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A parameterized view to the robust recoverable base problem of matroids under structural uncertainty

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ABSTRACT

We study a robust recoverable version of the matroid base problem where the uncertainty is imposed on combinatorial structures rather than on weights as studied in the literature. We prove that the problem is NP-hard even when a given matroid is uniform or graphic. On the other hand, we prove that the problem is fixed-parameter tractable with respect to the number of scenarios.

1. Introduction

Robust recoverable optimization, or recoverable robust optimization, is a field of mathematical optimization that deals with uncertainty, which was introduced by Liebchen et al. [24]. For example, when we want to construct a communication network, we solve a minimum-cost spanning tree problem in the traditional optimization framework. However, it may happen that some of the links will fail or communication cost will change in the future. In such a case, we may want to construct the network again, but at the same time we want to avoid computing the network from scratch because it may be costly.

To deal with such changes, in the framework of robust recoverable optimization, we take two-stage decision making. At the first stage, we construct a spanning tree that is not necessarily of minimum cost, but is possibly robust under future changes. Before the second stage, changes happen and we know all the changes. Then, at the second stage, we modify the spanning tree from the first stage to adapt the changes. The overall goal is to minimize the sum of the construction cost at the first stage and the modification cost at the second stage.

This setup arises not only in communication networks, but also in scheduling and railway optimization [24]. Recently, robust recoverable versions of standard combinatorial optimization problems have also been studied [5,6,9,10,15].

In this paper, we deal with *structural changes* in the future for uncertainty. Namely, we face with changes of combinatorial structures before the second stage. In the example of communication networks, this corresponds to link failure. We also assume that at the first stage, we know a finite number of *scenarios* that represent uncertainty, and each scenario corresponds to a change that happens before the second stage.

The simplest form of the problem we study in this paper can be described as follows. We are given an undirected graph $G = (V, E)$. Let s be the number of scenarios, and for each $i \in \{1, 2, \ldots, s\}$ we are given a subgraph $G_i = (V, E_i)$ of G as a scenario. Namely, in each scenario, the edges in $E \setminus E_i$ will be useless by failure. We assume that each subgraph G_i is connected, and thus contains a spanning tree. Then, we want to find a spanning tree $T = (V, B)$ of G and a spanning tree $T_i = (V, B_i)$ for each $i \in \{1, 2, \ldots, s\}$ such that $\max_i |B \triangle B_i|$ is as small as possible, where $\triangle$ denotes the symmetric difference. Note that $\max_i |B \triangle B_i|$ corresponds to the cost at the second stage since $|B \triangle B_i|$ is the "distance" between B and B_i. Further note that we ignore the first-stage cost since the cost of every spanning tree (i.e., the number of edges in every spanning tree) is identical.

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We actually study the following decision problem. Namely, instead of minimizing $\max_i |B \triangle B_i|$, we decide if there exist spanning trees $T, T_1, T_2, \ldots, T_s$ with $\max_i |B \triangle B_i| \leq 2k$ for a given natural number k. Note that $|B \triangle B_i|$ is always even since $|B| = |V| - 1 = |B_i|$. If we can solve this decision problem, then we can also solve the minimization problem by, for example, binary search. On the other hand, if we can solve the minimization problem, then we can also solve the decision problem by comparing the optimal value and $2k$. Therefore, the decision problem and the minimization problem are polynomial-time equivalent.

Notice that the existence of a spanning tree $T_i = (V, B_i)$ of G_i with $|B \triangle B_i| \leq 2k$ is equivalent to the condition that $|B \cap E_i| \geq |V| - k - 1$ (see Lemma 1).

The simplest form that we explained so far can be generalized to the following setup in terms of matroids (necessary definitions for matroids will be introduced in the next section). Let $\mathbf{M} = (E, \mathcal{I})$ be a matroid, where $\mathcal{I}$ is the family of independent sets. The family of bases of $\mathbf{M}$ is denoted by $\mathcal{B}(\mathbf{M})$. For a set $X \subseteq E$, we denote the rank of X by $\mathrm{rk}(X) = \max\{|I| \mid I \in \mathcal{I}, I \subseteq X\}$, and the rank of $\mathbf{M}$ is the size of its base.

This paper studies the following problem.

Robust Recoverable Matroid Base Problem

Input: A matroid $\mathbf{M} = (E, \mathcal{I})$ of rank r, s subsets $E_1, E_2, \ldots, E_s \subseteq E$, where $\mathrm{rk}(E_i) = r$ for each $i \in \{1, 2, \ldots, s\}$, and a positive integer k.

Output: A base $B \in \mathcal{B}(\mathbf{M})$ such that $|B \cap E_i| \geq r - k$ for each $i \in \{1, 2, \ldots, s\}$, or "no" if no such base B exists.

When a matroid $\mathbf{M}$ is obtained from a graph $G = (V, E)$, i.e., $\mathbf{M}$ is a graphic matroid, r is the number of edges in a maximal forest of G, $\mathcal{I}$ is the family of edge sets of forests of G, $\mathcal{B}(\mathbf{M})$ is the family of edge sets of maximal forests of G, $\mathrm{rk}(E_i)$ is the number of edges in a maximal forest of the subgraph $G_i = (V, E_i)$. Therefore, if $r = |V| - 1$, then the condition that $\mathrm{rk}(E_i) = r$ implies that the graph G_i contains a spanning tree, and Robust Recoverable Matroid Base Problem corresponds to the robust recoverable optimization problem that we introduced for communication networks above.

In general, the condition $\mathrm{rk}(E_i) = r$ means that the restriction of $\mathbf{M}$ to E_i contains a base of $\mathbf{M}$, which intuitively means that each scenario contains a feasible solution to the original setting.

The following are the results of this paper.

1. Robust Recoverable Matroid Base Problem is NP-hard even when $k \geq 1$ is constant, and $\mathbf{M}$ is a uniform matroid or a graphic matroid. Note that s is part of the input.
2. When s is a parameter and k is arbitrary, Robust Recoverable Matroid Base Problem is fixed-parameter tractable. In particular, if s is constant and k is arbitrary, Robust Recoverable Matroid Base Problem can be solved in polynomial time. Note that $\mathbf{M}$ does not have to be a uniform matroid or a graphic matroid, but $\mathbf{M}$ can be a general matroid.

Fixed-parameter tractability is defined as follows. We consider a problem that is associated with a number p, called a parameter, apart from the input (such a problem is sometimes called a parameterized problem). Then, the problem is *fixed-parameter tractable* if there exists a function $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that the problem can be solved in time $f(p)\mathrm{poly}(n)$, where n is the input size and poly is a polynomial. For example, $2^p n^2$ is allowed for such running time, but n^p is not. In particular, when p is constant, a fixed-parameter tractable problem can be solved in polynomial time, and in that case the degree of the polynomial running time does not depend on p.

The class of fixed-parameter tractable problems is often denoted by FPT. For details of fixed-parameter tractability, we refer to textbooks [26,11].

Note that Robust Recoverable Matroid Base Problem can easily be solved when $k = 0$. In such a case, we only require that $|B \cap E_i| \geq r$, but this is equivalent to $B \subseteq E_i$ since $|B| = r$. Therefore, we consider the restriction of $\mathbf{M}$ to $E_1 \cap E_2 \cap \cdots \cap E_s$, and find a base of that restriction. Then, we check that its size is equal to r. A base of the restriction of a matroid can be found in polynomial time by the greedy algorithm.

Related work

Robust recoverable matroid base problem has also been studied in the literature, but the authors in the literature mainly discuss the uncertainty for weights, i.e., the change of weights. Namely, we are given as input a matroid $\mathbf{M} = (E, \mathcal{I})$, $s + 1$ weights $w_e^0, w_e^1, \ldots, w_e^s \in \mathbb{R}_+$ for each element $e \in E$, and a natural number $k \in \mathbb{N}$. We want to find $s + 1$ bases $B_0, B_1, \ldots, B_s \in \mathcal{B}(\mathbf{M})$ such that $|B_0 \triangle B_i| \leq 2k$ for every $i \in \{1, 2, \ldots, s\}$ and

$$\sum_{e \in B_0} w_e^0 + \max_{i \in \{1,2,\ldots,s\}} \sum_{e \in B_i} w_e^i$$

is minimized. We call this variant the weight change version for short. The Robust Recoverable Matroid Base Problem of this paper can be cast to the weight change version by setting for each $e \in E$ and $i \in \{1, 2, \ldots, s\}$, $w_e^0 = 1$, and

$$w_e^i = \begin{cases} 1 & \text{if } e \in E_i, \\ \infty & \text{if } e \notin E_i, \end{cases}$$

where ∞ represents a sufficiently large positive constant.

Consider the case where $\mathbf{M}$ is a uniform matroid. Averbakh [7] proved that the weight change variant is weakly NP-hard when $s = 2$. Kasperski and Zieliński [20] proved that the weight change variant is strongly NP-hard when k and s are part of the input. Kasperski and Zieliński [21] proved that the weight change variant is strongly NP-hard when $k = 2$ (but s is part of the input). Kasperski, Kurpisz and Zieliński [19] proved that the weight change variant is NP-hard to approximate within any constant factor. An approximation algorithm of factor $\ln s$ is known [20,19].

Consider the case where $\mathbf{M}$ is a graphic matroid. Kasperski, Kurpisz, and Zieliński [18] proved that the weight change variant is weakly NP-hard even when $s = 2$ and $k = 0$. They also proved that the variant is strongly NP-hard when s and k are part of the input.

For general matroids, Büsing [8] proved that the weight change variant can be solved in polynomial time when $s = 1$ and k is an arbitrary constant. However, she did not show that the problem is fixed-parameter tractable with respect to the parameter k.

Table 1 summarizes the results in the literature and this paper. Note that the results of this paper are not obtained as consequences of those in the literature since the hardness results there use specific weights that do not correspond to our setting (see above).

In the literature, there have been papers on various versions of robust combinatorial optimization problems. Famous are the min-max and min-max regret versions [22,4]. We can cast the problem of finding a (minimum-cost) base in a matroid to these setups: In the min-max version, we are given a matroid $\mathbf{M}$ on a ground set E, cost $c_i(e)$ for each element $e \in E$ and each scenario $i \in \{1, \ldots, s\}$, and then we want to find a single base $B \in \mathcal{B}(\mathbf{M})$ that minimizes $\max\{\sum_{e \in B} c_i(e) \mid i \in \{1, \ldots, s\}\}$; In the min-max regret version, we are given the same input as the min-max version, and then we want to find a single base $B \in \mathcal{B}(\mathbf{M})$ that minimizes

Table 1
Results on robust recoverable matroid base problem. The mark $*$ represents the results of this paper.

			Change			
Matroid	#Scenarios	Robustness k	Weight		Structure	
Uniform	2	arbitrary	NP-hard	[7]		
Uniform	arbitrary	2	NP-hard	[21]		
Graphic	2	0	NP-hard	[18]		
General	1	constant	Poly	[8]		
Uniform	arbitrary	1			NP-hard	[$*$]
Graphic	arbitrary	1			NP-hard	[$*$]
General	parameter	arbitrary			FPT	[$*$]

$\max\{\sum_{e\in B} c_i(e) - \sum_{e\in B_i^*} c_i(e) \mid i \in \{1,\dots,s\}\}$, where $B_i^* \in \mathcal{B}(\mathbf{M})$ is the minimum-cost base in scenario i (i.e., with respect to c_i).

For the minimum-cost spanning tree problem, both the min-max version and the min-max regret version are NP-hard [22,3]. When the number of scenarios is constant, the min-max version and the min-max regret version have fully polynomial-time approximation schemes [3]. On the other hand, when the number of scenarios is non-constant, the min-max version and the min-max regret version have no polynomial-time approximation algorithms with approximation factor better than $3/2$ [3].

The Robust Recoverable Matroid Base Problem of this paper can be cast to the min-max version by setting for each $e \in E$ and $i \in \{1,2,\dots,s\}$, $c_i(e) = 1$ if $e \in E_i$, and 0 if $e \notin E_i$. If the minimum value is at least $r-k$, the output must be "Yes;" Otherwise the output must be "No." By the same reduction, the Robust Recoverable Matroid Base Problem can be cast to the min-max regret version. Here, we should point out that the known NP-hardness proofs for the min-max version are hard to adapt to Robust Recoverable Matroid Base Problem. In the proof of Kouvelis and Yu [22], the edge costs are not bound to zero or one. On the other hand, the proof by Aissi et al. [3] only uses the edge costs of zero and one, but with their proof we cannot directly guarantee that $\mathrm{rk}(E_i) = r$ and the hardness when $k = 1$ is hard to derive.

We also point out that the pseudo-polynomial-time algorithms by Aissi et al. [2] for the min-max version and the min-max regret version do not imply the fixed-parameter tractability of Robust Recoverable Matroid Base Problem when the number of scenario is a parameter since the running time of their algorithms has the number of scenarios in the exponent of the number of edges.

Another line of research studies the bulk-robust version [1]. If we cast the problem of finding a (minimum-cost) base in a matroid to the bulk-robust setting, we are given a matroid $\mathbf{M}$ on a ground set E, cost $c(e)$ for each element $e \in E$ and s subsets $S_1,\dots,S_s \subseteq E$ of the ground set, and then we want to find a minimum-cost set $F \subseteq E$ such that $F \setminus S_i$ contains a base of $\mathbf{M}$ for every $i \in \{1,\dots,s\}$. Even when $\mathbf{M}$ is a graphic matroid or a uniform matroid of rank one, the bulk-robust version is NP-hard while there exists a polynomial-time approximation algorithm with approximation ratio $O(\log(rs))$, where $r = \mathrm{rk}(\mathbf{M})$.

Yet another line of research pursues the demand-robust version [14,17]. If we cast the problem of finding a (minimum-cost) base in a matroid to the demand-robust setting, we are given a matroid $\mathbf{M} = (E,\mathcal{I})$, cost $c(e)$ for each element $e \in E$, s independent sets $I_1,\dots,I_s \in \mathcal{I}$, and a real number $\lambda_i > 1$ for each $i \in \{1,\dots,s\}$. Then, we want to find $s+1$ sets $F_0, F_1,\dots,F_s \subseteq E$ such that $I_i \subseteq F_0 \cup F_i$ for every $i \in \{1,\dots,s\}$. The objective is to minimize $\sum_{e\in F_0} c(e) + \max\{\sum_{e\in F_i} \lambda_i c(e) \mid i \in \{1,\dots,s\}\}$.

2. Preliminaries

An undirected graph $G = (V,E)$ is a pair of its vertex set V and its edge set E. In this paper, a graph is undirected and finite. For a graph $G = (V,E)$ and an edge subset $F \subseteq E$, if the graph (V,F) contains no cycle, then it is called a *forest*.

Let E be a finite set. A *matroid* on E is a set system $\mathbf{M} = (E,\mathcal{I})$ that consists of E and a family $\mathcal{I} \subseteq 2^E$ satisfying the following conditions:

(I1) $\emptyset \in \mathcal{I}$;
(I2) If $X \subseteq Y$ and $Y \in \mathcal{I}$, then $X \in \mathcal{I}$;
(I3) If $X, Y \in \mathcal{I}$ and $|X| > |Y|$, then there exists $e \in X \setminus Y$ such that $Y \cup \{e\} \in \mathcal{I}$.

A set $I \in \mathcal{I}$ is called an *independent set* of $\mathbf{M}$, and E is called the *ground set* of $\mathbf{M}$. We often write $\mathcal{I}(\mathbf{M})$ for $\mathcal{I}$ to emphasize $\mathcal{I}$ is the family of independent sets of the matroid $\mathbf{M}$.

For a matroid $\mathbf{M} = (E,\mathcal{I})$, a maximal independent set is called a *base* of $\mathbf{M}$. The family of bases of $\mathbf{M}$ is denoted by $\mathcal{B}(\mathbf{M})$. Bases in $\mathcal{B}(\mathbf{M})$ have the same size and their size is called the rank of $\mathbf{M}$. For a set $X \subseteq E$, we denote $\mathrm{rk}(X) = \max\{|I| \mid I \in \mathcal{I}, I \subseteq X\}$.

Restriction is an operation to create a matroid from another matroid, and defined as follows. Let $\mathbf{M} = (E,\mathcal{I})$ be a matroid, and $F \subseteq E$. Then, the *restriction* of $\mathbf{M}$ to F is the set system $\mathbf{M}|_F = (F, \mathcal{I}|_F)$, where $\mathcal{I}|_F = \{I \cap F \mid I \in \mathcal{I}\}$. It is known that $\mathbf{M}|_F$ is a matroid. The restriction $\mathbf{M}|_F$ is also called the *deletion* of $E \setminus F$ from $\mathbf{M}$.

A typical example of matroids is obtained from graphs. Let $G = (V,E)$ be a graph, and let $\mathcal{F}$ be the family of edge sets of forests in G. Namely, $\mathcal{F} = \{F \subseteq E \mid (V,F) \text{ is a forest}\}$. Then, $(E,\mathcal{F})$ is a matroid called a *graphic matroid* or a *cycle matroid*.

Another typical example of matroids is a uniform matroid. Let r be a non-negative integer, E be a finite set with $|E| \geq r$, and $\mathcal{I}$ be the family of all subsets of E of size at most r. Namely, $\mathcal{I} = \{X \subseteq E \mid |X| \leq r\}$. Then, $\mathbf{M} = (E,\mathcal{I})$ is a matroid of rank r called a uniform matroid.

Partition matroids are also used in this paper. A partition of a finite set E is a family $\{E_1, E_2,\dots,E_t\}$ of subsets of E such that $E_1 \cup E_2 \cup \cdots \cup E_t = E$ and $E_i \cap E_j = \emptyset$ for all $i \neq j$. A *partition matroid* defined on the partition $\{E_1,E_2,\dots,E_t\}$ of E is a matroid $\mathbf{M} = (E,\mathcal{I})$ such that there exist natural numbers $d_1, d_2,\dots,d_r$ for which

$$\mathcal{I} = \{X \subseteq E \mid |X \cap E_i| \leq d_i \ \forall i \in \{1,2,\dots,t\}\}.$$

Now, consider a general matroid $\mathbf{M}$ and fix an index $i \in \{1,2,\dots,s\}$. The requirement $|B \cap E_i| \geq r-k$ for the output of Robust Recoverable Matroid Base Problem is equivalent to the condition that there exists $B_i \in \mathcal{B}(\mathbf{M}|_{E_i})$ such that $|B \triangle B_i| \leq 2k$, as the next lemma shows.

Lemma 1. *Let $\mathbf{M} = (E,\mathcal{I})$ be a matroid of rank r, and consider $E_i \subseteq E$ with $\mathrm{rk}_{\mathbf{M}}(E_i) = r$ and $B \in \mathcal{B}(\mathbf{M})$. Then, the following two conditions are equivalent.*

1. *There exists $B_i \in \mathcal{B}(\mathbf{M}|_{E_i})$ such that $|B \triangle B_i| \leq 2k$.*
2. *$|B \cap E_i| \geq r-k$.*

The proof is postponed to the online appendix.

In this paper, when we take a matroid $\mathbf{M} = (E, \mathcal{I})$ as input to algorithms, we assume that $\mathbf{M}$ is given as an independence oracle. Namely, the oracle accepts a subset $X \subseteq E$ as a query, and decides whether $X \in \mathcal{I}$. For graphic matroids, uniform matroids and partition matroids, such oracles can be constructed in polynomial time concretely, in which each query can be processed in linear time.

The common independent set problem of two matroids can be solved in polynomial time [16]. This is a basic fact in matroid optimization. Let $\mathbf{M}_1 = (E, \mathcal{I}_1)$ and $\mathbf{M}_2 = (E, \mathcal{I}_2)$ be matroids. Their intersection is defined as $\mathbf{M}_1 \cap \mathbf{M}_2 = (E, \mathcal{I}_1 \cap \mathcal{I}_2)$. Note that $\mathbf{M}_1 \cap \mathbf{M}_2$ is not necessarily a matroid. However, the maximum-size set $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ can be obtained in polynomial time.

3. Hardness

In this section, we prove that Robust Recoverable Matroid Base Problem is NP-hard. The proofs are postponed to the online appendix.

Theorem 2. *The* Robust Recoverable Matroid Base Problem *is* NP-*hard even if a given matroid is uniform and $k \geq 1$ is constant.*

Note that a uniform matroid is not necessarily a graphic matroid.

Theorem 3. *The* Robust Recoverable Matroid Base Problem *is* NP-*hard even if a given matroid is graphic and $k \geq 1$ is constant.*

4. Fixed-parameter tractability: warm-up

We will prove that Robust Recoverable Matroid Base Problem is fixed-parameter tractable when the number s of scenarios is a parameter. Before that, in this section, we prove that Robust Recoverable Matroid Base Problem is fixed-parameter tractable when s and k are parameters. Then, in the next section we explain an algorithm when s is a parameter. We remind that a scenario is given as a subset $E_i \subseteq E$ of the ground set E.

For a positive integer s, let $[s] = \{1, 2, \ldots, s\}$. For a non-empty subset $X \subseteq [s]$, let

$$E_X = \left(\bigcap_{i \in X} E_i \right) \setminus \left(\bigcup_{i \in [s] \setminus X} E_i \right),$$

and let $E_\emptyset = E \setminus \bigcup_{X \subseteq [s], X \neq \emptyset} E_X$. Then, $\{E_X \mid X \subseteq [s]\}$ is a partition of E of size at most 2^s. Intuitively speaking, the set E_X collects the elements of E that are present exactly in the scenarios corresponding to X. If B is a solution to Robust Recoverable Matroid Base Problem, then

$$|B| = r \tag{1}$$

and for every $i \in [s]$

$$\sum_{X \subseteq [s]:\, i \in X} |B \cap E_X| = |B \cap E_i| \geq r - k. \tag{2}$$

Let $t_X = |B \cap E_X|$ for every $X \subseteq [s]$. Then, Eq. (1) can be rewritten as

$$\sum_{X \subseteq [s]} t_X = r, \tag{3}$$

and Eq. (2) can be rewritten as

$$\sum_{X \subseteq [s]:\, i \in X} t_X \geq r - k. \tag{4}$$

Our algorithm first lists all $\{t_X \mid X \subseteq [s]\}$ that satisfy Eqs. (3) and (4), and then for each candidate $\{t_X \mid X \subseteq [s]\}$ we check the existence of a base $B \in \mathcal{B}(\mathbf{M})$ that satisfies $|B \cap E_X| = t_X$ for every subset $X \subseteq [s]$. If such a base B exists for some $\{t_X \mid X \subseteq [s]\}$, then B is a solution to Robust Recoverable Matroid Base Problem. For a fixed $\{t_X \mid X \subseteq [s]\}$, such a base B can be found by solving the common independent set problem for the matroid $\mathbf{M}$ and the partition matroid defined over $\{E_X \mid X \subseteq [s]\}$. Since the size of $\{E_X \mid X \subseteq [s]\}$ is bounded by 2^s from above, the number of possible $\{t_X \mid X \subseteq [s]\}$ is bounded by r^{2^s} from above. Therefore, the running time of this algorithm is $O(r^{2^s} \mathrm{poly}(|E|))$. Below, to improve this running time, we give a better upper bound for the number of possible $\{t_X \mid X \subseteq [s]\}$.

Theorem 4. *The* Robust Recoverable Matroid Base Problem *can be solved in $O((sk)^{2^s} \mathrm{poly}(|E|))$ time.*

Proof. Let B be a solution to a given instance of Robust Recoverable Matroid Base Problem. We show that $|B \setminus E_{[s]}| \leq sk$. This means that $\sum_{X \subsetneq [s]} t_X \leq sk$, which further implies that the number of possible $\{t_X \mid X \subseteq [s]\}$ is bounded by $(sk)^{2^s} |E|$ from above. Then, the proof will be finished.

By the definition of our problem, it holds that

$$\sum_{i \in [s]} |B \cap E_i| \geq \sum_{i \in [s]} (r - k) \geq s(r - k).$$

On the other hand,

$$\begin{aligned} \sum_{i \in [s]} |B \cap E_i| &= \sum_{e \in B} |\{i \in [s] \mid e \in E_i\}| \\ &\leq s|B| - |B \setminus E_{[s]}| \\ &= sr - |B \setminus E_{[s]}|. \end{aligned}$$

By combining these two, we obtain $|B \setminus E_{[s]}| \leq sk$. $\square$

5. Fixed-parameter tractability: main result

In the previous section, we proved the fixed-parameter tractability with respect to s and k. In this section, we will prove the fixed-parameter tractability with respect to s only. To this end, we use a result by Edmonds for matroid polytopes.

Lemma 5 *(Edmonds [16]). Let $\mathbf{M} = (U, \mathcal{I})$ be a matroid and $I \subseteq U$. Then, $I \in \mathcal{I}$ if and only if $|I \cap U'| \leq \mathrm{rk}(U')$ for every subset $U' \subseteq U$.* $\square$

Let P be the set of $(t, x) \in \mathbb{Z}^{2^{[s]}} \times \{0, 1\}^E$ that satisfy the following conditions:

$$\sum_{X \subseteq [s]} t_X = r, \tag{5}$$

$$\sum_{X \subseteq [s]:\, i \in X} t_X \geq r - k \qquad (\forall i \in [s]), \tag{6}$$

$$\sum_{e \in E_X} x_e = t_X \qquad (\forall X \subseteq [s]), \tag{7}$$

$$\sum_{e \in E'} x_e \leq \mathrm{rk}(E') \qquad (\forall E' \subseteq E). \tag{8}$$

Lemma 6. *A solution to* Robust Recoverable Matroid Base Problem *exists if and only if $P \neq \emptyset$.*

Proof. Assume that B is a solution to ROBUST RECOVERABLE MATROID BASE PROBLEM. Then, let $t_X = |B \cap E_X|$ for each $X \subseteq [s]$. We already showed that Eqs. (5) and (6) are satisfied in Section 4. Let $x = \chi_B$, the characteristic vector of B. Then, Eq. (7) is satisfied since $t_X = |B \cap E_X|$. Furthermore, by Lemma 5, Eq. (8) is also satisfied. Hence, $(t, x) \in P$.

Assume that $P \neq \emptyset$, and let (t, x) be an element of P. Define B as the set of $e \in E$ with $x_e = 1$. Then, from Eqs. (5) and (7), it holds that $|B| = r$. Therefore, Eq. (8) and Lemma 5 imply that $B \in \mathcal{B}(\mathbf{M})$. Since $|B \cap E_X| = t_X$ for every $X \subseteq [s]$ from Eq. (7), it holds that for each $i \in [s]$

$$|B \cap E_i| = \sum_{X \subseteq [s]:\, i \in X} |B \cap E_X| = \sum_{X \subseteq [s]:\, i \in X} t_X.$$

Hence, by Eq. (6) it follows that $|B \cap E_i| \geq r - k$ for every $i \in [s]$, and B is a solution to ROBUST RECOVERABLE MATROID BASE PROBLEM. $\square$

From Lemma 6, to decide the existence of a solution to ROBUST RECOVERABLE MATROID BASE PROBLEM, it suffices to decide whether $P \neq \emptyset$. Furthermore, from the proof of Lemma 6, we can construct a solution to ROBUST RECOVERABLE MATROID BASE PROBLEM from an element of P provided that $P \neq \emptyset$.

For each subfamily $\mathcal{S} \subseteq 2^{[s]}$, we denote by $\delta(\mathcal{S})$ the set of $e \in E$ for which there exists $S \in \mathcal{S}$ such that $e \in E_S$. Namely, $\delta(\mathcal{S}) = \bigcup_{S \in \mathcal{S}} E_S$.

The following result by McDiarmid [25] is inevitable for our algorithm. To state the result, we introduce some terms and symbols.

A bipartite graph is denoted by $(A, B; F)$ with a bipartition $A \cup B$ of the vertex set and its edge set F. For a vertex $v \in A \cup B$, we denote by $\delta(v)$ the set of edges incident to v. For a vertex subset $X \subseteq A$ (or $X \subseteq B$), we denote by $\partial(X)$ the set of vertices adjacent to a vertex in X.

Lemma 7 *(McDiarmid [25, Proposition 2B]). Let $G = (A, B; F)$ be a bipartite graph, $\mathbf{M} = (B, \mathcal{J})$ be a matroid, and $y \in \mathbb{Z}_+^A$ be an integral vector. Then, there exist a vector $x \in \mathbb{Z}_+^B$ and a vector $z \in \mathbb{Z}_+^F$ such that*

$$\sum_{b \in B'} x_b \leq \mathrm{rk}(B') \quad (\forall B' \subseteq B),$$

$$\sum_{e \in \delta(a)} z_e = y_a \quad (\forall a \in A),$$

$$\sum_{e \in \delta(b)} z_e = x_b \quad (\forall b \in B)$$

if and only if

$$\sum_{a \in A'} y_a \leq \mathrm{rk}(\partial(A')) \quad (\forall A' \subseteq A).$$

The application of Lemma 7 to our situation immediately gives the following lemma, by using the bipartite graph $(A, B; F)$ defined by

$$A = 2^{[s]}, \quad B = E, \quad \text{and}$$

$$F = \{\{X, e\} \mid X \in 2^{[s]}, e \in E_X \subseteq E\},$$

and by setting $y = t$ and $z_{\{X,e\}} = x_e$ if $e \in E_X$; The detail is left to the reader.

Lemma 8. *Let $t \in \mathbb{Z}_+^{2^{[s]}}$. Then, there exists $x \in \{0, 1\}^E$ that satisfies Eqs. (7) and (8) if and only if*

$$\sum_{X \in \mathcal{S}} t_X \leq \mathrm{rk}(\delta(\mathcal{S})) \quad (\forall \mathcal{S} \subseteq 2^{[s]}). \qquad \square$$

By Lemma 8, P is non-empty if and only if there exists t that satisfies the following conditions:

$$\sum_{X \subseteq [s]} t_X = r, \tag{9}$$

$$\sum_{X \subseteq [s]:\, i \in X} t_X \geq r - k, \qquad (\forall i \in [s]), \tag{10}$$

$$\sum_{X \in \mathcal{S}} t_X \leq \mathrm{rk}(\delta(\mathcal{S})) \qquad (\forall \mathcal{S} \subseteq 2^{[s]}), \tag{11}$$

$$t \in \mathbb{Z}_+^{2^{[s]}}. \tag{12}$$

When such t exists, we compute a maximum-size common independent set I^* of $\mathbf{M}$ and the matroid $\mathbf{M}' = (E, \mathcal{I}')$ defined as

$$\mathcal{I}' = \{I \subseteq E \mid \forall X \subseteq [s], |I \cap E_X| \leq t_X\},$$

and define $x = \chi_{I^*}$, the characteristic vector of I^*. Then, $x \in \{0, 1\}^E$ satisfies Eqs. (7) and (8).

To decide whether there exists t that satisfies Eqs. (9)–(12), we use an algorithm for integer programming. As in the next lemma, the fixed-parameter tractability of integer programming is well-known.

Lemma 9 *(Lenstra [23]). Let U, V be finite sets, $A \in \mathbb{R}^{V \times U}$ a matrix, and $b \in \mathbb{R}^V$ a vector. If the rank of A is ℓ, then the problem of deciding whether the set*

$$\{x \in \mathbb{Z}^U \mid Ax \leq b\}$$

is empty is fixed-parameter tractable with respect to ℓ. $\square$

The running time of the current fastest algorithm to solve the problem in Lemma 9 is $2^{O(\ell \log \ell)}$ multiplied by a polynomial of the input size [12,13].

If we write Eqs. (9)–(12) in the form of Lemma 9, the coefficient matrix will have 2^s columns and $2 + s + 2^{2^s} + 2^s$ rows. Hence, we can apply Lemma 9 with $\ell \leq 2^s$, and obtain the following theorem.

Theorem 10. *The* ROBUST RECOVERABLE MATROID BASE PROBLEM *can be solved in $O(2^{O(s2^s)} \mathrm{poly}(|E|))$ time.* $\square$

In particular, when $s = O(\log \log |E|)$, ROBUST RECOVERABLE MATROID BASE PROBLEM can be solved in polynomial time.

6. Conclusion

Possible future work is to investigate other models of robustness. This paper concentrated on minimizing $\max_i |B \triangle B_i|$, but we may also minimize $\sum_i |B \triangle B_i|$, which corresponds to the expectation minimization of the second-stage modification cost.

Approximation should also be studied. For example, we may try to approximate the minimum possible value of k such that there exists a basis B with $|B \cap E_i| \geq r - k$ for all $i \in \{1, \dots, s\}$.

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Appendix A. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.orl.2022.05.001.

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