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Symmetry of the unsteady linearized Boltzmann equation in a fixed bounded domain

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Abstract A symmetric relation between time-dependent problems described by the linearized Boltzmann equation is obtained for a gas in a fixed bounded domain. General representations of the total mass, momentum, and energy in the domain, as well as their fluxes through the boundary, in terms of an appropriate Green function are derived from that relation. Several application examples are presented. Similarities to the fluctuation–dissipation theorem in the linear response theory and its generalization to gas systems of arbitrary Knudsen numbers are also discussed. The present paper is an extension of the previous contribution [S. Takata, J. Stat. Phys. **136**, 751–784 (2009)] to time-dependent problems.

Keywords Boltzmann equation · kinetic theory of gases · symmetry · reciprocity · fluctuation–dissipation theorem · linear response theory

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Contents

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1 Introduction

The linearized Boltzmann equation is widely used for the study of a slow rarefied gas flow or a gas in a micro scale system such as a micro channel, an aerosol particle, etc. One of the interesting features in such gas systems is cross effects between fluid-dynamical and thermodynamical phenomena (e.g., the Poiseuille flow vs. the thermal transpiration [1, 2], the thermophoresis [3, 4] vs. the thermal polarization [5]; see, e.g., [6–8]). In order to investigate what kind of relation holds in general between such independent problems, we recently derived in [9] a symmetric relation between two different boundary-value problems described by the steady linearized Boltzmann equation. In this reference, by considering a solution of the problem as a static response of the system against the perturbation from the surroundings through the boundary, we introduced the Green function for an elemental source on the boundary. Then, we derived a general expression of mass, momentum, and heat fluxes through the boundary in terms of the corresponding Green function. The expression is valid for the entire range of the Knudsen number Kn ($0 < \mathrm{Kn} < \infty$). As a natural consequence of the Green function approach, the reciprocity of the fluxes on the boundary caused by the Green functions was obtained. With the aid of this reciprocity, in [10], we discussed the Onsager–Casimir reciprocity for the systems of arbitrary Kn on the basis of the entropy production argument. In this reference, as in [11–15], we considered the total entropy production caused by the gas–gas and gas–surface interactions. Then, we obtained the Onsager–Casimir reciprocity in a way of point correspondence. We also pointed out that the real identity of the conventional-type Onsager–Casimir relation is the Green reciprocity.

In [9, 10], we have restricted ourselves to discussion of time-independent problems. In the present paper, we will extend our theory [9] to time-dependent problems. The existing theories on the Onsager–Casimir relation are based on the entropy production argument (e.g., [11–15]), in which the production caused by the gas–surface interaction is determined indirectly by the entropy balance at the steady state. On the other hand, we have shown in [9] that the meaningful cross effects can be deduced from the Green function approach, which is completely free from the entropy production argument. This feature allows us to develop a general framework for the cross effects between time-dependent problems described by the linearized Boltzmann equation. In the present paper, we restrict ourselves to a monatomic single component gas in a fixed bounded domain.

The paper is organized as follows. In Sect. 2, we first formulate the class of time-dependent problems to be discussed and then derive a symmetric relation between two problems described by the unsteady linearized Boltzmann equation. Next in Sect. 3, we introduce a few Green functions for the initial data and present a general expression of the total mass, momentum, and energy in the system for any time. In Sect. 4, we introduce a few Green functions for the inhomogeneous term, which represents the effect of a generalized weak external force, and present an expression corresponding to that derived in Sect. 3. By comparing two expressions, we show that the Green functions for the inhomogeneous term are a time integration of the corresponding Green function for the initial data. In Sect. 5, we introduce the Green functions for boundary data and derive a general expression of the mass, momentum, and heat fluxes through the boundary. This is an extension of the representation theorem in [9] to the case of unsteady systems. One may find a similarity of some results in Sects. 3 and 4 to those of the linear response theory [16, 17] for the systems without boundary effect (the so-called bulk systems, where Kn is supposed to be small), though our theory covers the entire range of the Knudsen number. In order to illustrate a point about the resemblance, in Sect. 6, we apply the symmetric relation to a gas in a periodic box and derive the expressions similar to the fluctuation–dissipation theorem and to those for the static admittance in the linear response theory for bulk systems [16]. We also discuss their extensions to the systems of arbitrary Kn.

2 Symmetry of the linearized Boltzmann equation in unsteady systems

2.1 Problem

Consider the time-dependent behavior of a single component monatomic rarefied gas that occupies a fixed bounded domain $\mathscr{D}$ (i.e., the shape of $\mathscr{D}$ does not change in time). The state of the gas is assumed to be close to the equilibrium state at rest with density ρ_0 and temperature T_0, so that the higher order effects of the deviation from this state will be neglected.

With a proper choice of the reference time t_0 and length L, we denote by $t_0 t$, $L\boldsymbol{x}$, $(2kT_0/m)^{1/2}\boldsymbol{\zeta}$, $\rho_0(2kT_0/m)^{-3/2}[1+\phi(t,\boldsymbol{x},\boldsymbol{\zeta})]E(\boldsymbol{\zeta})$, the time, the space coordinates, the molecular velocity, and the velocity distribution function of gas molecules, respectively. Here, k is the Boltzmann constant, m is the mass of a molecule, and $E(\boldsymbol{\zeta}) = \pi^{-3/2}\exp(-|\boldsymbol{\zeta}|^2)$. The domain of the dimensionless $\boldsymbol{x}$-space corresponding to $\mathscr{D}$ will be denoted by D. Then, the behavior of the gas is described by the following linearized Boltzmann equation:

$$\mathrm{Sh}\frac{\partial\phi}{\partial t} + \zeta_i\frac{\partial\phi}{\partial x_i} = \frac{2}{\sqrt{\pi}}\frac{1}{\mathrm{Kn}}\mathscr{L}(\phi) + I \qquad (\boldsymbol{\zeta}\in\mathbb{R}^3,\ \boldsymbol{x}\in D,\ t>0). \tag{1}$$

Here $\mathrm{Sh}[= L/t_0(2kT_0/m)^{1/2}]$ is the Strouhal number, $\mathrm{Kn}(=\ell_0/L)$ is the Knudsen number (ℓ_0 is the mean free path of a molecule in the reference equilibrium state at rest), $\mathscr{L}$ is the linearized collision operator and is required to have the properties summarized in Appendix A.1. For the sake of the latter discussions, the inhomogeneous term $I(t,\boldsymbol{x},\boldsymbol{\zeta})$ is added to the right-hand side of (1).[1]

The initial data is denoted by putting the subscripted "initial":

$$\phi(0,\boldsymbol{x},\boldsymbol{\zeta}) = \phi_{\mathrm{initial}}(\boldsymbol{x},\boldsymbol{\zeta}) \quad (\boldsymbol{\zeta}\in\mathbb{R}^3,\ \boldsymbol{x}\in D). \tag{2}$$

The boundary ∂D of the domain D is composed of two parts in general. One is a simple boundary (or a solid surface) or an interface with the condensed phase of the gas, which we generically call the real boundary and denote by ∂D_{w}. The other is an artificial boundary or a control surface set inside a gas region wider than D, which we generically call the imaginary boundary and denote by ∂D_{g}. The specular reflection and periodic boundaries are typical examples of the imaginary boundary.

First consider the real boundary ∂D_{w}. Let us denote the temperature of the boundary by $T_0(1+\tau_{\mathrm{w}})$, the corresponding saturation pressure of the gas by $p_0(1+P_{\mathrm{w}})$, and the velocity of the boundary by $(2kT_0/m)^{1/2}\boldsymbol{u}_{\mathrm{w}}$, where $p_0 = (\rho_0/m)kT_0$ is the pressure at the reference equilibrium state at rest. P_{w}, τ_{w}, and $\boldsymbol{u}_{\mathrm{w}}$ are, in general, a function of t and $\boldsymbol{x}$. Since the shape of D does not change in time, $\boldsymbol{u}_{\mathrm{w}}\cdot\boldsymbol{n}=0$. ϕ obeys the following condition on ∂D_{w}:

$$\phi = g_{\mathrm{w}} + \int_{\zeta_n^*<0}\frac{|\zeta_n^*|E(\boldsymbol{\zeta}^*)}{|\zeta_n|E(\boldsymbol{\zeta})}R(\boldsymbol{\zeta}^*,\boldsymbol{\zeta};\boldsymbol{x})(\phi^*-g_{\mathrm{w}}^*)\,\mathrm{d}\boldsymbol{\zeta}^* \quad \text{for } \zeta_n>0,\ \boldsymbol{x}\in\partial D_{\mathrm{w}},\ t>0, \tag{3}$$

[1] A typical example of I is the effect of weak external forces of $O(\phi)$, though I is not necessarily related to external forces. For instance, in the problem of the Poiseuille flow, the imposed pressure gradient can be treated as the inhomogeneous term (see Examples 2 and 3). In the present paper, we do not consider an external force of $O(1)$ for simplicity.

where

$$g_{\rm w} = P_{\rm w}(t,\boldsymbol{x}) + 2\zeta_i u_{{\rm w}i}(t,\boldsymbol{x}) + (|\boldsymbol{\zeta}|^2 - \frac{5}{2})\tau_{\rm w}(t,\boldsymbol{x}) \quad \text{for } \boldsymbol{\zeta} \in \mathbb{R}^3,\, \boldsymbol{x} \in \partial D_{\rm w},\, t > 0, \tag{4}$$

$\zeta_n = \boldsymbol{\zeta}\cdot\boldsymbol{n}$, and $\zeta_n^* = \boldsymbol{\zeta}^*\cdot\boldsymbol{n}$. The $(2kT_0/m)^{3/2}R$ is the reflection kernel of the boundary that is at rest with the reference temperature T_0. In what follows, if necessary, we shall denote R of a simple boundary by $R_{\rm CR}$ and that of an interface with the condensed phase by $R_{\rm PR}$. As to the descriptions with the notation R, they are valid both for $R_{\rm CR}$ and $R_{\rm PR}$. The $R_{\rm CR}$ and $R_{\rm PR}$ satisfy the properties summarized in Appendix A.2. Due to the fourth property of $R_{\rm CR}$, $P_{\rm w}$ in $g_{\rm w}$ is a fake parameter on a simple boundary, and one may put $P_{\rm w} = 0$ without loss of generality for $R_{\rm CR}$. It should be noted that $\langle \zeta_n \phi \rangle = 0$ holds on a simple boundary (see the second property of $R_{\rm CR}$). This equality does not hold in general for $R_{\rm PR}$.

Next consider the imaginary boundary $\partial D_{\rm g}$. We assume that the imaginary boundary is, in general, composed of two parts, say $\partial D_{\rm g}^{(1)}$ and $\partial D_{\rm g}^{(2)}$ (i.e., $\partial D_{\rm g} = \partial D_{\rm g}^{(1)} \cup \partial D_{\rm g}^{(2)}$):

(i) On $\partial D_{\rm g}^{(1)}$, ϕ obeys the following condition:

$$\phi(t,\boldsymbol{x},\boldsymbol{\zeta}) = h_{\rm in}(t,\boldsymbol{x},\boldsymbol{\zeta}) \quad \text{for } \zeta_n > 0,\, \boldsymbol{x} \in \partial D_{\rm g}^{(1)},\, t > 0. \tag{5a}$$

Here $h_{\rm in}$ is a given function for $\zeta_n > 0$. For the sake of later discussions, we denote this function extended to the whole range of $\boldsymbol{\zeta}$ by $h(t,\boldsymbol{x},\boldsymbol{\zeta})$. The way of extension may be arbitrary and does not influence the results that follow in the present paper.

(ii) On $\partial D_{\rm g}^{(2)}$, ϕ obeys the following condition:

$$\phi = h(t,\boldsymbol{x},\boldsymbol{\zeta}) + \int_{\partial D_{\rm g}^{(2)}} \int_{\zeta'_{n'}<0} P(\boldsymbol{x}',\boldsymbol{\zeta}',\boldsymbol{x},\boldsymbol{\zeta})(\phi' - h')\,\mathrm{d}\boldsymbol{\zeta}'\,\mathrm{dS}'$$
$$\text{for } \zeta_n > 0,\, \boldsymbol{x} \in \partial D_{\rm g}^{(2)},\, t > 0, \tag{5b}$$

where $h(t,\boldsymbol{x},\boldsymbol{\zeta})$ is a given function for $\boldsymbol{\zeta} \in \mathbb{R}^3$, $\boldsymbol{x} \in \partial D_{\rm g}^{(2)}$, and $t > 0$; $\zeta'_{n'} = \boldsymbol{\zeta}'\cdot\boldsymbol{n}'$; $\boldsymbol{n}'$ is the inward unit vector normal to $\partial D_{\rm g}^{(2)}$ at position $\boldsymbol{x}'$; dS' is the surface element of $\partial D_{\rm g}^{(2)}$ at position $\boldsymbol{x}'$; and $\phi' = \phi(t,\boldsymbol{x}',\boldsymbol{\zeta}')$ and $h' = h(t,\boldsymbol{x}',\boldsymbol{\zeta}')$. The kernel P is independent of t and its properties are summarized in Appendix A.3. It should be noted that the specular and periodic boundaries are a typical example of $\partial D_{\rm g}^{(2)}$.

Let us denote the mass density, flow velocity, and temperature of the gas by $\rho_0(1+\omega)$, $(2kT_0/m)^{1/2}u_i$, and $T_0(1+\tau)$. Let us denote the stress tensor, heat-flow vector, and specific energy of the gas by $p_0(\delta_{ij}+P_{ij})$, $p_0(2kT_0/m)^{1/2}Q_i$, and $\frac{3}{2}p_0(1+\mathscr{E})$. Then, they are defined as the moment of ϕ as follows:

$$\omega[\phi] = \langle \phi \rangle, \quad u_i[\phi] = \langle \zeta_i \phi \rangle, \quad \tau[\phi] = \frac{2}{3}\langle (|\boldsymbol{\zeta}|^2 - \frac{3}{2})\phi \rangle, \tag{6a}$$

$$\mathscr{E}[\phi] = \frac{2}{3}\langle |\boldsymbol{\zeta}|^2 \phi \rangle, \quad Q_i[\phi] = \langle \zeta_i (|\boldsymbol{\zeta}|^2 - \frac{5}{2})\phi \rangle, \quad P_{ij}[\phi] = \langle 2\zeta_i\zeta_j\phi \rangle. \tag{6b}$$

Here, the brackets $\langle\ \rangle$ represents the following moment:

$$\langle f \rangle(t,\boldsymbol{x}) := \int f(t,\boldsymbol{x},\boldsymbol{\zeta})E(\boldsymbol{\zeta})\,\mathrm{d}\boldsymbol{\zeta}.$$

In later discussions, we need to deal with the moments of different velocity distribution functions, say ϕ^A, ϕ^B. For the sake of later convenience, we have introduced the notation convention in (6) that the concerned velocity distribution function is indicated inside the solid parenthesis [ ] (e.g., $\omega[\phi^A]$, $\omega[\phi^B]$, etc.). If $[\cdot]$ is omitted, the quantity is defined as the corresponding moment of ϕ (e.g., $\omega = \omega[\phi]$). In what follows, we denote a quantity integrated in time from $t = 0$ by putting a line $^-$ over the quantity. For instance,

$$\overline{\omega}[\phi](t,\boldsymbol{x}) = \int_0^t \omega[\phi](r,\boldsymbol{x})\mathrm{d}r.$$

2.2 Preparations — basic lemmas

We introduce the following notation:

$$\begin{aligned}
f_\tau(t,\boldsymbol{x},\boldsymbol{\zeta}) &:= f(\tau+t,\boldsymbol{x},\boldsymbol{\zeta}) \quad (\tau>0, t\geq -\tau),\\
f_\tau^\flat(t,\boldsymbol{x},\boldsymbol{\zeta}) &:= f(\tau-t,\boldsymbol{x},\boldsymbol{\zeta}) \quad (\tau>0, t\leq \tau),\\
f^-(t,\boldsymbol{x},\boldsymbol{\zeta}) &:= f(t,\boldsymbol{x},-\boldsymbol{\zeta}),
\end{aligned}$$

and frequently use the following obvious properties in the sequel:

$$(\varPhi^-)^- = \varPhi, \quad \langle\varPhi\rangle = \langle\varPhi^-\rangle \quad \text{for any } \varPhi. \tag{7}$$

Below we show Lemmas 1–3, which will be the base in deriving the symmetric relation for unsteady systems.

Lemma 1 *Let ϕ^A be a solution of* (1) *with $I = I^A$. Let ϕ^B be a solution of* (1) *with $I = I^B$. Here,* Sh, Kn, *and $\mathscr{L}$ in* (1) *are common to ϕ^A and ϕ^B. If $\mathscr{L}$ satisfies the properties summarized in Appendix A.1, the following equality holds:*

$$\mathrm{Sh}\frac{\partial}{\partial t}\langle\phi_\tau^A\phi_s^{B\flat-}\rangle + \frac{\partial}{\partial x_i}\langle\zeta_i\phi_\tau^A\phi_s^{B\flat-}\rangle = \langle I_\tau^A\phi_s^{B\flat-}\rangle - \langle I_s^{B\flat-}\phi_\tau^A\rangle. \tag{8}$$

Proof Because of the definition of ϕ^A, ϕ_τ^A satisfies

$$\mathrm{Sh}\frac{\partial\phi_\tau^A}{\partial t} + \zeta_i\frac{\partial\phi_\tau^A}{\partial x_i} = \frac{2}{\sqrt{\pi}}\frac{1}{\mathrm{Kn}}\mathscr{L}(\phi_\tau^A) + I_\tau^A.$$

The integration over the whole space of $\boldsymbol{\zeta}$ after the multiplication of $\phi_s^{B\flat-}E$ yields

$$\langle\phi_s^{B\flat-}\mathrm{Sh}\frac{\partial\phi_\tau^A}{\partial t}\rangle + \langle\phi_s^{B\flat-}\zeta_i\frac{\partial\phi_\tau^A}{\partial x_i}\rangle = \langle\phi_s^{B\flat-}\frac{2}{\sqrt{\pi}}\frac{1}{\mathrm{Kn}}\mathscr{L}(\phi_\tau^A)\rangle + \langle\phi_s^{B\flat-}I_\tau^A\rangle. \tag{9}$$

On the other hand, because of the definition of ϕ^B, $\phi_s^{B\flat}$ satisfies

$$-\mathrm{Sh}\frac{\partial\phi_s^{B\flat}}{\partial t} + \zeta_i\frac{\partial\phi_s^{B\flat}}{\partial x_i} = \frac{2}{\sqrt{\pi}}\frac{1}{\mathrm{Kn}}\mathscr{L}(\phi_s^{B\flat}) + I_s^{B\flat}.$$

Thus the first term on the right-hand side of (9) is transformed as follows:

$$\begin{aligned}
\langle\phi_s^{B\flat-}\frac{2}{\sqrt{\pi}}\frac{1}{\mathrm{Kn}}\mathscr{L}(\phi_\tau^A)\rangle &= \langle\phi_\tau^{A-}\frac{2}{\sqrt{\pi}}\frac{1}{\mathrm{Kn}}\mathscr{L}(\phi_s^{B\flat})\rangle\\
&= -\langle\phi_\tau^{A-}\mathrm{Sh}\frac{\partial\phi_s^{B\flat}}{\partial t}\rangle + \langle\phi_\tau^{A-}\zeta_i\frac{\partial\phi_s^{B\flat}}{\partial x_i}\rangle - \langle\phi_\tau^{A-}I_s^{B\flat}\rangle\\
&= -\langle\phi_\tau^A\mathrm{Sh}\frac{\partial\phi_s^{B\flat-}}{\partial t}\rangle - \langle\phi_\tau^A\zeta_i\frac{\partial\phi_s^{B\flat-}}{\partial x_i}\rangle - \langle\phi_\tau^A I_s^{B\flat-}\rangle,
\end{aligned}$$

where in the first equality the self-adjointness of $\mathscr{L}$ and (7) have been used. Substitution into (9) eventually leads to (8). □

Lemma 2 *Let ϕ^A satisfy* (3) *with $g_{\rm w} = g_{\rm w}^A$. Let ϕ^B satisfy* (3) *with $g_{\rm w} = g_{\rm w}^B$. Here R in* (3) *is common to ϕ^A and ϕ^B and satisfies the properties summarized in Appendix A.2. Then, the following equality holds:*

$$\langle \zeta_n (\phi^A - g_{\rm w}^A)_\tau (\phi^B - g_{\rm w}^B)_s^{\flat -} \rangle = 0 \quad \text{on } \partial D_{\rm w}. \tag{10}$$

Proof Consider the integral $\int_{\zeta_n>0} \zeta_n (\phi^A - g_{\rm w}^A)_\tau (\phi^B - g_{\rm w}^B)_s^{\flat -} E \, {\rm d}\boldsymbol{\zeta}$. By the use of (3) for ϕ^A, this integral is expressed as

$$\begin{aligned}
&\int_{\zeta_n>0} \zeta_n (\phi^A - g_{\rm w}^A)_\tau (\phi^B - g_{\rm w}^B)_s^{\flat -} E \, {\rm d}\boldsymbol{\zeta} \\
&\qquad = \int_{\zeta_n>0} \int_{\zeta_n^*<0} |\zeta_n^*| E^* R(\boldsymbol{\zeta}^*, \boldsymbol{\zeta}; \boldsymbol{x}) (\phi^{A*} - g_{\rm w}^{A*})_\tau (\phi^B - g_{\rm w}^B)_s^{\flat -} \, {\rm d}\boldsymbol{\zeta}^* \, {\rm d}\boldsymbol{\zeta}.
\end{aligned}$$

Because of the detailed balance in Appendix A.2, the right-hand side is rewritten as

$$= \int_{\zeta_n>0} \int_{\zeta_n^*<0} |\zeta_n| E R(-\boldsymbol{\zeta}, -\boldsymbol{\zeta}^*; \boldsymbol{x}) (\phi^{A*} - g_{\rm w}^{A*})_\tau (\phi^B - g_{\rm w}^B)_s^{\flat -} \, {\rm d}\boldsymbol{\zeta}^* \, {\rm d}\boldsymbol{\zeta}.$$

This form can be further transformed by changing the variable of integration and substituting (3) for ϕ^B:

$$\begin{aligned}
&= \int_{\zeta_n<0} \int_{\zeta_n^*>0} |\zeta_n| E R(\boldsymbol{\zeta}, \boldsymbol{\zeta}^*; \boldsymbol{x}) (\phi^{A*} - g_{\rm w}^{A*})_\tau^- (\phi^B - g_{\rm w}^B)_s^{\flat} \, {\rm d}\boldsymbol{\zeta}^* \, {\rm d}\boldsymbol{\zeta} \\
&= \int_{\zeta_n^*>0} |\zeta_n^*| E^* (\phi^{A*} - g_{\rm w}^{A*})_\tau^- (\phi^{B*} - g_{\rm w}^{B*})_s^{\flat} \, {\rm d}\boldsymbol{\zeta}^* \\
&= \int_{\zeta_n<0} |\zeta_n| E (\phi^A - g_{\rm w}^A)_\tau (\phi^B - g_{\rm w}^B)_s^{\flat -} \, {\rm d}\boldsymbol{\zeta}.
\end{aligned}$$

Transposing the most right-hand side to the most left-hand side yields (10). □

Remark 1 Even when R is common to ϕ^A and ϕ^B only on a part of $\partial D_{\rm w}$, the equality (10) holds at every point on the common part.

Lemma 3 *Let ϕ^A satisfy* (5) *with $h = h^A$. Let ϕ^B satisfy* (5) *with $h = h^B$. Here P in* (5b) *is common to ϕ^A and ϕ^B and satisfies the properties summarized in Appendix A.3. Then, the following equality holds:*

$$\int_{\partial D_{\rm g}} \langle \zeta_n (\phi^A - h^A)_\tau (\phi^B - h^B)_s^{\flat -} \rangle \, {\rm dS} = 0. \tag{11}$$

Here, on $\partial D_{\rm g}^{(1)}$, h^A and h^B are respectively the extensions of $h_{\rm in}^A$ and $h_{\rm in}^B$ into the whole space of $\boldsymbol{\zeta}$.

Proof The proof is similar to that of Lemma 2 and is omitted here. □

2.3 Symmetric relation in unsteady systems

Lemmas 1–3 in Sect. 2.2 are the extensions of the corresponding lemmas for the steady systems in [9]. As in the case of the steady systems, we now derive the symmetric relation for the unsteady systems from Lemmas 1–3.

Proposition 1 (Symmetric relation for unsteady systems) *Let ϕ^A be a solution of the initial- and boundary-value problem* (1)–(5) *with $I = I^A$, $g_{\mathrm{w}} = g^A_{\mathrm{w}}$, $h = h^A$, and $\phi_{\mathrm{initial}} = \phi^A_{\mathrm{initial}}$. Let ϕ^B be a solution of the initial- and boundary-value problem* (1)–(5) *with $I = I^B$, $g_{\mathrm{w}} = g^B_{\mathrm{w}}$, $h = h^B$, and $\phi_{\mathrm{initial}} = \phi^B_{\mathrm{initial}}$. Here the bounded domain D, the Strouhal and Knudsen numbers* Sh *and* Kn, *the collision operator $\mathscr{L}$, and the kernels R and P are common to the problems of ϕ^A and ϕ^B. Then, if $\mathscr{L}$, R, and P satisfy the properties summarized in Appendix A, the following symmetric relation holds:*

$$\begin{aligned}
&\mathrm{Sh}\int_D \langle \phi^{B-}_{\mathrm{initial}}\phi^A\rangle(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_D \langle I^{B-} * \phi^A\rangle(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_{\partial D_{\mathrm{w}}} \langle \zeta_n (g^{B-}_{\mathrm{w}} * \phi^A)\rangle(t,\boldsymbol{x})\,\mathrm{dS}\\
&\qquad - \int_{\partial D_{\mathrm{g}}} \langle \zeta_n (h^{B-} * \phi^A)\rangle(t,\boldsymbol{x})\,\mathrm{dS} + \frac{1}{2}\int_{\partial D_{\mathrm{g}}} \langle \zeta_n (h^{B-} * h^A)\rangle(t,\boldsymbol{x})\,\mathrm{dS}\\
&= \mathrm{Sh}\int_D \langle \phi^{A-}_{\mathrm{initial}}\phi^B\rangle(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_D \langle I^{A-} * \phi^B\rangle(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_{\partial D_{\mathrm{w}}} \langle \zeta_n (g^{A-}_{\mathrm{w}} * \phi^B)\rangle(t,\boldsymbol{x})\,\mathrm{dS}\\
&\qquad - \int_{\partial D_{\mathrm{g}}} \langle \zeta_n (h^{A-} * \phi^B)\rangle(t,\boldsymbol{x})\,\mathrm{dS} + \frac{1}{2}\int_{\partial D_{\mathrm{g}}} \langle \zeta_n (h^{A-} * h^B)\rangle(t,\boldsymbol{x})\,\mathrm{dS}. \qquad (12)
\end{aligned}$$

Here, $f * g$ is a convolution of f and g with respect to time (thus $f * g = g * f$):

$$f * g(t,\cdot) \equiv \int_0^t f(r,\cdot)g(t-r,\cdot)\mathrm{d}r.$$

Proof First integrate (8) with respect to $\boldsymbol{x}$ over the domain D:

$$\mathrm{Sh}\frac{\partial}{\partial t}\int_D \langle \phi^A_\tau \phi^{B\flat-}_s\rangle\,\mathrm{d}\boldsymbol{x} - \int_{\partial D} \langle \zeta_n \phi^A_\tau \phi^{B\flat-}_s\rangle\,\mathrm{dS} = \int_D \langle I^A_\tau \phi^{B\flat-}_s\rangle\,\mathrm{d}\boldsymbol{x} - \int_D \langle I^{B\flat-}_s \phi^A_\tau\rangle\,\mathrm{d}\boldsymbol{x}. \qquad (13)$$

With the aid of (10), (11), and (4), the second term on the left-hand side is transformed as follows:

$$\begin{aligned}
\int_{\partial D} \langle \zeta_n \phi^A_\tau \phi^{B\flat-}_s\rangle\,\mathrm{dS} &= \int_{\partial D_{\mathrm{w}}} \langle \zeta_n \phi^A_\tau \phi^{B\flat-}_s\rangle\,\mathrm{dS} + \int_{\partial D_{\mathrm{g}}} \langle \zeta_n \phi^A_\tau \phi^{B\flat-}_s\rangle\,\mathrm{dS}\\
&= \int_{\partial D_{\mathrm{w}}} \langle \zeta_n g^A_{\mathrm{w}\tau} \phi^{B\flat-}_s\rangle\,\mathrm{dS} + \int_{\partial D_{\mathrm{w}}} \langle \zeta_n \phi^A_\tau g^{B\flat-}_{\mathrm{w}s}\rangle\,\mathrm{dS}\\
&\quad + \int_{\partial D_{\mathrm{g}}} \langle \zeta_n h^A_\tau \phi^{B\flat-}_s\rangle\,\mathrm{dS} + \int_{\partial D_{\mathrm{g}}} \langle \zeta_n \phi^A_\tau h^{B\flat-}_s\rangle\,\mathrm{dS} - \int_{\partial D_{\mathrm{g}}} \langle \zeta_n h^A_\tau h^{B\flat-}_s\rangle\,\mathrm{dS}.
\end{aligned}$$

Thus (13) is rewritten as

$$\begin{aligned}
\mathrm{Sh}\frac{\partial}{\partial t}\int_D \langle \phi^A_\tau \phi^{B\flat-}_s\rangle\,\mathrm{d}\boldsymbol{x} &= \int_D \langle I^{A-}_\tau \phi^{B\flat}_s\rangle\,\mathrm{d}\boldsymbol{x} - \int_{\partial D_{\mathrm{w}}} \langle \zeta_n g^{A-}_{\mathrm{w}\tau} \phi^{B\flat}_s\rangle\,\mathrm{dS}\\
&\quad - \int_{\partial D_{\mathrm{g}}} \langle \zeta_n h^{A-}_\tau \phi^{B\flat}_s\rangle\,\mathrm{dS} + \frac{1}{2}\int_{\partial D_{\mathrm{g}}} \langle \zeta_n h^{A-}_\tau h^{B\flat}_s\rangle\,\mathrm{dS} - \int_D \langle I^{B\flat-}_s \phi^A_\tau\rangle\,\mathrm{d}\boldsymbol{x}\\
&\quad + \int_{\partial D_{\mathrm{w}}} \langle \zeta_n g^{B\flat-}_{\mathrm{w}s} \phi^A_\tau\rangle\,\mathrm{dS} + \int_{\partial D_{\mathrm{g}}} \langle \zeta_n h^{B\flat-}_s \phi^A_\tau\rangle\,\mathrm{dS} - \frac{1}{2}\int_{\partial D_{\mathrm{g}}} \langle \zeta_n h^{B\flat-}_s h^A_\tau\rangle\,\mathrm{dS}. \qquad (14)
\end{aligned}$$

Now integrate (14) with respect to t from $t=-\tau$ to $t=s$. Then the integration of the left-hand side leads to

$$\mathrm{Sh}\Big(\int_D \langle \phi^{B-}_{\mathrm{initial}}\phi^A(\tau+s,\boldsymbol{x},\boldsymbol{\zeta})\rangle\,\mathrm{d}\boldsymbol{x} - \int_D \langle \phi^{A-}_{\mathrm{initial}}\phi^B(s+\tau,\boldsymbol{x},\boldsymbol{\zeta})\rangle\,\mathrm{d}\boldsymbol{x}\Big),$$

while the integration of each term of the right-hand side is transformed as follows:

$$\int_{-\tau}^{s} f_\tau g_s^\flat\,\mathrm{d}t = \int_{-\tau}^{s} f(\tau+t)g(s-t)\,\mathrm{d}t = \int_0^{\tau+s} f(t)g(s+\tau-t)\,\mathrm{d}t = (f*g)(s+\tau).$$

Denoting $\tau+s$ by t, we eventually obtain (12) from (14). □

Proposition 1 is an extension of the symmetric relation for steady systems (see Sect. 2.2.2 in [9]) and is the most general form proposed in the present contribution.[2] It should be noted that each term in the equation is a definite moment of ϕ^A or ϕ^B. This feature is significant and is due to the fact that the cross term of ϕ^A and ϕ^B is eliminated by Lemmas 2 and 3.

Remark 2 Consider the homogeneous linearized Boltzmann equation for the domain surrounded by a resting simple boundary. In the case, (12) is reduced to

$$\mathrm{Sh}\int_D \langle \phi^{B-}_{\mathrm{initial}}\phi^A\rangle\,\mathrm{d}\boldsymbol{x} - \int_{\partial D} \tau^B_{\mathrm{w}} * Q_n[\phi^A]\,\mathrm{dS} = \mathrm{Sh}\int_D \langle \phi^{A-}_{\mathrm{initial}}\phi^B\rangle\,\mathrm{d}\boldsymbol{x} - \int_{\partial D} \tau^A_{\mathrm{w}} * Q_n[\phi^B]\,\mathrm{dS}. \quad (16)$$

This equation is remarkably similar to the Green formula for the solutions of heat conduction equation. Let us consider two solutions of the heat conduction equation for a common domain D:

$$\partial_t T^A = \Delta T^A, \quad T^A(t=0,\boldsymbol{x}) = T^A_{\mathrm{initial}}(\boldsymbol{x}) \text{ in } D, \quad T^A = T^A_{\mathrm{w}}(t,\boldsymbol{x}) \text{ on } \partial D,$$
$$\partial_t T^B = \Delta T^B, \quad T^B(t=0,\boldsymbol{x}) = T^B_{\mathrm{initial}}(\boldsymbol{x}) \text{ in } D, \quad T^B = T^B_{\mathrm{w}}(t,\boldsymbol{x}) \text{ on } \partial D.$$

The Green formula for two functions $f(t,\boldsymbol{x})$ and $g(t,\boldsymbol{x})$ generally takes the form of

$$\int_D [(f*\Delta g)-(g*\Delta f)]\,\mathrm{d}\boldsymbol{x} = \int_{\partial D}[g*(\nabla f\cdot\boldsymbol{n}) - f*(\nabla g\cdot\boldsymbol{n})]\,\mathrm{dS}.$$

Substitution of T^A and T^B into f and g yields the relation

$$\int_D [(T^A*\partial_t T^B)-(T^B*\partial_t T^A)]\,\mathrm{d}\boldsymbol{x} = \int_{\partial D}[T^B_{\mathrm{w}}*(\nabla T^A\cdot\boldsymbol{n}) - T^A_{\mathrm{w}}*(\nabla T^B\cdot\boldsymbol{n})]\,\mathrm{dS}.$$

Since $\partial_t \int_D f*g\,\mathrm{d}\boldsymbol{x} = \int_D f g_{\mathrm{initial}}\,\mathrm{d}\boldsymbol{x} + \int_D f*\partial_t g\,\mathrm{d}\boldsymbol{x}$, we eventually obtain

$$\int_D T^B_{\mathrm{initial}}T^A\,\mathrm{d}\boldsymbol{x} + \int_{\partial D} T^B_{\mathrm{w}}*(\nabla T^A\cdot\boldsymbol{n})\,\mathrm{dS} = \int_D T^A_{\mathrm{initial}}T^B\,\mathrm{d}\boldsymbol{x} + \int_{\partial D} T^A_{\mathrm{w}}*(\nabla T^B\cdot\boldsymbol{n})\,\mathrm{dS}. \quad (17)$$

The correspondence to (16) is now obvious because the heat flow is of opposite sign of the temperature gradient. In the fluid-dynamic limit of the linearized Boltzmann equation, the temperature field is known to be expressed by the heat-conduction equation. In this sense, (16) may be regarded as the extension of (17) to the systems of arbitrary Knudsen number.

[2] In the case of steady systems, individual functions are time-independent, so that we may put $\phi^\alpha_{\mathrm{initial}} = \phi^\alpha$ ($\alpha = A,B$). Then, (12) divided by t is reduced to

$$\int_D \langle I^{B-}\phi^A\rangle\,\mathrm{d}\boldsymbol{x} - \int_{\partial D_{\mathrm{w}}}\langle \zeta_n g^{B-}_{\mathrm{w}}\phi^A\rangle\,\mathrm{dS} - \int_{\partial D_{\mathrm{g}}}\langle \zeta_n h^{B-}\phi^A\rangle\,\mathrm{dS} + \frac{1}{2}\int_{\partial D_{\mathrm{g}}}\langle \zeta_n h^{B-}h^A\rangle\,\mathrm{dS}$$
$$= \int_D \langle I^{A-}\phi^B\rangle\,\mathrm{d}\boldsymbol{x} - \int_{\partial D_{\mathrm{w}}}\langle \zeta_n g^{A-}_{\mathrm{w}}\phi^B\rangle\,\mathrm{dS} - \int_{\partial D_{\mathrm{g}}}\langle \zeta_n h^{A-}\phi^B\rangle\,\mathrm{dS} + \frac{1}{2}\int_{\partial D_{\mathrm{g}}}\langle \zeta_n h^{A-}h^B\rangle\,\mathrm{dS}. \quad (15)$$

This is identical to the symmetric relation for steady bounded-domain systems (see (14) in [9]).

Table 1 Green functions for initial data

Green function	corresponding element sources				note
$G^{(\mathrm{m})}(t,\boldsymbol{x},\boldsymbol{\zeta})$	$g_\mathrm{w}=0$	$h=0$	$G^{(\mathrm{m})}_{\mathrm{initial}}=1$	$I=0$	–
$G^{(\boldsymbol{\ell})}(t,\boldsymbol{x},\boldsymbol{\zeta})$[a]	$g_\mathrm{w}=0$	$h=0$	$G^{(\boldsymbol{\ell})}_{\mathrm{initial}}=\zeta_\ell$	$I=0$	$G^{(-\boldsymbol{\ell})}=-G^{(\boldsymbol{\ell})}$
$G^{(\mathrm{E})}(t,\boldsymbol{x},\boldsymbol{\zeta})$	$g_\mathrm{w}=0$	$h=0$	$G^{(\mathrm{E})}_{\mathrm{initial}}=\frac{2}{3}\lvert\boldsymbol{\zeta}\rvert^2$	$I=0$	–

[a] $\boldsymbol{\ell}$ is an arbitrary unit vector and $\zeta_\ell=\zeta_i\ell_i$.

In the same way as in [9], we can consider a point source in the initial data, inhomogeneous term, and boundary data and its corresponding Green functions. Then, various reciprocities in a way of point correspondence can be derived from Proposition 1. We shall not repeat all these processes here. Rather, we shall pick up only a few kinds of Green functions in Sects. 3–5 and present several interesting consequences obtained from Proposition 1. Further in Sect. 6, we will show that the expressions similar to the fluctuation–dissipation theorem and to those for the static admittance in the linear response theory for bulk systems [16] can be obtained from Proposition 1 by applying it to a gas in a periodic box. On the basis of this observation, we also discuss their extensions to the systems of arbitrary Kn.

3 Representation in terms of the Green functions for initial data

We consider an initial- and boundary-value problem (1)–(5) (the original problem, in short) and its associated problems (1)–(5) with the initial data, boundary data, and inhomogeneous term listed in Table 1. The latter problems are associated with the original one in the sense that the domain D, collision operator $\mathscr{L}$, kernels R and P, and the Strouhal and Knudsen numbers Sh and Kn are the same as those in the original problem. We denote the solutions of the associated problems by $G^{(\mathrm{m})}$, $G^{(\boldsymbol{\ell})}$, and $G^{(\mathrm{E})}$ and call them the Green function for the initial mass, momentum, and energy, respectively. $G^{(\mathrm{m})}$, $G^{(\boldsymbol{\ell})}$, and $G^{(\mathrm{E})}$ represent the response of the system against the initial uniform perturbation of mass, momentum, and energy. There is no perturbation through the boundary and the inhomogeneous term. By applying (12) to the pair of the solution ϕ of the original problem and any of the Green functions in Table 1, we obtain a general expression of the total mass, momentum, and energy in the domain D at any time for the original problem:

Proposition 2 *Consider the initial- and boundary-value problem* (1) – (5) *in the domain D. The total mass, momentum, and energy in the domain at any time are expressed in terms of the Green function for the initial mass, momentum, and energy as follows:*

$$\mathrm{Sh}\int_D \begin{bmatrix} \omega(t,\boldsymbol{x}) \\ u_\ell(t,\boldsymbol{x}) \\ \mathscr{E}(t,\boldsymbol{x}) \end{bmatrix} \mathrm{d}\boldsymbol{x} = \mathrm{Sh}\int_D \begin{bmatrix} \langle \phi^-_{\mathrm{initial}} G^{(\mathrm{m})}\rangle(t,\boldsymbol{x}) \\ \langle \phi^-_{\mathrm{initial}} G^{(-\boldsymbol{\ell})}\rangle(t,\boldsymbol{x}) \\ \langle \phi^-_{\mathrm{initial}} G^{(\mathrm{E})}\rangle(t,\boldsymbol{x}) \end{bmatrix} \mathrm{d}\boldsymbol{x} + \int_D \begin{bmatrix} \langle I^- * G^{(\mathrm{m})}\rangle(t,\boldsymbol{x}) \\ \langle I^- * G^{(-\boldsymbol{\ell})}\rangle(t,\boldsymbol{x}) \\ \langle I^- * G^{(\mathrm{E})}\rangle(t,\boldsymbol{x}) \end{bmatrix} \mathrm{d}\boldsymbol{x}$$
$$- \int_{\partial D_\mathrm{w}} \begin{bmatrix} \langle \zeta_n (g^-_\mathrm{w} * G^{(\mathrm{m})})\rangle(t,\boldsymbol{x}) \\ \langle \zeta_n (g^-_\mathrm{w} * G^{(-\boldsymbol{\ell})})\rangle(t,\boldsymbol{x}) \\ \langle \zeta_n (g^-_\mathrm{w} * G^{(\mathrm{E})})\rangle(t,\boldsymbol{x}) \end{bmatrix} \mathrm{dS} - \int_{\partial D_\mathrm{g}} \begin{bmatrix} \langle \zeta_n (h^- * G^{(\mathrm{m})})\rangle(t,\boldsymbol{x}) \\ \langle \zeta_n (h^- * G^{(-\boldsymbol{\ell})})\rangle(t,\boldsymbol{x}) \\ \langle \zeta_n (h^- * G^{(\mathrm{E})})\rangle(t,\boldsymbol{x}) \end{bmatrix} \mathrm{dS}. \quad (18)$$

Example 1 Consider a vapor in a domain D that is surrounded by its condensed phases I–IV at rest at uniform temperature T_0 (Fig. 1). The vapor is in phase equilibrium with the condensed phases (the reference equilibrium state). At $t=0$, the temperature of the condensed

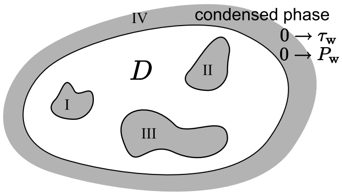

Fig. 1 A vapor in the domain D surrounded by its condensed phases (I $\sim$ IV). At $t = 0$, the temperature of the condensed phases instantaneously changes from T_0 to $T_0(1+\tau_w)$. The corresponding saturation pressure of the vapor changes from p_0 to $p_0(1+P_w)$.

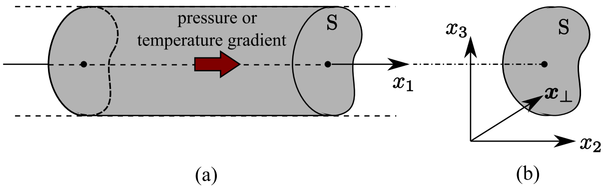

Fig. 2 Poiseuille and thermal transpiration flows. (a) Sketch of the problems. (b) Geometry of the pipe cross-section.

phases instantaneously changes from T_0 to $T_0(1+\tau_w)$, where τ_w may be nonuniform but is constant in time. We are interested in the time evolution of the total mass and energy in the domain D.

In the present case, there is no imaginary boundary, so that $\partial D = \partial D_w$. Let us denote the saturation pressure of the vapor at temperature $T_0(1+\tau_w)$ by $p_0(1+P_w)$ and the solution of the considered problem by ϕ. Since ϕ is the solution of the problem (1)–(3) with $\phi_{\text{initial}} = 0$, $I = 0$, and $g_w = P_w + (|\boldsymbol{\zeta}|^2 - \frac{5}{2})\tau_w$, we obtain by the use of the first and third equations of (18) the following expressions for the total mass and energy in the domain D:

$$\mathrm{Sh}\int_D \begin{bmatrix} \omega(t,\boldsymbol{x}) \\ \mathscr{E}(t,\boldsymbol{x}) \end{bmatrix} \mathrm{d}\boldsymbol{x} = -\int_{\partial D} \begin{bmatrix} P_w(\boldsymbol{x})\,\overline{u}_n[G^{(\mathrm{m})}](t,\boldsymbol{x}) + \tau_w(\boldsymbol{x})\,\overline{Q}_n[G^{(\mathrm{m})}](t,\boldsymbol{x}) \\ P_w(\boldsymbol{x})\,\overline{u}_n[G^{(\mathrm{E})}](t,\boldsymbol{x}) + \tau_w(\boldsymbol{x})\,\overline{Q}_n[G^{(\mathrm{E})}](t,\boldsymbol{x}) \end{bmatrix} \mathrm{dS}. \tag{19}$$

If the domain D is surrounded by a simple boundary, P_w is a fake parameter and disappears from the above expressions because the mass flux through the boundary vanishes. In the case, the first equation becomes trivial because of $G^{(\boldsymbol{m})} = 1$, so that (19) is reduced to

$$\mathrm{Sh}\int_D \mathscr{E}(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = -\int_{\partial D} \tau_w(\boldsymbol{x})\,\overline{Q}_n[G^{(\mathrm{E})}](t,\boldsymbol{x})\,\mathrm{dS}. \tag{20}$$

Example 2 Consider the time-dependent Poiseuille and thermal transpiration flows in a straight pipe (Fig. 2). The geometry of the pipe cross-section may be arbitrary.

The time-dependent problem of the Poiseuille flow can be formulated as the flow of a gas, which is initially in the thermal equilibrium at rest with the pipe wall of uniform temperature, caused by a uniform weak external force in the (negative) x_1-direction. Let us denote the perturbed velocity distribution function by $\phi^{\mathrm{P}}(t,\boldsymbol{x}_\perp,\boldsymbol{\zeta})$, where $\boldsymbol{x}_\perp = (x_2,x_3)$. The ϕ^{P} is a solution of (1)–(3) with $D = \mathrm{S}$, $\partial D = \partial D_w = \partial\mathrm{S}$, $I = -\zeta_1$, $g_w = 0$, and $\phi^{\mathrm{P}}_{\text{initial}} = 0$.

Next consider the time-dependent problem of the thermal transpiration and denote its perturbed velocity distribution function by $\phi^{\mathrm{T}} = x_1(|\boldsymbol{\zeta}|^2 - \frac{5}{2}) + \Phi^{\mathrm{T}}(t,\boldsymbol{x}_\perp,\boldsymbol{\zeta})$. Then, Φ^{T} is a solution of (1)–(3) with $D = \mathrm{S}$, $\partial D_w = \partial\mathrm{S}$, $\partial D_g = 0$, $I = -\zeta_1(|\boldsymbol{\zeta}|^2 - \frac{5}{2})$, $g_w = 0$, and a certain initial data $\Phi^{\mathrm{T}}_{\text{initial}}(\boldsymbol{x}_\perp,\boldsymbol{\zeta})$. [3]

[3] Natural initial data would be $\Phi^{\mathrm{T}}_{\text{initial}} = -\frac{\sqrt{\pi}}{2}\mathrm{Kn}\zeta_1 A(|\boldsymbol{\zeta}|)$, where A is the solution of $\mathscr{L}(\zeta_1 A(|\boldsymbol{\zeta}|^2)) = -\zeta_1(|\boldsymbol{\zeta}|^2 - \frac{5}{2})$ such that $\langle|\boldsymbol{\zeta}|^2 A(|\boldsymbol{\zeta}|^2)\rangle = 0$. Here, we simply assume that $\Phi^{\mathrm{T}}_{\text{initial}}$ is independent of x_1.

Table 2 Green functions for inhomogeneous term

Green function	corresponding element sources				note
$G^{(\mathrm{m};I)}(t,\boldsymbol{x},\boldsymbol{\zeta})$	$g_\mathrm{w}=0$	$h=0$	$G^{(\mathrm{m};I)}_\mathrm{initial}=0$	$I=1$	$-$
$G^{(\boldsymbol{\ell};I)}(t,\boldsymbol{x},\boldsymbol{\zeta})$	$g_\mathrm{w}=0$	$h=0$	$G^{(\boldsymbol{\ell};I)}_\mathrm{initial}=0$	$I=\zeta_\ell$	$G^{(-\boldsymbol{\ell};I)}=-G^{(\boldsymbol{\ell};I)}$
$G^{(\mathrm{E};I)}(t,\boldsymbol{x},\boldsymbol{\zeta})$	$g_\mathrm{w}=0$	$h=0$	$G^{(\mathrm{E};I)}_\mathrm{initial}=0$	$I=\frac{2}{3}\lvert\boldsymbol{\zeta}\rvert^2$	$-$

Substitution of ϕ^P and Φ^T into the second equation of (18) yields the following relations:

$$\mathrm{Sh}\int_\mathrm{S}\begin{bmatrix}u_1[\phi^\mathrm{P}](t,\boldsymbol{x}_\perp)\\ u_1[\phi^\mathrm{T}](t,\boldsymbol{x}_\perp)\end{bmatrix}\mathrm{d}\boldsymbol{x}_\perp=-\int_\mathrm{S}\begin{bmatrix}\overline{u}_1[G^{(\mathbf{e}_1)}](t,\boldsymbol{x}_\perp)\\ \mathrm{Sh}\langle\Phi^{\mathrm{T}-}_\mathrm{initial}G^{(\mathbf{e}_1)}\rangle(t,\boldsymbol{x}_\perp)+\overline{Q}_1[G^{(\mathbf{e}_1)}](t,\boldsymbol{x}_\perp)\end{bmatrix}\mathrm{d}\boldsymbol{x}_\perp,\quad(21)$$

where $\mathbf{e}_1$ is the unit vector in the x_1-direction. The relation $u_1[\phi^\mathrm{T}]=u_1[\Phi^\mathrm{T}]$ has been taken into account in the second equation. Thus, the mass flow (flux) through the pipe can be expressed for any time in terms of the Green function for the initial momentum in the x_1-direction. Incidentally, since $G^{(\mathbf{e}_1)}\to 0$ as $t\to\infty$, (21) is reduced in the same limit to

$$\int_\mathrm{S}\begin{bmatrix}u_1[\phi^\mathrm{P}](\boldsymbol{x}_\perp)\\ u_1[\phi^\mathrm{T}](\boldsymbol{x}_\perp)\end{bmatrix}\mathrm{d}\boldsymbol{x}_\perp=-\mathrm{Sh}^{-1}\int_\mathrm{S}\int_0^\infty\begin{bmatrix}u_1[G^{(\mathbf{e}_1)}](t,\boldsymbol{x}_\perp)\\ Q_1[G^{(\mathbf{e}_1)}](t,\boldsymbol{x}_\perp)\end{bmatrix}\mathrm{d}t\,\mathrm{d}\boldsymbol{x}_\perp.\quad(22)$$

The left-hand side is the mass flow of the steady Poiseuille and thermal transpiration flows.

4 Representation in terms of the Green function for inhomogeneous term

In the present section, for the original initial- and boundary-value problem (1) – (5), we consider the associated problems (1) – (5) with the initial data, boundary data, and inhomogeneous term listed in Table 2. We denote the solutions of the associated problems by $G^{(\mathrm{m};I)}$, $G^{(\boldsymbol{\ell};I)}$, $G^{(\mathrm{E};I)}$ and call them the Green function for the mass, momentum, and energy type inhomogeneous term, respectively. The total mass, momentum, and energy, which have already been studied in Sect. 3, can also be expressed for any time in terms of the Green function for the inhomogeneous term. These alternative expressions are obtained by applying the pair of the solution ϕ of the original problem and any of the Green functions in Table 2 to the symmetric relation (12). The results are as follows:

Proposition 3 *Consider the initial- and boundary-value problem* (1) – (5) *in the domain D. Then, the time integration of the total mass, momentum, and energy in the domain can be expressed in terms of the Green function for the mass, momentum, and energy type inhomogeneous term as follows:*

$$\begin{aligned}\int_D\begin{bmatrix}\overline{\omega}(t,\boldsymbol{x})\\ \overline{u}_\ell(t,\boldsymbol{x})\\ \overline{\mathscr{E}}(t,\boldsymbol{x})\end{bmatrix}\mathrm{d}\boldsymbol{x}=\mathrm{Sh}\int_D\begin{bmatrix}\langle\phi^-_\mathrm{initial}G^{(\mathrm{m};I)}\rangle(t,\boldsymbol{x})\\ \langle\phi^-_\mathrm{initial}G^{(-\boldsymbol{\ell};I)}\rangle(t,\boldsymbol{x})\\ \langle\phi^-_\mathrm{initial}G^{(\mathrm{E};I)}\rangle(t,\boldsymbol{x})\end{bmatrix}\mathrm{d}\boldsymbol{x}+\int_D\begin{bmatrix}\langle I^-*G^{(\mathrm{m};I)}\rangle(t,\boldsymbol{x})\\ \langle I^-*G^{(-\boldsymbol{\ell};I)}\rangle(t,\boldsymbol{x})\\ \langle I^-*G^{(\mathrm{E};I)}\rangle(t,\boldsymbol{x})\end{bmatrix}\mathrm{d}\boldsymbol{x}\\ -\int_{\partial D_\mathrm{w}}\begin{bmatrix}\langle\zeta_n(g^-_\mathrm{w}*G^{(\mathrm{m};I)})\rangle(t,\boldsymbol{x})\\ \langle\zeta_n(g^-_\mathrm{w}*G^{(-\boldsymbol{\ell};I)})\rangle(t,\boldsymbol{x})\\ \langle\zeta_n(g^-_\mathrm{w}*G^{(\mathrm{E};I)})\rangle(t,\boldsymbol{x})\end{bmatrix}\mathrm{dS}-\int_{\partial D_\mathrm{g}}\begin{bmatrix}\langle\zeta_n(h^-*G^{(\mathrm{m};I)})\rangle(t,\boldsymbol{x})\\ \langle\zeta_n(h^-*G^{(-\boldsymbol{\ell};I)})\rangle(t,\boldsymbol{x})\\ \langle\zeta_n(h^-*G^{(\mathrm{E};I)})\rangle(t,\boldsymbol{x})\end{bmatrix}\mathrm{dS}.\end{aligned}\quad(23)$$

The total mass, momentum, and energy in the domain can be obtained for any time by differentiating (23) *with respect to t.*

Example 3 Consider again the time-dependent Poiseuille and thermal transpiration flows studied in Example 2. As explained before, ϕ^{P} is the solution of (1)–(3) with $D = \mathrm{S}$, $\partial D_{\mathrm{w}} = \partial \mathrm{S}$, $I = -\zeta_1$, $g_{\mathrm{w}} = 0$, and $\phi^{\mathrm{P}}_{\mathrm{initial}} = 0$. Hence ϕ^{P} is no other than the Green function $G^{(-\mathbf{e}_1;I)}$, if we consider Φ^{T} to be the solution of the original problem in the domain S. Therefore, by applying the second equation of (23) with $\boldsymbol{\ell} = \mathbf{e}_1$ to Φ^{T}, we obtain the following relation:

$$\int_{\mathrm{S}} \overline{u}_1[\phi^{\mathrm{T}}](t,\boldsymbol{x}_\perp)\,\mathrm{d}\boldsymbol{x}_\perp = \mathrm{Sh}\int_{\mathrm{S}} \langle \Phi^{\mathrm{T}-}_{\mathrm{initial}}\phi^{\mathrm{P}}\rangle(t,\boldsymbol{x}_\perp)\,\mathrm{d}\boldsymbol{x}_\perp + \int_{\mathrm{S}} \overline{Q}_1[\phi^{\mathrm{P}}](t,\boldsymbol{x}_\perp)\,\mathrm{d}\boldsymbol{x}_\perp.$$

Here again we have used the fact that $u_1[\phi^{\mathrm{T}}] = u_1[\Phi^{\mathrm{T}}]$. Differentiating the above relation with respect to t yields

$$\int_{\mathrm{S}} u_1[\phi^{\mathrm{T}}](t,\boldsymbol{x}_\perp)\,\mathrm{d}\boldsymbol{x}_\perp = \mathrm{Sh}\,\partial_t\int_{\mathrm{S}} \langle \Phi^{\mathrm{T}-}_{\mathrm{initial}}\phi^{\mathrm{P}}\rangle(t,\boldsymbol{x}_\perp)\,\mathrm{d}\boldsymbol{x}_\perp + \int_{\mathrm{S}} Q_1[\phi^{\mathrm{P}}](t,\boldsymbol{x}_\perp)\,\mathrm{d}\boldsymbol{x}_\perp. \tag{24}$$

This is an extension of the reciprocal relation between the steady Poiseuille and thermal transpiration flows. In the limit $t \to \infty$, the first term on the right-hand side vanishes, because ϕ^{P} tends to a steady solution. Thus, in the same limit, (24) recovers the known reciprocity that the (dimensionless) heat flux of the Poiseuille flow is identical to the mass flux of the thermal transpiration flow (e.g., [6] and Example 5 in [9]). It should be noted that the same reciprocity as the steady case remains valid for any time, if $\Phi^{\mathrm{T}}_{\mathrm{initial}} = 0$.

By the comparison between (23) and (18), it is seen that the integration of the right-hand side of (18) from the initial to time t is the same as the right-hand side of (23) multiplied by Sh. Since ϕ_{initial} in (23) is arbitrary, we obtain the following:

Corollary 1 *The Green function for the inhomogeneous term is a time integration of the Green function for the corresponding initial data:*

$$G^{(\alpha;I)}(t,\boldsymbol{x},\boldsymbol{\zeta}) = \mathrm{Sh}^{-1}\int_0^t G^{(\alpha)}(r,\boldsymbol{x},\boldsymbol{\zeta})\mathrm{d}r \qquad (\alpha = \mathrm{m},\boldsymbol{\ell},\mathrm{E}). \tag{25}$$

Remark 3 (25) implies that $\partial_t G^{(\alpha;I)}$ multiplied by Sh solves the same initial- and boundary-value problem (1)–(5) as that for $G^{(\alpha)}$, which suggests the interesting behavior of $G^{(\alpha;I)}$. That is, even when $G^{(\alpha;I)}$ is continuous, $\partial_t G^{(\alpha;I)}$ has a discontinuity corresponding to that of $G^{(\alpha)}$ which is caused by the difference between the initial condition and the boundary condition for $G^{(\alpha)}$.

5 Representation in terms of the Green function for boundary data

In the present section, we consider the reaction of the system against the elemental source on the boundary. We will show that the mass, momentum, and energy transferred to the boundary can be expressed for any time in terms of the Green function for the element source. The discussions are essentially parallel to those for steady systems in [9]. We discuss the case of real boundary and that of imaginary boundary separately.

Table 3 Green functions for the boundary data on $\mathrm{S_w}$ of real boundary ∂D_w.

Green function	corresponding element sources			
$G^{(P;\mathrm{S_w})}(t,\boldsymbol{x},\boldsymbol{\zeta})$	$g_\mathrm{w} = \chi_{\mathrm{S_w}}$ [a]	$h=0$	$G^{(P;\mathrm{S_w})}_\mathrm{initial}=0$	$I=0$
$G^{(T;\mathrm{S_w})}(t,\boldsymbol{x},\boldsymbol{\zeta})$	$g_\mathrm{w} = (\lvert\boldsymbol{\zeta}\rvert^2-\frac{5}{2})\chi_{\mathrm{S_w}}$	$h=0$	$G^{(T;\mathrm{S_w})}_\mathrm{initial}=0$	$I=0$

[a] $\chi_{\mathrm{S_w}}$ is the characteristic function of $\mathrm{S_w}\subseteq\partial D_\mathrm{w}$, i.e., $\chi_{\mathrm{S_w}}=1$ on $\mathrm{S_w}$ and $\chi_{\mathrm{S_w}}=0$ otherwise.

5.1 Mass and heat fluxes on the real boundary

For the original initial- and boundary-value problem (1)–(5), we consider the associated problems (1)–(5) with the initial data, boundary data, and inhomogeneous term listed in Table 3. We denote the solutions of the associated problems by $G^{(P;\mathrm{S_w})}$ and $G^{(T;\mathrm{S_w})}$ and call them the Green functions for the pressure and temperature sources on the surface $\mathrm{S_w}$. By applying (12) to the pair of the solution ϕ of the original problem and any of the Green functions in Table 3, we obtain the representation of the mass and heat fluxes through $\mathrm{S_w}$ in terms of the Green functions. The results are summarized as follows:

Proposition 4 *Consider the initial- and boundary-value problem* (1) – (5) *in the domain D. The total mass and heat transferred from the surface* $\mathrm{S_w}$ *on* ∂D_w *to the gas up to time t can be expressed in terms of the Green function for the pressure and temperature sources on* $\mathrm{S_w}$*:*

$$\int_{\mathrm{S_w}}\begin{bmatrix}\overline{u}_n(t,\boldsymbol{x})\\ \overline{Q}_n(t,\boldsymbol{x})\end{bmatrix}\mathrm{dS} = -\mathrm{Sh}\int_D\begin{bmatrix}\langle\phi^-_\mathrm{initial}G^{(P;\mathrm{S_w})}\rangle(t,\boldsymbol{x})\\ \langle\phi^-_\mathrm{initial}G^{(T;\mathrm{S_w})}\rangle(t,\boldsymbol{x})\end{bmatrix}\mathrm{d}\boldsymbol{x} - \int_D\begin{bmatrix}\langle I^- * G^{(P;\mathrm{S_w})}\rangle(t,\boldsymbol{x})\\ \langle I^- * G^{(T;\mathrm{S_w})}\rangle(t,\boldsymbol{x})\end{bmatrix}\mathrm{d}\boldsymbol{x}$$
$$+\int_{\partial D_\mathrm{w}}\begin{bmatrix}\langle\zeta_n(g^-_\mathrm{w} * G^{(P;\mathrm{S_w})})\rangle(t,\boldsymbol{x})\\ \langle\zeta_n(g^-_\mathrm{w} * G^{(T;\mathrm{S_w})})\rangle(t,\boldsymbol{x})\end{bmatrix}\mathrm{dS} + \int_{\partial D_\mathrm{g}}\begin{bmatrix}\langle\zeta_n(h^- * G^{(P;\mathrm{S_w})})\rangle(t,\boldsymbol{x})\\ \langle\zeta_n(h^- * G^{(T;\mathrm{S_w})})\rangle(t,\boldsymbol{x})\end{bmatrix}\mathrm{dS}. \quad (26)$$

The mass and heat fluxes through $\mathrm{S_w}$ *are obtained for any time by differentiating* (26) *with respect to t.*

Example 4 Consider again the example studied in Example 1. We are interested in the mass and heat transfer from the condensed phase I (see Fig. 1). Denoting by $\mathrm{S_I}$ the interface between the vapor and the condensed phase I, the expression of the total mass and heat transferred from the condensed phase I up to time t is given by (26) as follows:

$$\int_{\mathrm{S_I}}\begin{bmatrix}\overline{u}_n(t,\boldsymbol{x})\\ \overline{Q}_n(t,\boldsymbol{x})\end{bmatrix}\mathrm{dS} = \int_{\partial D}\begin{bmatrix}P_\mathrm{w}(\boldsymbol{x})\,\overline{u}_n[G^{(P;\mathrm{S_I})}](t,\boldsymbol{x}) + \tau_\mathrm{w}(\boldsymbol{x})\,\overline{Q}_n[G^{(P;\mathrm{S_I})}](t,\boldsymbol{x})\\ P_\mathrm{w}(\boldsymbol{x})\,\overline{u}_n[G^{(T;\mathrm{S_I})}](t,\boldsymbol{x}) + \tau_\mathrm{w}(\boldsymbol{x})\,\overline{Q}_n[G^{(T;\mathrm{S_I})}](t,\boldsymbol{x})\end{bmatrix}\mathrm{dS}.$$

In particular, if $\mathrm{S_I}$ is a simple boundary, the above expression is reduced to

$$\int_{\mathrm{S_I}}\overline{Q}_n(t,\boldsymbol{x})\,\mathrm{dS} = \int_{\partial D}\tau_\mathrm{w}(\boldsymbol{x})\,\overline{Q}_n[G^{(T;\mathrm{S_I})}](t,\boldsymbol{x})\,\mathrm{dS}.$$

Example 5 Consider a gas in a straight channel, in which the bodies $1\sim4$ are periodically arranged (Fig. 3). The channel wall and the bodies are maintained at uniform temperature T_0 (the reference temperature). The gas is initially in the thermal equilibrium state with the channel wall and bodies. From $t=0$, a weak uniform external force acts on the gas in the x_1-direction. We are interested in the heat that the gas receives from the bodies.

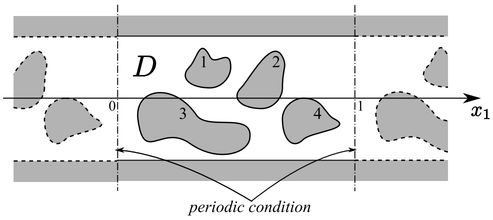

Fig. 3 A gas in a channel with periodically arranged bodies

The present problem is formulated by (1)–(5) with $\phi_{\text{initial}} = 0$, $I = \zeta_1$, and $g_{\text{w}} = h = 0$. Thus, the total heat that the gas receives from the bodies up to time t is expressed by the use of (26) as follows:

$$\int_{\partial B} \overline{Q}_n(t,\boldsymbol{x})\,\mathrm{dS} = \int_D \overline{u}_1[G^{(T;\partial B)}](t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{27}$$

where ∂B is the surface of the bodies. Thus, the heat transfer in the present problem can be computed by the flow in the channel when the bodies are uniformly heated.

The right-hand side of (27) can be transformed as follows:

$$\begin{aligned}\int_D \overline{u}_1[G^{(T;\partial B)}](t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= \int_D \Big(\frac{\partial x_1 \overline{u}_1[G^{(T;\partial B)}]}{\partial x_1} - x_1 \frac{\partial \overline{u}_1[G^{(T;\partial B)}]}{\partial x_1}\Big)\,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathrm{S}(x_1)} \overline{u}_1[G^{(T;\partial B)}]\,\mathrm{d}\boldsymbol{x}_\perp\Big|_{x_1=1} - \int_0^1 \int_{\mathrm{S}(x_1)} x_1(\mathrm{Sh}\langle G^{(T;\partial B)}\rangle - \nabla_\perp \cdot \overline{\boldsymbol{u}}_\perp[G^{(T;\partial B)}])\,\mathrm{d}\boldsymbol{x}_\perp \mathrm{d}x_1 \\ &= \int_{\mathrm{S}(x_1)} \overline{u}_1[G^{(T;\partial B)}]\,\mathrm{d}\boldsymbol{x}_\perp\Big|_{x_1=1} - \mathrm{Sh}\int_D x_1 \langle G^{(T;\partial B)}\rangle\,\mathrm{d}\boldsymbol{x},\end{aligned}$$

where $\mathrm{S}(x_1)$ is the cross section of D at the axial position x_1, $\boldsymbol{x}_\perp = (x_2, x_3)$, $\boldsymbol{u}_\perp = (u_2, u_3)$, and $\nabla_\perp = (\partial/\partial x_2, \partial/\partial x_3)$. In the above transformation, the second equality is due to the mass conservation law, while the third equality is due to the Gauss divergence theorem in the cross section and to no mass flow across the channel wall and body surfaces. Thus (27) is rewritten as follows:[4]

$$\int_{\partial B} \overline{Q}_n(t,\boldsymbol{x})\,\mathrm{dS} = -\mathrm{Sh}\int_D x_1 \langle G^{(T;\partial B)}\rangle(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{\mathrm{S}(x_1)} \overline{u}_1[G^{(T;\partial B)}](t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_\perp\Big|_{x_1=1}.$$

By taking the limit $t \to \infty$ after differentiating with respect to t, we obtain the following relation that holds at the final steady state:

$$\int_{\partial B} Q_n(\boldsymbol{x})\,\mathrm{dS} = \int_{\mathrm{S}(x_1)} u_1[G^{(T;\partial B)}](\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_\perp\Big|_{x_1=1}.$$

Therefore, if the bodies are arranged so that a steady one-way flow is not induced when uniformly heated, the steady flow induced by the uniform external force does not transfer the heat to the bodies in total.

5.2 Mass, momentum, and heat fluxes on the imaginary boundary

For the original initial- and boundary-value problem (1)–(5), we consider the associated problems (1)–(5) with the initial data, boundary data, and inhomogeneous term listed in

[4] If we introduce the function $\Phi = -x_1 + \phi$, Φ is a solution of (1)–(5) with $\Phi_{\text{initial}} = -x_1$, $I = 0$, $g_{\text{w}} = 0$, and $h = -1$. The present relation is obtained directly by applying (26) to Φ.

Table 4 Green functions for the boundary data on S_g of imaginary boundary.

Green function	corresponding elemental source				note
$G^{(P;S_g)}(t,\boldsymbol{x},\boldsymbol{\zeta})$	$g_w=0$	$h=\chi_{S_g}(\boldsymbol{x})$	$G^{(P;S_g)}_{\text{initial}}(\boldsymbol{x},\boldsymbol{\zeta})=0$	$I=0$	–
$G^{(\boldsymbol{\ell};S_g)}(t,\boldsymbol{x},\boldsymbol{\zeta})$	$g_w=0$	$h=2\zeta_i\ell_i\chi_{S_g}(\boldsymbol{x})$	$G^{(\boldsymbol{\ell};S_g)}_{\text{initial}}(\boldsymbol{x},\boldsymbol{\zeta})=0$	$I=0$	$G^{(-\boldsymbol{\ell};S_g)}=-G^{(\boldsymbol{\ell};S_g)}$
$G^{(T;S_g)}(t,\boldsymbol{x},\boldsymbol{\zeta})$	$g_w=0$	$h=(\lvert\boldsymbol{\zeta}\rvert^2-\frac{5}{2})\chi_{S_g}(\boldsymbol{x})$	$G^{(T;S_g)}_{\text{initial}}(\boldsymbol{x},\boldsymbol{\zeta})=0$	$I=0$	–

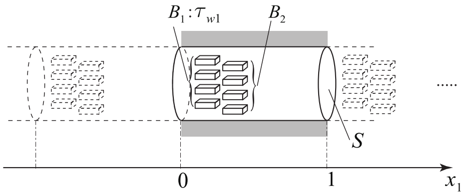

Fig. 4 Thermal edge pump [18]

Table 4. We denote the solutions of the associated problems by $G^{(P;S_g)}$, $G^{(\boldsymbol{\ell};S_g)}$, and $G^{(T;S_g)}$ and call them the Green functions for the pressure, velocity, and temperature sources on the surface S_g. By applying (12) to the pair of the solution ϕ of the original problem and any of the Green functions in Table 4, we obtain the representation of the mass, momentum, and heat fluxes through S_g in terms of the Green functions. The results are as follows:

Proposition 5 *Consider the initial- and boundary-value problem* (1) – (5) *in the domain D. The total mass, momentum, and heat transferred from the surface* S_g *on* ∂D_g *to the gas up to time t can be expressed in terms of the Green function for the pressure, velocity, and temperature sources on* S_g*:*

$$\int_{S_g}\begin{bmatrix}\overline{u}_n(t,\boldsymbol{x})\\ \overline{P}_{n\ell}(t,\boldsymbol{x})\\ \overline{Q}_n(t,\boldsymbol{x})\end{bmatrix}\mathrm{dS}=-\mathrm{Sh}\int_D\begin{bmatrix}\langle\phi^-_{\text{initial}}G^{(P;S_g)}\rangle(t,\boldsymbol{x})\\ \langle\phi^-_{\text{initial}}G^{(-\boldsymbol{\ell};S_g)}\rangle(t,\boldsymbol{x})\\ \langle\phi^-_{\text{initial}}G^{(T;S_g)}\rangle(t,\boldsymbol{x})\end{bmatrix}\mathrm{d}\boldsymbol{x}$$
$$-\int_D\begin{bmatrix}\langle I^-*G^{(P;S_g)}\rangle(t,\boldsymbol{x})\\ \langle I^-*G^{(-\boldsymbol{\ell};S_g)}\rangle(t,\boldsymbol{x})\\ \langle I^-*G^{(T;S_g)}\rangle(t,\boldsymbol{x})\end{bmatrix}\mathrm{d}\boldsymbol{x}+\int_{\partial D_w}\begin{bmatrix}\langle\zeta_n(g^-_w*G^{(P;S_g)})\rangle(t,\boldsymbol{x})\\ \langle\zeta_n(g^-_w*G^{(-\boldsymbol{\ell};S_g)})\rangle(t,\boldsymbol{x})\\ \langle\zeta_n(g^-_w*G^{(T;S_g)})\rangle(t,\boldsymbol{x})\end{bmatrix}\mathrm{dS}$$
$$+\int_{\partial D_g}\begin{bmatrix}\langle\zeta_n(h^-*G^{(P;S_g)})\rangle(t,\boldsymbol{x})\\ \langle\zeta_n(h^-*G^{(-\boldsymbol{\ell};S_g)})\rangle(t,\boldsymbol{x})\\ \langle\zeta_n(h^-*G^{(T;S_g)})\rangle(t,\boldsymbol{x})\end{bmatrix}\mathrm{dS}+\int_{S_g}\begin{bmatrix}\overline{u}_n[h](t,\boldsymbol{x})\\ \overline{P}_{n\ell}[h](t,\boldsymbol{x})\\ \overline{Q}_n[h](t,\boldsymbol{x})\end{bmatrix}\mathrm{dS}. \quad (28)$$

The mass, momentum, and heat fluxes through S_g *are obtained for any time by differentiating* (28) *with respect to t.*

Example 6 Consider a gas in a straight pipe, in which two arrays of plates, say B_1 and B_2, are periodically arranged (Fig. 4). Initially, the pipe wall and the plates are commonly at a uniform temperature T_0 and the gas is thermally in equilibrium with them. If the temperature of the arrays B_1 is changed to $T_0(1+\tau_{w1})$ (τ_{w1} is a small constant), a flow is induced to develop a steady one-way flow in the pipe (thermal edge pump [18]). When the change of the temperature occurs instantaneously at $t=0$, the problem is formulated by (1)–(5) with $\phi_{\text{initial}}=0$, $I=0$, $g_w=(\lvert\boldsymbol{\zeta}\rvert^2-\frac{5}{2})\tau_{w1}\chi_{\partial B_1}$, and $h=0$. Thus, the expression of the total mass passing through the pipe cross-section S up to time t in the x_1-direction is obtained from

(28) in the form:

$$\int_{\mathrm{S}} \overline{u}_1(t,\boldsymbol{x})\,\mathrm{dS} = -\tau_{\mathrm{w}1}\int_{\partial B_1} \overline{Q}_n[G^{(P;\mathrm{S})}](t,\boldsymbol{x})\,\mathrm{dS}.$$

Differentiating with respect to t yields the mass flux for any time:

$$\int_{\mathrm{S}} u_1(t,\boldsymbol{x})\,\mathrm{dS} = -\tau_{\mathrm{w}1}\int_{\partial B_1} Q_n[G^{(P;\mathrm{S})}](t,\boldsymbol{x})\,\mathrm{dS} \quad (t>0).$$

This is an extension of the formula in Example 6 of [9] to the time-dependent situation.

6 Similarity to fluctuation–dissipation theorem and symmetry of static admittance in the linear response theory for bulk systems

As mentioned at the end of Sects. 1 and 2.3, the symmetric relation (12) yields the expressions similar to the fluctuation–dissipation theorem and to those for the static admittance in the linear response theory for bulk systems (i.e., the systems for small Kn or without boundary effect) [16], when applied to a gas in a box with the periodic condition. In this section, we briefly discuss this issue and their extension to gas systems in a fixed bounded domain for arbitrary Kn. In rarefied gas dynamics, the fluctuation–dissipation theorem was sometimes used to evaluate the validity and/or accuracy of particle simulation methods like the DSMC (e.g., [19–21]). The point of the present argument is different from those studies.

Consider a gas in a box with the periodic condition (the periodic box D_{p}, in short). The gas in the periodic box is intended to represent a uniform expanse of the gas in a whole space, i.e., a bulk gas without boundary effect, so that Kn for the periodic box D_{p} is a fake parameter or may be considered small. Let us denote by ϕ^A the solution of (1), (2), and (5b) with $h=0$, $I=\zeta_1(|\boldsymbol{\zeta}|^2-\frac{5}{2})$, and $\phi^A_{\mathrm{initial}}=0$. Then, ϕ^A is independent of $\boldsymbol{x}$ and approaches $(\sqrt{\pi}/2)\mathrm{Kn}\zeta_1 A(|\boldsymbol{\zeta}|)$ as $t\to\infty$, where $A(|\boldsymbol{\zeta}|)$ is the solution of $\mathscr{L}(\zeta_i A(|\boldsymbol{\zeta}|)) = -\zeta_i(|\boldsymbol{\zeta}|^2-\frac{5}{2})$ with $\langle|\boldsymbol{\zeta}|^2 A(|\boldsymbol{\zeta}|)\rangle=0$ [22–24]. On the other hand, let us denote by ϕ^B the solution of (1), (2), and (5b) with $h=0$, $I=0$, and $\phi^B_{\mathrm{initial}}=\zeta_1(|\boldsymbol{\zeta}|^2-\frac{5}{2})$. Then, ϕ^B is also independent of $\boldsymbol{x}$ and tends to vanish as $t\to\infty$. Now applying (12) to the pair of ϕ^A and ϕ^B and dividing the resulting by D_{p}, we obtain[5]

$$\mathrm{Sh}\langle\zeta_1(|\boldsymbol{\zeta}|-\frac{5}{2})\phi^A(t,\boldsymbol{\zeta})\rangle = \int_0^t \langle\zeta_1(|\boldsymbol{\zeta}|-\frac{5}{2})\phi^B(s,\boldsymbol{\zeta})\rangle \mathrm{d}s. \tag{29}$$

Since $\phi^B_{\mathrm{initial}}=\zeta_1(|\boldsymbol{\zeta}|^2-\frac{5}{2})$ and that $\phi^A\to(\sqrt{\pi}/2)\mathrm{Kn}\zeta_1 A(|\boldsymbol{\zeta}|)$ as $t\to\infty$, taking the limit $t\to\infty$ in (29) leads to

$$\lambda = 2(k/m)p_0 t_0\int_0^\infty \langle\phi^B_{\mathrm{initial}}\phi^B(t)\rangle \mathrm{d}t. \tag{30}$$

Here, $\lambda[\equiv\sqrt{\pi}p_0(2kT_0/m)^{-1/2}(k/m)\ell_0\langle\zeta_1^2(|\boldsymbol{\zeta}|-\frac{5}{2})A(|\boldsymbol{\zeta}|)\rangle]$ is the thermal conductivity of the gas. The relation (30) means that the thermal conductivity is expressed by the time-correlation in the relaxation problem from the initial perturbation $\zeta_1(|\boldsymbol{\zeta}|^2-\frac{5}{2})$. In this sense, (30) is similar to the fluctuation–dissipation theorem for the thermal conductivity.

The corresponding expression for the viscosity can be obtained in the same way. Let us denote by ϕ^A the solution of (1), (2), and (5b) with $h=0$, $I=2\zeta_1\zeta_2$, and $\phi^A_{\mathrm{initial}}=0$. Let

[5] For the BGK (or BKW in [22]) model, the relation (29) is easily verified because ϕ^A and ϕ^B can be solved explicitly as $\phi^A=(\sqrt{\pi}/2)\mathrm{Kn}\zeta_1(|\boldsymbol{\zeta}|^2-\frac{5}{2})[1-\exp(-(2/\sqrt{\pi})(t/\mathrm{KnSh}))]$ and $\phi^B=\zeta_1(|\boldsymbol{\zeta}|^2-\frac{5}{2})\exp(-(2/\sqrt{\pi})(t/\mathrm{KnSh}))$. In the same way, (31) is easily verified for the BGK model, because ϕ^A and ϕ^B are given by $\phi^A=\sqrt{\pi}\mathrm{Kn}\zeta_1\zeta_2[1-\exp(-(2/\sqrt{\pi})(t/\mathrm{KnSh}))]$ and $\phi^B=2\zeta_1\zeta_2\exp(-(2/\sqrt{\pi})(t/\mathrm{KnSh}))$.

Table 5 Additional Green functions for initial data and inhomogeneous term

Green function	corresponding element sources			
$G^{(\boldsymbol{\ell}\otimes\boldsymbol{k};I)}(t,\boldsymbol{x},\boldsymbol{\zeta})$	$g_{\mathrm{w}}=0$	$h=0$	$G^{(\boldsymbol{\ell}\otimes\boldsymbol{k};I)}_{\mathrm{initial}}=0$	$I=2\zeta_k\zeta_\ell$
$G^{(H_\ell;I)}(t,\boldsymbol{x},\boldsymbol{\zeta})$	$g_{\mathrm{w}}=0$	$h=0$	$G^{(H_\ell;I)}_{\mathrm{initial}}=0$	$I=\zeta_\ell(\lvert\boldsymbol{\zeta}\rvert^2-\frac{5}{2})$
$G^{(\boldsymbol{\ell}\otimes\boldsymbol{k})}(t,\boldsymbol{x},\boldsymbol{\zeta})$	$g_{\mathrm{w}}=0$	$h=0$	$G^{(\boldsymbol{\ell}\otimes\boldsymbol{k})}_{\mathrm{initial}}=2\zeta_\ell\zeta_k$	$I=0$
$G^{(H_\ell)}(t,\boldsymbol{x},\boldsymbol{\zeta})$	$g_{\mathrm{w}}=0$	$h=0$	$G^{(H_\ell)}_{\mathrm{initial}}=\zeta_\ell(\lvert\boldsymbol{\zeta}\rvert^2-\frac{5}{2})$	$I=0$

us denote by ϕ^B the solution of (1), (2), and (5b) with $h=0$, $I=0$, and $\phi^B_{\mathrm{initial}}=2\zeta_1\zeta_2$. Here, D is the periodic box D_{p}. Then, both ϕ^A and ϕ^B are independent of $\boldsymbol{x}$, and $\phi^A\to(\sqrt{\pi}/2)\mathrm{Kn}\zeta_1\zeta_2B(\lvert\boldsymbol{\zeta}\rvert)$ and $\phi^B\to 0$ as $t\to\infty$, where $B(\lvert\boldsymbol{\zeta}\rvert)$ is the solution of $\mathscr{L}(\zeta_1\zeta_2B(\lvert\boldsymbol{\zeta}\rvert))=-2\zeta_1\zeta_2$. From (12), we obtain

$$\mathrm{Sh}\langle 2\zeta_1\zeta_2\phi^A(t,\boldsymbol{\zeta})\rangle=\int_0^t\langle 2\zeta_1\zeta_2\phi^B(s,\boldsymbol{\zeta})\rangle\mathrm{d}s, \tag{31}$$

and taking the limit $t\to\infty$ in this relation leads to

$$\mu=p_0t_0\int_0^\infty\langle\phi^B_{\mathrm{initial}}\phi^B(t,\boldsymbol{\zeta})\rangle\mathrm{d}t, \tag{32}$$

where $\mu[\equiv\sqrt{\pi}p_0(2kT_0/m)^{-1/2}\ell_0\langle\zeta_1^2\zeta_2^2B(\lvert\boldsymbol{\zeta}\rvert)\rangle]$ is the viscosity of gas. Thus, the viscosity is expressed by the time correlation in the relaxation problem from the initial perturbation $2\zeta_1\zeta_2$ (the similarity to the fluctuation–dissipation theorem for the viscosity).

Motivated by the above observations, we consider correlations between the Green functions for initial data and inhomogeneous term listed in Tables 1 and 5 for any fixed bounded domain D (and for any Kn). By applying the symmetric relation (12) to the pair of $G^{(\alpha;I)}$ and $G^{(\alpha)}$ $(\alpha=\boldsymbol{\ell},\boldsymbol{\ell}\otimes\boldsymbol{k},H_\ell)$ and taking the limit $t\to\infty$, we especially obtain

$$\lim_{t\to\infty}\int_D\begin{bmatrix}\langle\zeta_\ell G^{(\boldsymbol{\ell};I)}\rangle(t,\boldsymbol{x})\\ \langle 2\zeta_\ell\zeta_k G^{(\boldsymbol{\ell}\otimes\boldsymbol{k};I)}\rangle(t,\boldsymbol{x})\\ \langle\zeta_\ell(\lvert\boldsymbol{\zeta}\rvert^2-\frac{5}{2})G^{(H_\ell;I)}\rangle(t,\boldsymbol{x})\end{bmatrix}\mathrm{d}\boldsymbol{x}=\mathrm{Sh}^{-1}\int_D\int_0^\infty\begin{bmatrix}\langle G^{(\boldsymbol{\ell})}_{\mathrm{initial}}G^{(\boldsymbol{\ell})}\rangle(s,\boldsymbol{x})\\ \langle G^{(\boldsymbol{\ell}\otimes\boldsymbol{k})}_{\mathrm{initial}}G^{(\boldsymbol{\ell}\otimes\boldsymbol{k})}\rangle(s,\boldsymbol{x})\\ \langle G^{(H_\ell)}_{\mathrm{initial}}G^{(H_\ell)}\rangle(s,\boldsymbol{x})\end{bmatrix}\mathrm{d}s\,\mathrm{d}\boldsymbol{x}. \tag{33}$$

It should be noted that, if D is the periodic box D_{p}, the first equation becomes a trivial identity,[6] while the second and third become the expression corresponding to the fluctuation–dissipation theorem for the viscosity and the thermal conductivity, respectively. The expression (33) may be regarded as a generalization of the expression corresponding to the fluctuation–dissipation theorem to gas systems in a fixed bounded domain for arbitrary Kn. Incidentally, if the domain D is the cross-section S of the straight pipe discussed in Example 2, the first equation of (33) with $\boldsymbol{\ell}=\mathbf{e}_1$ is identical to the first equation of (22).

In the same way, we can discuss the extension of the static admittance in the linear response theory for bulk systems to the systems of arbitrary Kn. That is, by applying (12) to the pairs of $G^{(\boldsymbol{\ell};I)}$ and $G^{(H_\ell)}$ and of $G^{(H_\ell)}$ and $G^{(\boldsymbol{\ell};I)}$ and then taking the limit $t\to\infty$, we obtain

$$\lim_{t\to\infty}\int_D\begin{bmatrix}\langle\zeta_\ell G^{(H_\ell;I)}\rangle(t,\boldsymbol{x})\\ \langle\zeta_\ell(\lvert\boldsymbol{\zeta}\rvert^2-\frac{5}{2})G^{(\boldsymbol{\ell};I)}\rangle(t,\boldsymbol{x})\end{bmatrix}\mathrm{d}\boldsymbol{x}=\mathrm{Sh}^{-1}\int_D\int_0^\infty\begin{bmatrix}\langle G^{(H_\ell)}_{\mathrm{initial}}G^{(\boldsymbol{\ell})}\rangle(s,\boldsymbol{x})\\ \langle G^{(\boldsymbol{\ell})}_{\mathrm{initial}}G^{(H_\ell)}\rangle(s,\boldsymbol{x})\end{bmatrix}\mathrm{d}s\,\mathrm{d}\boldsymbol{x}.$$

[6] When $D=D_{\mathrm{p}}$, $G^{(\boldsymbol{\ell};I)}=\zeta_\ell t/\mathrm{Sh}$ and $G^{(\boldsymbol{\ell})}=\zeta_\ell$.

The left-hand side is the static admittance of the systems against the weak external force represented by the inhomogeneous term for any fixed bounded domain D. On the other hand, application of (12) to the pair of $G^{(\boldsymbol{\ell})}$ and $G^{(H_\ell)}$ yields

$$\int_D \langle G^{(\boldsymbol{\ell})}_{\text{initial}} G^{(H_\ell)} \rangle \, \mathrm{d}\boldsymbol{x} = \int_D \langle G^{(H_\ell)}_{\text{initial}} G^{(\boldsymbol{\ell})} \rangle \, \mathrm{d}\boldsymbol{x}.$$

Thus, the above static admittances are reciprocal each other:

$$\lim_{t\to\infty} \int_D \langle \zeta_\ell G^{(H_\ell;I)} \rangle (t,\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \lim_{t\to\infty} \int_D \langle \zeta_\ell (|\boldsymbol{\zeta}|^2 - \frac{5}{2}) G^{(\boldsymbol{\ell};I)} \rangle (t,\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}. \tag{34}$$

When D is the periodic box D_{p}, both sides vanish[7] and (34) loses its meaning. However, in general, the above reciprocity is physically meaningful. To illustrate it, let us consider Example 2 again, in which the domain D is the cross-section of the straight pipe. Then (34) with $\boldsymbol{\ell} = \mathbf{e}_1$ is identical to (24) with $t \to \infty$. Thus, the latter is an example of the symmetry of the static admittance generalized to gas systems in a fixed bounded domain for arbitrary Kn. As is mentioned in Example 3, (24) with $t \to \infty$ is a known relation between the net mass flow of the steady thermal transpiration and the net heat flow of the steady Poiseuille flow, both of which are known to be induced as a gas rarefaction effect. The present discussion gives an alternative view of the reciprocity between the steady systems for arbitrary Kn.

7 Conclusion

In the present paper, we first established a symmetric relation between two problems described by the unsteady linearized Boltzmann equation for a gas in a fixed bounded domain. Then, we introduced several Green functions for the initial data, inhomogeneous term, and boundary data and derived general expressions for the total mass, momentum, and energy in the system and those for the mass, momentum, and heat fluxes through the boundary for any time in terms of the Green function. Several examples of the application of these expressions have been presented. Finally, the expressions similar to the fluctuation–dissipation theorem and to the static admittance in the linear response theory for bulk systems were presented. Further, their extensions to the systems of arbitrary Knudsen number were discussed.

A Linearized collision operator $\mathscr{L}$ and the kernels R and P

A.1 Properties of $\mathscr{L}$

1. $\mathscr{L}(\Phi)^- = \mathscr{L}(\Phi^-)$ for any Φ, where $\Psi^-(\boldsymbol{x},\boldsymbol{\zeta}) \equiv \Psi(\boldsymbol{x},-\boldsymbol{\zeta})$.
2. $\langle \Phi \mathscr{L}(\Psi) \rangle = \langle \Psi \mathscr{L}(\Phi) \rangle$ for any Φ and Ψ, where $\langle \Phi \rangle = \int \Phi(\boldsymbol{\zeta}) E(\boldsymbol{\zeta}) \, \mathrm{d}\boldsymbol{\zeta}$. (self-adjointness)
3. $\mathscr{L}(\Phi) = 0$ holds if and only if Φ is a linear combination of 1, $\boldsymbol{\zeta}$, and $|\boldsymbol{\zeta}|^2$.
4. $\langle \Phi \mathscr{L}(\Phi) \rangle \leq 0$ for any Φ, where the equality holds if and only if Φ is a linear combination of 1, $\boldsymbol{\zeta}$, and $|\boldsymbol{\zeta}|^2$.

A.2 Properties of R

We summarize the properties of R_{CR} and R_{PR} separately (see Appendix A.9 in [22]).

[7] When $D = D_{\mathrm{p}}$, $G^{(H_\ell;I)} \to (\sqrt{\pi}/2)\mathrm{Kn}\,\zeta_\ell A(|\boldsymbol{\zeta}|)$ as $t \to \infty$ and $G^{(\boldsymbol{\ell};I)} = \zeta_\ell t/\mathrm{Sh}$.

Properties of R_{CR}

1. $R_{\mathrm{CR}}(\boldsymbol{\zeta}^*,\boldsymbol{\zeta};\boldsymbol{x}) \geq 0 \quad$ for $\zeta_n^* < 0$, $\zeta_n > 0$. (non-negativity)
2. $\int_{\zeta_n>0} R_{\mathrm{CR}}(\boldsymbol{\zeta}^*,\boldsymbol{\zeta};\boldsymbol{x})\,\mathrm{d}\boldsymbol{\zeta} = 1 \quad$ for $\zeta_n^* < 0$. (condition of no mass flow across the boundary)
3. $|\zeta_n^*| R_{\mathrm{CR}}(\boldsymbol{\zeta}^*,\boldsymbol{\zeta};\boldsymbol{x}) E(\boldsymbol{\zeta}^*) = |\zeta_n| R_{\mathrm{CR}}(-\boldsymbol{\zeta},-\boldsymbol{\zeta}^*;\boldsymbol{x}) E(\boldsymbol{\zeta}) \quad$ for $\zeta_n > 0$, $\zeta_n^* < 0$. (the detailed balance)
4. Let φ be $\varphi = c_0 + c_i\zeta_i + c_4|\boldsymbol{\zeta}|^2$, where c_0, c_i, and c_4 are independent of $\boldsymbol{\zeta}$. Among such φ, only $\varphi = c_0$ satisfies the following relation: (uniqueness condition)

$$\varphi(\boldsymbol{x},\boldsymbol{\zeta})E(\boldsymbol{\zeta}) = \int_{\zeta_n^*<0} \frac{|\zeta_n^*|}{|\zeta_n|} R_{\mathrm{CR}}(\boldsymbol{\zeta}^*,\boldsymbol{\zeta};\boldsymbol{x})\varphi(\boldsymbol{x},\boldsymbol{\zeta}^*)E(\boldsymbol{\zeta}^*)\,\mathrm{d}\boldsymbol{\zeta}^* \quad \text{for } \zeta_n > 0.$$

Properties of R_{PR}

1. $R_{\mathrm{PR}}(\boldsymbol{\zeta}^*,\boldsymbol{\zeta};\boldsymbol{x}) \geq 0 \quad$ for $\zeta_n^* < 0$, $\zeta_n > 0$. (non-negativity)
2. There exists a given function $g_0(\boldsymbol{x},\boldsymbol{\zeta}) \geq 0$ defined in $\zeta_n > 0$ such that

$$E(\boldsymbol{\zeta}) = g_0(\boldsymbol{x},\boldsymbol{\zeta}) + \int_{\zeta_n^*<0} \frac{|\zeta_n^*|}{|\zeta_n|} R_{\mathrm{PR}}(\boldsymbol{\zeta}^*,\boldsymbol{\zeta};\boldsymbol{x})E(\boldsymbol{\zeta}^*)\,\mathrm{d}\boldsymbol{\zeta}^* \quad \text{for } \zeta_n > 0.$$

3. $|\zeta_n^*| R_{\mathrm{PR}}(\boldsymbol{\zeta}^*,\boldsymbol{\zeta};\boldsymbol{x}) E(\boldsymbol{\zeta}^*) = |\zeta_n| R_{\mathrm{PR}}(-\boldsymbol{\zeta},-\boldsymbol{\zeta}^*;\boldsymbol{x}) E(\boldsymbol{\zeta}) \quad$ for $\zeta_n > 0$, $\zeta_n^* < 0$. (the detailed balance)
4. Let φ be $\varphi = c_0 + c_i\zeta_i + c_4|\boldsymbol{\zeta}|^2$, where c_0, c_i, and c_4 are independent of $\boldsymbol{\zeta}$. Among such φ, only $\varphi = 0$ satisfies the following relation: (uniqueness condition)

$$\varphi(\boldsymbol{x},\boldsymbol{\zeta})E(\boldsymbol{\zeta}) = \int_{\zeta_n^*<0} \frac{|\zeta_n^*|}{|\zeta_n|} R_{\mathrm{PR}}(\boldsymbol{\zeta}^*,\boldsymbol{\zeta};\boldsymbol{x})\varphi(\boldsymbol{x},\boldsymbol{\zeta}^*)E(\boldsymbol{\zeta}^*)\,\mathrm{d}\boldsymbol{\zeta}^* \quad \text{for } \zeta_n > 0.$$

A.3 Properties of P

(a) $P(\boldsymbol{x}',\boldsymbol{\zeta}',\boldsymbol{x},\boldsymbol{\zeta}) \geq 0$ and is not identically zero.
(b) $|\zeta_n| E(\boldsymbol{\zeta}) P(\boldsymbol{x}',\boldsymbol{\zeta}',\boldsymbol{x},\boldsymbol{\zeta}) = |\zeta'_{n'}| E(\boldsymbol{\zeta}') P(\boldsymbol{x},-\boldsymbol{\zeta},\boldsymbol{x}',-\boldsymbol{\zeta}') \quad$ for $\zeta_n > 0$ and $\zeta'_{n'} < 0$.
(c) There exists a given function $g_0(\boldsymbol{x},\boldsymbol{\zeta}) \geq 0$ defined in $\zeta_n > 0$ and $\boldsymbol{x} \in \partial D_{\mathrm{g}}^{(2)}$ such that

$$1 = g_0(\boldsymbol{x},\boldsymbol{\zeta}) + \int_{\partial D_{\mathrm{g}}^{(2)}} \int_{\zeta'_{n'}<0} P(\boldsymbol{x}',\boldsymbol{\zeta}',\boldsymbol{x},\boldsymbol{\zeta})\,\mathrm{d}\boldsymbol{\zeta}'\,\mathrm{dS}' \quad \text{for } \zeta_n > 0.$$

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