SIAM J. DISCRETE MATH.
Vol. 36, No. 2, pp. 1102–1123

SHORTEST RECONFIGURATION OF PERFECT MATCHINGS VIA ALTERNATING CYCLES*

TAKEHIRO ITO†, NAONORI KAKIMURA‡, NAOYUKI KAMIYAMA§, YUSUKE KOBAYASHI¶, AND YOSHIO OKAMOTO‖

Abstract. Motivated by adjacency in perfect matching polytopes, we study the shortest reconfiguration problem of perfect matchings via alternating cycles. Namely, we want to find a shortest sequence of perfect matchings which transforms one given perfect matching to another given perfect matching such that the symmetric difference of each pair of consecutive perfect matchings is a single cycle. The problem is equivalent to the combinatorial shortest path problem in perfect matching polytopes. We prove that the problem is NP-hard even when a given graph is planar or bipartite, but it can be solved in polynomial time when the graph is outerplanar.

Key words. graph algorithms, matching, combinatorial reconfiguration, combinatorial shortest paths

AMS subject classifications. 05C85, 52B05

DOI. 10.1137/20M1364370

1. Introduction.

Combinatorial reconfiguration is a fundamental research subject that sheds light on solution spaces of combinatorial (search) problems and connects various concepts, such as optimization, counting, enumeration, and sampling. In its general form, combinatorial reconfiguration is concerned with properties of the configuration space of a combinatorial problem. The configuration space of a combinatorial problem is often represented as a graph, but its size is usually exponential in the instance size. Thus, algorithmic problems on combinatorial reconfiguration are not trivial and require novel tools for resolution. For recent surveys, see [28, 15].

Two basic questions have been encountered in the study of combinatorial reconfiguration. The first question concerns the existence of a path between two given solutions in the configuration space, namely, the *reachability* of the two solutions. The second question concerns the shortest length of a path between two given solutions,

*Received by the editors September 3, 2020; accepted for publication (in revised form) November 2, 2021; published electronically April 28, 2022. A preliminary version has appeared in the Proceedings of the 27th Annual European Symposium on Algorithms (ESA 2019).

https://doi.org/10.1137/20M1364370

Funding: The first author was partially supported by JST CREST grant JPMJCR1402 and JSPS KAKENHI grants JP18H04091, JP19K11814, and JP20H05793, Japan. The second author was partially supported by JSPS KAKENHI grants JP17K00028, JP18H05291, and JP20H05795, Japan. The third author was partially supported by JST PRESTO grant JPMJPR1753, Japan. The fourth author was partially supported by JSPS KAKENHI grants JP16K16010, JP17K19960, JP18H05291, and JP20H05795, Japan. The fifth author was partially supported by JSPS KAKENHI grants JP15K00009, JP20K11679, and JP20H05795; JST CREST grant JPMJCR1402; and the Kayamori Foundation of Informational Science Advancement, Japan.

†Graduate School of Information Sciences, Tohoku University, Sendai, Miyagi, 980-8579, Japan (takehiro@tohoku.ac.jp).

‡Faculty of Science and Technology, Keio University, Yokohama, 223-8522, Japan (kakimura@math.keio.ac.jp).

§Institute of Mathematics for Industry, Kyushu University, Fukuoka, 819-0395, Japan (kamiyama@imi.kyushu-u.ac.jp).

¶Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606-8502, Japan (yusuke@kurims.kyoto-u.ac.jp).

‖Graduate School of Informatics and Engineering, University of Electro-Communications, Chofu, Tokyo, 182-8585, Japan (okamotoy@uec.ac.jp).

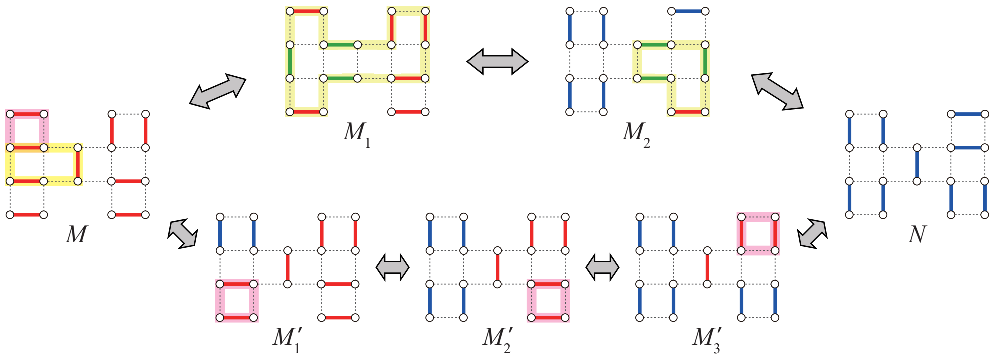


Fig. 1. *Two sequences of perfect matchings between M and N under the alternating cycle model. The sequence $\langle M, M_1, M_2, N\rangle$ is shortest even though it touches the edge in $M \cap N$ twice. On the other hand, $\langle M, M_1', M_2', M_3', N\rangle$ is not shortest, although it touches only the edges in $M \triangle N$.*

if it exists. The second question is usually referred to as a *shortest reconfiguration problem*.

In this paper, we focus on reconfiguration problems of matchings, namely, sets of independent edges. There are several ways of defining the configuration space for matchings, and some of them have already been studied in the literature [16, 20, 14, 6, 4]. We will explain them in section 1.1.

We study yet another configuration space for matchings which we call the *alternating path/cycle* model. The model is motivated by adjacency in matching polytopes, which we will see soon. In the model, we are given an undirected and unweighted graph G and also an integer $k \geq 0$. The vertex set of the configuration space consists of the matchings in G of size at least k. Two matchings M and N in G are adjacent in the configuration space if and only if their symmetric difference $M \triangle N := (M \cup N) \setminus (M \cap N)$ is a single path or cycle. In particular, we are interested in the case where $k = |V(G)|/2$, namely, the reconfiguration of *perfect matchings*. In that case, the model is simplified to the *alternating cycle* model since $M \triangle N$ cannot have a path. See Figure 1 as an example.

The reachability of two perfect matchings is trivial under the alternating cycle model: The answer is always yes. This is because the symmetric difference of two perfect matchings always consists of vertex-disjoint cycles. Therefore, we focus on the shortest perfect matching reconfiguration under the alternating cycle model.

1.1. Related work.

Other configuration spaces for matchings. As mentioned, reconfiguration problems of matchings have already been studied under different models [16, 20, 14, 6, 4]. These models chose more elementary changes as the adjacency on the configuration space. Then the situation changes drastically under such models: even the reachability of two matchings is not guaranteed.

Matching reconfiguration was initiated by the work of Ito et al. [16]. They proposed the *token addition/removal* model of reconfiguration, in which we are also given an integer $k \geq 0$, and the vertex set of the configuration space consists of the matchings of size at least k.[1] Two matchings M and N are adjacent if and only if they differ

[1]Precisely, their model is defined in a slightly different way, but it is essentially the same as this definition.

in only one edge. Ito et al. [16] proved that the reachability of two given matchings can be checked in polynomial time.

Another model of reconfiguration is *token jumping*, introduced by Kamiński et al. [20]. In the token jumping model, we are also given an integer $k \geq 0$, and the vertex set of the configuration space consists of the matchings of size exactly k. Two matchings M and N are adjacent if and only if they differ in only two edges. Kamiński et al. [20, Theorem 1] proved that the token jumping model is equivalent to the token addition/removal model when two given matchings have the same size. Thus, using the result by Ito et al. [16], the reachability can be checked in polynomial time also under the token jumping model [20, Corollary 2].

On the other hand, the shortest matching reconfiguration is known to be hard. Gupta et al. [14] and Bousquet et al. [6] independently proved that the problem is NP-hard under the token jumping model. Then the problem is also NP-hard under the token addition/removal model because the shortest lengths are preserved under the two models [20, Theorem 1].

Recently, Bonamy et al. [4] studied the reachability of two perfect matchings under a model close to ours, namely, the alternating cycle model *restricted to length four*. In the model, two perfect matchings M and N are adjacent if and only if their symmetric difference $M \triangle N$ is a cycle of length four. Then the answer to the reachability is not always yes, and Bonamy et al. [4] proved that the reachability problem is PSPACE-complete under this restricted model.

Relation to matching polytopes. Our alternating cycle model (without any restriction of cycle length) for the perfect matching reconfiguration is natural when we see the connection with the simplex methods for linear optimization, or combinatorial shortest paths of the graphs of convex polytopes.

In the combinatorial shortest path problem of a convex polytope, we are given a convex polytope P, explicitly or implicitly, and two vertices v, w of P. Then we want to find a shortest sequence $u_0, u_1, \ldots, u_t$ of vertices of P such that $u_0 = v, u_t = w$ and $\overline{u_i u_{i+1}}$ forms an edge of P for every $i = 0, 1, \ldots, t-1$. Often, we are only interested in the length of such a shortest sequence, and we are also interested in the maximum shortest path length among all pairs of vertices, which is known as the combinatorial diameter of the polytope P. The combinatorial diameter of a polytope has attracted much attention in the optimization community from the motivation of better understanding of simplex methods. Simplex methods for linear optimization start at a vertex of the feasible region, follow edges, and arrive at an optimal vertex. Therefore, the combinatorial diameter dictates the best-case behavior of such methods. The famous Hirsch conjecture states that every d-dimensional convex polytope with n facets has the combinatorial diameter at most $n - d$. This has been disproved by Santos [34], and the current best upper bound of $(n-d)^{\log_2 O(d/\log d)}$ for the combinatorial diameter was given by Sukegawa [35]. On the other hand, for 0/1-polytopes (i.e., polytopes in which the coordinates of all vertices belong to $\{0, 1\}$), the Hirsch conjecture holds [27].

The shortest perfect matching reconfiguration under the alternating cycle model can be seen as the combinatorial shortest path problem of a perfect matching polytope. The *perfect matching polytope* of a graph G is defined as follows. The polytope lives in $\mathbb{R}^{E(G)}$; namely, each coordinate corresponds to an edge of G. Each vertex v of the polytope corresponds to a perfect matching M of G as $v_e = 1$ if $e \in M$ and $v_e = 0$ if $e \notin M$. The polytope is defined as the convex hull of those vertices. It is known that two vertices u, v of the perfect matching polytope form an edge if and only if the corresponding perfect matchings M, N have the property that $M \triangle N$ contains only one cycle [9]. This means that the graph of the perfect matching polytope is exactly the configuration space for perfect matchings under the alternating cycle model.

Further related work. As mentioned before, the matching reconfiguration has been studied by several authors [16, 20, 14, 6, 4]. Extension to b-matchings has been considered, too [26, 17].

Shortest reconfiguration has attracted considerable attention. Starting from an old work on the 15-puzzle [32], we see the work on pancake sorting [8], triangulations of point sets [22, 30] and simple polygons [2] under flip distances, and also independent set reconfigurations [36], satisfiability reconfiguration [25], coloring reconfiguration [19], and token swapping problems [38, 24, 39, 5, 37, 21]. A tantalizing open problem is to determine the complexity of computing the rotation distance of two rooted binary trees (or equivalently the flip distance of two triangulations of a convex polygon, or the combinatorial shortest path of an associahedron).

The computational aspect of the combinatorial shortest path problem on convex polytopes is not well investigated. It is known that the combinatorial diameter is hard to determine [11] even for fractional matching polytopes [33]. In the literature, we find many papers on the adjacency of convex polytopes arising from combinatorial optimization problems [13, 23, 3, 10]. Among others, Papadimitriou [29] proved that determining whether two given vertices are adjacent in a traveling salesman polytope is coNP-complete. This implies that computing the combinatorial shortest path between two vertices of a traveling salesman polytope is NP-hard. However, to the best of the authors' knowledge, all known combinatorial polytopes with such adjacency hardness stem from NP-hard combinatorial optimization problems, and the associated polytopes have exponentially many facets. We also point out the work on a randomized algorithm to compute a combinatorial "short" path [7].

1.2. Our contribution. To the best of the authors' knowledge, known results under different models do not have direct relations to our alternating cycle model because their configuration spaces are different. In this paper, we thus investigate the polynomial-time solvability of the shortest perfect matching reconfiguration under the alternating cycle model. The results of our paper are twofold.

1. The shortest perfect matching reconfiguration under the alternating cycle model can be solved in polynomial time if the input graph is outerplanar.
2. The shortest perfect matching reconfiguration under the alternating cycle model is NP-hard even when the input graph is planar or bipartite.

Since outerplanar graphs form a natural and fundamental subclass of planar graphs, our results exhibit a tractability border among planar graphs.

The hardness result for bipartite graphs implies that the computation of a combinatorial shortest path in a convex polytope is NP-hard even when an inequality description is explicitly given. This is because a polynomial-size inequality description of the perfect matching polytope can be explicitly written down from a given bipartite graph.

We point out that the hardness results have been independently obtained by Aichholzer et al. [1]. Indeed, they proved the hardness for planar bipartite graphs (i.e., an input graph is planar *and* bipartite).

Technical key points. Compared to recent algorithmic developments on reachability problems, only a few polynomial-time solvable cases are known for shortest reconfiguration problems. We now explain two technical key points, especially for algorithmic results on shortest reconfiguration problems.

The first point is the symmetric difference of two given solutions. Under several known models (not only for matchings) that employ elementary changes as the adjacency on the configuration space, the symmetric difference gives a (good) lower bound on the shortest reconfiguration. This is because any reconfiguration sequence (i.e., a path in the configuration space) between two given solutions must touch all elements in their symmetric difference at least once. For example, in Figure 1, the symmetric difference of two perfect matchings M and N consists of 16 edges, and hence it gives the lower bound of $16/4 = 4$ under the alternating cycle model restricted to length four [4]. In such a case, if we can find a reconfiguration sequence touching only the elements in the symmetric difference (e.g., the sequence $\langle M, M_1', M_2', M_3', N\rangle$ in Figure 1), then it is automatically the shortest under that model. However, this useful property does not hold under our alternating cycle model because the length of an alternating cycle for reconfiguration is not fixed.

The second point is the characterization of *unhappy moves* that touch elements contained commonly in two given solutions. For example, the shortest reconfiguration sequence $\langle M, M_1, M_2, N\rangle$ in Figure 1 has an unhappy move since it touches the edge in $M \cap N$ twice. In general, analyzing a shortest reconfiguration becomes much more difficult if such unhappy moves are required. A well-known example is the (generalized) 15-puzzle [32], in which the reachability can be determined in polynomial time, while the shortest reconfiguration is NP-hard. As illustrated in Figure 1, the shortest perfect matching reconfiguration requires unhappy moves even for outerplanar graphs, and hence we need to characterize the unhappy moves to develop a polynomial-time algorithm.

2. Problem definition. In this paper, a graph always refers to an undirected graph that might have parallel edges and does not have loops. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. An edge subset $M \subseteq E$ is called a *matching* in G if no two edges in M share the end vertices. A matching M is *perfect* if $|M| = |V(G)|/2$.

A graph is *planar* if it can be drawn on the plane without edge crossing. Such a drawing is called a *plane* drawing of the planar graph. A *face* of a plane drawing is a maximal region of the plane that contains no point used in the drawing. There is a unique unbounded face which is called the *outer face*. A planar graph is *outerplanar* if it has an *outerplane* drawing, i.e., a plane drawing in which all vertices are incident to the outer face.

For a matching M in a graph G, a cycle C in G is called M*-alternating* if edges in M and $E(G) \setminus M$ alternate in C. We identify a cycle with its edge set to simplify the notation. We say that two perfect matchings M and N are *reachable* (under the alternating cycle model) if there exists a sequence $\langle M_0, M_1, \ldots, M_t\rangle$ of perfect matchings in G such that

(i) $M_0 = M$ and $M_t = N$;

(ii) $M_i = M_{i-1} \triangle C_i$ for some M_{i-1}-alternating cycle C_i for each $i = 1, \ldots, t$.

Such a sequence is called a *reconfiguration sequence* between M and N, and its *length* is defined as t.

For two perfect matchings M and N, the subgraph $M \triangle N$ consists of disjoint M-alternating cycles $C_1, \ldots, C_t$. Thus, it is clear that M and N are always reachable for any two perfect matchings M and N by setting $M_i = M_{i-1} \triangle C_i$ for $i = 1, \ldots, t$. In this paper, we are interested in finding a *shortest* reconfiguration sequence of perfect matchings. That is, the problem is defined as follows:

SHORTEST PERFECT MATCHING RECONFIGURATION
Input: A graph G and two perfect matchings M and N in G
Find: A shortest reconfiguration sequence between M and N.

We denote by a tuple $I = (G, M, N)$ an instance of SHORTEST PERFECT MATCHING RECONFIGURATION. Also, we denote by $\mathrm{OPT}(I)$ the length of a shortest reconfiguration sequence of an instance I. We note that it may happen that $\mathrm{OPT}(I)$ is much shorter than the number of disjoint M-alternating cycles in $M \triangle N$ (see Figure 1).

3. Polynomial-time algorithm for outerplanar graphs.

In this section, we prove that there exists a polynomial-time algorithm for SHORTEST PERFECT MATCHING RECONFIGURATION on an outerplanar graph as follows.

THEOREM 3.1. SHORTEST PERFECT MATCHING RECONFIGURATION *on outerplanar graphs G can be solved in $O(|V(G)|^5)$ time.*

We give such an algorithm in this section. Let $I = (G, M, N)$ be an instance of the problem such that $G = (V, E)$ is an outerplanar graph. We first observe that it suffices to consider the case when G is 2-connected.

LEMMA 3.2. *Let $I = (G, M, N)$ be an instance of* SHORTEST PERFECT MATCHING RECONFIGURATION *and $G_1, \dots, G_p$ be the 2-connected components of G. Furthermore, for every $i = 1, \dots, p$, let $I_i = (G_i, M \cap E(G_i), N \cap E(G_i))$ be an instance of* SHORTEST PERFECT MATCHING RECONFIGURATION. *Then* $\mathrm{OPT}(I) = \sum_{i=1}^{p} \mathrm{OPT}(I_i)$.

Proof. Let $G_1, \dots, G_p$ be 2-connected components in G. Then, since any M'-alternating cycle is contained in some G_i for a perfect matching M' of G, it suffices to solve the problem for each G_i. Specifically, it holds that $\mathrm{OPT}(I) = \sum_{i=1}^{p} \mathrm{OPT}(I_i)$, where $I_i = (G_i, M \cap E(G_i), N \cap E(G_i))$. ∎

Since the 2-connected components of a graph can be found in linear time, the reduction to 2-connected outerplanar graphs can be done in linear time, too.

We fix an outerplane drawing of a given 2-connected outerplanar graph G and identify G with the drawing for the sake of convenience. We denote by C_{out} the outer face boundary. Then C_{out} is a cycle since G is 2-connected. We denote the set of the *inner edges of* G by $E_{\mathrm{in}} = E \setminus C_{\mathrm{out}}$. In other words, E_{in} is the set of chords of C_{out}.

3.1. Technical highlight.

As mentioned in the introduction, two technical results are required to develop a polynomial-time algorithm for SHORTEST PERFECT MATCHING RECONFIGURATION: a lower bound on the length of a shortest reconfiguration sequence and the characterization of unhappy moves. We here explain our ideas roughly and will give detailed descriptions in the next subsections.

Since G is planar, we can define its "dual-like" graph G^*. Then G^* forms a tree since G is outerplanar and 2-connected. (The definition of G^* will be given in section 3.2, and an example is given in Figure 2.) We make a correspondence between an edge in G^* and a set of edges in G. Then we will define the length $\ell(e^*)$ of each edge e^* in G^* so that it represents the "gap" between M and N when we are restricted to the edges in the corresponding set of e^*. It is important to notice that any cycle C in G corresponds to a subtree of G^*, and vice versa. Indeed, we focus on a cut C^* of G^* clipping the subtree from G^*, that is, the set of edges in G^* leaving the subtree. If we apply an M-alternating cycle C to a perfect matching M of G, then it changes lengths $\ell(e^*)$ of the edges e^* in the corresponding cut C^*.

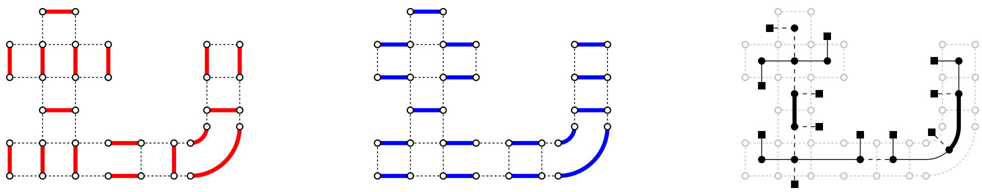
For our algorithm, we need a (good) lower bound for the length of a shortest reconfiguration sequence between two given perfect matchings M and N. Recall that $|M \triangle N|$ does not give a good lower bound under the alternating cycle model. This is because we can take a cycle of an arbitrary (non-fixed) length, and hence $|M \triangle N|$ can decrease drastically by only a single alternating cycle. Furthermore, no matter how we define the length $\ell(e^*)$ of each edge e^* in G^*, the total length of *all* edges in G^* does not give a good lower bound. This is because a cycle C of nonfixed length in G may correspond to a cut C^* having many edges in G^*, and hence it can change the total length drastically. Our key idea is to focus on the total length of each *path* in G^*; that is, we take the *diameter* of G^* (with respect to length ℓ) as a lower bound. Then, because G^* is a tree, any path in G^* can contain at most two edges from the corresponding cut C^*. Therefore, regardless of the cycle length, the diameter of G^* can be changed by only these two edges. By carefully setting the length $\ell(e^*)$ as in (1), we will prove that the diameter of G^* is not only a lower bound but indeed gives the shortest length under the assumption that $E_{\text{in}} \cap M \cap N$ is empty. Therefore, the real difficulty arises when $E_{\text{in}} \cap M \cap N$ is not empty.

In the latter case, we will characterize the unhappy moves. Assume that we know the set $F \subseteq E_{\text{in}} \cap M \cap N$ of chords that are *not* touched in a shortest reconfiguration sequence between M and N; in other words, *all* chords in $(E_{\text{in}} \cap M \cap N) \setminus F$ must be touched for unhappy moves in that sequence. Then we subdivide a given outerplanar graph G into subgraphs $G_1, \ldots, G_{|F|+1}$ along the chords in F. Notice that each edge in F appears on the outer face boundaries in two of these subgraphs. Furthermore, each chord e in these subgraphs satisfies $e \in (E_{\text{in}} \cap M \cap N) \setminus F$ if $e \in M \cap N$. Therefore, *all* chords in these subgraphs are touched for unhappy moves as long as they are in $M \cap N$. Under this assumption, we will prove that the diameter of G_i^* gives the shortest length of a reconfiguration sequence between $M \cap E(G_i)$ and $N \cap E(G_i)$. Thus, we can solve the problem in polynomial time if we know F, which yields a shortest reconfiguration sequence between M and N. Finally, to find such a set F of chords, we construct a polynomial-time algorithm which employs a dynamic programming method along the tree G^*.

3.2. Preliminaries: Constructing a dual graph. Let $I = (G, M, N)$ be an instance of SHORTEST PERFECT MATCHING RECONFIGURATION such that G is a 2-connected outerplanar graph. Since G is planar, we can define the *dual* of G. In fact, we here construct a graph G^* obtained from the dual by applying a slight modification as follows. The construction is illustrated in Figure 2. Let V^* be the set of faces (without the outer face) of G. For a face $v^* \in V^*$, let $E_{v^*} \subseteq E(G)$ be the set of edges around v^*. We denote the set of faces touching the outer face by U^*, i.e., $U^* = \{v^* \in V^* \mid E_{v^*} \cap C_{\text{out}} \neq \emptyset\}$. We make a copy of U^*, denoted by $\tilde{U}^*$. We set the vertex set of G^* to be $V^* \cup \tilde{U}^*$. For v^*, w^* in V^*, an edge v^*w^* in G^* exists if and only if the faces v^* and w^* share an edge in E_{in}, i.e., $|E_{v^*} \cap E_{w^*}| = 1$. Also, G^* has an edge between u^* and $\tilde{u}^*$ for every $u^* \in U^*$, where $\tilde{u}^* \in \tilde{U}^*$ is the copy of u^*. Thus the edge set of G^* is given by

$$E(G^*) = \{v^*w^* \mid v^*, w^* \in V^*,\ |E_{v^*} \cap E_{w^*}| = 1\} \cup \{u^*\tilde{u}^* \mid u^* \in U^*\}.$$

The first part is denoted by E^*_{in}, and the second part is denoted by $\tilde{E}^*$. We observe that G^* is a tree since G is 2-connected and outerplanar. A face of G that touches only one face (other than the outer face) is called a *leaf in* $G^* - \tilde{U}^*$. We note that there is a one-to-one correspondence between edges in E_{in} of G and E^*_{in} of G^*. Thus, for each $e \in E_{\text{in}}$, we denote by e^* the corresponding edge in E^*_{in}, and vice versa. We



(a) G and M. (b) G and N. (c) G^*.

FIG. 2. *The construction of G^* and the length function ℓ. In* (c), *the edge lengths are depicted by different styles: Thick solid lines represent edges of length two, thin solid lines represent edges of length one, and dotted lines represent edges of length zero.*

extend this correspondence to $\tilde{E}^*$; that is, $u^*\tilde{u}^* \in \tilde{E}^*$ corresponds to the edge set $E_{u^*} \cap C_{\text{out}}$ for $u^* \in U^*$, and vice versa.

It follows from the duality that there is a relationship between a cut in G^* and a cycle in G. Suppose that we are given a cycle C in G. Then, since G is outerplanar, the cycle C surrounds a set X^* of faces such that X^* does not have the outer face. The set X^* induces a connected graph (subtree) in G^*, and the set of edges leaving from X^* yields a cut $C^* = \{e^* = v^*w^* \mid v^* \in X^*, w^* \in V(G^*) \setminus X^*\}$. Conversely, let $X^* \subseteq V^*$ be a vertex subset of G^* such that the subgraph induced by X^* is connected. Then the set of edges leaving from X^* yields a cut C^* in G^*, which corresponds to a cycle in G.

We classify faces in U^* into two groups. For a face u^* in U^*, the edge set $E_{u^*} \cap C_{\text{out}}$ forms a family $\mathcal{P}_{u^*}$ of disjoint paths. Since M and N are perfect matchings, each path P in $\mathcal{P}_{u^*}$ is both M-alternating and N-alternating. In addition, P satisfies either

(i) $E(P) \subseteq M \triangle N$ or

(ii) $(M \triangle N) \cap E(P) = \emptyset$.

Furthermore, we observe that either (i) holds for *every* path P in $\mathcal{P}_{u^*}$ or (ii) holds for *every* path P in $\mathcal{P}_{u^*}$. Indeed, since $M \triangle N$ consists of disjoint cycles, if some path P in $\mathcal{P}_{u^*}$ satisfies (i), then P is included in a cycle C in $M \triangle N$ that separates u^* from the outer face. Since the other paths in $\mathcal{P}_{u^*}$ touch the outer face, they are on C. Thus, every path satisfies (i), which shows the observation. We divide U^* into two groups U_1^* and U_2^*, where each face in U_1^* satisfies (i) for every path, while each face in U_2^* satisfies (ii) for every path.

For an edge e^* in $E(G^*)$, we define the length $\ell(e^*)$ to be

$$\ell(e^*) = \begin{cases} |M \cap \{e\}| + |N \cap \{e\}| & \text{if } e^* \in E^*_{\text{in}}; \\ 1 & \text{if } e^* = u^*\tilde{u}^* \in \tilde{E}^* \text{ such that } u^* \in U_1^*; \\ 0 & \text{if } e^* = u^*\tilde{u}^* \in \tilde{E}^* \text{ such that } u^* \in U_2^*. \end{cases} \tag{1}$$

See Figure 2 for an example. Let $\ell(u^*, v^*)$ be the length of the (unique) path from u^* to v^* in G^*. We define the *gap* between M and N in the graph G as the diameter of G^*; that is, we define

$$\text{gap}(I) = \max\{\ell(u^*, v^*) \mid u^*, v^* \in V(G^*)\}.$$

This value is simply denoted by $\text{gap}(M, N)$ if G is clear from the context.

3.3. Characterization for the disjoint case. Let $I = (G, M, N)$ be an instance of SHORTEST PERFECT MATCHING RECONFIGURATION such that G is a 2-connected outerplanar graph. In this subsection, we show that if $E_{\text{in}} \cap M \cap N$ is empty, we can characterize the optimal value with $\mathrm{gap}(I)$, which leads to a simple polynomial-time algorithm for this case. We note that if $E_{\mathrm{in}} \cap M \cap N$ is empty, then no edge in E_{in} belongs to both M and N, and hence $\ell(e^*)$ can only take values 0 or 1.

LEMMA 3.3. *It holds that* $\mathrm{gap}(M, N)$ *is even.*

Proof. Consider a path P^* whose length is equal to $\mathrm{gap}(M, N)$ in G^*. We may assume that the end vertices of P^* are in $\tilde{U}^*$, as otherwise we can extend the path to some vertex in $\tilde{U}^*$ without decreasing the length. Let $\tilde{u}, \tilde{v} \in \tilde{U}^*$ be the end vertices of P^*. This means that the faces u and v touch the outer face. Take arbitrary edges $e_u \in E_u \cap C_{\mathrm{out}}$ and $e_v \in E_v \cap C_{\mathrm{out}}$. Then $(P \cap E_{\mathrm{in}}) \cup \{e_u, e_v\}$ forms a cut C in G by the duality. By the definition of ℓ, for $w \in \{u, v\}$, it holds that $\ell(w, \tilde{w}) = 0$ if and only if $|M \cap \{e_w\}| = |N \cap \{e_w\}|$. Hence, the parity of $\sum_{e^* \in E(P^*)} \ell(e^*)$ is the same as that of $|M \cap C| + |N \cap C|$. Since M and N are perfect matchings, the parities of $|M \cap C|$ and $|N \cap C|$ are the same. Therefore, $|M \cap C| + |N \cap C|$ is even, and thus $\mathrm{gap}(M, N)$ is also even. ◻

A main theorem of this subsection is to give a characterization of the optimal value with $\mathrm{gap}(M, N)$.

THEOREM 3.4. *Let* $I = (G, M, N)$ *be an instance of* SHORTEST PERFECT MATCHING RECONFIGURATION *such that* G *is a* 2*-connected outerplanar graph. If* $E_{\mathrm{in}} \cap M \cap N$ *is empty, then it holds that* $\mathrm{OPT}(I) = \mathrm{gap}(M, N)/2$.

Proof. To show the theorem, we first prove the following claim.

CLAIM 1. *For any* M*-alternating cycle* C*, it holds that*

$$\mathrm{gap}(M, N) \leq \mathrm{gap}(M \triangle C, N) + 2.$$

Proof of Claim 1. By the duality, the cycle C in G corresponds to a cut C^* in G^* such that the inside is connected. Such a cut intersects with any path in G^* at most twice, as G^* is a tree, and only the intersected edges can change the length by one. Therefore, the distance can be decreased by at most 2. ◻

Consider a shortest reconfiguration sequence $\langle M_0, M_1, \ldots, M_t \rangle$ from $M_0 = M$ to $M_t = N$. Then $t = \mathrm{OPT}(I)$. For each $i = 1, \ldots, t$, it then holds that $\mathrm{gap}(M_{i-1}, N) \leq \mathrm{gap}(M_i, N) + 2$. By repeatedly applying the above inequalities, we obtain

$$\mathrm{gap}(M, N) = \mathrm{gap}(M_0, N) \leq \mathrm{gap}(M_t, N) + 2t = 2t = 2\mathrm{OPT}(I)$$

since $\mathrm{gap}(M_t, N) = 0$. Hence, it holds that $\mathrm{OPT}(I) \geq \mathrm{gap}(M, N)/2$.

It remains to show that $\mathrm{OPT}(I) \leq \mathrm{gap}(M, N)/2$. We prove the following claim.

CLAIM 2. *There exists an* M*-alternating cycle* C *such that*

$$\mathrm{gap}(M, N) = \mathrm{gap}(M \triangle C, N) + 2. \tag{2}$$

Proof of Claim 2. We prove the claim by induction on the number of edges.

We first observe that we may assume that $E_{\mathrm{in}} \setminus (M \cup N) = \emptyset$. Otherwise, we can just delete all the edges in $E_{\mathrm{in}} \setminus (M \cup N)$ and apply the induction to find an M-alternating cycle C that satisfies (2) for the modified graph. Since the deleted edges are neither in M nor in N, by (1) the deletion does not change the gap. Thus, C is a desired cycle in G as well. Therefore, we may assume that all the edges in E^*_{in} have length one.

In addition, we may assume that any leaf u^* in $G^* - \tilde{U}^*$ belongs to U^*_1. In other words, M and N are distinct in $E_{u^*} \cap C_{\mathrm{out}}$. Indeed, suppose that there exists a leaf

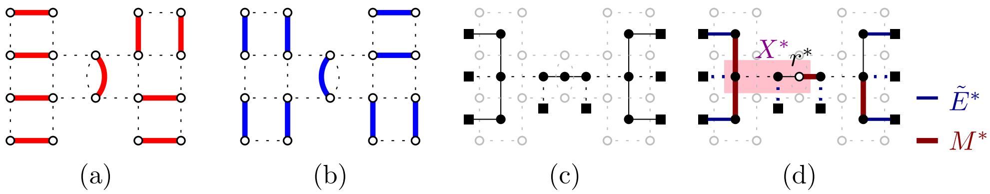


FIG. 3. *Illustration of the proof of Claim* 2. (a) *The perfect matching* M *is shown in bold red.* (b) *The perfect matching* N *is shown in bold blue.* (c) *The graph* G^*. (d) *The center* r *and the chosen set* X^*.

u^* in U_2^*. Then $\ell(u^*, \tilde{u}^*) = 0$. Since any chord is in either M or N by the above observation and the assumption that $E_{\text{in}} \cap M \cap N = \emptyset$, $\ell(u^*, v^*) = 1$, where v^* is the unique neighbor to u^* in $G^* - \tilde{U}^*$. We delete $E_{u^*} \setminus E_{\text{in}}$ from G, M, and N and then delete all the isolated vertices. We denote the obtained graph by G'. This corresponds to deleting the leaf u^* with $\tilde{u}^*$ from G^* and adding $\tilde{v}^*$ to G^* if necessary. We can see that, in the modified graph $(G')^*$, we have $\ell(v^*, \tilde{v}^*) = 1$, as $E_{u^*} \cap E_{v^*}$ is in either M or N. Hence, this deletion preserves $\text{gap}(M, N)$. We then apply the induction to G' to find an M-alternating cycle C that satisfies (2). This cycle is a desired one in G. Thus, we may assume that any leaf u^* in $G^* - \tilde{U}^*$ belongs to U_1^*.

Since $\text{gap}(M, N)$ is even by Lemma 3.3, we have $\text{gap}(M, N) = 2d$ for some positive integer d. Let $u_1^*, u_2^* \in V(G^*)$ be a pair of vertices such that $\ell(u_1^*, u_2^*) = \text{gap}(M, N) = 2d$, and let $r^* \in V^*$ be the middle point of the unique u_1^*-u_2^* path in G^*. Note that such r^* always exists because $\ell(u_1^*, u_2^*)$ is even and all the edges in G^* have length one. By the maximality of $\ell(u_1^*, u_2^*)$, for every $v^* \in V(G^*)$, the r^*-v^* path has length at most d. Let $X^* \subseteq V^*$ be a minimal vertex subset of G^* such that

- $r^* \in X^*$;
- the subgraph induced by X^* is connected in G^*;
- the cut $C^* = \{e^* = u^*v^* \mid u^* \in X^*, v^* \in V(G^*) \setminus X^*\}$ has only edges in $M^* \cup \tilde{E}^*$, where $M^* = \{e^* \in E_{\text{in}}^* \mid e \in M \cap E_{\text{in}}\}$. Note that C^* may contain edges in $\tilde{E}^*$, whereas M^* is defined as a subset of E_{in}^*.

Such X^* always exists, as V^* satisfies all the conditions. The cut C^* corresponds to a cycle C in G. An example is given in Figure 3.

We claim that C is M-alternating. Assume not. Then there exist two consecutive edges $e = uv$, $e' = vw$ in C such that $e, e' \notin M$, which implies that $e, e' \in C_{\text{out}}$ as $E(C^*) \subseteq M^* \cup \tilde{E}^*$. Since M is a perfect matching, the vertex v is incident to another edge f in M. Since G is 2-connected and outerplanar, f is a chord of C. However, this contradicts that X^* was chosen to be minimal. Thus, C is an M-alternating cycle.

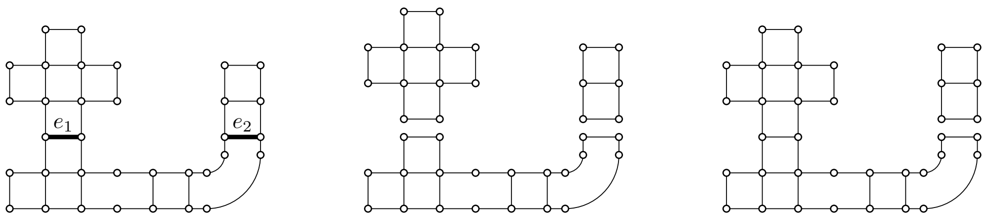
Consider taking $M \triangle C$. Let ℓ' be the length defined by (1) with $M \triangle C$ and N. It follows that, for an edge $e^* \in E(G^*)$,

$$\ell'(e^*) = \begin{cases} \ell(e^*) & \text{if } e^* \notin C^*, \\ 1 - \ell(e^*) & \text{if } e^* \in C^*. \end{cases}$$

We will show that, for any vertex $\tilde{v}^*$ in $\tilde{U}^*$, we have $\ell'(r^*, \tilde{v}^*) \leq d - 1$. This proves the claim, as, for any two vertices $\tilde{u}^*, \tilde{v}^*$ in $\tilde{U}^*$, it holds that

$$\ell'(\tilde{u}^*, \tilde{v}^*) \leq \ell'(r^*, \tilde{u}^*) + \ell'(r^*, \tilde{v}^*) \leq 2d - 2.$$

Since $r^* \in X^*$ and no vertex in $\tilde{U}^*$ is in X^*, the r^*-$\tilde{v}^*$ path P intersects with C^* exactly once. Hence, the length of P is changed by 1 by taking $M \triangle C$. So, if



(a) G and $E'_{\rm in} = \{e_1, e_2\}$. (b) G_1, G_2, G_3 when $F = E'_{\rm in}$. (c) G_1, G_2 when $F = \{e_2\}$.

FIG. 4. *Decomposition of the outerplanar graph in Figure* 2. *The edges in* $E'_{\rm in}$ *are shown with bold lines.*

$\ell(r^*, \tilde{v}^*) \le d-2$, then $\ell'(r^*, \tilde{v}^*) \le d-1$. Thus, it suffices to consider the case when $\ell(r^*, \tilde{v}^*) \ge d-1$, i.e., $\ell(r^*, \tilde{v}^*) = d-1$ or d.

Assume that $\ell(v^*, \tilde{v}^*) = 0$, which implies that $v^* \in U_2^*$, and hence v^* is not a leaf in $G^* - \tilde{U}^*$. In this case, there exists a leaf u^* in $G^* - \tilde{U}^*$ such that $\ell(r^*, u^*) \ge \ell(r^*, v^*) + 1$. Since $u^* \in U_1^*$, we obtain

$$\ell(r^*, \tilde{u}^*) = \ell(r^*, u^*) + 1 \ge \ell(r^*, v^*) + 2 = \ell(r^*, \tilde{v}^*) + 2 \ge d+1,$$

which is a contradiction.

Thus, we may assume that $\ell(v^*, \tilde{v}^*) = 1$. If the r^*-$\tilde{v}^*$ path P intersects $C^* \cap M^*$, then the intersected cut edge has length one, and hence we see that $\ell'(r^*, \tilde{v}^*) = \ell(r^*, \tilde{v}^*) - 1 \le d-1$. Otherwise, that is, if P intersects with $C^* \cap \tilde{E}^*$, then the intersected cut edge is $(v^*, \tilde{v}^*)$, and hence $\ell'(r^*, \tilde{v}^*) = \ell(r^*, \tilde{v}^*) - 1 \le d-1$. Thus, $\ell'(r^*, \tilde{v}^*) \le d-1$ in each case. □

For a perfect matching M_{i-1} in G, it follows from Claim 2 that there exists an M_{i-1}-alternating cycle C_i such that $\mathrm{gap}(M_{i-1}, N) = \mathrm{gap}(M_{i-1} \triangle C_i, N) + 2$. Define $M_i = M_{i-1} \triangle C_i$, and repeat finding an alternating cycle satisfying the above equation. The repetition ends when $\mathrm{gap}(M_i, N) = 0$, which means that $M_i = N$. The number of repetitions is equal to $\mathrm{gap}(M, N)/2$, and therefore we have $\mathrm{OPT}(I) \le \mathrm{gap}(M, N)/2$. Thus, the proof is complete. □

3.4. General case. Let $I = (G, M, N)$ be an instance of SHORTEST PERFECT MATCHING RECONFIGURATION such that G is a 2-connected outerplanar graph. Define $E'_{\rm in} = E_{\rm in} \cap M \cap N$. In this subsection, we deal with the general case; that is, $E'_{\rm in}$ is not necessarily empty. Then there is a case when changing an edge in $E'_{\rm in}$ reduces the number of reconfiguration steps as in Figure 1. We call such a move an *unhappy move*. The key idea of our algorithm is to detect a set of edges necessary for unhappy moves.

Since G is outerplanar and 2-connected, any $F \subseteq E'_{\rm in}$ divides the inner region of $C_{\rm out}$ into $|F|+1$ parts $R_1, \ldots, R_{|F|+1}$. For each $i = 1, \ldots, |F|+1$, let G_i be the subgraph of G consisting of all the vertices and the edges in R_i and its boundary. Thus, each edge $e \in F$ appears on the outer face boundaries in two of these subgraphs. See Figure 4. Let $\mathcal{G}_F = \{G_1, \ldots, G_{|F|+1}\}$. Note that each graph in $\mathcal{G}_F$ is outerplanar and 2-connected. For each $H \in \mathcal{G}_F$, let $I_H = (H, M \cap E(H), N \cap E(H))$. We now show the following theorem.

THEOREM 3.5. $\mathrm{OPT}(I) = \dfrac{1}{2} \min\limits_{F \subseteq E'_{\rm in}} \sum\limits_{H \in \mathcal{G}_F} \mathrm{gap}(I_H)$.

Proof. Let $\langle M_0, M_1, \dots, M_t\rangle$ be a shortest reconfiguration sequence from $M_0 = M$ to $M_t = N$. We denote by C_i the M_{i-1}-alternating cycle with $M_i = M_{i-1} \triangle C_i$. Define

$$F_{\text{opt}} = \{e \in E'_{\text{in}} \mid e \notin C_i \ \forall i\},$$

which is the set of edges in E'_{in} that are not touched in the shortest reconfiguration sequence; in other words, all edges in $E'_{\text{in}} \setminus F_{\text{opt}}$ are touched for unhappy moves in the sequence. Then C_i is contained in some $H \in \mathcal{G}_{F_{\text{opt}}}$ because $F_{\text{opt}} \subseteq M_{i-1}$, C_i is an M_{i-1}-alternating cycle, and $C_i \cap F_{\text{opt}} = \emptyset$. Thus, C_i can be used to obtain a reconfiguration sequence from $M \cap E(H)$ to $N \cap E(H)$ in H. Therefore, we have

$$\text{OPT}(I) = \sum_{H \in \mathcal{G}_{F_{\text{opt}}}} \text{OPT}(I_H). \tag{3}$$

We can also see that

$$\text{OPT}(I) \le \sum_{H \in \mathcal{G}_F} \text{OPT}(I_H) \tag{4}$$

for any $F \subseteq E'_{\text{in}}$.

To evaluate $\text{OPT}(I_H)$ for $H \in \mathcal{G}_F$, we slightly modify the instance I_H by replacing every inner edge of H contained in $M \cap N$ by two parallel edges each in M and N, respectively. The obtained graph is denoted by H', and the corresponding instance is denoted by $I_{H'}$. Since a reconfiguration sequence for $I_{H'}$ can be converted to one for I_H (by identifying the parallel edges), it holds that $\text{OPT}(I_H) \le \text{OPT}(I_{H'})$, and hence

$$\text{OPT}(I) \le \sum_{H \in \mathcal{G}_F} \text{OPT}(I_H) \le \sum_{H \in \mathcal{G}_F} \text{OPT}(I_{H'}) \tag{5}$$

holds for any $F \subseteq E'_{\text{in}}$ by (4). Moreover, by the definition of F_{opt}, there exists an index i such that $e \in C_i$ for any $e \in E'_{\text{in}} \setminus F_{\text{opt}}$. Therefore, for $H \in \mathcal{G}_{F_{\text{opt}}}$, the shortest reconfiguration sequence for I_H can be converted to a reconfiguration sequence for $I_{H'}$. Thus, $\text{OPT}(I_H) \ge \text{OPT}(I_{H'})$ holds for $H \in \mathcal{G}_{F_{\text{opt}}}$, and hence

$$\text{OPT}(I) = \sum_{H \in \mathcal{G}_{F_{\text{opt}}}} \text{OPT}(I_H) \ge \sum_{H \in \mathcal{G}_{F_{\text{opt}}}} \text{OPT}(I_{H'}) \tag{6}$$

by (3). By (5) and (6), we obtain

$$\text{OPT}(I) = \min_{F \subseteq E'_{\text{in}}} \sum_{H \in \mathcal{G}_F} \text{OPT}(I_{H'}), \tag{7}$$

and F_{opt} is a minimizer of the right-hand side.

By (7) and Theorem 3.4, we obtain

$$\text{OPT}(I) = \frac{1}{2} \min_{F \subseteq E'_{\text{in}}} \sum_{H \in \mathcal{G}_F} \text{gap}(I_{H'}) \tag{8}$$

because each $I_{H'}$ satisfies the condition in Theorem 3.4. Since $(H')^*$ is obtained from H^* by subdividing some edges of length two into two edges of length one, the diameter of $(H')^*$ is equal to that of H^*; that is, $\text{gap}(I_{H'}) = \text{gap}(I_H)$. Therefore, we obtain the theorem by (8). ∎

As an example, we apply this theorem to the instance in Figure 2. See Figure 4(c). If F consists of only the right thick edge in Figure 2(c), then $\mathcal{G}_F$ consists of two graphs G_1 and G_2 such that $\mathrm{gap}(I_{G_1}) = 6$ and $\mathrm{gap}(I_{G_2}) = 2$. Since we can check that such F attains the minimum in the right-hand side of Theorem 3.5, we obtain $\mathrm{OPT}(I) = 4$ by Theorem 3.5.

In order to compute the value in Theorem 3.5 efficiently, we reduce the problem to Min-Sum Diameter Decomposition, whose definition will be given later.

For $F \subseteq E'_{\mathrm{in}}$, let F^* be the edge subset of E^*_{in} corresponding to F, and let $\mathcal{G}_F = \{G_1, \ldots, G_{|F|+1}\}$. Then $G^* - F^*$ consists of $|F|+1$ components $T_1, T_2, \ldots, T_{|F|+1}$ such that T_i coincides with G^*_i (except that some edges of length zero are missing) for $i = 1, \ldots, |F|+1$. In particular, for each i, we have $\mathrm{gap}(I_{G_i}) = \max\{\ell(u^*, v^*) \mid u^*, v^* \in V(T_i)\}$, where ℓ is the length function on $E(G^*)$ defined by the instance $I = (G, M, N)$. We call $\max\{\ell(u^*, v^*) \mid u^*, v^* \in V(T_i)\}$ the *diameter* of T_i, which is denoted by $\mathrm{diam}_\ell(T_i)$. Then Theorem 3.5 shows that

$$\mathrm{OPT}(I) = \frac{1}{2} \min_{F \subseteq E'_{\mathrm{in}}} \sum_{i=1}^{|F|+1} \mathrm{diam}_\ell(T_i). \tag{9}$$

Therefore, we can compute $\mathrm{OPT}(I)$ by solving the following problem in which $T = G^*$ and $E_0 = (E'_{\mathrm{in}})^*$.

Min-Sum Diameter Decomposition

Input: A tree T, an edge subset $E_0 \subseteq E(T)$, and a length function $\ell : E(T) \to \mathbb{Z}_{\geq 0}$

Find: An edge set $F \subseteq E_0$ that minimizes $\sum_{T'} \mathrm{diam}_\ell(T')$, where the sum is taken over all the components T' of $T - F$.

In the subsequent subsection, we show that Min-Sum Diameter Decomposition can be solved in time polynomial in $|V(T)|$ and $L := \sum_{e \in E(T)} \ell(e)$.

Theorem 3.6. Min-Sum Diameter Decomposition *can be solved in* $O(|V(T)|L^4)$ *time, where* $L := \sum_{e \in E(T)} \ell(e)$.

Since (9) shows that Shortest Perfect Matching Reconfiguration on outerplanar graphs is reduced to Min-Sum Diameter Decomposition in which $L = O(|V(T)|)$, we obtain Theorem 3.1.

3.5. Algorithm for Min-Sum Diameter Decomposition. The remaining task is to show Theorem 3.6, that is, to give an algorithm for Min-Sum Diameter Decomposition that runs in $O(|V(T)|L^4)$ time. For this purpose, we adopt a dynamic programming approach.

We choose an arbitrary vertex r of a given tree T and regard T as a rooted tree with the root r. For each vertex v of T, we denote by T_v the subtree of T, which is rooted at v and is induced by all descendants of v in T (see Figure 5(a)). Thus, $T = T_r$ for the root r. Let $w_1, w_2, \ldots, w_q$ be the children of v, ordered arbitrarily. For each $j \in \{1, 2, \ldots, q\}$, we denote by T^j_v the subtree of T induced by $\{v\} \cup V(T_{w_1}) \cup V(T_{w_2}) \cup \cdots \cup V(T_{w_j})$. For example, in Figure 5(b), the subtree T^j_v is surrounded by a thick dotted rectangle. For notational convenience, we denote by T^0_v the tree consisting of a single vertex v. Then $T_v = T^0_v$ for each leaf v of T. Our algorithm computes and extends partial solutions for subtrees T^j_v from the leaves to the root r of T by keeping the information required for computing (the sum of) diameters of a partial solution.

We now define partial solutions for subtrees. For a subtree T^j_v and an edge subset $F' \subseteq E_0 \cap E(T^j_v)$, the *frontier* for F' is the component (subtree) in $T^j_v - F'$ that

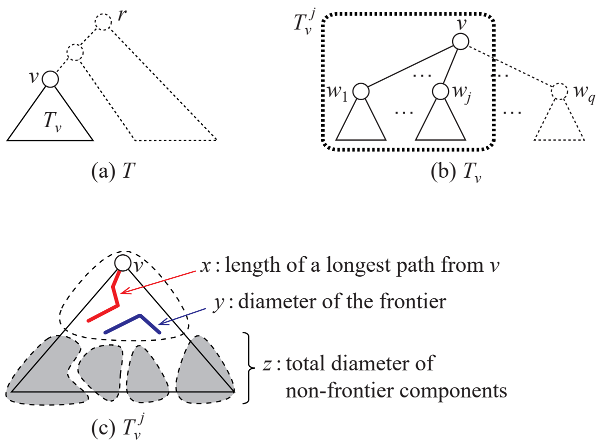


FIG. 5. (a) *Subtree T_v in the whole tree T,* (b) *subtree T_v^j in T_v, and* (c) *an (x, y, z)-separator of T_v^j.*

contains the root v of T_v^j. We sometimes call it the *v-frontier* for F' to emphasize the root v. For three integers $x, y, z \in \{0, 1, \dots, L\}$, the edge subset F' is called an *(x, y, z)-separator* of T_v^j if the following three conditions hold (see also Figure 5(c)):

- $x = \max\{\ell(v, u) \mid u \in V(T_{F'})\}$, where $T_{F'}$ is the v-frontier for F'. That is, the longest path from v to a vertex in $T_{F'}$ is of length x.
- $y = \mathrm{diam}_\ell(T_{F'})$. That is, y denotes the diameter of the v-frontier $T_{F'}$ for F'.
- $z = \sum_{T'} \mathrm{diam}_\ell(T')$, where the sum is taken over all the components T' of $(T - F') \setminus T_{F'}$.

Note that $x \leq y$ always holds for an (x, y, z)-separator of T_v^j. We then define the following function: For a subtree T_v^j and two integers $x, y \in \{0, 1, \dots, L\}$, we let

$$f(T_v^j; x, y) = \min\left\{z \mid T_v^j \text{ has an } (x, y, z)\text{-separator}\right\}.$$

Note that $f(T_v^j; x, y)$ is defined as $+\infty$ if T_v^j does not have an (x, y, z)-separator for any $z \in \{0, 1, \dots, L\}$. Then the optimal objective value to MIN-SUM DIAMETER DECOMPOSITION can be computed as $\min\{y + f(T; x, y) \mid x, y \in \{0, 1, \dots, L\}\}$.

For a given tree T, our algorithm computes $f(T_v^j; x, y)$ for all possible triplets (T_v^j, x, y) from the leaves to the root r of T as follows.

Initialization. We first compute $f(T_v^0; x, y)$ for all vertices $v \in V(T)$ (including internal vertices in T). Recall that T_v^0 consists of a single vertex v. Therefore, we have

$$f(T_v^0; x, y) = \begin{cases} 0 & \text{if } x = y = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Notice that we have computed $f(T_v; x, y)$ for all leaves v of T since $T_v = T_v^0$ if v is a leaf.

Update. We now consider the case where $j \geq 1$. To compute $f(T_v^j; x, y)$, we classify (x, y, z)-separators of T_v^j into the following two groups (a) and (b).

(a) The vertices v and w_j are contained in the same component (see also Figure 6(a)).

In this case, the edge vw_j is not deleted, and the v-frontier for an (x, y, z)-separator of T_v^j contains both v and w_j. Therefore, we can obtain the v-frontier

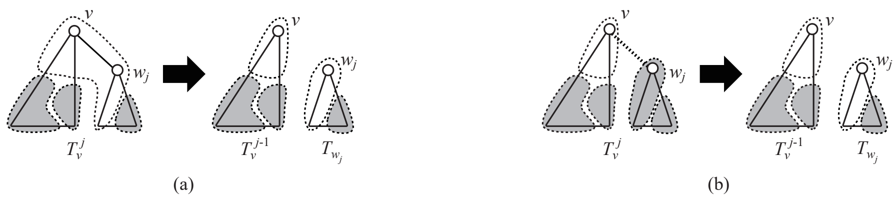


FIG. 6. *(x, y, z)-separators of a subtree T_v^j and their restrictions to subtrees T_v^{j-1} and T_{w_j}.*

for an (x, y, z)-separator of T_v^j by merging the v-frontier for some (x', y', z')-separator of T_v^{j-1} with the w_j-frontier for some (x'', y'', z'')-separator of T_{w_j}. Thus, we define

$$f^{\mathrm{a}}(T_v^j; x, y) := \min \left\{ f(T_v^{j-1}; x', y') + f(T_{w_j}; x'', y'') \right\},$$

where the minimum is taken over all integers $x', y', x'', y'' \in \{0, 1, \ldots, L\}$ such that $x = \max\{x', x'' + \ell(vw_j)\}$ and $y = \max\{y', y'', x' + \ell(vw_j) + x''\}$.

(b) The vertices v and w_j are contained in different components (see also Figure 6(b)). In this case, the edge vw_j is deleted, and hence this case happens only when $vw_j \in E_0$. Then the v-frontier for an (x, y, z)-separator of T_v^j is the v-frontier for some (x', y', z')-separator of T_v^{j-1}. Note that w_j is contained in a nonfrontier component for the (x, y, z)-separator of T_v^j, but the component forms the w_j-frontier for some (x'', y'', z'')-separator of T_{w_j}, as illustrated in Figure 6(b). Thus, we need to take the diameter of the w_j-frontier into account when we compute $f(T_v^j; x, y)$ from $f(T_v^{j-1}; x', y')$ and $f(T_{w_j}; x'', y'')$. Therefore, we define

$$f^{\mathrm{b}}(T_v^j; x, y) := \min \left\{ f(T_v^{j-1}; x', y') + f(T_{w_j}; x'', y'') + y'' \right\},$$

where the minimum is taken over all integers $x', y', x'', y'' \in \{0, 1, \ldots, L\}$ such that $x = x'$ and $y = y'$.

Then we can compute $f(T_v^j; x, y)$ as follows:

$$f(T_v^j; x, y) = \begin{cases} \min \left\{ f^{\mathrm{a}}(T_v^j; x, y), f^{\mathrm{b}}(T_v^j; x, y) \right\} & \text{if } vw_j \in E_0, \\ f^{\mathrm{a}}(T_v^j; x, y) & \text{otherwise.} \end{cases}$$

Since $x', y', x'', y'' \in \{0, 1, \ldots, L\}$, this update can be done in $O(L^4)$ time for each subtree T_v^j. The number of subtrees T_v^j is equal to $|V(T)| + |E(T)| = 2|V(T)| - 1$. Therefore, this algorithm runs in $O(|V(T)|L^4)$ time in total.

Note that we can easily modify the algorithm so that we obtain not only the optimal value but also an optimal solution. This completes the proof of Theorem 3.6.

We note here that the algorithm can be modified so that the running time is bounded by a polynomial in $|V(T)|$ by replacing the domain $\{0, 1, \ldots, L\}$ of x and y with $D := \{\ell(u, v) \mid u, v \in V(T)\}$. This modification is valid because $f(T_v^j; x, y) = +\infty$ unless $x, y \in D$. Since $|D| = O(|V(T)|^2)$, the modified algorithm runs in $O(|V(T)||D|^4) = O(|V(T)|^9)$ time. Note that, although this bound is polynomial only in $|V(T)|$, it is worse than $O(|V(T)|L^4)$ when $L = O(|V(T)|)$.

4. NP-hardness for planar graphs and bipartite graphs.

In this section, we prove that SHORTEST PERFECT MATCHING RECONFIGURATION is NP-hard even when the input graph is planar or bipartite.

THEOREM 4.1. SHORTEST PERFECT MATCHING RECONFIGURATION *is* NP-*hard even for planar graphs of maximum degree three.*

We reduce the HAMILTONIAN CYCLE PROBLEM, which is known to be NP-complete even when a given graph is 3-regular and planar [12].

HAMILTONIAN CYCLE PROBLEM

Input: A 3-regular planar graph $H = (V, E)$

Question: Decide whether H has a Hamiltonian cycle, i.e., a cycle that goes through all the vertices exactly once.

Proof. Let H be a 3-regular planar graph, which is an instance of the HAMILTONIAN CYCLE PROBLEM. We assume that $|V(H)| \geq 3$; otherwise, the problem is trivial. For each vertex $v \in V(H)$, we define a 8-vertex graph D_v (see also the top right in Figure 7):

$$V(D_v) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\},$$
$$E(D_v) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_4v_5, v_5v_7, v_3v_6, v_6v_8\}.$$

The 4-cycle formed by v_1, v_2, v_3, v_4 is denoted by C_v.

We construct an instance $I = (G, M, N)$ of our problem as follows (see Figure 7 as an example). We subdivide each edge $e = uv$ in H twice, and the obtained vertices are denoted by u_e and v_e, where u_e is closer to u. Then, for each vertex $v \in V(H)$, we replace v with the graph D_v and connect v_7 to $v_{e_v^{(1)}}$ and $v_{e_v^{(2)}}$, v_8 to $v_{e_v^{(2)}}$ and $v_{e_v^{(3)}}$, where $e_v^{(1)}$, $e_v^{(2)}$, $e_v^{(3)}$ are edges incident to v and the order follows the plane drawing of H. Let $E_v = \{v_7v_{e_v^{(1)}}, v_7v_{e_v^{(2)}}, v_8v_{e_v^{(2)}}, v_8v_{e_v^{(3)}}\}$. The resulting graph is denoted by G; i.e., G is defined as follows:

$$V(G) = \left(\bigcup_{v \in V(H)} V(D_v)\right) \cup \left(\bigcup_{e=uv \in E(H)} \{u_e, v_e\}\right),$$
$$E(G) = \left(\bigcup_{v \in V(H)} E(D_v) \cup E_v\right) \cup \{u_ev_e \mid e \in E(H)\}.$$

It follows that G is a planar graph of maximum degree three. Furthermore, we define initial and target perfect matchings M and N in G, respectively, to be

$$M = \{v_1v_2, v_3v_4, v_5v_7, v_6v_8 \mid v \in V(H)\} \cup \{u_ev_e \mid e \in E(H)\},$$
$$N = \{v_1v_4, v_2v_3, v_5v_7, v_6v_8 \mid v \in V(H)\} \cup \{u_ev_e \mid e \in E(H)\}.$$

This completes the construction of our corresponding instance $I = (G, M, N)$. The construction can be done in polynomial time.

We then give the following claims. Recall that t^* is the length of a shortest reconfiguration sequence for the constructed instance I.

CLAIM 3. *It holds that* $t^* \geq 2$.

Proof of Claim 3. We observe that if $t^* = 1$, then $M \triangle N$ must consist of a single M-alternating cycle, but it is not true for our instance I. Thus, the length of a reconfiguration sequence is at least two. ∎

We remark that G has an M-alternating path from $v_e^{(x)}$ to $v_e^{(y)}$ for any $x, y \in \{1, 2, 3\}$ with $x \neq y$. This implies that, for a cycle C in H, there exists a corresponding M-alternating cycle C' in G such that it goes through vertices of D_v for every $v \in V(C)$ and edges u_ev_e for every $e \in E(C)$.

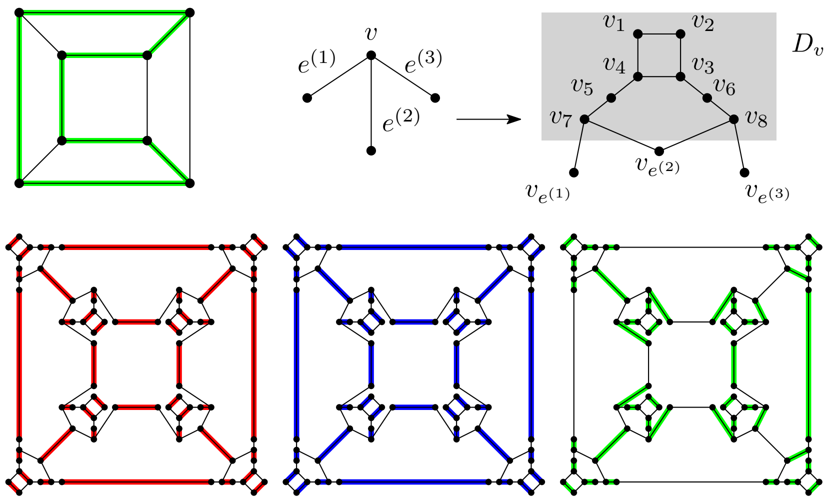


FIG. 7. *Reduction for planar graphs of maximum degree three. Top left: a yes instance H of the* HAMILTONIAN CYCLE PROBLEM *with a thick (green) Hamiltonian cycle. Top right: the constructed fragment D_v. Bottom left: the initial perfect matching M (red). Bottom middle: the target perfect matching N (blue). Bottom right: the perfect matching obtained as $M \triangle C$, where C corresponds to the Hamiltonian cycle of H.*

CLAIM 4. *If H has a Hamiltonian cycle C, then it holds that $t^* = 2$.*

Proof of Claim 4. We see that G has an M-alternating cycle C', corresponding to C of H, that has one edge v_3v_4 of C_v for each $v \in V(C)$. Then $M' = M \triangle C'$ is a perfect matching. In a similar way, G has an M'-alternating cycle C'', corresponding to C, that uses three edges v_3v_2, v_2v_1, and v_1v_4 of C_v for each $v \in V(C)$. Then $M' \triangle C''$ is equal to N. Thus, we can find a reconfiguration sequence of length two, which is shortest by Claim 3. □

CLAIM 5. *If $t^* = 2$, then H has a Hamiltonian cycle.*

Proof of Claim 5. We denote by $\langle M, M', N\rangle$ a shortest reconfiguration sequence of I. Let $C = M \triangle M'$. If $C = C_v$ for some $v \in V(H)$, then $M' \triangle N$ consists of more than one cycle as $|V(H)| \geq 3$, which contradicts that M' and N are adjacent. Therefore, we may assume that C is not C_v for any $v \in V(H)$. We will prove that the edge subset $F = \{e \in E(H) \mid u_ev_e \in C\}$ forms a Hamiltonian cycle in H. We denote by W_C the set of vertices in H used in F. Let $\overline{W_C} = V(H) \setminus W_C$. Since $M' \cap C_v$ and $N \cap C_v$ are distinct for $v \in \overline{W_C}$, the symmetric difference $M' \triangle N$ has at least $|\overline{W_C}|$ disjoint M'-alternating cycles. Moreover, for a vertex $v \in W_C$, we see that $M' \cap C_v = \{v_1v_2\}$ and $N \cap C_v = \{v_1v_4, v_2v_3\}$, which are distinct. Hence, $M' \triangle N$ has at least one M'-alternating cycle disjoint from $\bigcup_{v \in \overline{W_C}} V(D_v)$. Therefore, we have at least $|\overline{W_C}| + 1$ disjoint M'-alternating cycles. However, $M' \triangle N$ must consist of one cycle (see the proof of Claim 3), implying that $\overline{W_C} = \emptyset$. This means that C goes through C_v for every v, and hence C' is a Hamiltonian cycle in H. Thus, the claim holds. □

Therefore, it follows that H has a Hamiltonian cycle if and only if $t^* = 2$. This completes the proof of Theorem 4.1. □

The hardness for bipartite graphs of maximum degree three can be obtained with a similar proof.

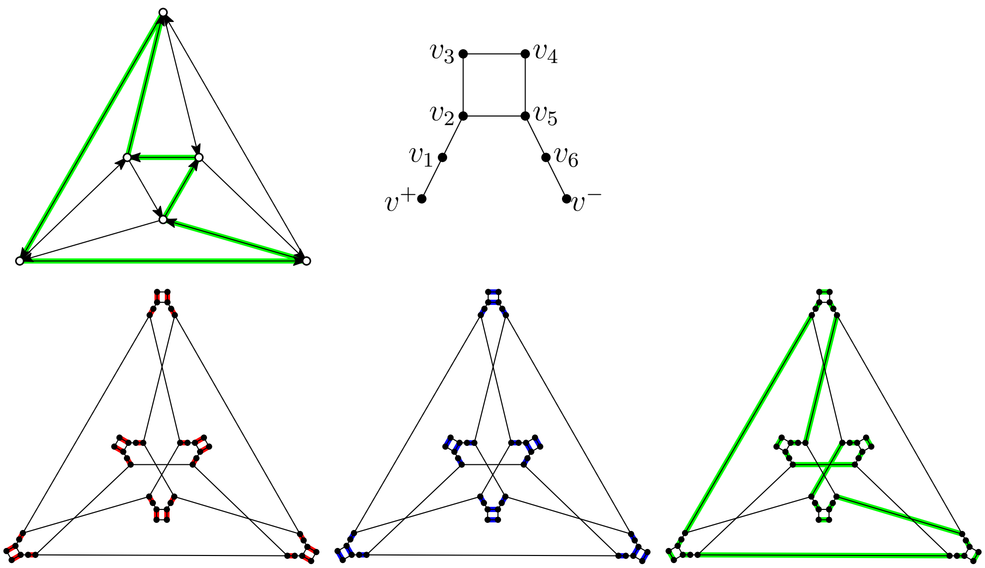


Fig. 8. *Reduction for bipartite graphs of maximum degree three. Top left: a yes instance H of the* Directed Hamiltonian Cycle Problem *with a thick (green) directed Hamiltonian cycle. Top middle: the constructed fragment D_v. Bottom left: the initial perfect matching M (red). Bottom middle: the target perfect matching N (blue). Bottom right: the perfect matching obtained as $M \triangle C$, where C corresponds to the directed Hamiltonian cycle of H.*

Theorem 4.2. Shortest Perfect Matching Reconfiguration *is* NP*-hard even for bipartite graphs of maximum degree three.*

We reduce the directed Hamiltonian cycle problem, which is known to be NP-complete even if digraphs have maximum in- and out-degree two [31].

Directed Hamiltonian Cycle Problem

Input: A digraph $H = (V, E)$ of maximum in- and out-degree two

Question: Decide whether H has a directed Hamiltonian cycle, i.e., a directed cycle that goes through all the vertices exactly once.

Proof. Let H be a digraph, which is an instance of the directed Hamiltonian cycle problem. We assume that $|V(H)| \geq 3$; otherwise, the problem is trivial. For each vertex $v \in V(H)$, we define a 6-vertex graph D_v (see the top right in Figure 8):

$$V(D_v) = \{v^+, v^-, v_1, v_2, v_3, v_4, v_5, v_6\},$$
$$E(D_v) = \{v^+v_1, v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_2, v_5v_6, v_6v^-\}.$$

The cycle of length four formed by v_2, v_3, v_4, v_5 is denoted by C_v.

We construct an instance $I = (G, M, N)$ of our problem as follows. The vertex set and the edge set of G are defined as

$$V(G) = \bigcup_{v \in V(H)} V(D_v), \qquad E(G) = \bigcup_{v \in V(H)} E(D_v) \cup \{u^-v^+ \mid uv \in E(H)\},$$

respectively. Namely, for each directed edge from u to v in H, we add an undirected edge to G between u^- and v^+. This finishes the construction of G. Note that G is bipartite and that its maximum degree is at most three as both the maximum in-degree and the maximum out-degree of H are at most two. Let M and N be defined as

$$M = \bigcup_{v \in V(H)} \{v^+v_1, v_2v_3, v_4v_5, v_6v^-\},$$
$$N = \bigcup_{v \in V(H)} \{v^+v_1, v_2v_5, v_3v_4, v_6v^-\}.$$

Refer to Figure 8 for the illustration. Let t^* be the length of a shortest reconfiguration sequence for I.

CLAIM 6. *It holds that $t^* \geq 2$.*

Proof. If $t^* = 1$, then $M \triangle N$ must consist of one M-alternating cycle, but this is not the case for our instance I. Thus, the length of a reconfiguration sequence is at least two. ∎

CLAIM 7. *If H has a directed Hamiltonian cycle C, then it holds that $t^* = 2$.*

Proof. We see that G has an M-alternating cycle C', corresponding to C of H, that has four edges $v^+v_1, v_2v_3, v_4v_5, v_6v^-$ of D_v for each $v \in V(C)$. Then $M' = M \triangle C'$ is a perfect matching. In a similar way, G has an M'-alternating cycle C'', corresponding to C, that uses three edges v^+v_1, v_2v_5, and v_6v^- of C_v for each $v \in V(C)$. Then $M' \triangle C''$ is equal to N. Thus, we can find a reconfiguration sequence of length two, which is the shortest by Claim 6. ∎

CLAIM 8. *If $t^* = 2$, then H has a directed Hamiltonian cycle.*

Proof. Let $\langle M, M', N\rangle$ be a shortest reconfiguration sequence of I. Let $C = M \triangle M'$. If $C = C_v$ for some $v \in V(H)$, then $M' \triangle N$ consists of more than one cycle as $|V(H)| \geq 3$, which contradicts that M' and N are adjacent. Therefore, we may assume that C is not C_v for any $v \in V(H)$. We will prove that the edge subset $F = \{uv \in E(H) \mid u^-v^+ \in C\}$ forms a Hamiltonian cycle in H. We denote by W_C the set of vertices in H used in F. Let $\overline{W_C} = V(H) \setminus W_C$. Since $M' \cap C_v$ and $N \cap C_v$ are distinct for $v \in \overline{W_C}$, the symmetric difference $M' \triangle N$ has at least $|\overline{W_C}|$ disjoint M'-alternating cycles. Moreover, for a vertex $v \in W_C$, we see that $M' \cap C_v = \{v_3v_4\}$ and $N \cap C_v = \{v_3v_4, v_2v_5\}$, which are distinct. Hence, $M' \triangle N$ has at least one M'-alternating cycle disjoint from $\bigcup_{v \in \overline{W_C}} V(D_v)$. Therefore, we have at least $|\overline{W_C}| + 1$ disjoint M'-alternating cycles. However, $M' \triangle N$ must consist of one cycle (see the proof of Claim 6), implying that $\overline{W_C} = \emptyset$. This means that C goes through C_v for every v, and hence C' is a Hamiltonian cycle in H. Thus, the claim holds. ∎

Therefore, it follows that H has a directed Hamiltonian cycle if and only if $t^* = 2$. This completes the proof. ∎

Note that the reduction does not produce a planar graph even when the input digraph has a planar underlying graph. The example in Figure 8 contains a K_5-minor.

The proofs actually show that SHORTEST PERFECT MATCHING RECONFIGURATION is NP-hard to approximate within a factor of less than 3/2.

5. Conclusion.

In this paper, we studied the shortest reconfiguration problem of perfect matchings under the alternating cycle model, which is equivalent to the combinatorial shortest path problem on perfect matching polytopes. We prove that the problem can be solved in polynomial time for outerplanar graphs, but it is NP-hard and even APX-hard for planar graphs and bipartite graphs.

Several questions remain unsolved. For polynomial-time solvability, our algorithm runs only for outerplanar graphs, and it looks difficult to extend the algorithm to other graph classes. A next step would be to try k-outerplanar graphs for fixed $k \geq 2$.

One way to tackle NP-hard cases is approximation. We only know the NP-hardness of approximating within a factor of less than 3/2. We believe the existence of a polynomial-time constant-factor approximation. Note that we do not obtain a constant-factor approximation by flipping alternating cycles in the symmetric difference of two given perfect matchings one by one.

This paper was mainly concerned with reconfiguration of perfect matchings. Alternatively, we may consider reconfiguration of maximum matchings, or maximum-weight matchings. In those cases, we need to adopt the alternating path/cycle model. Then the question is related to the combinatorial shortest path problem on faces of matching polytopes. Note that the perfect matching polytope is also a face of the matching polytope. Therefore, the study on maximum-weight matchings will be a generalization of this paper.

To the best of the authors' knowledge, the combinatorial shortest path problem of 0/1-polytopes has not been well investigated, while the adjacency in 0/1-polytopes has been extensively studied in the literature. This paper opens up a new perspective for the study of combinatorial and computational aspects of polytopes and connects them with the study of combinatorial reconfiguration.

Acknowledgment. The authors thank anonymous referees of the preliminary version [18] and of this journal version for their helpful suggestions.

REFERENCES

[1] O. Aichholzer, J. Cardinal, T. Huynh, K. Knauer, T. Mütze, R. Steiner, and B. Vogtenhuber, *Flip distances between graph orientations*, Algorithmica, 83 (2021), pp. 116–143, https://doi.org/10.1007/s00453-020-00751-1.

[2] O. Aichholzer, W. Mulzer, and A. Pilz, *Flip distance between triangulations of a simple polygon is NP-complete*, Discrete Comput. Geom., 54 (2015), pp. 368–389, https://doi.org/10.1007/s00454-015-9709-7.

[3] A. Y. Alfakih and K. G. Murty, *Adjacency on the constrained assignment problem*, Discrete Appl. Math., 87 (1998), pp. 269–274, https://doi.org/10.1016/S0166-218X(98)00063-8.

[4] M. Bonamy, N. Bousquet, M. Heinrich, T. Ito, Y. Kobayashi, A. Mary, M. Mühlenthaler, and K. Wasa, *The perfect matching reconfiguration problem*, in 44th International Symposium on Mathematical Foundations of Computer Science, MFCS 2019, Aachen, Germany, P. Rossmanith, P. Heggernes, and J.-P. Katoen, eds., LIPIcs 138, Schloss Dagstuhl—Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 2019, pp. 80:1–80:14, https://doi.org/10.4230/LIPIcs.MFCS.2019.80, http://www.dagstuhl.de/dagpub/978-3-95977-117-7.

[5] É. Bonnet, T. Miltzow, and P. Rzążewski, *Complexity of token swapping and its variants*, Algorithmica, 80 (2018), pp. 2656–2682, https://doi.org/10.1007/s00453-017-0387-0.

[6] N. Bousquet, T. Hatanaka, T. Ito, and M. Mühlenthaler, *Shortest reconfiguration of matchings*, in Graph-Theoretic Concepts in Computer Science—45th International Workshop, WG 2019, Vall de Núria, Spain, Revised Papers, I. Sau and D. M. Thilikos, eds., Lecture Notes in Computer Science 11789, Springer-Verlag, Berlin, 2019, pp. 162–174, https://doi.org/10.1007/978-3-030-30786-8_13.

[7] T. Brunsch and H. Röglin, *Finding short paths on polytopes by the shadow vertex algorithm*, in Automata, Languages, and Programming—40th International Colloquium, ICALP 2013, Riga, Latvia, Proceedings, Part I, F. V. Fomin, R. Freivalds, M. Z. Kwiatkowska, and D. Peleg, eds., Lecture Notes in Computer Science 7965, Springer-Verlag, Berlin, 2013, pp. 279–290, https://doi.org/10.1007/978-3-642-39206-1_24.

[8] L. Bulteau, G. Fertin, and I. Rusu, *Pancake flipping is hard*, J. Comput. Syst. Sci., 81 (2015), pp. 1556–1574, https://doi.org/10.1016/j.jcss.2015.02.003.

[9] V. Chvátal, *On certain polytopes associated with graphs*, J. Comb. Theory Ser. B, 18 (1975), pp. 138–154, https://doi.org/10.1016/0095-8956(75)90041-6.

[10] S. Fiorini, *A combinatorial study of partial order polytopes*, Eur. J. Comb., 24 (2003), pp. 149–159, https://doi.org/10.1016/S0195-6698(03)00009-X.

[11] A. M. Frieze and S.-H. Teng, *On the complexity of computing the diameter of a polytope*, Comput. Complexity, 4 (1994), pp. 207–219, https://doi.org/10.1007/BF01206636.

[12] M. R. Garey, D. S. Johnson, and R. E. Tarjan, *The planar Hamiltonian circuit problem is NP-complete*, SIAM J. Comput., 5 (1976), pp. 704–714, https://doi.org/10.1137/0205049.

[13] D. Geist and E. Y. Rodin, *Adjacency of the* 0-1 *knapsack problem*, Comput. Oper. Res., 19 (1992), pp. 797–800, https://doi.org/10.1016/0305-0548(92)90019-2.

[14] M. Gupta, H. Kumar, and N. Misra, *On the complexity of optimal matching reconfiguration*, in SOFSEM 2019: Theory and Practice of Computer Science—45th International Conference on Current Trends in Theory and Practice of Computer Science, Nový Smokovec, Slovakia, Proceedings, B. Catania, R. Královic, J. R. Nawrocki, and G. Pighizzini, eds., Lecture Notes in Computer Science 11376, Springer-Verlag, Berlin, 2019, pp. 221–233, https://doi.org/10.1007/978-3-030-10801-4_18.

[15] J. van den Heuvel, *The complexity of change*, in Surveys in Combinatorics 2013, S. R. Blackburn, S. Gerke, and M. Wildon, eds., London Mathematical Society Lecture Note Series 409, Cambridge University Press, Cambridge, 2013, pp. 127–160, https://doi.org/10.1017/CBO9781139506748.005.

[16] T. Ito, E. D. Demaine, N. J. A. Harvey, C. H. Papadimitriou, M. Sideri, R. Uehara, and Y. Uno, *On the complexity of reconfiguration problems*, Theoret. Comput. Sci., 412 (2011), pp. 1054–1065, https://doi.org/10.1016/j.tcs.2010.12.005.

[17] T. Ito, N. Kakimura, N. Kamiyama, Y. Kobayashi, and Y. Okamoto, *Reconfiguration of maximum-weight b-matchings in a graph*, J. Comb. Optim., 37 (2019), pp. 454–464, https://doi.org/10.1007/s10878-018-0289-3.

[18] T. Ito, N. Kakimura, N. Kamiyama, Y. Kobayashi, and Y. Okamoto, *Shortest reconfiguration of perfect matchings via alternating cycles*, in 27th Annual European Symposium on Algorithms, ESA 2019, Munich/Garching, Germany, M. A. Bender, O. Svensson, and G. Herman, eds., LIPIcs 144, Schloss Dagstuhl—Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 2019, pp. 61:1–61:15, https://doi.org/10.4230/LIPIcs.ESA.2019.61, http://www.dagstuhl.de/dagpub/978-3-95977-124-5.

[19] M. Johnson, D. Kratsch, S. Kratsch, V. Patel, and D. Paulusma, *Finding shortest paths between graph colourings*, Algorithmica, 75 (2016), pp. 295–321, https://doi.org/10.1007/s00453-015-0009-7.

[20] M. Kamiński, P. Medvedev, and M. Milanič, *Complexity of independent set reconfigurability problems*, Theoret. Comput. Sci., 439 (2012), pp. 9–15, https://doi.org/10.1016/j.tcs.2012.03.004.

[21] J. Kawahara, T. Saitoh, and R. Yoshinaka, *The time complexity of permutation routing via matching, token swapping and a variant*, J. Graph Algorithms Appl., 23 (2019), pp. 29–70, https://doi.org/10.7155/jgaa.00483.

[22] A. Lubiw and V. Pathak, *Flip distance between two triangulations of a point set is NP-complete*, Comput. Geom., 49 (2015), pp. 17–23, https://doi.org/10.1016/j.comgeo.2014.11.001.

[23] T. Matsui, *NP-completeness of non-adjacency relations on some* 0-1 *polytopes*, Mathematical Engineering Technical Reports METR-94-12, University of Tokyo, 1994.

[24] T. Miltzow, L. Narins, Y. Okamoto, G. Rote, A. Thomas, and T. Uno, *Approximation and hardness of token swapping*, in 24th Annual European Symposium on Algorithms, ESA 2016, Aarhus, Denmark, P. Sankowski and C. D. Zaroliagis, eds., LIPIcs 57, Schloss Dagstuhl—Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 2016, pp. 66:1–66:15, https://doi.org/10.4230/LIPIcs.ESA.2016.66.

[25] A. E. Mouawad, N. Nishimura, V. Pathak, and V. Raman, *Shortest reconfiguration paths in the solution space of Boolean formulas*, SIAM J. Discrete Math., 31 (2017), pp. 2185–2200, https://doi.org/10.1137/16M1065288.

[26] M. Mühlenthaler, *Degree-constrained subgraph reconfiguration is in P*, in Mathematical Foundations of Computer Science 2015—40th International Symposium, MFCS 2015, Milan, Italy, Proceedings, Part II, G. F. Italiano, G. Pighizzini, and D. Sannella, eds., Lecture Notes in Computer Science 9235, Springer-Verlag, Berlin, 2015, pp. 505–516, https://doi.org/10.1007/978-3-662-48054-0_42.

[27] D. Naddef, *The Hirsch conjecture is true for* $(0,1)$*-polytopes*, Math. Program., 45 (1989), pp. 109–110, https://doi.org/10.1007/BF01589099.

[28] N. Nishimura, *Introduction to reconfiguration*, Algorithms, 11 (2018), 52, https://doi.org/10.3390/a11040052.

[29] C. H. Papadimitriou, *The adjacency relation on the traveling salesman polytope is NP-complete*, Math. Program., 14 (1978), pp. 312–324, https://doi.org/10.1007/BF01588973.

[30] A. Pilz, *Flip distance between triangulations of a planar point set is APX-hard*, Comput. Geom., 47 (2014), pp. 589–604, https://doi.org/10.1016/j.comgeo.2014.01.001.

[31] J. Plesník, *The NP-completeness of the Hamiltonian cycle problem in planar digraphs with degree bound two*, Inform. Process. Lett., 8 (1979), pp. 199–201, https://doi.org/10.1016/0020-0190(79)90023-1.

[32] D. Ratner and M. K. Warmuth, *Finding a shortest solution for the $N \times N$ extension of the 15-PUZZLE is intractable*, in Proceedings of the 5th National Conference on Artificial Intelligence, Philadelphia, 1986, Volume 1: Science, T. Kehler, ed., Morgan Kaufmann, Burlington, MA, 1986, pp. 168–172.

[33] L. Sanità, *The diameter of the fractional matching polytope and its hardness implications*, in 59th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2018, Paris, M. Thorup, ed., IEEE Computer Society, Washington, DC, 2018, pp. 910–921, https://doi.org/10.1109/FOCS.2018.00090.

[34] F. Santos, *A counterexample to the Hirsch conjecture*, Ann. Math., 176 (2012), pp. 383–412, https://doi.org/10.4007/annals.2012.176.1.7.

[35] N. Sukegawa, *An asymptotically improved upper bound on the diameter of polyhedra*, Discrete Comput. Geom., 62 (2019), pp. 690–699, https://doi.org/10.1007/s00454-018-0016-y.

[36] T. Yamada and R. Uehara, *Shortest reconfiguration of sliding tokens on subclasses of interval graphs*, Theoret. Comput. Sci., 863 (2021), pp. 53–68, https://doi.org/10.1016/j.tcs.2021.02.019.

[37] K. Yamanaka, E. D. Demaine, T. Horiyama, A. Kawamura, S.-I. Nakano, Y. Okamoto, T. Saitoh, A. Suzuki, R. Uehara, and T. Uno, *Sequentially swapping colored tokens on graphs*, J. Graph Algorithms Appl., 23 (2019), pp. 3–27, https://doi.org/10.7155/jgaa.00482.

[38] K. Yamanaka, E. D. Demaine, T. Ito, J. Kawahara, M. Kiyomi, Y. Okamoto, T. Saitoh, A. Suzuki, K. Uchizawa, and T. Uno, *Swapping labeled tokens on graphs*, Theoret. Comput. Sci., 586 (2015), pp. 81–94, https://doi.org/10.1016/j.tcs.2015.01.052.

[39] K. Yamanaka, T. Horiyama, J. M. Keil, D. G. Kirkpatrick, Y. Otachi, T. Saitoh, R. Uehara, and Y. Uno, *Swapping colored tokens on graphs*, Theoret. Comput. Sci., 729 (2018), pp. 1–10, https://doi.org/10.1016/j.tcs.2018.03.016.