}{\mu(K_{w^{i-1}})\mu(K_{w^i})} \frac{1}{\prod_{i=1}^{L-1} \mu(K_{w^i})} = \frac{\mu(K_v)\mu(K_w)}{\mu(K_{w^{i-1}})\mu(K_{w^i})} = 1.$$

Furthermore, the Ahlfors regularity of μ (more precisely, the volume doubling property of μ) implies that there exists a constant $c_1 > 0$ depending only on $C_{\mathrm{AR}}, \alpha, c$ such that $\mu(K_z) \le c_1 \mu(B(x, ca^{-n}))$ for any $n \in \mathbb{N}$, $z \in W_n$ and $x \in K$. Thus, it follows from (5.21) that

$$\begin{aligned}
&A^{(n)}_{p,\beta_p}(f) \\
&\le \bigl(c_1 L^{p-1} D_*^{L}\bigr) a^{\beta_p n} \sum_{(v,w)\in E_n} \int_{K_w} \fint_{K_v} |f(x) - f(y)|^p \, d\mu(y) d\mu(x) \\
&\le c_{5.17} \Biggl(a^{\beta_p n} \sum_{v \in W_n} \int_{K_v} |f(x) - M_n f(v)|^p \, d\mu(x) \\
&\qquad\qquad + \rho_p^n \sum_{(v,w)\in E_n} |M_n f(v) - M_n f(w)|^p \Biggr),
\end{aligned}$$

where $c_{5.17} \coloneqq c_1 (2L)^{p-1} D_*^{L+1}$. This proves (5.17).

Next let us prove (5.18). Let $\beta > \alpha$, let $p > 1$ and let $f \in C(K)$. For $n \in \mathbb{N}$, $(v, w) \in E_n$, $x \in K_v$ and $y \in K_w$, we see that

$$\begin{aligned}
&|M_n f(v) - M_n f(w)|^p \\
&\le 3^{p-1} \bigl(|M_n f(v) - f(x)|^p + |f(x) - f(y)|^p + |M_n f(w) - f(y)|^p \bigr).
\end{aligned}$$

Integrating this over K_v and K_w, we obtain

$$\begin{aligned}
&a^{(\beta-\alpha)n} \cdot |M_n f(v) - M_n f(w)|^p \\
&\le 3^{p-1} \Biggl(a^{\beta n} \int_{K_v} |M_n f(v) - f(x)|^p \, d\mu(x) \\
&\quad + a^{\beta n} \int_{K_w} \fint_{K_v} |f(x) - f(y)|^p \, d\mu(x) d\mu(y) + a^{\beta n} \int_{K_w} |M_n f(w) - f(y)|^p \, d\mu(y) \Biggr).
\end{aligned}$$

Summing over $(v, w) \in E_n$, we obtain

$$\begin{aligned}
&a^{(\beta-\alpha)n} \cdot \mathcal{E}_p^{G_n}(M_n f) \\
&\le 2 \cdot 3^{p-1} D_* \Biggl(a^{\beta n} \sum_{v \in W_n} \int_{K_v} |M_n f(v) - f(x)|^p \, d\mu(x) \\
&\qquad\qquad + a^{\beta n} \sum_{(v,w)\in E_n} \int_{K_w} \fint_{K_v} |f(x) - f(y)|^p \, d\mu(x) d\mu(y) \Biggr).
\end{aligned}$$

A bound of the first term in the right-hand side is obtained in Lemma 5.14. Noting that $K_v \subseteq B(y, ca^{-n})$ for $(v, w) \in E_n$ and $y \in K_w$, we can estimate the second term as follows:

$$a^{\beta n} \sum_{(v,w)\in E_n} \int_{K_w} \fint_{K_v} |f(x) - f(y)|^p \, d\mu(x) d\mu(y)$$

$$\leq c_2 a^{\beta n} \sum_{(v,w)\in E_n} \int_{K_w} \fint_{B(y,ca^{-n})} |f(x)-f(y)|^p \, d\mu(x)d\mu(y) \leq c_2 c^{-\beta} D_* S^{(n)}_{p,\beta}(f),$$

where $c_2 > 0$ depends only on C_{AR}, c. This proves (5.18) and finishes the proof. □

As an immediate consequence of Theorem 5.15, we have a characterization of β_p as *critical Besov exponents*. For details on critical Besov exponents, see [3, 24] for example. This result is well-known when $p = 2$ (see [24, Theorem 4.6]).

Corollary 5.16. *It holds that* $\beta_p = p \sup\{s > 0 \mid \Lambda^s_{p,\infty} \neq \{constant\}\}$.

Proof. Note that $\Lambda^{\beta'/p}_{p,\infty} \subseteq \Lambda^{\beta/p}_{p,\infty}$ for any $\beta \leq \beta'$. It is immediate that $\beta_p \leq p \sup\{s > 0 \mid \Lambda^s_{p,\infty} \neq \{\text{constant}\}\}$. To prove the converse, let $\beta > \beta_p$. If $f \in C(K)$ is not constant, then there exists $N \in \mathbb{N}$ such that $\widetilde{\mathcal{E}}^{G_N}_p(M_N f) > 0$. By the weak monotonicity (Corollary 4.17), for any $n \geq N$,

$$a^{(\beta-\alpha)n} \cdot \mathcal{E}^{G_n}_p(M_n f) = a^{(\beta-\alpha)n}\rho_p^{-n} \cdot \widetilde{\mathcal{E}}^{G_n}_p(M_n f) \geq C_{\mathrm{WM}}^{-1} a^{(\beta-\alpha)n}\rho_p^{-n} \cdot \widetilde{\mathcal{E}}^{G_N}_p(M_N f).$$

Letting $n \to \infty$ in this inequality, we obtain $\varlimsup_{n\to\infty} a^{(\beta-\alpha)n} \cdot \mathcal{E}^{G_n}_p(M_n f) = \infty$ since $\rho_p^{-1} a^{\beta-\alpha} > 1$. By (5.18), we conclude that $|f|_{\Lambda^{\beta/p}_{p,\infty}} = \infty$ whenever $\beta > \beta_p$ and $f \in C(K)$ is not constant. This proves our assertion. □

6 Construction of a canonical scaling limit of p-energies

This section aims to prove Theorem 2.20. To construct a canonical "Dirichlet form" on fractals, there is already an established way as appeared in [47, proof of Theorem 6.9]. However, in the original argument of [47], the Markov property of their "Dirichlet form" was not clarified. In [39], Kigami has pointed out this gap and filled this in an abstract way, that is, he has proven the existence of the desired Dirichlet forms by an argument using some fixed point theorem. An approach based on the combination of Kusuoka–Zhou's and Kigami's arguments is enough to construct a p-energy satisfying all the properties in Theorem 2.17 (see [38, Theorem 9.1]). Regrettably, this abstract way is insufficient to prove our main result about $\mathcal{E}_p$-energy measures: Theorem 2.22. After Kigami's work, another very simple way to check the Markov property is given by Barlow, Bass, Kumagai, and Teplyaev (see [8, proof of Theorem 2.1]). This method deduces that the Dirichlet forms of Kusuoka and Zhou in [47] have the Markov property, but it very heavily relies on being Dirichlet forms, that is, the use of bilinearity (and locality) is essential to follow [8, proof of Theorem 2.1]. To overcome these difficulties, we will introduce a new series of graphs $\mathbb{G}_n$ approximating the Sierpiński carpet in subsection 6.1. Then in subsection 6.2 we directly construct p-energy $\mathcal{E}_p$ as a subsequential scaling limit of discrete p-energies on this new series of graphs.

Throughout this section, we suppose Assumption 4.2 holds.

6.1 Behavior of p-energies on modified Sierpiński carpet graphs

Recall that $p_i \in K$ denotes the fixed point of F_i for each $i \in S = \{1, \dots, N_*\}$, where $N_* = 8$. We use $\widehat{p}_{2m+1}$ to denote the middle point between p_{2m} and $p_{2(m+1)}$ for each $m = 0, \dots, 3$, where $p_0 \coloneqq p_8$. One can show that $\widehat{p}_m \in K$ by a direct calculation. Set $V(\mathbb{G}_1) \coloneqq \{p_{2m}, \widehat{p}_m\}_{m=0}^{3}$ and $V(\mathbb{G}_n)$ is defined inductively by

$$V(\mathbb{G}_n) \coloneqq \{F_i(x) \mid i \in S, x \in V(\mathbb{G}_{n-1})\}.$$

Next, the edge set $E(\mathbb{G}_n)$ is defined inductively as follows:

$$E(\mathbb{G}_1) \coloneqq \{(p_{2m}, \widehat{p}_m), (\widehat{p}_m, p_{2m+2}), (\widehat{p}_m, p_{2m}), (p_{2m+2}, \widehat{p}_m)\}_{m=0}^{3},$$

and

$$E(\mathbb{G}_n) \coloneqq \bigl\{\bigl(F_i(x), F_i(y)\bigr) \bigm| i \in S, (x, y) \in E(\mathbb{G}_{n-1})\bigr\}.$$

We will consider a new finite connected graph defined by $\mathbb{G}_n \coloneqq (V(\mathbb{G}_n), E(\mathbb{G}_n))$ (see Figure 8). Then it is immediate that

$$D_*(\{\mathbb{G}_n\}_{n\ge 1}) \coloneqq \sup_{n\in\mathbb{N}} \max_{x\in V(\mathbb{G}_n)} \#\{y \in V(\mathbb{G}_n) \mid (x, y) \in E(\mathbb{G}_n)\} = 4.$$

For any $n, m \in \mathbb{N}$ and $w \in W_n$, we define a subset $V_w(\mathbb{G}_m)$ of $V(\mathbb{G}_{n+m})$ by setting $V_w(\mathbb{G}_m) \coloneqq \{F_w(x) \mid x \in V(\mathbb{G}_m)\}$, and define a subgraph $F_w(\mathbb{G}_m) \coloneqq \bigl(V_w(\mathbb{G}_m), E(\mathbb{G}_{n+m})^{V_w(\mathbb{G}_m)}\bigr)$.

For simplicity, we write $\mathcal{R}_p^{\mathbb{G}_n}(x, y)$ to denote the reciprocal of $C_p^{\mathbb{G}_n}(\{x\}, \{y\})$, that is,

$$\mathcal{R}_p^{\mathbb{G}_n}(x, y) \coloneqq \sup\left\{ \frac{|f(x) - f(y)|^p}{\mathcal{E}_p^{\mathbb{G}_n}(f)} \;\middle|\; f\colon V(\mathbb{G}_n) \to \mathbb{R} \text{ is not constant} \right\},$$

for each $x, y \in V(\mathbb{G}_n)$. Then one of the key ingredients is that $\mathcal{R}_p^{\mathbb{G}_n}(x, y)$ behaves like $\mathcal{R}_p^{(n)}$.

Proposition 6.1. *There exists a positive constant $C_{6.1}$ (depending only on p, D_*, ρ_p, a, C_{UH} and C_{Mult}) such that, for any $m \in \{0, \dots, 3\}$ and $n \in \mathbb{N}$,*

$$\mathcal{R}_p^{\mathbb{G}_n}(p_{2m}, \widehat{p}_m) \le C_{6.1}\rho_p^n. \tag{6.1}$$

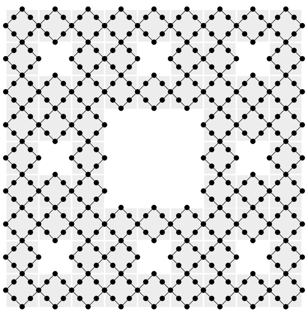

Figure 8: Modified Sierpiński carpet graph $\{\mathbb{G}_n\}_{n\ge 1}$ (This figure draws $\mathbb{G}_3$)

Remark 6.2. When $p = 2$, such point-to-point estimates on a series of Sierpiński carpet graphs are proved in [49, Appendix], where a uniform Hanarck inequality by Barlow and Bass (see also Remark 4.18) is used. In [38, Lemma 8.5], Kigami also shows similar estimates for all $p > \dim_{\mathrm{ARC}}$ assuming p-conductive homogeneity, where he also uses some uniform Hölder estimate. Our proof also relies on the uniform Hölder estimate: Corollary 4.16.

To prove this proposition, we need an estimate of p-conductance between points on the original graph $\{G_n\}_{n\geq 1}$. Let $w^i(n)$ denote a unique element of W_n such that $p_i \in K_{w^i(n)}$ for each $i \in S$ and $n \in \mathbb{N}$. Then we can show the following lemma in a similar way to the "chain argument" in the proof of Theorem 4.13.

Lemma 6.3. *There exists a constant $C_{6.3} \geq 1$ (depending only on p, ρ_p, a, C_{UH} and C_{Mult}) such that, for any $i \neq j \in S$ and $n \in \mathbb{N}$,*

$$C_{6.3}^{-1}\rho_p^{-n} \leq \mathcal{C}_p^{G_n}(\{w^i(n)\}, \{w^j(n)\}) \leq C_{6.3}\rho_p^{-n}. \tag{6.2}$$

Proof. An upper bound

$$\mathcal{C}_p^{G_n}(\{w^i(n)\}, \{w^j(n)\}) \leq C_{\mathrm{Mult}}\rho_p \cdot \rho_p^{-n} \tag{6.3}$$

is an immediate consequence of Proposition 2.12 and Theorem 3.4.

To prove the converse, we first consider the case $|i - j| = 1$. It will suffice to treat the case $i = 1$ and $j = 2$ by the symmetries of the Sierpiński carpet. Let $f_n \colon W_n \to \mathbb{R}$ satisfy $f_n(w^i(n)) = i - 1$ for $i = 1, 2$ and $\mathcal{E}_p^{G_n}(f_n) = \mathcal{C}_p^{G_n}(\{w^1(n)\}, \{w^2(n)\})$. Note that f_n is $[0, 1]$-valued. In view of (4.19) in Corollary 4.16 and (6.3), we choose $l \in \mathbb{N}$ such that $C_{\mathrm{UH}}C_{\mathrm{Mult}}\rho_p a^{-(\beta_p - \alpha)l} \leq 4^{-1}$. We also set $v^k := [w^k(n)]_l \in W_l$ for each k. Then it follows from (4.19) that

$$\max_{v^1 \cdot W_{n-l}} f_n \leq \frac{1}{4} \quad \text{and} \quad \min_{v^2 \cdot W_{n-l}} f_n \geq \frac{3}{4}.$$

Now we define g_n by setting $g_n := 2\bigl((f_n \vee 1/4) \wedge 3/4\bigr)$. We easily see that there exist $3^l \leq L \leq 2 \cdot 3^l$ and a horizontal chain $[z^1, \dots, z^L]$ in G_l such that $z^1 = v^1$, $z^L = v^2$, $z^k \in \partial_* G_l$ for any $k = 1, \dots, L$ and $(z^k, z^{k+1}) \in \widetilde{E}_l$ for each $k = 1, \dots, L - 1$. Recall the definition of $\mathcal{C}_p^{(n,L)}$ in subsection 4.2. Then we have that

$$\mathcal{C}_p^{(n-l,L)} \leq \mathcal{E}_p^{G_n}(g_n) \leq 2^p \mathcal{E}_p^{G_n}(f_n).$$

From this estimate and Lemma 4.8, there exists a constant $C(L) > 0$ (depending only on p, N_*, D_*, L) such that $\mathcal{C}_p^{G_n}(\{w^1(n)\}, \{w^2(n)\}) \geq C(L)^{-1}\mathcal{C}_p^{(n-l)}$. The submultiplicative inequality of $\mathcal{C}_p^{(n)}$ (Theorem 3.4) gives our assertion when $|i - j| = 1$.

Finally, note that $(x, y) \mapsto \mathcal{C}_p^G(\{x\}, \{y\})^{-1/p}$ is a metric on a graph G. Indeed, this fact immediately follows from the representation:

$$\mathcal{C}_p^G(\{x\}, \{y\})^{-1/p} = \max\left\{ \frac{|f(x) - f(y)|}{\mathcal{E}_p^G(f)^{1/p}} \,\middle|\, f \colon V \to \mathbb{R} \text{ with } \mathcal{E}_p^G(f) > 0 \right\}.$$

Applying the triangle inequality of this metric, we get the desired results for $i \neq j \in S$. □

Remark 6.4. It is known that $(x,y) \mapsto C_p^G(\{x\},\{y\})^{-1/(p-1)}$ also becomes a metric, and thus $C_p^G(\,\cdot\,,\,\cdot\,)^{-1/(p-1)}$ gives a generalization of the *resistance metric*. This fact is proved in [1, Theorem 8], where G is a finite graph. One can check the case of infinite graphs in [50, Theorem 4.3], which is mainly based on the author's Master thesis.

Now we are ready to prove Proposition 6.1.

Proof of Proposition 6.1. Enumerate the elements W_n as $W_n = \{v_n(1),\dots,v_n(N_*^n)\}$ for each $n \in \mathbb{N}$. Note that $V(\mathbb{G}_n) = \bigcup_{w \in W_{n-1}} V_w(\mathbb{G}_1)$. Define $\varphi_n \colon V(\mathbb{G}_n) \to W_n$ as follows:

$$\varphi_n(F_{v_{n-1}(j)}(\widehat{p}_m)) := v_{n-1}(j)\cdot(2m+1),$$

and

$$\varphi_n(F_{v_{n-1}(j)}(p_{2m})) := v_{n-1}(j)\cdot(2m) \quad \text{if } F_{v_{n-1}(j)}(p_{2m}) \notin \bigcup_{i<j} K_{v_{n-1}(i)},$$

where we set $0 := 8 \in S$ for convention. Then it is immediate that $\{\varphi_n\}_{n\ge1}$ is a uniform rough isometry from $\{\mathbb{G}_n\}_{n\ge1}$ to $\{G_n\}_{n\ge1}$. (See Figure 9 for an illustration of $\{\varphi_n\}_{n\ge1}$.) Note that $\varphi_n(\widehat{p}_m) = w^{2m-1}(n)$ and $\varphi_n(p_{2m}) = w^{2m}(n)$ for each $m \in \{0,\dots,3\}$. Thus Lemma 4.11 yields that

$$C_p^{G_n}(\{w^{2m}(n)\},\{w^{2m-1}(n)\}) \le C_{\mathrm{URI}} C_p^{\mathbb{G}_n}(\{p_{2m}\},\{\widehat{p}_m\}),$$

which implies (6.1). □

Next, we define rescaled p-energy $\widetilde{\mathcal{E}}_p^{\mathbb{G}_n} \colon C(K) \to \mathbb{R}$ on $\mathbb{G}_n$ by setting

$$\widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(f) := \frac{\rho_p^n}{2} \sum_{(x,y)\in E(\mathbb{G}_n)} |f(x)-f(y)|^p,$$

for each $f \in C(K)$. Then the following lemma, especially statements (3) and (4) below, is a collection of benefits of the new graphical approximation $\{\mathbb{G}_n\}_{n\ge1}$.

Lemma 6.5. *The following statements hold.*

(1) *If $f \in C(K)$, then $\sup_{n\in\mathbb{N}} \widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(f) = 0$ if and only if f is constant.*

(2) *If $T \in \mathrm{Sym}(K)$ and $f \in C(K)$, then $\widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(f \circ T) = \widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(f)$.*

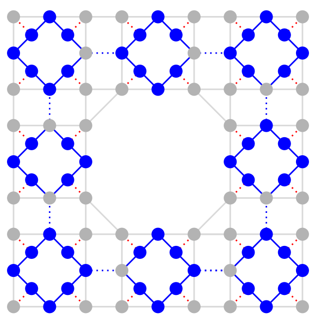

Figure 9: Uniform rough isometry from $\mathbb{G}_n$ (blue) to G_n (gray) as an embedding

(3) *If* $f \in C(K)$, *then*

$$\widetilde{\mathcal{E}}_p^{\mathbb{G}_{n+1}}(f) = \rho_p \sum_{i \in S} \widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(F_i^* f). \tag{6.4}$$

(4) *If* $f \in C(K)$ *and* $\varphi \colon \mathbb{R} \to \mathbb{R}$ *with* $\mathrm{Lip}(\varphi) \le 1$, *then* $\widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(\varphi \circ f) \le \widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(f)$.

Proof. (1) It is immediate that $\sup_{n \in \mathbb{N}} \widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(f) = 0$ if $f \in C(K)$ is constant. We easily see that $\overline{\bigcup_{n \in \mathbb{N}} V(\mathbb{G}_n)}^K = K$. Thus, if $f \in C(K)$ satisfies $\mathcal{E}_p^{\mathbb{G}_n}(f) = 0$ for any $n \in \mathbb{N}$, then f is constant.

(2) This immediately follows from the symmetries of $\mathbb{G}_n$.

(3) Let $f \in C(K)$ and $n \in \mathbb{N}$. Then, by the self-similarity of K, we have that

$$\begin{aligned} \widetilde{\mathcal{E}}_p^{\mathbb{G}_{n+1}}(f) &= \rho_p \sum_{i \in S} \left(\frac{\rho_p^n}{2} \sum_{\substack{(x,y) \in E(\mathbb{G}_{n+1}); \\ x,y \in K_i}} \left| f(F_i(F_i^{-1}(x))) - f(F_i(F_i^{-1}(y))) \right|^p \right) \\ &= \rho_p \sum_{i \in S} \widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(F_i^* f), \end{aligned}$$

which proves (6.4).

(4) The required estimate immediately follows from the fact that $|\varphi(a) - \varphi(b)| \le |a - b|$ for any $a, b \in \mathbb{R}$ whenever $\varphi \colon \mathbb{R} \to \mathbb{R}$ satisfies $\mathrm{Lip}(\varphi) \le 1$. □

We also have the weak monotonicity of $\widetilde{\mathcal{E}}_p^{\mathbb{G}_n}$ as follows. Its proof is similar to [25, Theorem 7.1], where 2-energies are considered.

Lemma 6.6. *There exists a constant* $C_{\mathrm{WM}}(\{\mathbb{G}_n\}_{n \ge 1}) > 0$ *(depending only on* $p, D_*, \rho_p, a, C_{\mathrm{UH}}, C_{\mathrm{Mult}}$*) such that*

$$\widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(f) \le C_{\mathrm{WM}}(\{\mathbb{G}_n\}_{n \ge 1}) \widetilde{\mathcal{E}}_p^{\mathbb{G}_{n+m}}(f), \tag{6.5}$$

for every $n, m \in \mathbb{N}$ *and* $f \in C(K)$. *In particular, for any* $f \in C(K)$,

$$\sup_{n \in \mathbb{N}} \widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(f) \le C_{\mathrm{WM}}(\{\mathbb{G}_n\}_{n \ge 1}) \varliminf_{n \to \infty} \widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(f).$$

Proof. Let $(x, y) \in E(\mathbb{G}_n)$. Then there exists $w \in W_{n-1}$ such that $x, y \in K_w$. Furthermore, there exists $l = \{0, \dots, 3\}$ such that $x = F_w(p_{2l})$ and $y = F_w(\widehat{p}_l)$. Now we define

$$\begin{aligned} &\mathcal{R}_p^{F_w(\mathbb{G}_{m+1})}(z_1, z_2) \\ &\quad := \sup \left\{ \frac{|f(z_1) - f(z_2)|^p}{\mathcal{E}_p^{F_w(\mathbb{G}_{m+1})}(f)} \;\middle|\; f \in C(K) \text{ with } f|_{V_w(\mathbb{G}_{m+1})} \text{ is not constant} \right\}. \end{aligned}$$

Then we see from Proposition 6.1 that

$$|f(x) - f(y)|^p \le \mathcal{R}_p^{F_w(\mathbb{G}_{m+1})}\bigl(F_w(p_{2l}), F_w(\widehat{p}_l)\bigr) \mathcal{E}_p^{F_w(\mathbb{G}_{m+1})}(f)$$

$$\le \mathcal{R}_p^{\mathbb{G}_{m+1}}(p_{2l}, \widehat{p_l})\mathcal{E}_p^{F_w(\mathbb{G}_{m+1})}(f)$$
$$\le C_{6.1}\rho_p^m \mathcal{E}_p^{F_w(\mathbb{G}_{m+1})}(f),$$

where we used the cutting law ([50, Proposition 3.18]) and the self-similarity of $\mathbb{G}_{n+m}$ in the second line. Summing over $(x, y) \in E(\mathbb{G}_n)$, we obtain $\mathcal{E}_p^{\mathbb{G}_n}(f) \le 2^{-1}C_{6.1}\rho_p^m\mathcal{E}_p^{\mathbb{G}_{n+m}}(f)$, which deduces our assertion. □

The rest of this subsection is devoted to proving the following lemma.

Lemma 6.7. *There exists a constant $C_{6.7} \ge 1$ such that*

$$C_{6.7}^{-1}\,|f|_{\mathcal{F}_p}^p \le \sup_{n\in\mathbb{N}} \widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(f) \le C_{6.7}\,|f|_{\mathcal{F}_p}^p \quad \textit{for any } f \in L^p(K,\mu). \tag{6.6}$$

Remark 6.8. Such discrete characterizations of Lipschitz–Besov space are treated in [11], but we need some modification as stated in [26, Remark 1 in Section 3]. To be self-contained, we give complete proofs in a similar way to [25, Theorems 3.5 and 3.6 and Proposition 11.1], where they give discrete characterizations of $\mathcal{F}_2$ on the SC.

By virtue of Theorem 2.18, it will suffice to show that $\sup_{n\in\mathbb{N}} \widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(f)$ and $|f|_{\Lambda_{p,\infty}^{\beta_p/p}}$ are comparable. To this end, we introduce a simplified version $\mathbb{G}_n^*$ of $\mathbb{G}_n$. Set $V(\mathbb{G}_1^*) := \{p_{2m}\}_{m=1}^4$ and define inductively

$$V(\mathbb{G}_n^*) := \{F_i(x) \mid i \in S, x \in V(\mathbb{G}_{n-1})\}.$$

Set $E(\mathbb{G}_1^*) := \{(p_{2m}, p_{2m+2}), (p_{2m+2}, p_{2m})\}_{m=1}^4$, where we write $p_{10} = p_2$ for convenience, and define inductively

$$E(\mathbb{G}_n^*) := \bigl\{(F_i(x), F_i(y)) \bigm| i \in S, (x, y) \in E(\mathbb{G}_{n-1})\bigr\}.$$

Then we define $\mathbb{G}_n^*$ by $\mathbb{G}_n^* := (V(\mathbb{G}_n^*), E(\mathbb{G}_n^*))$. Note that $V(\mathbb{G}_n^*)$ is a subset of $V(\mathbb{G}_n)$. We also define $V_w(\mathbb{G}_n^*)$ in the same way as $V_w(\mathbb{G}_n)$. Similarly to [11], we will apply an argument using discrete approximations of measure μ with respect to the weak convergence of probability measures (see Lemma 6.10-(2)). For each $x \in K$ and $n \in \mathbb{N}$, we define $U_1^{(n)}(x)$ by setting

$$U_1^{(n)}(x) := \bigcup\{K_w \mid w \in W_n \text{ with } K_v \cap K_w \ne \emptyset \text{ for some } v \in W_n \text{ such that } x \in K_v\}.$$

(See [41, Definition 2.3.5].) Then the following lemma is elementary (see also [11, Lemma 3.12], [26, Remark 1 in Section 3]).

Lemma 6.9. *If a sequence of probability measures $\{\mu_n\}_{n\ge 1}$ on K is given by*

$$\mu_n := \frac{1}{\#V(\mathbb{G}_n^*)} \sum_{x\in V(\mathbb{G}_n^*)} \delta_x,$$

where δ_x denotes the Dirac measure with support $\{x\}$ for each $x \in K$, then μ_n converges weakly to μ as $n \to \infty$. Moreover, for any $m \in \mathbb{N}$, $p > 1$ and $f \in C(K)$,

$$\begin{aligned}\lim_{n\to\infty} \int_K \int_{U_1^{(m)}(x)} |f(x) - f(y)|^p \, d\mu_n(y)d\mu_n(x) \\ = \int_K \int_{U_1^{(m)}(x)} |f(x) - f(y)|^p \, d\mu(y)d\mu(x).\end{aligned} \tag{6.7}$$

Proof. Since K is compact, by Prokhorov's theorem (see [9, Theorem 5.1] for example), there exist a subsequence $\{n_k\}_{k\geq 1}$ and a Borel probability measure $\widetilde{\mu}$ on K such that μ_{n_k} converges weakly to $\widetilde{\mu}$ as $k \to \infty$. From the definition of $\mathbb{G}_n^*$, we easily see that $\#V(\mathbb{G}_1) = 4$ and

$$\#V(\mathbb{G}_n^*) = 8 \cdot \#V(\mathbb{G}_{n-1}^*) - 8 \cdot 3^{n-2},$$

for each $n \geq 2$. In particular, $\#V(\mathbb{G}_n^*) = \frac{12}{5} \cdot 8^{n-1} + \frac{8}{5} \cdot 3^{n-1}$ for any $n \in \mathbb{N}$, and thus

$$\frac{3}{10} a^{\alpha n} \leq \#V(\mathbb{G}_n^*) \leq \frac{1}{2} a^{\alpha n}. \tag{6.8}$$

This implies that $\widetilde{\mu}$ is α-Ahlfors regular. By [27, Exercise 8.11] and Lemma 2.4, we get $\widetilde{\mu}(\partial K_w) = 0$ for each $w \in W_\#$. Moreover, for any $n, m \in \mathbb{N}$ and $w \in W_m$, we have that

$$\mu_n(K_w) = \frac{\frac{12}{5} \cdot 8^{n-m-1} + \frac{8}{5} \cdot 3^{n-m-1}}{\frac{12}{5} \cdot 8^{n-1} + \frac{8}{5} \cdot 3^{n-1}} \to 8^{-m} = N_*^{-m} \quad (n \to \infty).$$

By the portmanteau theorem (see [9, Theorem 2.1] for example), we conclude that $\widetilde{\mu}(K_w) = N_*^{-|w|}$ for all $w \in W_\#$. A simple application of Dynkin's π-λ theorem yields that $\widetilde{\mu} = \mu$. Therefore, μ_n converges weakly to μ as $n \to \infty$.

Next we will prove (6.7). Let $f \in C(K)$ be not constant. Then there exists $N \geq 1$ such that $f|_{V(\mathbb{G}_N)}$ is not constant. Define

$$c_n := \int_K \int_K |f(x) - f(y)|^p \, d\mu_n(y) d\mu_n(x), \quad c := \int_K \int_K |f(x) - f(y)|^p \, d\mu(y) d\mu(x),$$

for each $n \geq N$. Since $f|_{V(\mathbb{G}_N)}$ is not constant and f is bounded, we have that $c, c_n \in (0, \infty)$. The weak convergence of $\mu_n \times \mu_n$ to $\mu \times \mu$ (see [9, Theorem 2.8] for example) implies that $c_n \to c$ as $n \to \infty$. Next, we define $\nu_n(dx \otimes dy) := c_n^{-1} |f(x) - f(y)|^p \, d\mu_n(x) d\mu_n(y)$. Since f is continuous, we easily see that ν_n converges weakly to a probability measure ν on $K \times K$ given by $\nu(dx \otimes dy) := c^{-1} |f(x) - f(y)|^p \, d\mu(x) d\mu(y)$. From Lemma 2.4, it is immediate that $\nu(\partial(K \times U_1^{(m)}(x))) = 0$. Again by the portmanteau theorem, we deduce (6.7). □

Then, we can show some (p, p)-Poincaré type inequalities (in the sense of Kumagai and Sturm) in this context. The statement (2) in the lemma below is similar to Lemma 5.12, but we do not have any uniform Hölder estimate like Corollary 4.16 for the new rescaled p-energy $\widetilde{\mathcal{E}}_p^{\mathbb{G}_n}$. For this reason, its proof will be a bit complicated.

Lemma 6.10. *Suppose that $\beta > \alpha$ and $p > 1$ in* (1) *and that Assumption 4.2 holds in* (2). *Then there exists a positive constant $C_{6.10}$ (depending only on p, β, α, a,* diam(K), *C_{AR} and ρ_p) such that the following hold:*

(1) *for every $n \in \mathbb{N}$ and $f \in L^p(K, \mu)$,*

$$a^{\beta n} \sum_{w \in W_n} \sum_{x \in V_w(\mathbb{G}_1)} \int_{K_w} |f(x) - f(z)|^p \, d\mu(z) \leq C_{6.10} \, |f|_{\Lambda_{p,\infty}^{\beta/p}}^p ; \tag{6.9}$$

(2) *for every* $n, m \in \mathbb{N}$ *and* $f \in C(K)$,

$$a^{\beta_p n} \sum_{w \in W_n} \sum_{x \in V_w(\mathbb{G}_1^*)} \int_{K_w} |f(x) - f(z)|^p \, d\mu_{n+m}(z) \\ \leq C_{6.10} \sup_{n \in \mathbb{N}} a^{(\beta_p - \alpha)n} \mathcal{E}_p^{\mathbb{G}_n^*}(f). \tag{6.10}$$

Proof. (1) A slight change in the proof of Lemma 5.14 shows our assertion. Indeed, let $w = w_1 \cdots w_{n-1} \in W_{n-1}$, let $x \in V_w(\mathbb{G}_1)$ and fix $k \in \mathbb{N}$. Let $\omega \in \pi^{-1}(\{x\}) \cap \Sigma_w$ (i.e. $\pi(\omega) = x$ and $[\omega]_{n-1} = w$). For each $l \in \mathbb{Z}_{\geq 0}$, define $w(l) := [\omega]_{n+kl-1} \in W_{n+kl-1}$. Then, similarly to (5.11), we have that, for $z_l \in K_{w(l)}$ ($l = 0, \dots, n$),

$$|f(x) - f(z_0)|^p \leq 2^{p-1} |f(x) - f(z_n)|^p + 2^{p-1} \sum_{l=0}^{n-1} 2^{l(p-1)} |f(z_l) - f(z_{l+1})|^p .$$

By integrating this inequality over $z_l \in K_{w(l)}$, we obtain

$$\begin{aligned} &⨍_{K_w} |f(x) - f(z)|^p \, d\mu(z) \\ &\leq 2^{p-1} ⨍_{K_{w(n)}} |f(x) - f(z_n)|^p \, d\mu(z_n) \\ &\qquad + 2^{2(p-1)} \sum_{l=0}^{n-1} 2^{l(p-1)} ⨍_{K_{w(l)}} ⨍_{K_{w(l+1)}} |f(z_l) - f(z_{l+1})|^p \, d\mu(z_{l+1}) d\mu(z_l). \end{aligned}$$

The rest of the proof is essentially the same as the proof of Lemma 5.14.

(2) The idea is essentially the same as [25, Theorem 3.6]. Let $w \in W_n$, $m \in \mathbb{N}$ and choose $x_w \in V_w(\mathbb{G}_1^*)$ such that

$$\int_{K_w} |f(x_w) - f(z)|^p \, d\mu_{n+m}(z) = \max_{x \in V_w(\mathbb{G}_1^*)} \int_{K_w} |f(x) - f(z)|^p \, d\mu_{n+m}(z)$$

By Lemma 6.9, there exists a positive constant c_1 such that $\mu_l(\{y\}) \leq c_1 a^{-\alpha l}$ for any $l \in \mathbb{N}$ and $y \in V(\mathbb{G}_l^*)$, and thus

$$\begin{aligned} &\int_{K_w} |f(x_w) - f(z)|^p \, d\mu_{n+m}(z) \\ &= \sum_{z \in K_w \cap V(\mathbb{G}_{n+m}^*)} |f(x_w) - f(z)|^p \, \mu_{n+m}(\{z\}) \\ &\leq c_1 a^{-\alpha(n+m)} \sum_{z \in K_w \cap V(\mathbb{G}_{n+m}^*)} |f(x_w) - f(z)|^p . \end{aligned} \tag{6.11}$$

For each $z \in K_w \cap V(\mathbb{G}_{n+m}^*) = V_w(\mathbb{G}_m^*)$, fix a choice of $\omega^z \in \pi^{-1}(\{z\}) \cap \Sigma_w$. Define $w^z(l) := [\omega^z]_{n+l} \in W_{n+l}$ for each $l \in \mathbb{Z}_{\geq 0}$. We frequently omit the dependence on z to simplify notations, i.e. we also write $w(l)$ to denote $w^z(l)$. To estimate the right-hand side of (6.11), we will find $v(l+1) \in w(l) \cdot W_1$ and a sequence $(q_0, q_1, \dots, q_{2m})$ such that $(v(l+1), w(l+1)) \in \widetilde{E}_{n+l+1} \cup \{(v,v) \mid v \in W_{n+l+1}\}$, $q_0 = x_w$ and

$$(q_{2l}, q_{2l+1}), (q_{2l+1}, q_{2l+2}) \in E(\mathbb{G}^*_{n+l+1})^{V_{w(l)}(\mathbb{G}^*_1) \cup V_{v(l)}(\mathbb{G}^*_1)} \cup \mathsf{D}^{(l+1)}_w \tag{6.12}$$

for each $l \in \{0, \dots, m-1\}$, where $\mathsf{D}^{(l+1)}_w := \{(x,x) \mid x \in V_w(\mathbb{G}^*_{l+1})\}$. Define $V_{w(0)}(\mathbb{G}^*_1) \cup V_{v(0)}(\mathbb{G}^*_1) := V_{w(0)}(\mathbb{G}^*_1)$ for convention. First, set $q_0 := x_w$, pick $v(1) \in w \cdot W_1$ satisfying $(v(1), w(1)) \in \widetilde{E}_{n+1} \cup \{(v,v) \mid v \in W_{n+1}\}$ and define $q_2 \in V_{w(0)}(\mathbb{G}^*_1)$ as the element such that

$$\{q_2\} = (K_{w(1)} \cup K_{v(1)}) \cap V_{w(0)}(\mathbb{G}^*_1).$$

Then there exists $q_1 \in V_{w(0)}(\mathbb{G}^*_1)$ satisfying (6.12). Inductively, we define $v(l+1)$ and q_{2l+2} as follows: Choose $v(l+1) \in w(l) \cdot W_1$ such that $d_{n+l+1}(q_{2l}, q_{2l+2}) = 2$, where q_{2l+2} is defined by

$$\{q_{2l+2}\} = (K_{w(l+1)} \cup K_{v(l+1)}) \cap V_{w(l)}(\mathbb{G}^*_1)$$

and d_{n+l+1} is the graph distance of $\mathbb{G}^*_{n+l+1}$. Then we pick $q_{2l+1} \in V_w(\mathbb{G}_{l+1})$ satisfying (6.12). As a consequence, we obtain the desired objects. Since $(v(m), w(m)) \in \widetilde{E}_{n+m} \cup \{(v,v) \mid v \in W_{n+m}\}$ and the diameter of $\mathbb{G}^*_1$ is 2, we can modify the choice of $v(m)$ and q_{2m} so that

$$(q_{2m}, z) \in E(\mathbb{G}^*_{n+m})^{V_{w(m-1)}(\mathbb{G}^*_1)} \cup \{(x,x) \mid x \in V_{w(m-1)}(\mathbb{G}^*_1)\}.$$

Set $q_{2m+1} := z$. Note that this choice of a sequence $(q_0, q_1, \dots, q_{2m}, q_{2m+1})$ depends on z.

By Hölder's inequality, for each $l \in \{0, \dots, m\}$,

$$\begin{aligned} &|f(x_w) - f(q_{2l})|^p \\ &\qquad \le a^{p-1}\bigl(|f(q_{2l}) - f(q_{2l+1})|^p + |f(q_{2l+1}) - f(q_{2l+2})|^p + |f(x_w) - f(q_{2l+2})|^p \bigr). \end{aligned}$$

Iterating this estimate, we obtain

$$|f(x_w) - f(q_{2m})|^p \le \sum_{l=0}^{m-1} a^{(p-1)(l+1)}\bigl(|f(q_{2l}) - f(q_{2l+1})|^p + |f(q_{2l+1}) - f(q_{2l+2})|^p \bigr).$$

Therefore, there exists a positive constant c_2 depending only on a, p such that

$$\begin{aligned} &c_2^{-1} |f(x_w) - f(z)|^p \\ &\le |f(q_{2m}) - f(q_{2m+1})|^p + \sum_{l=0}^{m-1} a^{(p-1)l}\bigl(|f(q_{2l}) - f(q_{2l+1})|^p + |f(q_{2l+1}) - f(q_{2l+2})|^p \bigr). \end{aligned}$$

In view of the fact that $\#\{x \in K_w \cap V(\mathbb{G}^*_{n+m}) \mid x \in K_{[\omega^z]_{n+l}}\} = \#V(\mathbb{G}^*_{m-l})$ for any $z \in V(\mathbb{G}^*_{n+m})$, we see from (6.12) and (6.8) that

$$\begin{aligned} a^{-\alpha(n+m)} &\sum_{z \in K_w \cap V(\mathbb{G}^*_{n+m})} |f(x_w) - f(z)|^p \\ &\le 2c_2 a^{-\alpha(n+m)} \sum_{l=0}^{m-1} a^{(p-1)l} \#V(\mathbb{G}^*_{m-l}) \cdot \mathcal{E}_p^{V_{w(l)}(\mathbb{G}^*_1) \cup V_{v(l)}(\mathbb{G}^*_1)}(f) \end{aligned} \tag{6.13}$$

$$\leq c_2 a^{-\alpha n} \sum_{l=0}^{m-1} a^{(p-1-\alpha)l} \mathcal{E}_p^{V_w(\mathbb{G}_{l+1}^*)}(f).$$

Combining (6.11) and (6.13), we have that

$$\begin{aligned} a^{\beta_p n} \sum_{w \in W_n} \sum_{x \in V_w(\mathbb{G}_1^*)} \int_{K_w} |f(x) - f(z)|^p \, d\mu_{n+m}(z) \\ &\leq 4c_1 c_2 a^{(\beta_p - \alpha)n} \sum_{w \in W_n} \sum_{l=0}^{m-1} a^{(p-1-\alpha)l} \mathcal{E}_p^{V_w(\mathbb{G}_{l+1}^*)}(f) \\ &\leq 4c_1 c_2 a^{-(\beta_p - \alpha)} \left(\sum_{l=0}^{\infty} a^{-(\beta_p - p+1)l} \right) \sup_{n \in \mathbb{N}} a^{(\beta_p - \alpha)n} \mathcal{E}_p^{\mathbb{G}_n^*}(f). \end{aligned}$$

Since $\beta_p \geq p$ by Proposition 3.6, it holds that $\sum_{l=0}^{\infty} a^{-(\beta_p - p+1)l} < \infty$. Hence we obtain

$$a^{\beta_p n} \sum_{w \in W_n} \sum_{x \in V_w(\mathbb{G}_1^*)} \int_{K_w} |f(x) - f(z)|^p \, d\mu_{n+m}(z) \leq c_3 \sup_{n \in \mathbb{N}} a^{(\beta_p - \alpha)n} \mathcal{E}_p^{\mathbb{G}_n^*}(f),$$

where $c_3 = 4c_1 c_2 a^{-(\beta_p - \alpha)} \sum_{l=0}^{\infty} a^{-(\beta_p - p+1)l}$.

For each $n \in \mathbb{N}$, define $\varphi_n \colon V(\mathbb{G}_n^*) \to V(\mathbb{G}_n)$ by setting $\varphi_n(x) := x$ for any $x \in V(\mathbb{G}_n^*)$. Then we easily see that $\{\varphi_n\}_{n \geq 1}$ is a uniform rough isometry from $\{\mathbb{G}_n^*\}_{n \geq 1}$ to $\{\mathbb{G}_n\}_{n \geq 1}$. Applying Lemma 4.11 (especially (4.11)), we complete the proof. □

Now, we are ready to prove Lemma 6.7.

Proof of Lemma 6.7. Let $\beta > \alpha$. We will prove the following two bounds:

$$\text{Upper bound: } \sup_{n \in \mathbb{N}} a^{(\beta - \alpha)n} \cdot \mathcal{E}^{\mathbb{G}_n}(f) \leq c_{\mathrm{U}} |f|_{\Lambda_{p,\infty}^{\beta/p}}^p \quad \text{for every } f \in L^p(K, \mu),$$

$$\text{Lower bound: } \sup_{n \in \mathbb{N}} a^{(\beta_p - \alpha)n} \cdot \mathcal{E}^{\mathbb{G}_n}(f) \geq c_{\mathrm{L}} |f|_{\Lambda_{p,\infty}^{\beta/p}}^p \quad \text{for every } f \in C(K),$$

for some positive constants $c_{\mathrm{U}}, c_{\mathrm{L}}$ (without depending on f). Note that, from Theorem 2.18, the case $\beta = \beta_p$ in these bounds includes our assertion.

Upper bound. For $f \in L^p(K, \mu)$, we easily see that

$$a^{(\beta - \alpha)n} \cdot \mathcal{E}_p^{\mathbb{G}_n}(f) \leq 2^{p-2} D_*(\{\mathbb{G}_n\}_{n \geq 1}) a^{\beta n} \sum_{w \in W_{n-1}} \sum_{x \in V_w(\mathbb{G}_1)} \int_{K_w} |f(x) - f(z)|^p \, d\mu(z).$$

Applying Lemma 6.10, we get the desired upper bound.[4]

Lower bound. Let $f \in C(K)$ and let $n, m \in \mathbb{N}$. Then we see that

$$\begin{aligned} \int_K \int_{U_1^{(n)}(x)} |f(x) - f(y)|^p \, d\mu_m(y) d\mu_m(x) \\ \leq D_* \sum_{w \in W_n} \int_{K_w} \int_{U_1^{(n)}(x)} |f(x) - f(y)|^p \, d\mu_m(y) d\mu_m(x) \end{aligned} \tag{6.14}$$

[4] The constant c_{U} depends only on $p, D_*(\{\mathbb{G}_n\}_{n \geq 1}), C_{6.10}$.

$$\leq D_* \sum_{w \in W_n} \sum_{\substack{v \in W_n; \\ d_{G_n}(v,w) \leq 2}} \int_{K_w} \int_{K_v} |f(x) - f(y)|^p \, d\mu_m(y) d\mu_m(x).$$

For $v, w \in W_n$ with $d_{G_n}(v, w) \leq 2$, there exists a path $[v^0, \dots, v^l]$ in $\bigl(W_n, \widetilde{E_n}\bigr)$ with $l \leq 4$ such that $v^0 = v$ and $v^l = w$. Let $x_{2i-1} \in K_{v^{i-1}} \cap K_{v^i} \cap V(\mathbb{G}^*_{n+1})$ and $x_{2i} \in K_{v^i} \cap V(\mathbb{G}^*_{n+1})$ such that $x_{2i-1} = x_{2i}$ or $(x_{2i-1}, x_{2i}) \in E(\mathbb{G}_{n+1})$ for each $i = 1, \dots, l$. Then, for any $x \in K_v$ and $y \in K_w$,

$$\begin{aligned} &|f(x) - f(y)|^p \\ &\leq (2l)^{p-1} \left(|f(x) - f(x_1)|^p + |f(y) - f(x_{2l-1})|^p + \sum_{j=1}^{2l-2} \left| f(x_j) - f(x_{j+1}) \right|^p \right). \end{aligned}$$

For any $m > n$ and $j = 1, \dots, 2l - 2$, we see that

$$\begin{aligned} \int_{K_w} \int_{K_v} \left| f(x_j) - f(x_{j+1}) \right|^p \, d\mu_m(y) d\mu_m(x) &= \left| f(x_j) - f(x_{j+1}) \right|^p \mu_m(K_v) \mu_m(K_w) \\ &\leq \frac{25}{9} a^{-2\alpha n} \left| f(x_j) - f(x_{j+1}) \right|^p. \end{aligned}$$

Hence, from (6.14), Lemma 6.10-(2) and Lemma 4.11, we have that

$$\begin{aligned} &a^{(\alpha+\beta_p)n} \int_K \int_{U_1^{(n)}(x)} |f(x) - f(y)|^p \, d\mu_m(y) d\mu_m(x) \\ &\leq c_1 \left(a^{\beta_p n} \sum_{v \in W_n} \sum_{z \in V_v(\mathbb{G}_1^*)} \int_{K_v} |f(x) - f(z)|^p \, d\mu_m(x) + a^{(\beta_p - \alpha)n} \cdot \mathcal{E}_p^{\mathbb{G}^*_{n+1}}(f) \right) \\ &\leq c_2 \sup_{n \in \mathbb{N}} a^{(\beta_p - \alpha)n} \cdot \mathcal{E}_p^{\mathbb{G}_n}(f), \end{aligned}$$

where $c_1 := 2(2l)^{p-1} (5D_*/3)^3$ and $c_2 := c_1 C_{6.10} C_{\mathrm{RUI}}$. Letting $m \to \infty$ in this inequality, we see from Lemma 6.9 that

$$a^{(\alpha+\beta_p)n} \int_K \int_{U_1^{(n)}(x)} |f(x) - f(y)|^p \, d\mu(y) d\mu(x) \leq c_2 \sup_{n \in \mathbb{N}} a^{(\beta_p - \alpha)n} \cdot \mathcal{E}_p^{\mathbb{G}_n}(f),$$

for every $n \in \mathbb{N}$. By Lemma 2.7, it is immediate that there exists $c > 0$ (depending only on C_{AD}) such that $B(x, ca^{-n}) \subseteq U_1^{(n)}(x)$ for any $x \in K$ and $n \in \mathbb{N}$. Therefore, we conclude that $|f|_{\Lambda_{p,\infty}^{\beta_p/p}} \leq c_{\mathrm{L}} \sup_{n \in \mathbb{N}} a^{(\beta_p - \alpha)n} \cdot \mathcal{E}_p^{\mathbb{G}_n}(f)$ for all $f \in C(K)$.[5] This completes the proof. □

6.2 Proof of Theorem 2.20

Now, we construct the desired p-energy $\mathcal{E}_p$ on the Sierpiński carpet.

Proof of Theorem 2.20. From the identity (6.4) and Lemma 6.7, we immediately conclude that

$$\mathcal{F}_p = \{ f \in C(K) \mid F_i^* f \in \mathcal{F}_p \text{ for any } i \in S \}.$$

[5] The constant c_{L} depends only on $p, a, \alpha, D_*, C_{6.10}, C_{\mathrm{AR}}, C_{\mathrm{AD}}$.

Define

$$\overline{\mathcal{E}}_{p,n}(f) := \frac{1}{n}\sum_{l=1}^{n}\widetilde{\mathcal{E}}_p^{\mathbb{G}_l}(f) = \sum_{l=1}^{n}\frac{1}{2}\sum_{(x,y)\in E(\mathbb{G}_l)}\left|\frac{\rho_p^{l/p}}{n^{1/p}}f(x) - \frac{\rho_p^{l/p}}{n^{1/p}}f(y)\right|^p, \tag{6.15}$$

for each $n \in \mathbb{N}$ and $f \in C(K)$. Then we can show that $\overline{\mathcal{E}}_{p,n}(\,\cdot\,)^{1/p}$ gives a semi-norm satisfying Clarkson's inequality. Indeed, $\left(\overline{\mathcal{E}}_{p,n}(f)\right)^{1/p}$ can be regarded as a ℓ^p-norm of $(c_1\nabla^{\mathbb{G}_l}f|_{V_1},\dots,c_n\nabla^{\mathbb{G}_l}f|_{V_n})$ on $E(\mathbb{G}_1)\times\cdots\times E(\mathbb{G}_n)$, where $c_l := \rho_p^{l/p}/n^{1/p}$ and $\nabla^{\mathbb{G}_l}f|_{V_l}\big((x,y)\big) := f(y)-f(x)$ for each $(x,y)\in E(\mathbb{G}_l)$.

Since $\mathcal{F}_p$ is separable by Theorem 5.10, there exists a countable dense subset $\mathcal{F}_p^0 = \{f_j\}_{j\ge1}$ of $\mathcal{F}_p$. Since $\overline{\mathcal{E}}_{p,n}(f) \le C_{6.7}\,|f|_{\mathcal{F}_p}^p$ from Lemma 6.7, we have that $\left\{\overline{\mathcal{E}}_{p,n}(f)\right\}_{n\ge1}$ is bounded for each $f\in\mathcal{F}_p$. By the diagonal argument, we can take a subsequence $\{n_k\}_{k\ge1}$ such that $\left\{\overline{\mathcal{E}}_{p,n_k}(f_j)\right\}_{k\ge1}$ converges for each $j\ge1$ as $k\to\infty$. Let $f\in\mathcal{F}_p$, let $\varepsilon>0$ and let $f_*\in\mathcal{F}_p^0$ such that $\|f-f_*\|_{\mathcal{F}_p}<\varepsilon$. For $k,l\ge1$,

$$\begin{aligned}
&\left|\overline{\mathcal{E}}_{p,n_k}(f)^{1/p} - \overline{\mathcal{E}}_{p,n_l}(f)^{1/p}\right| \\
&\le \left|\overline{\mathcal{E}}_{p,n_k}(f)^{1/p} - \overline{\mathcal{E}}_{p,n_k}(f_*)^{1/p}\right| \\
&\qquad + \left|\overline{\mathcal{E}}_{p,n_k}(f_*)^{1/p} - \overline{\mathcal{E}}_{p,n_l}(f_*)^{1/p}\right| + \left|\overline{\mathcal{E}}_{p,n_l}(f)^{1/p} - \overline{\mathcal{E}}_{p,n_l}(f_*)^{1/p}\right| \\
&\le 2C_{6.7}\,|f-f_*|_{\mathcal{F}_p} + \left|\overline{\mathcal{E}}_{p,n_k}(f_*)^{1/p} - \overline{\mathcal{E}}_{p,n_l}(f_*)^{1/p}\right|,
\end{aligned}$$

and hence we obtain $\overline{\lim}_{k\wedge l\to\infty}\left|\overline{\mathcal{E}}_{p,n_k}(f)^{1/p} - \overline{\mathcal{E}}_{p,n_l}(f)^{1/p}\right| \le 2C_{6.7}\varepsilon$. Thus a sequence $\left\{\overline{\mathcal{E}}_{p,n_k}(f)\right\}_{k\ge1}$ is Cauchy. We conclude that $\lim_{k\to\infty}\overline{\mathcal{E}}_{p,n_k}(f)$ exists for all $f\in\mathcal{F}_p$. Denote this limit by $\mathcal{E}_p(f)$. Then we can easily show that $\mathcal{E}_p(\,\cdot\,)^{1/p}$ is a semi-norm. Furthermore, this limit diverges to ∞ for every $f\notin\mathcal{F}_p$. By Lemmas 6.6 and 6.7, we deduce that a semi-norm $\mathcal{E}_p(\,\cdot\,)^{1/p}$ is equivalent to $|\cdot|_{\mathcal{F}_p}$. Since $\overline{\mathcal{E}}_{p,n}(\,\cdot\,)^{1/p}$ is regarded as the ℓ^p-norm, $\mathcal{E}_p(\,\cdot\,)^{1/p}$ satisfies Clarkson's inequality. Similarly, a norm $\|\cdot\|_{\mathcal{F}_p}$ also satisfies Clarkson's inequality.

Next, we prove the properties (1)-(5) of $\mathcal{E}_p$.

(1) Let $f\in C(K)$ be constant. Then we easily see that $\widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(f)=0$ for any $n\in\mathbb{N}$. Thus $\mathcal{E}_p(f)=0$. Conversely, if $\mathcal{E}_p(f)=0$, then $\widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(f)=0$ for any $n\in\mathbb{N}$. By Lemma 6.5-(1), $f|_{\bigcup_{n\in\mathbb{N}}V(\mathbb{G}_n)}$ is constant. Since $\bigcup_{n\in\mathbb{N}}V(\mathbb{G}_n)$ is dense in K, we conclude that f is constant. Next, let $f\in\mathcal{F}_p$ and let $a\in\mathbb{R}$. Then it is immediate that $\widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(f)=\widetilde{\mathcal{E}}_p^{\mathbb{G}_n}(f+a\mathbb{1}_K)$ for any $n\in\mathbb{N}$. Hence $\mathcal{E}_p(f)=\mathcal{E}_p(f+a\mathbb{1}_K)$.

(2) This is proved in Theorem 5.5.

(3) Note that $\|\varphi(f)\|_{L^p}^p \le 2^{p-1}(|\varphi(0)|^p+\|f\|_{L^p}^p)$ whenever $\varphi\colon\mathbb{R}\to\mathbb{R}$ satisfies $\mathrm{Lip}(\varphi)\le1$. Since $\widetilde{\mathcal{E}}_p^{\mathbb{G}_n}$ has the Markov property (Lemma 6.5-(4)), we see that $\overline{\mathcal{E}}_{p,n}$ also has the same property. This immediately deduces the Markov property of $\mathcal{E}_p$.

(4) For the same reason as (3), $\mathcal{E}_p$ has the required symmetries (see Lemma 6.5-(2)).

(5) We see from (6.4) that, for $f\in\mathcal{F}_p$,

$$\rho_p\sum_{i\in S}\mathcal{E}_p(F_i^*f) = \lim_{k\to\infty}\frac{1}{n_k}\sum_{l=1}^{n_k}\left(\rho_p\sum_{i\in S}\widetilde{\mathcal{E}}_p^{\mathbb{G}_l}(F_i^*f)\right)$$

$$= \lim_{k\to\infty} \left(\overline{\mathcal{E}}_{p,n_k}(f) - \frac{1}{n_k}\widetilde{\mathcal{E}}_p^{\mathbb{G}_1}(f) + \frac{1}{n_k}\widetilde{\mathcal{E}}_p^{\mathbb{G}_{n_k+1}}(f) \right) = \mathcal{E}_p(f).$$

(6) Let $A_1 \coloneqq \mathrm{supp}[f]$ and let $A_2 \coloneqq \mathrm{supp}[g - a\mathbb{1}_K]$. Since $\mathrm{dist}(A_1, A_2) > 0$, there exists $N \in \mathbb{N}$ such that $\sup_{n\geq N} \max_{w\in W_n} \mathrm{diam}\, K_w < \mathrm{dist}(A_1, A_2)$. Then $F_w^* f$ or $F_w^*(g - a\mathbb{1}_K)$ is equal to 0 for any $w \in W_n$ and $n \geq N$. From the self-similarity, we deduce that, for $n \geq N$,

$$\begin{aligned}
\mathcal{E}_p(f+g) &= \mathcal{E}_p(f + g - a\mathbb{1}_K) \\
&= \rho_p^n \sum_{w\in W_n} \mathcal{E}_p\big(F_w^* f + F_w^*(g - a\mathbb{1}_K)\big) \\
&= \rho_p^n \sum_{w\in W_n; K_w\cap A_1\neq\emptyset} \mathcal{E}_p\big(F_w^* f\big) + \rho_p^n \sum_{w\in W_n; K_w\cap A_2\neq\emptyset} \mathcal{E}_p\big(F_w^*(g - a\mathbb{1}_K)\big) \\
&= \mathcal{E}_p(f) + \mathcal{E}_p(g).
\end{aligned}$$

This completes the proof. □

Remark 6.11. (1) Since $\mathcal{E}_p(\,\cdot\,)^{1/p}$ satisfies Clarkson's inequality, $\mathcal{E}_p(\,\cdot\,)^{1/p}$ is strictly convex, that is, if $\lambda \in (0, 1)$ $f, g \in \mathcal{F}_p$ with $f - g$ is not constant, then

$$\mathcal{E}_p\big(\lambda f + (1-\lambda)g\big)^{1/p} < \lambda\, \mathcal{E}_p(f)^{1/p} + (1-\lambda)\mathcal{E}_p(g)^{1/p}.$$

The convexity of $\mathcal{E}_p(\,\cdot\,)$ is also immediate from the construction. Moreover, from the convexity of $x \mapsto |x|^p$, we can show that $\mathcal{E}_p(\,\cdot\,)$ is strictly convex.

(2) The framework in [47] includes not only the standard planar Sierpiński carpet but also Sierpiński gaskets and other self-similar sets (nested fractals and generalized Sierpiński carpets for example). A recent paper by Kigami [38] gives a more general framework to construct canonical p-energy on p-conductively homogeneous compact metric spaces, which includes new results even when $p = 2$.

Using the self-similarity of $\mathcal{E}_p$, we obtain the following property, which is called the strong locality when $p = 2$.

Corollary 6.12. *If $f, g \in \mathcal{F}_p$ satisfy $\mathrm{supp}[f] \cap \mathrm{supp}[g - a\mathbb{1}_K] = \emptyset$ for some $a \in \mathbb{R}$, then $\mathcal{E}_p(f+g) = \mathcal{E}_p(f) + \mathcal{E}_p(g)$.*

Proof. Let $A_1 \coloneqq \mathrm{supp}[f]$ and let $A_2 \coloneqq \mathrm{supp}[g - a\mathbb{1}_K]$. Since $\mathrm{dist}(A_1, A_2) > 0$, there exists $N \in \mathbb{N}$ such that $\sup_{n\geq N} \max_{w\in W_n} \mathrm{diam}\, K_w < \mathrm{dist}(A_1, A_2)$. Then $F_w^* f$ or $F_w^*(g - a\mathbb{1}_K)$ is equal to 0 for any $w \in W_n$ and $n \geq N$. From the and self-similarity, we deduce that, for $n \geq N$,

$$\begin{aligned}
\mathcal{E}_p(f+g) &= \mathcal{E}_p(f + g - a\mathbb{1}_K) \\
&= \rho_p^n \sum_{w\in W_n} \mathcal{E}_p\big(F_w^* f + F_w^*(g - a\mathbb{1}_K)\big) \\
&= \rho_p^n \sum_{w\in W_n; K_w\cap A_1\neq\emptyset} \mathcal{E}_p\big(F_w^* f\big) + \rho_p^n \sum_{w\in W_n; K_w\cap A_2\neq\emptyset} \mathcal{E}_p\big(F_w^*(g - a\mathbb{1}_K)\big) \\
&= \mathcal{E}_p(f) + \mathcal{E}_p(g).
\end{aligned}$$

This completes the proof. □

For future work, it might be useful to provide the following estimate concerning products of functions in $\mathcal{F}_p$. When $p = 2$, this result is standard (see [23, Theorem 1.4.2-(ii)] for example). See also [38, Lemma 6.17-(2)].

Proposition 6.13. *For any* $f, g \in \mathcal{F}_p$,

$$\mathcal{E}_p(f \cdot g) \leq 2^{p-1}\bigl(\|f\|_{C(K)}^p \mathcal{E}_p(g) + \|g\|_{C(K)}^p \mathcal{E}_p(f)\bigr).$$

In particular, $f \cdot g \in \mathcal{F}_p$.

Proof. For any $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathcal{E}_p^{\mathbb{G}_n}(f \cdot g) &\leq 2^{p-1}\frac{1}{2}\sum_{(x,y)\in E(\mathbb{G}_n)}\bigl(|g(x)|^p\,|f(x)-f(y)|^p + |f(y)|^p\,|g(x)-g(y)|^p\bigr)\\ &\leq 2^{p-1}\bigl(\|g\|_{C(K)}^p \mathcal{E}_p^{\mathbb{G}_n}(f) + \|f\|_{C(K)}^p \mathcal{E}_p^{\mathbb{G}_n}(g)\bigr).\end{aligned}$$

In view of the proof of Theorem 2.20, this immediately implies our assertion. □

6.3 Proof of Theorem 2.19

We conclude this section by proving Theorem 2.19: a strict inequality $\beta_p > p$. Our argument is similar to [33, Section 3]. The key to the proof is the notion of $\mathcal{E}_p$*-harmonicity*.

Definition 6.14. Let U be a non-empty open subset of K. We define

(6.16) $$C^U := \{f \in \mathcal{F}_p \mid \mathrm{supp}[f] \subseteq U\}, \quad \text{and} \quad \mathcal{F}_p^U := \overline{C^U}^{\|\cdot\|_{\mathcal{F}_p}}.$$

Proposition 6.15. *It holds that*

(6.17) $$\mathcal{F}_p^U = \{f \in \mathcal{F}_p \mid f(x) = 0 \text{ for any } x \in K \setminus U\}.$$

Proof. It is easy to show that $\mathcal{F}_p^U \subseteq \{f \in \mathcal{F}_p \mid f(x) = 0$ for any $x \in K \setminus U\}$. To prove the converse, let $\{\varphi_w\}_{w\in W_\#}$ be the partition of unity in Lemma 5.4 and let $f \in \mathcal{F}_p$ with $f(x) = 0$ for any $x \in K \setminus U$. For each $n \in \mathbb{N}$, define

$$f_n(x) := \sum_{w\in W_n;\, U_1^{(n)}(w)\cap(K\setminus U)=\emptyset} M_n f(w)\varphi_w(x), \quad x \in K.$$

Then it is clear that $f_n \in \mathcal{F}_p$ and $\mathrm{supp}[f_n] \subseteq U$. By Lemma 2.7, we easily see that $f_n \to f$ in $C(K)$ as $n \to \infty$. A similar argument in the proof of Theorem 5.10 deduces that there exist a subsequence $\{n_k\}_{k\geq 1}$ and a sequence $\{g_m\}_{m\geq 1}$ from

$$\left\{\sum_{k=1}^{N} a_k f_{n_k} \;\middle|\; N \in \mathbb{N}, a_k \geq 0 \text{ for each } k = 1, \ldots, N\right\}$$

such that $g_m \to f$ in $\mathcal{F}_p$ as $m \to \infty$. Since $\mathrm{supp}[g_m] \subseteq U$ for any $m \in \mathbb{N}$, we obtain (6.17). □

Definition 6.16. Let U be a non-empty open subset of K. Then $h \in \mathcal{F}_p$ is said to be *$\mathcal{E}_p$-harmonic* on U if and only if the following condition holds:

$$\mathcal{E}_p(h) = \inf\{\mathcal{E}_p(f) \mid f \in \mathcal{F}_p, f = h \text{ on } K \setminus U\}. \tag{6.18}$$

Proposition 6.17. *Let U be a non-empty open subset of K with $U \neq K$ and let $g \in \mathcal{F}_p$. Then there exists a unique function $h \in \mathcal{F}_p$ that is $\mathcal{E}_p$-harmonic on U and $h|_{K\setminus U} \equiv g|_{K\setminus U}$.*

Proof. If $g|_{K\setminus U} \equiv a$ for some $a \in \mathbb{R}$, then $h := a$ is the required function. Suppose that $g \in \mathcal{F}_p$ is not constant on $K \setminus U$. Since g is bounded and $\mathcal{E}_p(f + a\mathbb{1}_K) = \mathcal{E}_p(f)$ for any $f \in \mathcal{F}_p$ and $a \in \mathbb{R}$, we may assume that $0 \leq g \leq 1$. Clearly, $\{f \in \mathcal{F}_p \mid f|_{K\setminus U} \equiv g|_{K\setminus U}\}$ is non-empty. For each $\lambda \geq 0$, define

$$\mathsf{c}_\lambda := \inf\{\mathcal{E}_p(f) + \lambda \|f\|_{L^p}^p \mid f \in \mathcal{F}_p \text{ with } f|_{K\setminus U} \equiv g|_{K\setminus U}\}.$$

Note that $\mathsf{c}_\lambda < \infty$. Let $f \in \mathcal{F}_p$ satisfy $f|_{K\setminus U} \equiv g|_{K\setminus U}$. Set $f^\# := (f \vee 0) \wedge 1 \in \mathcal{F}_p$. Then, it follows from $0 \leq g \leq 1$ that $f^\#|_{K\setminus U} = g|_{K\setminus U}$. Thus,

$$\mathcal{E}_p(f) \geq \mathcal{E}_p(f^\#) + \lambda \left\|f^\#\right\|_{L^p}^p - \lambda \geq \mathsf{c}_\lambda - \lambda,$$

which implies that $\mathsf{c}_0 \geq \mathsf{c}_\lambda - \lambda$ for any $\lambda \geq 0$. For each $n \in \mathbb{N}$, we can choose $f_n \in \mathcal{F}_p$ with $f_n|_{K\setminus U} \equiv g|_{K\setminus U}$ such that

$$\mathcal{E}_p(f_n) + n^{-1} \|f_n\|_{L^p}^p < \mathsf{c}_{n^{-1}} + n^{-1}.$$

Then $\mathcal{E}_p(f_n^\#) \leq \mathsf{c}_0 + 2n^{-1}$ for each $n \in \mathbb{N}$, where $f_n^\# := (f_n \vee 0) \wedge 1 \in \mathcal{F}_p$. Since $\left\|f_n^\#\right\|_{L^p}^p \leq 1$ for any $n \in \mathbb{N}$, there exist $h \in L^p(K, \mu)$ and a subsequence $\{n_k\}_{k\geq 1}$ such that $\{f_{n_k}^\#\}_{k\geq 1}$ converges weakly to h in L^p. Applying Mazur's lemma, we find convex combinations $u_k = \sum_{j=k}^{N_k} a_{k,j} f_{n_j}^\#$ (i.e. $N_k \in \mathbb{N}$, $a_{k,j} \geq 0$ and $\sum_{j=k}^{N_k} a_{k,j} = 1$) such that u_k converges to h in L^p as $k \to \infty$. Note that $f_n^\#|_{K\setminus U} \equiv g|_{K\setminus U}$ and thus $u_k|_{K\setminus U} \equiv g|_{K\setminus U}$. Also, we obtain $h|_{K\setminus U} = g|_{K\setminus U}$ μ-a.e. since $u_k \to h$ in L^p as $k \to \infty$. By the triangle inequality of $\mathcal{E}_p(\,\cdot\,)^{1/p}$, we have that $\mathcal{E}_p(u_k) \in [\mathsf{c}_0, \mathsf{c}_0 + 2n_k^{-1})$, which together with Clarkson's inequality implies that $\lim_{k\wedge l\to\infty} \mathcal{E}_p(u_k - u_l) = 0$. Indeed, when $p < 2$,

$$\begin{aligned}
\mathcal{E}_p(u_k - u_l)^{\frac{1}{p-1}} &\leq 2\bigl(\mathcal{E}_p(u_k) + \mathcal{E}_p(u_l)\bigr)^{\frac{1}{p-1}} - \mathcal{E}_p(u_k + u_l)^{\frac{1}{p-1}} \\
&\leq 2\bigl(2\mathsf{c}_0 + 2n_k^{-1} + 2n_l^{-1}\bigr)^{\frac{1}{p-1}} - 2^{\frac{p}{p-1}} \mathsf{c}_0^{\frac{1}{p-1}} \\
&= 2\bigl(2n_k^{-1} + 2n_l^{-1}\bigr)^{\frac{1}{p-1}} \underset{k\wedge l\to\infty}{\longrightarrow} 0.
\end{aligned}$$

The case $p \geq 2$ is similar to the above. Therefore, $\{u_k\}_{k\geq 1}$ is a Cauchy sequence in $\mathcal{F}_p$. Since $\mathcal{F}_p$ is a Banach space, we see that $h \in \mathcal{F}_p$ and u_k converges to h in $\mathcal{F}_p$. Moreover, by $\lim_{k\to\infty} \|u_k\|_{L^p} = \|h\|_{L^p}$, we conclude that $\mathcal{E}_p(h) = \lim_{k\to\infty} \mathcal{E}_p(u_k) = \mathsf{c}_0$, that is, h is a minimizer of $\inf\{\mathcal{E}_p(f) \mid f \in \mathcal{F}_p \text{ with } f|_{K\setminus U} \equiv g|_{K\setminus U}\}$.

Lastly, we prove the uniqueness. Let $h_i \in \mathcal{F}_p$ $(i = 1, 2)$ be $\mathcal{E}_p$-harmonic on U with $h_i|_{K\setminus U} \equiv g|_{K\setminus U}$. When $p < 2$, by Clarkson's inequality of $\mathcal{E}_p$,

$$\mathcal{E}_p(h_1 - h_2)^{\frac{1}{p-1}} \leq 2\bigl(\mathcal{E}_p(h_1) + \mathcal{E}_p(h_2)\bigr)^{\frac{1}{p-1}} - \mathcal{E}_p(h_1 + h_2)^{\frac{1}{p-1}}$$

$$\leq 2^{1+\frac{1}{p-1}}\mathsf{c}_0^{\frac{1}{p-1}} - 2^{\frac{p}{p-1}}\mathsf{c}_0^{\frac{1}{p-1}} = 0.$$

Thus $h_1 - h_2$ is constant by Theorem 2.20-(1). Since $K \setminus U$ is not empty, we have that $h_1 = h_2$. The case $p \leq 2$ is similar. □

Recall the definitions of $K^{\mathrm{L}}, K^{\mathrm{R}}$ (see (4.8)) and define

$$K^{\mathrm{T}} := \{(x, y) \in K \mid y = 1/2\}, \quad K^{\mathrm{B}} := \{(x, y) \in K \mid y = -1/2\}. \tag{6.19}$$

It is immediate from Theorems 2.17 and 2.20-(3) that $\{f \in \mathcal{F}_p \mid f|_{K^{\mathrm{L}}} \equiv 1, f|_{K^{\mathrm{R}}} \equiv 0\} \neq \emptyset$. Thus we have the following lemma by applying Proposition 6.17.

Lemma 6.18. *There exists a function $h_0 \in \mathcal{F}_p$ such that $h_0|_{K^{\mathrm{L}}} \equiv 1$, $h_0|_{K^{\mathrm{R}}} \equiv 0$ and h_0 is $\mathcal{E}_p$-harmonic on $K \setminus (K^{\mathrm{L}} \cup K^{\mathrm{R}})$.*

Let h_0 be as in Lemma 6.18. Since $\mathcal{E}_p(f) = 0$ if and only if f is constant, we immediately have that $\mathcal{E}_p(h_0) > 0$. Inductively, we define

$$\begin{aligned} h_n := &\sum_{i \in \{1,7,8\}} (F_i)_*\bigl(a^{-1}h_{n-1}\bigr) \\ &+ \sum_{i \in \{2,6\}} (F_i)_*\bigl(a^{-1}(h_{n-1}+1)\bigr) + \sum_{i \in \{3,4,5\}} (F_i)_*\bigl(a^{-1}(h_{n-1}+2)\bigr). \end{aligned} \tag{6.20}$$

Then the following proposition is clear by its definition and self-similarity of $\mathcal{E}_p$.

Lemma 6.19. *For any $n \in \mathbb{Z}_{\geq 0}$, it holds that $h_n \in \mathcal{F}_p$, $h_n|_{K^{\mathrm{L}}} \equiv 1$, and $h_n|_{K^{\mathrm{R}}} \equiv 0$.*

The following lemma is a key to prove Theorem 2.19.

Lemma 6.20. *The function h_2 is not $\mathcal{E}_p$-harmonic on $K \setminus (K^{\mathrm{L}} \cup K^{\mathrm{R}})$.*

Proof. Suppose to the contrary that h_2 were $\mathcal{E}_p$-harmonic on $K \setminus (K^{\mathrm{L}} \cup K^{\mathrm{R}})$. We claim that then a contradiction that $h_0|_{K^{\mathrm{T}}} \equiv 0$ would be implied. Let $\varphi \in \mathcal{F}_p^{K \setminus (K^{\mathrm{L}} \cup K^{\mathrm{B}})}$. We set $\varphi_{\mathrm{h}} := \varphi \circ T_{\mathrm{h}}$. Define $\varphi_0 \in C(K)$ by

$$\varphi_0(x) := \begin{cases} a^{-2}\varphi(F_{83}^{-1}(x)) & \text{if } x \in K_{83}, \\ a^{-2}\varphi_{\mathrm{h}}(F_{84}^{-1}(x)) & \text{if } x \in K_{84}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that φ_0 is well-defined from the definitions of φ_{h}. Furthermore, we have $\varphi_0 \in \mathcal{F}_p$ since $F_w^*\varphi_0 \in \mathcal{F}_p$ for any $w \in W_2$ (see Theorem 2.20-(4), (5)). Since $h_2 + \varphi_0 = h_2$ on $K^{\mathrm{L}} \cup K^{\mathrm{R}}$, the uniqueness of $\mathcal{E}_p$-harmonic function yields that $\mathcal{E}_p(h_2 + \varphi_0) > \mathcal{E}_p(h_2)$ unless $\varphi \equiv 0$. Using Theorem 2.20-(4) and (5), we see that

$$\begin{aligned} &\mathcal{E}_p(h_2 + \varphi_0) - \mathcal{E}_p(h_2) \\ &= \rho_p^2 \sum_{w \in W_2} \bigl(\mathcal{E}_p(F_w^*h_2 + F_w^*\varphi_0) - \mathcal{E}_p(F_w^*h_2)\bigr) \end{aligned}$$

$$= \rho_p^2 \sum_{w\in\{83,84\}} \bigl(\mathcal{E}_p(F_w^* h_2 + F_w^*\varphi_0) - \mathcal{E}_p(F_w^* h_2)\bigr) \quad (\text{by } F_w^*\varphi_0 \equiv 0 \text{ for } w \notin \{83,84\})$$

$$= 2\rho_p^2 a^{-2p} \bigl(\mathcal{E}_p(h_0+\varphi) - \mathcal{E}_p(h_0)\bigr) \quad (\text{by } h_0 \circ T_{\mathrm{h}} \equiv h_0).$$

Hence, we conclude that $\mathcal{E}_p(h_0+\varphi) > \mathcal{E}_p(h_0)$ for any $\varphi \in \mathcal{F}_p^{K\setminus(K^{\mathrm{L}}\cup K^{\mathrm{B}})} \setminus \{0\}$. This implies that h_0 is the minimizer of $\inf\{\mathcal{E}_p(f) \mid f \in \mathcal{F}_p \text{ with } f|_{K^{\mathrm{L}}\cup K^{\mathrm{B}}} \equiv h_0|_{K^{\mathrm{L}}\cup K^{\mathrm{B}}}\}$.

Next, we define $\widetilde{h}_0 \colon K \to \mathbb{R}$ by

$$\widetilde{h}_0(\mathbf{x}) := \begin{cases} h_0(\mathbf{x}) & \text{if } \mathbf{x} \in K \cap \{(x,y)\in\mathbb{R}^2 \mid y \le -x\} \\ h_0(T_-(\mathbf{x})) & \text{otherwise.} \end{cases} \tag{6.21}$$

Then it is clear that $\widetilde{h}_0 \in C(K)$ and $\widetilde{h}_0|_{K^{\mathrm{L}}\cup K^{\mathrm{B}}} \equiv h_0|_{K^{\mathrm{L}}\cup K^{\mathrm{B}}}$. Moreover, we can verify that $\widetilde{h}_0 \in \mathcal{F}_p$ and $\mathcal{E}_p(\widetilde{h}_0) = \mathcal{E}_p(h_0)$. Indeed, for any $n \in \mathbb{N}$,

$$\mathcal{E}_p^{\mathbb{G}_n}(h_0) = \mathcal{E}_p^{\mathbb{G}_n^1}(h_0) + \mathcal{E}_p^{\mathbb{G}_n^2}(h_0),$$

where $\mathbb{G}_n^i = (\mathbb{V}_n^i, \mathbb{E}_n^i)$ $(i = 1,2)$ are given by

$$\mathbb{V}_n^1 = \{(x,y)\in V(\mathbb{G}_n) \mid y \le -x\}, \qquad \mathbb{E}_n^1 = \{(v,w)\in E(\mathbb{G}_n) \mid v,w \in \mathbb{V}_n^1\},$$
$$\mathbb{V}_n^2 = \{(x,y)\in V(\mathbb{G}_n) \mid y \ge -x\}, \qquad \mathbb{E}_n^2 = \{(v,w)\in E(\mathbb{G}_n) \mid v,w \in \mathbb{V}_n^2\}.$$

Since $(1-h_0)\circ R_+^2 = \widetilde{h}_0$ on $\mathbb{V}_n^2$, we have $\mathcal{E}_p^{\mathbb{G}_n^2}(h_0) = \mathcal{E}_p^{\mathbb{G}_n^2}(\widetilde{h}_0)$, and thus $\mathcal{E}_p^{\mathbb{G}_n}(h_0) = \mathcal{E}_p^{\mathbb{G}_n}(\widetilde{h}_0)$ for all $n\in\mathbb{N}$. Therefore, we conclude that $\mathcal{E}_p(\widetilde{h}_0) = \mathcal{E}_p(h_0)$.

By Proposition 6.17, we have $h_0 = \widetilde{h}_0$. Hence $h_0|_{K^{\mathrm{T}}} \equiv 1$, which contradicts the fact that $h_0(p_5) = 0$. We complete the proof. □

Now we are ready to prove Theorem 2.19.

Proof of Theorem 2.19. By Lemmas 6.19 and 6.20, we obtain $\mathcal{E}_p(h_2) > \mathcal{E}_p(h_0)$. Since

$$\mathcal{E}_p(h_2) = \rho_p^2 \sum_{w\in W_2} \mathcal{E}_p(F_w^* h_2) = \rho_p^2 a^{-2p} \sum_{w\in W_2} \mathcal{E}_p(h_0),$$

we conclude that $\rho_p^2 a^{-2p} N_*^2 > 1$, which proves our assertion for $p > \dim_{\mathrm{ARC}}(K,d)$. We know that β_p/p is monotonically non-increasing by [41, Lemma 4.7.4], and thus we obtain the desired result. □

Remark 6.21. A recent study of L^p Besov critical exponent in [3] implies a partial result of Theorem 2.19. Indeed, β_p is characterized as the L^p Besov critical exponent in Corollary 5.16. Thus a critical exponent $\alpha_p^{\#}$ in [3, equation (7)] coincides with $\beta_p/(p\beta_2)$. Therefore, [3, Theorem 3.11] gives a lower bound of β_p:

- $\beta_p \ge \frac{p\beta_2}{2}$ for $p \in (\dim_{\mathrm{ARC}}(K,d), 2)$;
- $\beta_p \ge (p-2)(\beta_2 - \alpha) + \beta_2$ for $p \ge 2$.

Moreover, this bound implies that $\beta_p > p$ for $p \in \bigl(\dim_{\mathrm{ARC}}(K,d), (2\alpha-\beta_2)/(\alpha-\beta_2+1)\bigr)$.

7 Construction and basic properties of $\mathcal{E}_p$-energy measures

In this section, we construct $\mathcal{E}_p$-energy measures in the same way as Hino's work [31, Lemma 4.1]. We also investigate some properties, especially the chain rule of $\mathcal{E}_p$-energy measures.

Let $(\mathcal{E}_p, \mathcal{F}_p)$ be the p-energy in subsection 6.2, and let $f \in \mathcal{F}_p$. For each $n \in \mathbb{N}$, we define a measure $\mathfrak{m}^{p,n}_{\langle f \rangle}$ on W_n (equipped with the σ-algebra 2^{W_n}) by setting

$$\mathfrak{m}^{p,n}_{\langle f \rangle}(A) \coloneqq \rho_p^n \sum_{w \in A} \mathcal{E}_p(F_w^* f), \quad A \subseteq W_n.$$

Then we easily see that the total mass of $\mathfrak{m}^{p,n}_{\langle f \rangle}$ is equal to $\mathcal{E}_p(f) < \infty$. Furthermore, it follows from the self-similarity of $\mathcal{E}_p$ that, for any $A \subseteq W_n$,

$$\mathfrak{m}^{p,n+1}_{\langle f \rangle}(A \cdot W_1) = \rho_p^n \sum_{w \in A} \rho_p \sum_{i \in S} \mathcal{E}_p(F_{wi}^* f) = \rho_p^n \sum_{w \in A} \mathcal{E}_p(F_w^* f) = \mathfrak{m}^{p,n}_{\langle f \rangle}(A).$$

Therefore, $\{\mathfrak{m}^{p,n}_{\langle f \rangle}\}_{n \geq 1}$ satisfies the consistency condition, and hence Kolmogorov's extension theorem (see [22, Theorem 12.1.2] for example) yields a unique Borel finite measure $\mathfrak{m}^{p}_{\langle f \rangle}$ on Σ such that $\mathfrak{m}^{p}_{\langle f \rangle}(\Sigma_w) = \mathfrak{m}^{p,|w|}_{\langle f \rangle}(\{w\})$ for every $w \in W_\#$. Then we define $\mu^{p}_{\langle f \rangle} \coloneqq \pi_* \mathfrak{m}^{p}_{\langle f \rangle}$, where π is the natural projection (recall Proposition 2.3). Note that $\mu^{p}_{\langle f \rangle}$ is Borel regular (see [22, Theorem 7.1.3] for example).

Before proving Theorem 2.22, we observe two fundamental properties of $\mu^{p}_{\langle f \rangle}$.

Proposition 7.1. *Let $f \in \mathcal{F}_p$. Then $\mu^{p}_{\langle f \rangle} \equiv 0$ if and only if f is constant.*

Proof. It is immediate from $\mu^{p}_{\langle f \rangle}(K) = \mathcal{E}_p(f)$ and Theorem 2.20-(1). □

Proposition 7.2. *For every $f, g \in \mathcal{F}_p$ and $A \in \mathcal{B}(K)$, it holds that*

$$\left| \mu^{p}_{\langle f \rangle}(A)^{1/p} - \mu^{p}_{\langle g \rangle}(A)^{1/p} \right| \leq \mu^{p}_{\langle f-g \rangle}(A)^{1/p}. \tag{7.1}$$

In particular, if $f_n \in \mathcal{F}_p$ converges to f in $\mathcal{F}_p$, then $\mu^{p}_{\langle f_n \rangle}(A) \to \mu^{p}_{\langle f \rangle}(A)$ for every $A \in \mathcal{B}(K)$.

Proof. Since $\mu^{p}_{\langle f \rangle}$ is Borel regular, it will suffice to prove (7.1) when A is a closed set. Let A be a closed set of K and define $C_l \coloneqq \{w \in W_l \mid \Sigma_w \cap \pi^{-1}(A) \neq \emptyset\}$ for each $l \in \mathbb{N}$. Then, as proved in [31, proof of Lemma 4.1], one can show that $\{\Sigma_{C_l}\}_{l \geq 1}$ is a decreasing sequence and $\bigcap_{l \in \mathbb{N}} \Sigma_{C_l} = \pi^{-1}(A)$, where $\Sigma_{C_l} \coloneqq \{\omega \in \Sigma \mid [\omega]_l \in C_l\}$.

Recall that $\mathcal{E}_p$ is obtained as a subsequential limit of $\{\overline{\mathcal{E}}_{p,n}\}_{n \geq 1}$, where $\overline{\mathcal{E}}_{p,n}$ is given in (6.15). We may assume that $\mathcal{E}_p(f) = \lim_{n \to \infty} \overline{\mathcal{E}}_{p,n}(f)$ for every $f \in \mathcal{F}_p$. For each $l, n \in \mathbb{N}$, by choosing suitable constants $c_{w,i}$ $(w \in C_l, i \in \{1, \dots, n\})$, we can regard $\left(\sum_{w \in C_l} \overline{\mathcal{E}}_{p,n}(F_w^* f)\right)^{1/p}$ as a ℓ^p-norm of $(c_{w,i} \nabla^{\mathbb{G}_{|w|}} F_w^* f)_{w \in C_l, 1 \leq i \leq n}$. Consequently, we have that

$$\left| \left(\rho_p^l \sum_{w \in C_l} \overline{\mathcal{E}}_{p,n}(F_w^* f)\right)^{1/p} - \left(\rho_p^l \sum_{w \in C_l} \overline{\mathcal{E}}_{p,n}(F_w^* g)\right)^{1/p} \right|$$

$$\leq \left(\rho_p^l \sum_{w \in C_l} \overline{\mathcal{E}}_{p,n}\big(F_w^*(f-g)\big) \right)^{1/p}.$$

Letting $n \to \infty$ in this inequality, we conclude that

$$\left| \mathfrak{m}^p_{\langle f \rangle}(\Sigma_{C_l})^{1/p} - \mathfrak{m}^p_{\langle g \rangle}(\Sigma_{C_l})^{1/p} \right| \leq \mathfrak{m}^p_{\langle f-g \rangle}(\Sigma_{C_l})^{1/p},$$

for any $l \in \mathbb{N}$. Letting $l \to \infty$, we obtain (7.1) for any closed set A. □

First, we prove Theorem 2.22-(2) and (3)

Theorem 7.3 (Theorem 2.22-(2))**.** *For any* $\Phi \in C^1(\mathbb{R})$ *and* $f \in \mathcal{F}_p$*,*

$$\mu^p_{\langle \Phi(f) \rangle}(dx) = |\Phi'(x)|^p \, \mu^p_{\langle f \rangle}(dx). \tag{7.2}$$

Proof. Note that $\Phi \circ f \in \mathcal{F}_p$ for any $f \in \mathcal{F}_p$ by the Markov property of $\mathcal{E}_p$ (Theorem 2.20-(c)) and the compactness of $f(K)$.

First, we prove (7.2) when Φ is a polynomial. Let Φ be a polynomial and let $f \in \mathcal{F}_p$. Since $f(K)$ is compact, for any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\left| \left| \frac{\Phi(f(y)) - \Phi(f(y'))}{f(y) - f(y')} \right|^p - |\Phi'(f(x))|^p \right| < \varepsilon$$

whenever $x, y, y' \in K_w$ for some $w \in W_n$ with $n \geq N(\varepsilon)$. Thus, for any $m \in \mathbb{N}$, $n \geq N(\varepsilon)$, $(y, y') \in E(\mathbb{G}_m)$, $w \in W_n$ and $x \in K_w$,

$$\begin{aligned} &\Big| |\Phi(f(F_w(y))) - \Phi(f(F_w(y')))|^p - |\Phi'(x)|^p \, |f(F_w(y)) - f(F_w(y'))|^p \Big| \\ &\leq \varepsilon \, |f(F_w(y)) - f(F_w(y'))|^p, \end{aligned}$$

and we conclude that

$$\left| \widetilde{\mathcal{E}}_p^{\mathbb{G}_m}\big(F_w^*(\Phi \circ f)\big) - |\Phi'(f(x))|^p \, \widetilde{\mathcal{E}}_p^{\mathbb{G}_m}\big(F_w^* f\big) \right| \leq \varepsilon \, \widetilde{\mathcal{E}}_p^{\mathbb{G}_m}\big(F_w^* f\big).$$

Taking an appropriate limit, we have

$$\left| \mathcal{E}_p\big(F_w^*(\Phi \circ f)\big) - |\Phi'(f(x))|^p \, \mathcal{E}_p\big(F_w^* f\big) \right| \leq \varepsilon \, \mathcal{E}_p\big(F_w^* f\big), \tag{7.3}$$

whenever $w \in W_n$ and $n \geq N(\varepsilon)$. For any $m \in \mathbb{N}$, $w \in W_m$ and $n \geq N(\varepsilon)$, we see from the self-similarity of $\mathcal{E}_p$ that

$$\begin{aligned} &\left| \mathfrak{m}^p_{\langle \Phi(f) \rangle}(\Sigma_w) - \int_{\Sigma_w} |\Phi'(f(\pi(\omega)))|^p \, d\mathfrak{m}^p_{\langle f \rangle}(\omega) \right| \\ &\leq \sum_{v \in W_n} \int_{\Sigma_{wv}} \left| \frac{\mathcal{E}_p\big(F_{wv}^*(\Phi(f))\big)}{\mathcal{E}_p\big(F_{wv}^* f\big)} - |\Phi'(f(\pi(\omega)))|^p \right| d\mathfrak{m}^p_{\langle f \rangle}(\omega) \\ &\leq \varepsilon \, \mathfrak{m}^p_{\langle f \rangle}(\Sigma_w). \end{aligned}$$

Hence, for any $w \in W_{\#}$,

$$\mathfrak{m}^p_{\langle \Phi(f) \rangle}(\Sigma_w) = \int_{\Sigma_w} |\Phi'(f(\pi(\omega)))|^p \, d\mathfrak{m}^p_{\langle f \rangle}(\omega).$$

By Dynkin's π-λ theorem, we get $\mathfrak{m}^p_{\langle \Phi(f) \rangle}(d\omega) = |\Phi'(f(\pi(\omega)))| \, \mathfrak{m}^p_{\langle f \rangle}(d\omega)$. By the change of variable formula (see [22, Theorem 4.1.11] for example), we have (7.2) in this case.

Next, let $\Phi \in C^1(\mathbb{R})$. Then, by applying Weierstrass' approximation theorem for Φ', we can obtain a sequence of polynomials $\{\Phi_k\}_{k \geq 1}$ with $\Phi_k(0) = \Phi(0)$ such that $\Phi_k \to \Phi$ and $\Phi'_k \to \Phi'$ uniformly on $f(K)$. By the argument in the last paragraph, we know that

$$\mu^p_{\langle \Phi_k(f) \rangle}(dx) = \left|\Phi'_k(f(x))\right|^p \mu^p_{\langle f \rangle}(dx), \tag{7.4}$$

for every $k \in \mathbb{N}$. For any $\widetilde{\Phi} \in C^1(\mathbb{R})$, it is immediate that

$$\left|\widetilde{\Phi}(f(F_w(y))) - \widetilde{\Phi}(f(F_w(y')))\right| \leq \sup_{s \in f(K_w)} \left|\widetilde{\Phi}'(s)\right| |f(F_w(y)) - f(F_w(y'))|,$$

and hence,

$$\widetilde{\mathcal{E}}^{\mathbb{G}_m}_p\big(F^*_w(\widetilde{\Phi}(f))\big) \leq \sup_{s \in f(K_w)} \left|\widetilde{\Phi}'(s)\right| \widetilde{\mathcal{E}}^{\mathbb{G}_m}_p\big(F^*_w f\big).$$

From the construction in subsection 6.2 and the self-similarity of $\mathcal{E}_p$, we conclude that

$$\mathcal{E}_p\big(\widetilde{\Phi}(f)\big) \leq \rho^n_p \sum_{w \in W_n} \sup_{s \in f(K_w)} \left|\widetilde{\Phi}'(s)\right| \widetilde{\mathcal{E}}^{\mathbb{G}_m}_p\big(F^*_w f\big), \tag{7.5}$$

for every $n \in \mathbb{N}$. From (7.5) and the self-similarity of $\mathcal{E}_p$, since the convergence $\Phi'_k \to \Phi'$ is uniform, we obtain $\lim_{k \to \infty} \mathcal{E}_p\big(\Phi(f) - \Phi_k(f)\big) = 0$. We deduce our assertion by letting $k \to \infty$ in (7.4) and applying Proposition 7.2. □

Theorem 7.4 (Theorem 2.22-(3))**.** *For any $n \in \mathbb{N}$ and $f \in \mathcal{F}_p$,*

$$\mu^p_{\langle f \rangle}(dx) = \rho^n_p \sum_{w \in W_n} (F_w)_* \mu^p_{\langle F^*_w f \rangle}(dx). \tag{7.6}$$

Proof. Let $n, m \in \mathbb{N}$, let $w \in W_m$ and let $f \in \mathcal{F}_p$. If $m \leq n$, then we see that

$$\begin{aligned}
\rho^n_p \sum_{v \in W_n} (\sigma_v)_* \mathfrak{m}^p_{\langle F^*_v f \rangle}(\Sigma_w) &= \rho^n_p \sum_{v \in w \cdot W_{n-m}} (\sigma_v)_* \mathfrak{m}^p_{\langle F^*_v f \rangle}(\Sigma_w) \\
&= \rho^n_p \sum_{v \in w \cdot W_{n-m}} \mathfrak{m}^p_{\langle F^*_v f \rangle}(\Sigma) \\
&= \rho^{n+m}_p \sum_{v \in w \cdot W_{n-m}} \mathcal{E}_p\big(F^*_v f\big) = \mathfrak{m}^p_{\langle f \rangle}(\Sigma_w).
\end{aligned}$$

If $m \leq n$, then we have that

$$\begin{aligned}
\rho^n_p \sum_{v \in W_n} (\sigma_v)_* \mathfrak{m}^p_{\langle F^*_v f \rangle}(\Sigma_w) &= \rho^n_p \, (\sigma_{[w]_n})_* \mathfrak{m}^p_{\langle F^*_{[w]_n} f \rangle}(\Sigma_w) \\
&= \rho^m_p \, \mathcal{E}_p\big(F^*_w f\big) = \mathfrak{m}^p_{\langle f \rangle}(\Sigma_w).
\end{aligned}$$

Therefore, by Dynkin's π-λ theorem, we deduce that

$$\mathfrak{m}^p_{\langle f\rangle}(d\omega) = \rho_p^n \sum_{w\in W_n} (\sigma_w)_*\mathfrak{m}^p_{\langle F_w^* f\rangle}(d\omega),$$

for every $n \in \mathbb{N}$. By Proposition 2.3, we have the desired result. □

As an immediate consequence of Theorems 7.3 and 7.4, we can prove the following *energy image density property* (we borrow this naming from [13, Theorem I.7.1.1]).

Proposition 7.5. *For any $f \in \mathcal{F}_p$, it holds that the image measure of $\mu^p_{\langle f\rangle}$ by f is absolutely continuous with respect to the one-dimensional Lebesgue measure $\mathscr{L}^1$ on $\mathbb{R}$. In particular, $\mu^p_{\langle f\rangle}(\{x\}) = 0$ for any $x \in K$.*

Proof. We follow [19, Theorem 4.3.8]. It will suffice to show that $f_*\mu^p_{\langle f\rangle}(F) = 0$ whenever $f \in \mathcal{F}_p$ and F is a compact subset of $\mathbb{R}$ with $\mathscr{L}^1(F) = 0$. We can choose a sequence $\{\varphi_n\}_{n\geq 1}$ from continuous functions on $\mathbb{R}$ with compact supports such that $|\varphi_n| \leq 1$, $\lim_{n\to\infty}\varphi_n(x) = \mathbb{1}_K(x)$ for each $x \in \mathbb{R}$, and

$$\int_0^\infty \varphi_n(t)\,dt = \int_{-\infty}^0 \varphi_n(t)\,dt = 0,$$

for each $n \in \mathbb{N}$. Define $\Phi_n(x) := \int_0^x \varphi_n(t)\,dt$ for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then we easily see that $\Phi_n \in C^1(\mathbb{R})$ with compact support, $\Phi_n(0) = 0$, and $\left|\Phi_n'\right| \leq 1$ for each $n \in \mathbb{N}$. By the dominated convergence theorem, it is immediate that $\lim_{n\to\infty}\Phi_n(x) = 0$ for each $x \in \mathbb{R}$ and $\Phi_n(f)$ converges to 0 in $L^p(K,\mu)$. Since $\mathcal{E}_p(\Phi_n(f)) \leq \mathcal{E}_p(f)$ by the Markov property of $\mathcal{E}_p$, we deduce that $\{\Phi_n\}_{n\geq 1}$ is $\mathcal{F}_p$-bounded. Therefore, there exists a subsequence $\{n_k\}_{k\geq 1}$ such that $\{\Phi_{n_k}(f)\}_{k\geq 1}$ converges to 0 weakly in $\mathcal{F}_p$. By Mazur's lemma, there exist $N(l) \in \mathbb{N}$ and $\{a(l)_k\}_{k=l}^{N(l)}$ with $a(l)_k \geq 0$ and $\sum_{k=l}^{N(l)} a(l)_k = 1$ such that $\Psi_l \circ f := \sum_{k=l}^{N(l)} a(l)_k\Phi_{n_k} \circ f$ converges to 0 in $\mathcal{F}_p$ as $l \to \infty$. Then, by Fatou's lemma and the change of variable formula, we conclude that

$$\begin{aligned} f_*\mu^p_{\langle f\rangle}(F) &= \int_{\mathbb{R}} \lim_{l\to\infty}\left|\sum_{k=l}^{N(l)} a(l)_k\Phi_{n_k}'(t)\right|^p f_*\mu^p_{\langle f\rangle}(dt) \\ &\leq \varliminf_{l\to\infty}\int_K \left|\Psi_l'(f(x))\right|^p \mu^p_{\langle f\rangle}(dx) \\ &= \varliminf_{l\to\infty}\mu^p_{\langle \Psi_l(f)\rangle}(K) = \varliminf_{l\to\infty}\mathcal{E}_p\bigl(\Psi_l(f)\bigr) = 0. \end{aligned}$$ □

Finally, we prove Theorem 2.20-(1).

Theorem 7.6 (Theorem 2.22-(1))**.** *Let $f, g \in \mathcal{F}_p$. If $(f-g)|_A$ is constant for some Borel subset A of K, then $\mu^p_{\langle f\rangle}(A) = \mu^p_{\langle g\rangle}(A)$.*

Proof. Let $f \in \mathcal{F}_p$ and let $A \in \mathcal{B}(K)$. Suppose that $f|_A = c$ for some $c \in \mathbb{R}$. Then, by Proposition 7.5, we have $\mu^p_{\langle f\rangle}(f^{-1}(\{c\})) = 0$, which implies that $\mu^p_{\langle f\rangle}(A) = 0$. Combining this result and Proposition 7.2, we finish the proof. □

We conclude this paper by seeing some consequences of the symmetries of $\mathcal{E}_p$.

Proposition 7.7. *For any $f \in \mathcal{F}_p$ and $T \in \mathrm{Sym}(K)$, it holds that $T_*\mu^p_{\langle f\rangle} = \mu^p_{\langle T^*f\rangle}$.*

Proof. Let $A \in \mathcal{B}(K)$ be a closed set and let $T \in \mathrm{Sym}(K)$. By the symmetries of $\mathbb{G}_l$, there exists a graph automorphism $\tau^{(l)}_T$ of $\mathbb{G}_l$ (i.e. $\tau^{(l)}_T\colon V(\mathbb{G}_l) \to V(\mathbb{G}_l)$ is a bijection, and $(x, y) \in E(\mathbb{G}_l)$ if and only if $(\tau^{(l)}_T(x), \tau^{(l)}_T(y)) \in E(\mathbb{G}_l)$) such that $T(K_w) = K_{\tau^{(l)}_T(w)}$ for any $w \in W_l$. For each $l \in \mathbb{N}$, define

$$C_l \coloneqq \{w \in W_l \mid \Sigma_w \cap \pi^{-1}(A) \neq \emptyset\} \quad \text{and} \quad C^T_l \coloneqq \{w \in W_l \mid \Sigma_w \cap \pi^{-1}(T^{-1}(A)) \neq \emptyset\}.$$

Then we easily see that $\tau^{(l)}_T$ gives a bijection between C_l and C^T_l. Hence, for any $n \in \mathbb{N}$,

$$\sum_{w\in C_l} \widetilde{\mathcal{E}}^{\mathbb{G}_n}_p(F^*_w(T^*f)) = \sum_{w\in C_l} \widetilde{\mathcal{E}}^{\mathbb{G}_n}_p((T \circ F_w)^*f) = \sum_{w\in C^T_l} \widetilde{\mathcal{E}}^{\mathbb{G}_n}_p(F^*_w f).$$

In view of (6.15), we get $\sum_{w\in C_l} \overline{\mathcal{E}}_{p,n}(F^*_w(T^*f)) = \sum_{w\in C^T_l} \overline{\mathcal{E}}_{p,n}(F^*_w f)$, and thus

$$\mathfrak{m}^p_{\langle T^*f\rangle}(\Sigma_{C_l}) = \sum_{w\in C_l} \mathcal{E}_p(T^*f) = \sum_{w\in C^T_l} \mathcal{E}_p(f) = \mathfrak{m}^p_{\langle f\rangle}(\Sigma_{C^T_l}).$$

Letting $l \to \infty$, we obtain $\mu^p_{\langle T^*f\rangle}(A) = T_*\mu^p_{\langle f\rangle}(A)$ because $\bigcap_{l\in\mathbb{N}} C_l = \pi^{-1}(A)$ and $\bigcap_{l\in\mathbb{N}} C^T_l = \pi^{-1}(T^{-1}(A))$ as seen in the proof of Proposition 7.2. Since both of these measures $\mu^p_{\langle T^*f\rangle}$ and $T_*\mu^p_{\langle f\rangle}$ are Borel regular, we complete the proof. □

Remark 7.8. Applying Proposition 7.7, we can provide an alternative proof of $\mathcal{E}_p(h_0) = \mathcal{E}_p(\widetilde{h}_0)$ in the proof of Lemma 6.20. Recall the definitions of h_0 and $\widetilde{h}_0$ (see Lemma 6.18 and (6.21)). Let $A_1 \coloneqq \{(x, y) \in K \mid y > -x\}$, let $A_2 \coloneqq \{(x, y) \in K \mid y < -x\}$ and let $A_3 \coloneqq \{(x, y) \in K \mid y = -x\}$. Then $\{A_i\}^3_{i=1}$ are disjoint and $K = \bigcup^3_{i=1} A_i$. Note that $\widetilde{h}_0|_{A_2\cup A_3} = h_0|_{A_2\cup A_3}$ and $(\widetilde{h}_0 \circ T_+)|_{A_1} = (1 - h_0)|_{A_1}$. We immediately have $\mu^p_{\langle \widetilde{h}_0\rangle}(A_2 \cup A_3) = \mu^p_{\langle h_0\rangle}(A_2 \cup A_3)$ by Theorem 7.6. Since $T_+(A_1) = A_1$, we see from Proposition 7.7 and Theorem 7.6 that

$$\mu^p_{\langle \widetilde{h}_0\rangle}(A_1) = \mu^p_{\langle T^*_+\widetilde{h}_0\rangle}(A_1) = \mu^p_{\langle 1-h_0\rangle}(A_1) = \mu^p_{\langle h_0\rangle}(A_1).$$

Hence we conclude that $\mu^p_{\langle \widetilde{h}_0\rangle}(K) = \mu^p_{\langle h_0\rangle}(K)$, which implies that $\mathcal{E}_p(h_0) = \mathcal{E}_p(\widetilde{h}_0)$.

A Miscellaneous facts

A.1 Proof of Lemma 4.11

This lemma is obtained by observing that the estimates in [50, Lemma 8.4] depend only on the constants controlling rough isometries. For the reader's convenience, we give a complete proof.

Lemma A.1. *For each $i = 1, 2$, let $\{G_n^i = (V_n^i, E_n^i)\}_{n\geq 1}$ be a series of finite graphs with*

$$D_*^i := \sup_{n\in\mathbb{N}} \max_{x\in V_n^i} \#\{y \in V_n^i \mid (x, y) \in E_n^i\} < \infty,$$

and let $\varphi_n \colon V_n^1 \to V_n^2$ be a uniform rough isometry from $\{G_n^1\}_{n\geq 1}$ to $\{G_n^2\}_{n\geq 1}$. Then there exists a positive constant C_{URI} (depending only on C_1, C_2 in Definition 4.9, D_*^1 and p) such that

$$\mathcal{E}_p^{G_n^1}(f \circ \varphi_n) \leq C_{\mathrm{URI}}\, \mathcal{E}_p^{G_n^2}(f),$$

for every $n \in \mathbb{N}$ and $f \colon V_n^2 \to \mathbb{R}$. In particular,

$$C_p^{G_n^1}(\varphi_n^{-1}(A_n), \varphi_n^{-1}(B_n)) \leq C_{\mathrm{URI}} C_p^{G_n^2}(A_n, B_n)$$

for every $n \in \mathbb{N}$, where A_n, B_n are disjoint subsets of V_n^2.

Proof. Let $n \in \mathbb{N}$, let $f \colon V_n^2 \to \mathbb{R}$ and let $(x, y) \in E_n^1$. We set $x' = \varphi_n(x)$ and $y' = \varphi_n(y)$. Let C_i ($i = 1, \dots 4$) be constants in the definition of uniform rough isometry. Then we get

$$0 \leq d_{G_n^2}(x', y') \leq C_1 + C_2.$$

We set $L \in \mathbb{N}$ such that $L - 1 < C_1 + C_2 \leq L$. Since $d_{G_n^2}(x', y') \leq L$, there exist $l \leq L$ and a path $[z_0, z_1, \dots, z_l]$ in G_n^2 from x' to y', that is $z_0 = x'$, $z_l = y'$ and $(z_{i-1}, z_i) \in E_n^2$ for each $i = 1, \dots, l$. Now, by Hölder's inequality, we have that

$$\begin{aligned} |f \circ \varphi_n(y) - f \circ \varphi_n(x)| = |f(x') - f(y')| &\leq \sum_{i=1}^{l} |f(z_{i-1}) - f(z_i)| \\ &\leq L^{(p-1)/p} \left(\sum_{i=1}^{l} |f(z_{i-1}) - f(z_i)|^p \right)^{1/p}. \end{aligned}$$

In particular, it follows that

$$|f \circ \varphi_n(y) - f \circ \varphi_n(x)|^p \leq L^{p-1} \sum_{i=1}^{l} |f(z_{i-1}) - f(z_i)|^p. \tag{A.1}$$

For each $(x, y) \in E_n^1$, fix a path $\gamma'_{xy} := [z'_0, \dots, z'_l]$ in G_n^2 from $\varphi_n(x)$ to $\varphi_n(y)$ with $l \leq L$. For each $(v, w) \in E_n^2$, we set

$$M_{(v,w)} := \#\{(x, y) \in E_n^1 \mid \text{path } \gamma'_{xy} \text{ contains } (v, w)\}.$$

We also define $\mathcal{A}_v^{(n)} := \{x \in V_n^1 \mid \varphi_n(x) \in B_{d_{G_n^2}}(v, L)\}$ for each $y \in V_n^2$. Then, for any $a, b \in \mathcal{A}_v^{(n)}$,

$$C_1^{-1} d_{G_n^1}(a, b) - C_2 \leq d_{G_n^2}(\varphi_n(a), \varphi_n(b)) \leq d_{G_n^2}(\varphi_n(a), v) + d_{G_n^2}(\varphi_n(b), v) \leq 2L.$$

Therefore, we have $\mathrm{diam}(\mathcal{A}_v^{(n)}, d_{G_n^1}) \leq C_1(2L + C_2) =: L_*$, which implies that $\#\mathcal{A}_v^{(n)} \leq L_* D_*^1$ for any $n \in \mathbb{N}$ and $v \in V_n^2$. Now, since the length of $\gamma_{\varphi_n(x)\varphi_n(y)}$ is less than L, it follows that

$$\{(x, y) \in E_n^1 \mid \text{a path } \gamma'_{xy} \text{ contains a edge } (v, w) \in E_n^2\} \subseteq \mathcal{A}_v^{(n)} \times \mathcal{A}_v^{(n)}.$$

This yields that $\# M_{(v,w)} \leq (L_* D_*^1)^2$ for any $n \in \mathbb{N}$ and $(v, w) \in E_n^2$. Using this bound and summing (A.1) over $(x, y) \in E_n^1$, we conclude that

$$\mathcal{E}_p^{G_n^1}(f \circ \varphi_n) \leq L^{p-1} (L_* D_*^1)^2\, \mathcal{E}_p^{G_n^2}(f). \qquad \square$$

A.2 Proof of Lemma 5.13

We prove Lemma 5.13 in a metric measure space setting by extending [24, Theorem 4.11-(iii)]. Let $(\mathcal{X}, d, \mu)$ be α-Ahlfors regular, that is, $(\mathcal{X}, d)$ is a non-empty metric space, μ is a Borel regular measure on $(\mathcal{X}, d)$ without point mass, and there exist $\alpha > 0$ and $C_{\mathrm{AR}} \geq 1$ such that

$$C_{\mathrm{AR}}^{-1} r^{\alpha} \leq \mu(B_d(x,r)) \leq C_{\mathrm{AR}} r^{\alpha}$$

for every $x \in X$ and $r \in (0, \operatorname{diam}(\mathcal{X}, d))$. Note that $\dim_{\mathrm{H}}(\mathcal{X}, d) = \alpha$.

Lemma A.2. *Let $\beta > \alpha$ and $p > 1$. Then there exists a positive constant $C_{5.13}$ (depending only on $p, \beta, \alpha, C_{\mathrm{AR}}$) such that*

$$\begin{aligned}&|f(x) - f(y)|^p \\ &\leq C_{5.13} d(x,y)^{\beta-\alpha} \sup_{r \in (0, 3d(x,y)]} r^{-\beta} \int_{\mathcal{X}} \fint_{B_d(z,r)} |f(z) - f(z')|^p \, d\mu(z') d\mu(z),\end{aligned}$$

for every $f \in \Lambda^{\beta/p}_{p,\infty}$ and μ-a.e. $x, y \in \mathcal{X}$.

Proof. For $f \in L^1_{\mathrm{loc}}(\mathcal{X}, \mu)$, $x \in \mathcal{X}$ and $r > 0$, we set $f_{B_d(x,r)} := \fint_{B_d(x,r)} f(z)\, d\mu(z)$. Let $f \in \Lambda^{\beta/p}_{p,\infty}$, let $x \neq y \in \mathcal{X}$ and $r > 0$ such that $d(x,y) \leq r$. By Fubini's theorem, we see that

$$f_{B_d(x,r)} = \frac{1}{\mu(B_d(x,r))\mu(B_d(y,r))} \int_{B_d(x,r)} \int_{B_d(y,r)} f(z)\, d\mu(z') d\mu(z).$$

Also, we have

$$f_{B_d(y,r)} = \frac{1}{\mu(B_d(x,r))\mu(B_d(y,r))} \int_{B_d(x,r)} \int_{B_d(y,r)} f(z')\, d\mu(z') d\mu(z).$$

From these identities, we have that

$$\begin{aligned}
&\left| f_{B_d(x,r)} - f_{B_d(y,r)} \right|^p \\
&= \left| \frac{1}{\mu(B_d(x,r))\mu(B_d(y,r))} \int_{B_d(x,r)} \int_{B_d(y,r)} (f(z) - f(z'))\, d\mu(z') d\mu(z) \right|^p \\
&\leq \frac{1}{\mu(B_d(x,r))\mu(B_d(y,r))} \int_{B_d(x,r)} \int_{B_d(y,r)} |f(z) - f(z')|^p \, d\mu(z') d\mu(z) \\
&\leq \frac{1}{\mu(B_d(x,r))\mu(B_d(y,r))} \int_{X} \int_{B_d(z,3r)} |f(z) - f(z')|^p \, d\mu(z') d\mu(z) \\
&\leq c_1 r^{-\alpha} \int_{\mathcal{X}} \fint_{B_d(z,3r)} |f(z) - f(z')|^p \, d\mu(z') d\mu(z) \\
&\leq c_1 r^{-\alpha+\beta} \sup_{\rho \in (0,3r]} \rho^{-\beta} \int_{\mathcal{X}} \fint_{B_d(z,\rho)} |f(z) - f(z')|^p \, d\mu(z') d\mu(z),
\end{aligned} \tag{A.2}$$

where we used Hölder's inequality in the third line and the Ahlfors regularity in the fifth line. (c_1 is a positive constant depending only on C_{AR}.) Similarly, we obtain

$$\left| f_{B_d(x,2r)} - f_{B_d(x,r)} \right|^p \tag{A.3}$$

$$\le c_1 r^{-\alpha+\beta} \sup_{\rho\in(0,3r]} \rho^{-\beta} \int_{\mathcal{X}} \fint_{B_d(z,\rho)} |f(z)-f(z')|^p \, d\mu(z')d\mu(z).$$

Next, let $\mathcal{X}_*$ be the set of Lebesgue points with respect to f and let $r > 0$. By Lebesgue's differential theorem on Ahlfors regular metric measure space (see [27, Theorem 1.8] for example), it holds that $\mu(\mathcal{X} \setminus \mathcal{X}_*) = 0$. Set $r_k := 2^{-k} r$ for any $k \in \mathbb{Z}_{\ge 0}$. Then, for any $x \in \mathcal{X}_*$ and any $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that $\left|f(x) - f_{B_d(x,r_k)}\right| < \epsilon$ for all $k \ge K$. Now we have that

$$\begin{aligned}
&\left|f(x) - f_{B_d(x,r)}\right| \\
&\le \left|f(x) - f_{B_d(x,r_k)}\right| + \left|f_{B_d(x,r_K)} - f_{B_d(x,r_0)}\right| \le \varepsilon + \sum_{k=0}^{\infty} \left|f_{B_d(x,r_k)} - f_{B_d(x,r_{k+1})}\right|.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\left|f(x) - f_{B_d(x,r)}\right| \le \sum_{k=0}^{\infty} \left|f_{B_d(x,r_k)} - f_{B_d(x,r_{k+1})}\right|,$$

for any $x \in \mathcal{X}_*$. From this inequality and (A.3), we see that

$$\begin{aligned}
&\left|f(x) - f_{B_d(x,r)}\right| \\
&\le \sum_{k=0}^{\infty} \left|f_{B_d(x,r_k)} - f_{B_d(x,2r_k)}\right| \\
&\le c_2 r^{(\beta-\alpha)/p} \left(\sup_{\rho\in(0,3r]} \rho^{-\beta} \int_{\mathcal{X}} \fint_{B_d(z,\rho)} |f(z)-f(z')|^p \, d\mu(z')d\mu(z) \right)^{1/p},
\end{aligned} \tag{A.4}$$

where $c_2 := c_1^{1/p} \sum_{k=0}^{\infty} 2^{-k(\beta-\alpha)/p}$.

Since μ has no point mass, it holds that $\mathcal{X}_* \setminus \{x\} \neq \emptyset$ for any $x \in \mathcal{X}_*$. Let $y \in \mathcal{X}_* \setminus \{x\}$ and set $r := d(x,y) > 0$. From (A.2) and (A.4), we conclude that

$$\begin{aligned}
|f(x) - f(y)| &\le \left|f(x) - f_{B_d(x,r)}\right| + \left|f_{B_d(x,r)} - f_{B_d(y,r)}\right| + \left|f(y) - f_{B_d(y,r)}\right| \\
&\le c_3 r^{(\beta-\alpha)/p} \left(\sup_{\rho\in(0,3r]} \rho^{-\beta} \int_{\mathcal{X}} \fint_{B_d(z,\rho)} |f(z)-f(z')|^p \, d\mu(z')d\mu(z) \right)^{1/p},
\end{aligned}$$

where $c_3 := c_1^{1/p} + 2c_2$. This proves our assertion. □

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