A result of the Gauss-Bonnet theorem for coherent tangent bundles over surfaces with boundary

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1 Introduction

In this paper, I present an original generalization of Fact3.3 of [2] to the case of 2-manifolds with boundary. The classical Gauss-Bonnet theorem is discussed on regular surfaces, and it is well known that there are two types, the local and global theorem. The following theorem is a generalization of the classical global Gauss-Bonnet's theorem for coherent tangent bundles.

Theorem 1.1 (Fact3.3 of [2]) *Let $(M, \mathcal{E}, \langle\ ,\ \rangle, D, \varphi)$ be a coherent tangent bundle over a compact oriented 2-dimensional manifold M, and suppose that the singular set $\Sigma(\varphi)$ consists of singular points of the first and admissible second kind. We denote by K the Gaussian curvature of the induced metric $ds^2 = \varphi^*\langle\ ,\ \rangle$. Then it holds that*

$$\frac{1}{2\pi}\int_M K d\hat{A} = \chi(M^+) - \chi(M^-) + \#S^+ - \#S^-, \tag{1}$$

$$2\pi\chi(M) = \int_M K dA + 2\int_{\Sigma(\varphi)} \kappa_s ds, \tag{2}$$

where $\#S^+$ (resp. $\#S^-$) are the numbers of positive (resp. negative) singular points of the second kind, and κ_s is the singular curvature of $\Sigma(\varphi)$.

The concepts described in the above theorem are defined precisely in §2. I discuss a generalization of (1) and (2) for manifolds with boundary (See Theorem3.3 for details).

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2 Preliminaries

From now on, let M be a compact oriented 2-manifold with boundary.

The set of C^∞- functions on M is denoted by $C^\infty(M)$, and the set of C^∞-sections of a vector bundle $\mathcal{E}$ is denoted by $\Gamma(\mathcal{E})$. Let $\mathcal{E}^*$ be a dual vector bundle of $\mathcal{E}$. $\langle\ ,\ \rangle \in \Gamma(\mathcal{E}^* \otimes \mathcal{E}^*)$ is called an *inner product* on $\mathcal{E}$, when $\langle\ ,\ \rangle$ defines a positive definite inner product on each fiber $\mathcal{E}_p$ $(p \in M)$. A map $D : \Gamma(TM) \times \Gamma(\mathcal{E}) \ni (X, \xi) \mapsto D_X\xi \in \Gamma(\mathcal{E})$ is called a *connection* on $\mathcal{E}$ if, for any $f, g \in C^\infty(M), X, Y \in \Gamma(TM), \xi, \eta \in \Gamma(\mathcal{E})$, the following conditions hold.

(1) $D_{fX+gY}\xi = fD_X\xi + gD_Y\xi$,

(2) $D_X(\xi + \eta) = D_X\xi + D_X\eta$,

(3) $D_X(f\xi) = (Xf)\xi + fD_X\xi$.

A connection D on $\mathcal{E}$ with an inner product is called a *metric connection* if it holds that

$$X\langle \xi, \eta\rangle = \langle D_X\xi, \eta\rangle + \langle \xi, D_X\eta\rangle \ (X \in \Gamma(TM), \xi, \eta \in \Gamma(\mathcal{E})).$$

Definition 2.1 Let $\mathcal{E}$ a vector bundle of rank 2 with an inner product $\langle\ ,\ \rangle$, $D : \Gamma(TM) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ a metric connection on $\mathcal{E}$, and $\varphi : TM \to \mathcal{E}$ a bundle homomorphism. A 5-tuple $(M, \mathcal{E}, \langle\ ,\ \rangle, D, \varphi)$ is called a *coherent tangent bundle* if, for any $X, Y \in \Gamma(TM)$, it holds that

$$D_X\varphi(Y) - D_Y\varphi(X) - \varphi([X, Y]) = 0.$$

Example 2.2 Let (M, g) be a 2-dimensional Riemannian manifold and $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ a Levi-Civita connection on TM. If we take an identity map $\mathrm{id} : TM \to TM$ as a bundle homomorphism, then a 5-tuple $(M, TM, g, \nabla, \mathrm{id})$ is a coherent tangent bundle. This shows that the concept of coherent tangent bundles is a generalization of Riemannian manifolds.

From now on, let $\mathcal{E}$ be a oriented vector bundle.

If, for each point $p \in M$, $(ds^2)_p : T_pM \times T_pM \to \mathbb{R}$ is defined by

$$(ds^2)_p(\boldsymbol{v}_1, \boldsymbol{v}_2) = \langle \varphi_p(\boldsymbol{v}_1), \varphi_p(\boldsymbol{v}_2)\rangle \ (\boldsymbol{v}_1, \boldsymbol{v}_2 \in T_pM),$$

ds^2 is called the *first fundamental form* of φ. A point $p \in M$ is called a *regular point* of φ if $\varphi_p : T_pM \to \mathcal{E}_p$ is a linear isomorphism. A point $p \in M$ is called a *singular point* of φ if φ_p is not a linear isomorphism. The set of singular points of φ is denoted by $\Sigma(\varphi)$.

Let $(U; u, v)$ be a positive local coordinate system of M, $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ an positive orthonormal basis field on $\mathcal{E}|_U$, and $\{\omega_1, \omega_2\}$ a dual basis field of $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. If a section $\mu : U \to \mathcal{E}^*|_U \wedge \mathcal{E}^*|_U$ is defined by

$$\mu := \omega_1 \wedge \omega_2,$$

then μ is independent of the choice of positive orthonormal basis fields of $\mathcal{E}|_U$. Therefore, μ is defined on M. A differential 2 form $d\hat{A}$ on M is called a *signed area form* of $\mathcal{E}$ if, for each point $p \in M$, $(d\hat{A})_p : T_pM \times T_pM \to \mathbb{R}$ is defined by

$$(d\hat{A})_p(\boldsymbol{v}_1, \boldsymbol{v}_2) = \mu(\varphi_p(\boldsymbol{v}_1), \varphi_p(\boldsymbol{v}_2)) \ (\boldsymbol{v}_1, \boldsymbol{v}_2 \in T_pM).$$

If a *signed area density function* $\lambda \in C^\infty(U)$ is defined by

$$\lambda := \mu\left(\varphi\left(\frac{\partial}{\partial u}\right), \varphi\left(\frac{\partial}{\partial v}\right)\right),$$

then it holds that

$$\Sigma(\varphi) \cap U = \{p \in U \mid \lambda(p) = 0\}, \ d\hat{A} = \lambda du \wedge dv.$$

In particular, $d\hat{A}$ defines a C^∞-differential 2 form on M. On the other hand, $dA := |\lambda| du \wedge dv$ does not depend on the choice of positive local coordinate systems of M. Thus, dA defines a continuous differential 2 form on M, and dA is called an *area form* of $\mathcal{E}$. If we set

$$M^+ := \left\{p \in M \backslash \Sigma(\varphi) \mid dA_p = d\hat{A}_p\right\}, \ M^- := \left\{p \in M \backslash \Sigma(\varphi) \mid dA_p = -d\hat{A}_p\right\},$$

then $\Sigma(\varphi)$ is coincide with ∂M^+ and ∂M^-, respectively. Using a signed area density function λ, we have

$$M^+ \cap U = \{p \in U \mid \lambda(p) > 0\}, \ M^- \cap U = \{p \in M \mid \lambda(p) < 0\}.$$

Since $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is the positive orthonormal basis field on $\mathcal{E}|_U$, it holds that

$$2\langle D_X \boldsymbol{e}_i, \boldsymbol{e}_i\rangle = \langle D_X \boldsymbol{e}_i, \boldsymbol{e}_i\rangle + \langle \boldsymbol{e}_i, D_X \boldsymbol{e}_i\rangle = X\langle \boldsymbol{e}_i, \boldsymbol{e}_i\rangle = 0 \ (X \in \Gamma(TU)).$$

Thus, $D_X \boldsymbol{e}_1$ is orthogonal to $\boldsymbol{e}_1$, and a C^∞-differential 1 form ω on U is defined by

$$D_X \boldsymbol{e}_1 = -\omega(X)\boldsymbol{e}_2, \ D_X \boldsymbol{e}_2 = \omega(X)\boldsymbol{e}_1.$$

The exterior derivative $d\omega$ does not depend on the choice of positive orthonormal basis fields of $\mathcal{E}|_U$. Therefore, $d\omega$ defines a C^∞-differential 2 form on M. Let K be the Gaussian curvature of ds^2 on the set of regular points of φ. Then it holds that

$$d\omega = K d\hat{A} = \begin{cases} K dA & (\text{on } M^+ \backslash \Sigma(\varphi)), \\ -K dA & (\text{on } M^- \backslash \Sigma(\varphi)). \end{cases}$$

A singular point $p \in \Sigma(\varphi) \cap U$ is called *non-degenerate* if it holds that

$$(\lambda_u(p), \lambda_v(p)) \neq (0, 0).$$

If $p \in \Sigma(\varphi) \cap U$ is non-degenerate, by the implicit function theorem, there exists a regular curve $\gamma(t)$ on U through the point p such that it holds $\lambda(\gamma(t)) = 0$. This curve γ is called a *singular curve*, the tangent vector $\dot{\gamma}(t)$ is called a *singular vector*, and a 1-dimensional vector space generated by $\dot{\gamma}(t)$ is called a *singular* *direction.* A tangent vector $\boldsymbol{v} \in T_pU\setminus\{\boldsymbol{0}\}$ at $p \in \Sigma(\varphi)\cap U$ is called a *null vector* if it holds that $\varphi_p(\boldsymbol{v}) = \boldsymbol{0}$, and a vector space generated by the null vector is called a *null direction.* We remark that a null direction at a non-degenerate singular point is 1-dimensional.

From now on, we assume that $\Sigma(\varphi)$ consists of non-degenerate singular points of φ. A singular point of $\Sigma(\varphi)$ is called a *singular point of the first kind* (resp. *singular point of the second kind*) if the singular and null direction at the point are different (resp. same). If there are only singular points of the first kind around a singular point of the second kind of φ, the point is called an *admissible singular point of the second kind.*

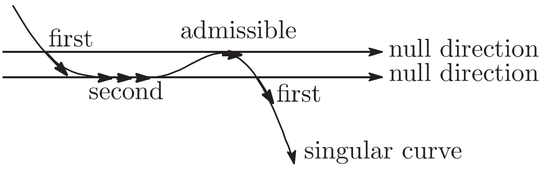

Figure 1:

From now on, we assume that $\Sigma(\varphi)$ consists of singular points of the first kind and admissible second kind.

If a singular point $p \in \Sigma(\varphi)\cap U$ is a first kind, there is a singular curve $\gamma(t)$ through the point p. Since the point p is the first kind, the singular and null directions at the point p are different, and the singular and null directions along γ change continuously, so we retake $(U; u, v)$ such that $\Sigma(\varphi)\cap U$ consists of singular points of the first kind. Then we set

$$\kappa_s(t) := \operatorname{sgn}(\lambda_\eta(t))\frac{\mu\left(\varphi_{\gamma(t)}(\dot\gamma(t)), D_{\dot\gamma(t)}(\varphi\circ\dot\gamma)\right)}{|\varphi_{\gamma(t)}(\dot\gamma(t))|^3}, \tag{3}$$

where $\eta(t)$ is a null vector at each point $\gamma(t)$ such that $\{\dot\gamma(t), \eta(t)\}$ is positively oriented, and we set

$$\lambda_\eta(t) := (d\lambda)_{\gamma(t)}(\eta(t)),\ |\varphi_{\gamma(t)}(\dot\gamma(t))| := \sqrt{\langle\varphi_{\gamma(t)}(\dot\gamma(t)), \varphi_{\gamma(t)}(\dot\gamma(t))\rangle}.$$

The equality (3) is called a *singular curvature*. We remark that κ_s is independent of the choice of parameters of γ, the orientation of γ, the orientation of M, and the orientation of $\mathcal{E}$. If a point $p \in M$ be an admissible singular point of the second kind and $\gamma(t)$ is a singular curve through p, $\kappa_s(t)ds$ $\left(ds := |\varphi_{\gamma(t)}(\dot\gamma(t))|dt\right)$ defines a bounded differential 1 form on $\gamma(t)$.

Consider triangulating M. We triangulate $M^\pm \cup \Sigma(\varphi)$ such that vertexes are singular points of the second kind. Then we triangulate M by subdividing such that their triangles on $M^\pm$ are properly congruent on $\Sigma(\varphi)$. As a result, such a triangulation has the following properties.

- Singular points appear on edges of a triangle, and the interior of the triangle consists of regular points. A singular point appears on an edge other than a vertex only if the edge consists of singular points (Such an edge is called a *singular edge*).

- A triangle has at most one singular edge.
- All edges other than a singular edge are gathered at singular vertexes from directions other than a null direction.
- There is no possibility that points at both ends of a singular edge are singular points of the second kind at the same time.

Triangles obtained by triangulation are classified into the following four types.

(1) A triangle consists of regular points.

(2) A triangle with one singular vertex consists of regular points except for that vertex.

(3) A triangle with one singular edge consists of regular points except for that edge, and does not have singular points of the second kind.

(4) A triangle with one singular edge consists of regular points except for that edge, and have a singular point of the second kind.

We remark that, at a non-degenerate singular point p, there exists a positive local coordinate system (u, v) with the origin at point p such that the null direction is parallel to the u-axis along the singular curve passing through point p. If the vertex A of $\triangle ABC$ is a singular point and we take the above local coordinate system (u, v) around A, the angle $\angle A$ is defined as follows:

$$\angle A := \begin{cases} \pi & \text{if the } u\text{-axis passes through the interior of } \triangle ABC, \\ 0 & \text{otherwise.} \end{cases}$$

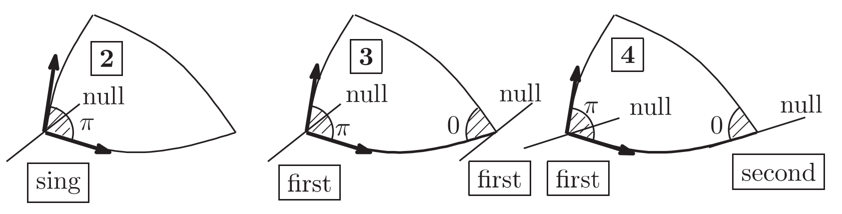

Figure 2:

3 The Gauss-Bonnet Theorem

Let $\gamma(t)$ be the parameterization of one of the sides of the triangles obtained by triangulating M. We define the *geometric curvature* $\tilde{\kappa}_g$ of γ as follows:

$$\tilde{\kappa}_g(t) = \begin{cases} \kappa_g(t) & \text{if } \gamma(t) \in M^+, \\ -\kappa_g(t) & \text{if } \gamma(t) \in M^-, \\ \kappa_s(t) & \text{if } \gamma(t) \in \Sigma(\varphi), \end{cases} \qquad \kappa_g(t) := \frac{\mu\left(\varphi_{\gamma(t)}(\dot{\gamma}(t)), D_{\dot{\gamma}(t)}(\varphi \circ \dot{\gamma})\right)}{|\varphi_{\gamma(t)}(\dot{\gamma}(t))|^3}.$$

Theorem 3.1 (Local Gauss-Bonnet Theorem) Let $(M, \mathcal{E}, \langle\ ,\ \rangle, D, \varphi)$ be a coherent tangent bundle over a compact oriented 2-manifold. If $\triangle ABC$ is a triangle obtained by triangulating M, then it holds that

$$\int_{\partial \triangle ABC} \tilde{\kappa}_g ds + \int_{\triangle ABC} K dA = \angle A + \angle B + \angle C - \pi.$$

Let $p \in \Sigma(\varphi)$ be an admissible singular point of the second kind. The sum of interior angles on side M^+ (resp. M^-) at p is denoted by $\alpha_+(p)$ (resp. $\alpha_-(p)$). Then it holds that

$$\alpha_+(p) + \alpha_-(p) = 2\pi,\ \alpha_+(p) - \alpha_-(p) \in \{-2\pi, 0, 2\pi\}.$$

A point p is called *positive* (resp. *null*, *negative*), If $\alpha_+(p) - \alpha_-(p) > 0$ (resp. $\alpha_+(p) - \alpha_-(p) = 0$, $\alpha_+(p) - \alpha_-(p) < 0$).

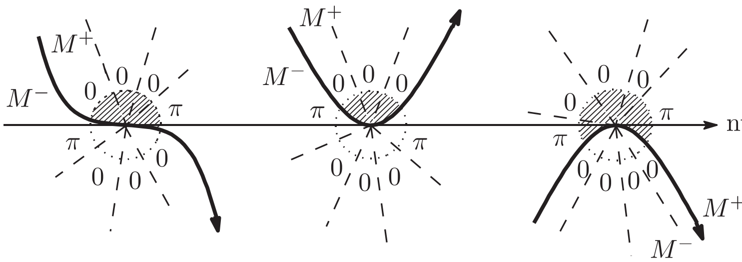

Figure 3:

Suppose that $\Sigma(\varphi)$ and ∂M satisfy the following conditions:

(1) $\Sigma(\varphi)$ and ∂M are transversal.

(2) All singular points on $\Sigma(\varphi) \cap \partial M$ are the first kind.

(3) The null direction of a singular point on $\Sigma(\varphi) \cap \partial M$ is not tangent to ∂M.

(1) implies that tangent directions of $\Sigma(\varphi)$ and ∂M at a singular point on ∂M are different. (2) implies that a null direction at the singular point on ∂M is not tangent to $\Sigma(\varphi)$.

Definition 3.2 We take a local coordinate system (u, v) around a singular point p on ∂M such that the null direction along the singular curve is parallel to the u-axis. The point p is called *positive* (resp. *negative*) if the u-axis passes through M^+ (resp. M^-)

The sum of interior angles on side M^+ (resp. M^-) at a singular point p on ∂M is denoted by $\beta_+(p)$ (resp. $\beta_-(p)$). Then it holds that

$$\beta_+(p) + \beta_-(p) = \pi,\ \beta_+(p) - \beta_-(p) \in \{-\pi, \pi\}.$$

By Definition3.2, a point p is positive (resp. negative) if and only if it holds that $\beta_+(p) - \beta_-(p) = \pi$ (resp. $\beta_+(p) - \beta_-(p) = -\pi$).

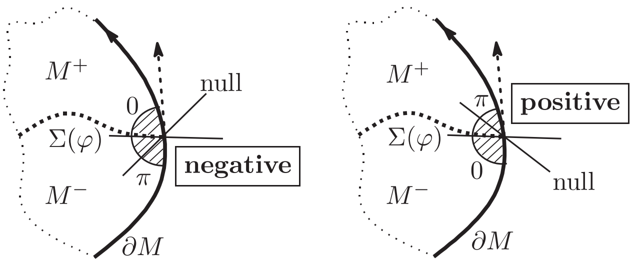

Figure 4:

Theorem 3.3 Let $(M, \mathcal{E}, \langle\ ,\ \rangle, D, \varphi)$ be a coherent tangent bundle over a compact oriented 2-manifold with boundary M, and suppose that the singular set $\Sigma(\varphi)$ consists of singular points of the first kind and admissible second kind. We set

$$\int_M K dA := \int_{M^+} K dA + \int_{M^-} K dA.$$

Then it holds that

$$\begin{aligned}\int_M K dA + 2\int_{\Sigma(\varphi)} \kappa_s ds + \int_{\partial M} \tilde{\kappa}_g ds &= 2\pi\chi(M) - \pi(\#(\Sigma(\varphi) \cap \partial M)),\\ \int_{\partial M} \kappa_g ds + \int_M K d\hat{A} &= 2\pi\left\{\chi(M^+) - \chi(M^-)\right\} + 2\pi(\#S^+ - \#S^-)\\ &\quad +\pi(\#(\Sigma(\varphi) \cap \partial M)^+ - \#(\Sigma(\varphi) \cap \partial M)^-),\end{aligned}$$

where $\#S^+$ (resp. $\#S^-$) are the numbers of positive (resp. negative) singular points of the second kind, $\#(\Sigma(\varphi) \cap \partial M)$ is the numbers of singular points on ∂M, and $\#(\Sigma(\varphi) \cap \partial M)^+$ (resp. $\#(\Sigma(\varphi) \cap \partial M)^-$) are the numbers of positive (resp. negative) singular points on ∂M .

Remark 3.4 The result presented here is closely related to Theorem 2.20 of [4]. Their theorem is a generalization of Theorem B of [1] to the case of manifolds with boundary, and it is generalized by considering peaks instead of admissible singular points of the second kinds. Several applications are presented in [4].

References

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