APPENDIX TO "DIFFEOMORPHISM CLASSES OF THE DOUBLING CALABI-YAU THREEFOLDS WITH PICARD NUMBER TWO"

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1. Introduction

This is an appendix to the author's paper entitled "Diffeomorphism classes of the doubling Calabi-Yau threefolds with Picard number two [Y21]" where he proved that any two of the doubling Calabi-Yau 3-folds with Picard number 2 are not diffeomorphic to each other when the underlying Fano 3-folds are distinct. We refer the reader to [Y21] for background on the problem and terminology discussed in this note.

As listed in Table 1 below, there are 8 doubling Calabi-Yau 3-folds M with Picard number 2 which have the same Hodge numbers $(h^{1,1}(M), h^{2,1}(M))$. These 8 overlapping Hodge numbers $(h^{1,1}(M), h^{2,1}(M))$ are listed with ✓ on the table. Furthermore, in Table 1, V denote the underlying Fano 3-folds which are the ingredients for the doubling construction of Calabi-Yau 3-folds in [DY14]. See [DY14, Section 6], for more details. This note aims to summarize computational details of

(i) the cubic forms, and
(ii) the λ-invariants

which we will use for the proof of Theorem 1.1 in [Y21].

Table 1. The doubling Calabi-Yau 3-folds with Picard number 2 and the underlying Fano 3-folds with Picard number 1

ID in [FG]	$-K_V^3$	$h^{1,2}(V)$	$(h^{1,1}(M), h^{2,1}(M))$
1-1	2	52	$(2,128)$
1-2	4	30	✓ $(2,86)$
1-3	6	20	$(2,68)$
1-4	8	14	✓ $(2,58)$
1-5	10	10	$(2,52)$
1-6	12	7	$(2,48)$
1-7	14	5	$(2,46)$
1-8	16	3	✓ $(2,44)$
1-9	18	2	✓ $(2,44)$
1-10	22	0	✓ $(2,44)$
1-11	8	21	$(2,72)$
1-12	16	10	✓ $(2,58)$
1-13	24	5	$(2,56)$
1-14	32	2	✓ $(2,58)$
1-15	40	0	$(2,62)$
1-16	54	0	$(2,76)$
1-17	64	0	✓ $(2,86)$

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2. $(h^{1,1}(M), h^{2,1}(M)) = (2, 86)$ CASE

These doubling Calabi-Yau 3-folds are listed in Table 1 with the underlying Fano 3-folds, (a) ID 1-2 and (b) ID 1-17. Geometric description of the corresponding Fano 3-folds are

(a) a quartic hypersurface in $\mathbb{C}P^4$; $V(4) \subset \mathbb{C}P^4$, and
(b) the projective space $\mathbb{C}P^3$.

2.1. **ID 1-2:** $V(4) \subset \mathbb{C}P^4$ **case.** Let V be a quartic hypersurface in $\mathbb{C}P^4$. Note that V is the Fano 3-fold with $-K_V^3 = 4$ (see [IsPr99, p.215]). By Lefschetz Hyperplane Theorem, we have more specific description of V such as

$$h^{p,q}(V) = \begin{matrix} &&&1&&& \\ &&0&&0&& \\ &0&&1&&0& \\ 0&&30&&30&&0 \\ &0&&1&&0& \\ &&0&&0&& \\ &&&1&&& \end{matrix}, \qquad g = g(V) = \frac{H^3}{2} + 1 = \frac{-K_V^3}{2} + 1 = 3$$

where g denotes the genus of Fano variety. In particular, $H^3 = 4$ for the ample generator $H \in H^2(V, \mathbb{Z})$. Let $D \in |-K_V|$ be a smooth anticanonical divisor and let $C \in |\mathcal{O}_D(1)|$ be a smooth curve in D which represents the intersection class of $D \cdot D$. Then the degree of C is $2g - 2$ and this is the reason why $g = \frac{-K_V^3}{2} + 1$ is called the *genus* of a Fano 3-fold [IsPr99, p.32]. Taking Y_i to be the blow-ups $\mathrm{Bl}_C(V)$ of V along C, we again denote the exceptional divisors by E_i for $i = 1, 2$. Then the cohomology rings of Y_i are

$$H^2(Y_i) = \mathbb{C}\langle \pi_i^*(H), E_i \rangle = \mathbb{C}\langle H_i, E_i \rangle$$

and the proper transforms D_i of D in Y_i are $H_i - E_i$. Let $\delta = \langle -D_1, D_2 \rangle = \langle E_1 - H_1, H_2 - E_2 \rangle$. Then we see that any element in $H^2(Y_1, \mathbb{Z}) \times H^2(Y_2, \mathbb{Z})$ is written as

$$(aH_1 + bE_1, cE_2 + (a + b - c)H_2) = (a + b)(H_1, H_2) - (b + c)(H_1 - E_1, 0) - c\delta.$$

Thus we conclude that

$$H^2(M, \mathbb{Z}) \cong \langle (H_1, H_2), (H_1 - E_1, 0) \rangle$$

up to torsion. Hence in this case, we take $e_1 = (H_1, H_2)$ and $e_2 = (H_1 - E_1, 0)$ as generators of $H^2(M, \mathbb{Z})$.

Now we compute the cubic products of e_i in $H^6(M, \mathbb{Z})$. Let us denote by $\pi_i : Y_i = \mathrm{Bl}_C(V) \dashrightarrow V$ two copies of the blow-ups of V along C for $i = 1, 2$. Let L be a fiber over a point on C under the blow-up π_i. Since the intersection number is preserved by the total transform, we see that $H_i^3 = (\pi_i^* H)^3 = H^3 = 4$. Moreover, $H_i L = 0$ and $E_i L = -1$. Let d be the degree of C. Since a hyperplane in V will intersect C in d points, its inverse image H_i in Y_i will meet the exceptional divisor E_i in d fibers. Thus

$$H_i E_i = dL = (2g - 2)L = 4L \qquad \text{and} \qquad E_i^2 = -4H_i^2 + 8L.$$

Then we see that

$$H_i^2 E_i = 4H_i L = 0, \qquad H_i E_i^2 = 4E_i L = -4 \qquad \text{and}$$
$$E_i^3 = -4H_i^2 E_i + 8LE_i = -8.$$

In sum, we find the following table of the multiplication of the intersection forms on $H^{2*}(Y_i, \mathbb{Z})$:

	H_i^2	L	$H^4(Y_i, \mathbb{Z})$
H_i	4	0	
E_i	0	-1	
$H^2(Y_i, \mathbb{Z})$			

	H_i	E_i	$H^2(Y_i, \mathbb{Z})$
H_i	H_i^2	$4L$	
E_i	$4L$	$-4H_i + 8L$	
$H^2(Y_i, \mathbb{Z})$			

Plugging these values into the products, we find that

$$\begin{aligned} e_1^3 &= (H_1, H_2)^3 = H_1^3 + H_2^3 = 8, \\ e_1^2 e_2 &= (H_1, H_2)^2 (H_1 - E_1, 0) = H_1^3 - H_1^2 E_1 = 4, \\ e_1 e_2^2 &= (H_1, H_2)(H_1 - E_1, 0)^2 = H_1^3 - 2H_1^2 E_1 + H_1 E_1^2 = 4 - 4 = 0, \\ e_2^3 &= (H_1 - E_1, 0)^3 = H_1^3 - 3H_1^2 E_1 + 3H_1 E_1^2 - E_1^3 = 4 - 0 + 3 \cdot (-4) - (-8) = 0. \end{aligned}$$

Next we calculate the λ-invariant of the resulting doubling Calabi-Yau 3-fold M. Since V is a degree 4 smooth hypersurface in $\mathbb{C}P^4$, the total Chern classes of V are given by the formula

$$\frac{(1+H)^5}{(1+4H)} = (1 + 5H + 10H^2)(1 - 4H + 16H^2) + O(H^3) = 1 + H + 6H^2 + O(H^3).$$

Hence we find that the second Chern classes of Y_i are given by

$$c_2(Y_i) = \pi_i^*(c_2(V) + \eta_C) - \pi_i^*(c_1(V)) \cdot E_i = 7H_i^2 - H_i E_i \tag{2.1}$$

by [GH, p.610], where η_C denotes the class of the blow-up center $C \in |\mathcal{O}_D(1)|$. Then the products of $c_2(M)$ and e_i $(i = 1, 2)$ are

$$\begin{aligned} e_1 \cdot c_2(M) &= 7H_1^3 - H_1^2 E_1 + 7H_2^3 - H_2^2 E_2 = 56 = 8 \cdot 7, \\ e_2 \cdot c_2(M) &= (7H_1^2 - H_1 E_1)(H_1 - E_1) \\ &= 7H_1^3 - H_1^2 E_1 - 7H_1^2 E_1 + H_1 E_1^2 \\ &= 7 \cdot 4 - 4 = 24 = 8 \cdot 3. \end{aligned}$$

Since the subgroup $\{\, e \in \langle e_1, e_2 \rangle \mid e \cdot c_2(M) = 0 \,\}$ of $H^2(M, \mathbb{Z})$ is generated by a single element $3e_1 - 7e_2$, the λ-invariant of M is

$$\begin{aligned} \lambda(M) &= |(3e_1 - 7e_2)^3| = |27e_1^3 - 189e_1^2 e_2 + 441 e_1 e_2^2 - 343 e_2^3| \\ &= |27 \cdot 8 - 189 \cdot 4| = 540. \end{aligned}$$

2.2. **ID** 1-17**:** $\mathbb{C}P^3$ **case.** The detailed calculations are written in [Y21]. Hence this subsection only collects the most basic part of computation on the cubic forms and the λ-invariant.

We set $V = \mathbb{C}P^3$, $D \in |\mathcal{O}_V(4)|$, $C \in |\mathcal{O}_D(4)|$ and $\pi_i : Y_i = \mathrm{Bl}_C(V) \dashrightarrow V$ for $i = 1, 2$, respectively. Then we have $H^2(Y_i) = \mathbb{C}\langle H_i, E_i \rangle$ with $E_i = \pi_i^{-1}(C)$ and $H_i = \pi_i^*(H) \subset Y_i$ for $H \in H^2(V, \mathbb{Z})$. Furthermore, the proper transform D_i of D in Y_i is $4H_i - E_i$ for each i. Then the straightforward computation shows that any element in $H^2(Y_1, \mathbb{Z}) \times H^2(Y_2, \mathbb{Z})$ can be expressed as

$$(a + 4b)(H_1, H_2) - (b + c)(4H_1 - E_1, 0) - c\delta, \qquad \delta := \langle E_1 - 4H_1, 4H_2 - E_2 \rangle.$$

This yields that

$$H^2(M, \mathbb{Z}) \cong \langle (H_1, H_2), (4H_1 - E_1, 0) \rangle$$

up to torsion. Taking $e_1 = (H_1, H_2)$ and $e_2 = (4H_1 - E_1, 0)$ as generators of $H^i(M, \mathbb{Z})$, we see that

$$\begin{aligned} e_1^3 &= (H_1, H_2)^3 = H_1^3 + H_2^3 = 2, \\ e_1^2 e_2 &= (H_1, H_2)^2 (4H_1 - E_1, 0) = 4H_1^3 - H_1^2 E_1 = 4, \\ e_1 e_2^2 &= (H_1, H_2)(4H_1 - E_1, 0)^2 = 16H_1^3 - 8H_1^2 E_1 + H_1 E_1^2 = 0, \\ e_2^3 &= (4H_1 - E_1)^3 = 64H_1^3 - 48H_1^2 E_1 + 12H_1 E_1^2 - E_1^3 = 0. \end{aligned}$$

As we have seen in Section 2.1, the second Chern class of Y_i is $c_2(Y_i) = 22H_i^2 - 4H_i E_i$ for each i. Thus the subgroup $\{\, e \in \langle e_1, e_2 \rangle \mid e \cdot c_2(M) = 0 \,\}$ of $H^2(M, \mathbb{Z})$ is generated by $6e_1 - 11e_2$. Then the λ-invariant is $\lambda(M) = |(6e_1 - 11e_2)^3| = 4320$.

3. $(h^{1,1}(M), h^{2,1}(M)) = (2, 44)$ CASE

In this case, the corresponding doubling Calabi-Yau 3-folds are listed in Table 1 with the underlying Fano 3-folds, (a) ID 1-8, (b) ID 1-9 and (c) ID 1-10. We remark that these Fano 3-folds have the following geometric description:

(a) a section of Plücker embedding of $\mathrm{SGr}(3,6)$ by codimension 3 subspace, where $\mathrm{SGr}(3,6)$ is the Lagrangian Grassmannian; $V(1,1,1) \hookrightarrow \mathrm{SGr}(3,6)$,
(b) a section of $\mathrm{G_2Gr}(2,7)$ by codimension 2 subspace; $V(1,1) \hookrightarrow \mathrm{G_2Gr}(2,7)$, and
(c) the zero locus of $\left(\bigwedge^2 \mathcal{V}^\vee\right)^{\oplus 3}$ on $\mathrm{Gr}(3,7)$ where $\mathcal{V} \to \mathrm{Gr}(3,7)$ is the tautological rank 3 vector bundle over the Grassmannian $\mathrm{Gr}(3,7)$.

In the above description (b), $\mathrm{G_2Gr}(2,7)$ denotes the adjoint $\mathrm{G_2}$-Grassmannian which is the zero locus of the section $s \in \bigwedge^3 \mathbb{C}^7$ corresponding to the $\mathrm{G_2}$-invariant 3-form. See [FG], [IsPr99, Chapter 4], [D08, Section 5] for more details. Systematically, all of these Fano 3-folds are expressed as anticanonically embedded Fano 3-folds $V = V_{2g-2} \subset \mathbb{C}P^{g+1}$ with Picard number 1 and genus g. Moreover, we may assume that $\mathrm{Pic}(V) = H \cdot \mathbb{Z}$ where H is the unique generator of $H^2(V,\mathbb{Z})$ and $H = -K_V$ for each case (a) $g = 9 : V_{16} \subset \mathbb{C}P^{10}$, (b) $g = 10 : V_{18} \subset \mathbb{C}P^{11}$ and (c) $g = 12 : V_{22} \subset \mathbb{C}P^{13}$, respectively.

3.1. **ID** 1-9**:** $V_{18} \subset \mathbb{C}P^{11}$ **case.** Firstly, we consider case (b). Let $V = V_{18} \subset \mathbb{C}P^{11}$ be an anticanonically embedded Fano 3-fold with genus $g = 10$, $\mathrm{Pic}(V) = \mathbb{Z} \cdot H$ and $-K_V = H$. Here and hereafter, we use the same notation as in Section 2. According to [FG], we have $-K_V^3 = 18$ and

$$h^{p,q}(V) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 2 & & 2 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}. \tag{3.1}$$

Let $D \in |\mathcal{O}_V(1)|$ be an anticanonical divisor and $C \in |\mathcal{O}_D(1)|$ a smooth curve in D. Setting Y_i to be two copies of the blow-up $\mathrm{Bl}_C(V)$ for $i = 1, 2$, we see that $H^2(Y_i) = \mathbb{C}\langle H_i, E_i\rangle$ and $H^2(M,\mathbb{Z}) \cong \langle (H_1, H_2), (H_1 - E_1, 0)\rangle$ up to torsion. This yields that generators of $H^2(M,\mathbb{Z})$ are given by $e_1 = (H_1, H_2)$ and $e_2 = (H_1 - E_1, 0)$.

In the same manner as the previous computation in Section 2.1, we find that $H_i^3 = 18$, $H_i L = 0$ and $E_i L = -1$ where L is a fiber over a point on C under the blow-up. Moreover, for $d = \deg C$, we have

$$\begin{aligned} H_i E_i &= dL = (2g-2)L = 18L \qquad \text{and} \\ H_i^2 E_i &= H_i(H_i E_i) = 18 H_i L = 0. \end{aligned}$$

Let $\tau = 2g$ be the number of branches of the double curve $Y_i \supset \widetilde{C} \xrightarrow{2:1} C \subset V$. By the list in [GH, p.623], we see that

$$\begin{aligned} E_i^2 &= -dH_i^2 + (4d + 2g - 2 - 2\tau)L \\ &= -18H_i^2 + (72 + 20 - 2 - 40)L = -18H_i^2 + 50L, \\ H_i E_i^2 &= H_i(-18H_i^2 + 50L) = -18H_i^3 + 50H_i L = -18 \cdot 18 = -324, \\ E_i^3 &= E_i(-18H_i^2 + 50L) = -18E_i H_i^2 + 50E_i L = -50. \end{aligned}$$

Consequently, we have the following table of the multiplication of the intersection forms on $H^{2*}(Y_i,\mathbb{Z})$:

	H_i^2	L	$H^4(Y_i,\mathbb{Z})$
H_i	18	0	
E_i	0	-1	
$H^2(Y_i,\mathbb{Z})$			

	H_i	E_i	$H^2(Y_i,\mathbb{Z})$
H_i	H_i^2	$18L$	
E_i	$18L$	$-18H_i^2+50L$	
$H^2(Y_i,\mathbb{Z})$			

Substituting these values into the cubic products, we see that

$$\begin{aligned} e_1^3 &= (H_1,H_2)^3 = H_1^3 + H_2^3 = 36, \\ e_1^2 e_2 &= (H_1,H_2)^2(H_1-E_1,0) = H_1^3 - H_1^2E_1 = 18, \\ e_1e_2^2 &= (H_1,H_2)(H_1-E_1,0)^2 = H_1^3 - 2H_1^2E_1 + H_1E_1^2 = -306, \\ e_2^3 &= (H_1-E_1,0)^3 = H_1^3 - 3H_1^2E_1 + 3H_1E_1^2 - E_1^3 = -904. \end{aligned}$$

Next we compute the λ-invariant of the doubling Calabi-Yau 3-fold M. Since $V = V_{18} \subset \mathbb{C}P^{11}$ is an anticanonically embedded Fano 3-fold with $-K_V = H$, we see that the first Chern class of V is given by $c_1(V) = H$. In order to find the second Chern class of V, we use the Riemann-Roch-Hirzebruch formula

$$\sum_{q=0}^{n}(-1)^q \dim H^q(V,\Omega^p) = \int_V td(V)\, ch\Big(\bigwedge^p T^*V\Big) \tag{3.2}$$

for $n=3$ and $p=0$. This yields the equality

$$\begin{aligned} &\sum_{q=0}^{3}(-1)^q \dim H^q(V,\Omega^0) = \int_V \left(1 + \frac{1}{2}c_1(V) + \frac{1}{12}(c_1(V)^2 + c_2(V)) + \frac{1}{24}c_1(V)c_2(V)\right) ch\Big(\bigwedge^0 T^*V\Big) \\ \Leftrightarrow \quad & h^{0,0} - h^{0,1} + h^{0,2} - h^{0,3} = \frac{1}{24}\int_V c_1(V)c_2(V) \end{aligned} \tag{3.3}$$

Suppose that $c_2(V) = aH^2$ for $a \in \mathbb{Q}$. Then the Hodge diamond (3.1) and the equality (3.3) imply that

$$\frac{1}{24}\int_V aH^3 = 1 \quad \Leftrightarrow \quad a = \frac{4}{3}$$

by $\int_V H^3 = (-K_V^3) = 18$. Thus, we find $c_2(V) = \frac{4}{3}H^2$. As we have seen in (2.1), the second Chern classes of Y_i are given by

$$\begin{aligned} c_2(Y_i) &= \pi_i^*(c_2(V) + \eta_C) - \pi_i^*(c_1(V)) \cdot E_i \\ &= \pi_i^*\left(\frac{4}{3}H^2 + H^2\right) - H_iE_i = \frac{7}{3}H_i^2 - H_iE_i. \end{aligned}$$

Then the products of $c_2(M)$ and e_i are

$$\begin{aligned} e_1 \cdot c_2(M) &= \frac{7}{3}H_1^3 - H_1^2E_1 + \frac{7}{3}H_2^3 - H_2^2E_2 = 84 = 6 \cdot 14, \\ e_2 \cdot c_2(M) &= (H_1 - E_1)c_2(Y_1) = (H_1 - E_1)\left(\frac{7}{3}H_1^2 - H_1E_1\right) \\ &= \frac{7}{3}H_1^3 + H_1E_1^2 = \frac{7}{3}\cdot 18 + (-324) = -282 = -6 \cdot 47. \end{aligned}$$

Since the subgroup $\{\, e \in \langle e_1, e_2\rangle \mid e \cdot c_2(M) = 0 \,\}$ of $H^2(M,\mathbb{Z})$ is generated by $47e_1 + 14e_2$, we see that the λ-invariant of M is given by

$$\lambda(M) = |(47e_1 + 14e_2)^3| = |47^3e_1^3 + 3\cdot 47^2 \cdot 14 \cdot e_1^2e_2 + 3 \cdot 47 \cdot 14^2 e_1e_2^2 + 14^3e_2^3| = 5529560.$$

3.2. **ID** 1-8**:** $V_{16} \subset \mathbb{C}P^{10}$ **case.** Secondly, we shall consider case (a). We refer the reader to [Y21] for details. The most essential part of the calculation can be summarized as follows.

We suppose that $V = V_{16} \subset \mathbb{C}P^{10}$, $g = 9$, $\mathrm{Pic}(V) = \mathbb{Z} \cdot H$ and $-K_V = H$. Furthermore, we have $-K_V^3 = 16$ and

$$h^{p,q}(V) = \begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 3 & & 3 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix}.$$

Setting $D \in |\mathcal{O}_V(1)|$, $C \in |\mathcal{O}_D(1)|$ and $\pi_i : Y_i = \mathrm{Bl}_C(V) \dashrightarrow V$ for $i = 1, 2$, we see that $H^2(Y_i) = \mathbb{C}\langle H_i, E_i\rangle$ and $H^2(M, \mathbb{Z}) \cong \langle (H_1, H_2), (H_1 - E_1, 0)\rangle$ up to torsion. Hence two generators of $H^2(M, \mathbb{Z})$ are taken as $e_1 = (H_1, H_2)$ and $e_2 = (H_1 - E_1, 0)$. Consequently, we find the values of the cubic forms as follows:

$$\begin{aligned} e_1^3 &= (H_1, H_2)^3 = H_1^3 + H_2^3 = 32, \\ e_1^2 e_2 &= (H_1, H_2)^2 (H_1 - E_1, 0) = H_1^3 - H_1^2 E_1 = 16, \\ e_1 e_2^2 &= (H_1, H_2)(H_1 - E_1, 0)^2 = H_1^3 - 2H_1^2 E_1 + H_1 E_1^2 = -240, \\ e_2^3 &= (H_1 - E_1, 0)^3 = H_1^3 - 3H_1^2 E_1 + 3H_1 E_1^2 - E_1^3 = -708. \end{aligned}$$

As we computed in Section 3.1, the second Chern class of V is calculated by the Riemann-Roch-Hirzebruch formula (3.2), from which we conclude that $c_2(V) = \frac{3}{2}H^2$. Thus the second Chern classes of Y_i are

$$c_2(Y_i) = \pi_i^* \left(\frac{3}{2} H^2 + H^2 \right) - H_i E_i = \frac{5}{2} H_i^2 - H_i E_i$$

for $i = 1, 2$. Then the subgroup $\{\, e \in \langle e_1, e_2\rangle \mid e \cdot c_2(M) = 0 \,\}$ of $H^2(M, \mathbb{Z})$ is generated by $27e_1 + 10e_2$. This implies that the λ-invariant is $\lambda(M) = |(27e_1 + 10e_2)^3| = 1672224$.

3.3. **ID** 1-10**:** $V_{22} \subset \mathbb{C}P^{13}$ **case.** Finally, we consider case (c), that is, $V = V_{22} \subset \mathbb{C}P^{13}$ is an anticanonically embedded Fano 3-fold with genus $g = 12$, $\mathrm{Pic}(V) = \mathbb{Z} \cdot H$ and $-K_V = H$. Note that the unique such 3-fold with $\mathrm{Aut}(V) = \mathrm{PGL}(2, \mathbb{C})$ is called the Mukai-Umemura 3-fold, and we refer the reader to [D08, Ti97] and references therein for more details.

As one can see in [FG], the Hodge diamond of V is

$$h^{p,q}(V) = \begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix} \tag{3.4}$$

and $-K_V^3 = 22$. Let $D \in |\mathcal{O}_V(1)|$ be an anticanonical divisor, $C \in |\mathcal{O}_D(1)|$ a smooth curve in D and Y_i two copies of the blow-up $\mathrm{Bl}_C(V)$ as usual. Then we see that $H^2(Y_i) = \mathbb{C}\langle H_i, E_i\rangle$ and $H^2(M, \mathbb{Z}) \cong \langle (H_1, H_2), (H_1 - E_1, 0)\rangle$ up to torsion. Hence two generators of $H^2(M, \mathbb{Z})$ are given by $e_1 = (H_1, H_2)$ and $e_2 = (H_1 - E_1, 0)$. The straightforward computation shows that $H_i^3 = 22$, $H_i L = 0$ and $E_i L = -1$. Furthermore, we have

$$\begin{aligned} H_i E_i &= dL = (2g - 2)L = 22L \qquad \text{and} \\ H_i^2 E_i &= H_i (H_i E_i) = 22 H_i L = 0. \end{aligned}$$

Again, let $\tau = 2g$ be the number of branches of the double curve $\widetilde{C} \xrightarrow{2:1} C \subset V$. Then we see that

$$\begin{aligned} E_i^2 &= -dH_i^2 + (4d + 2g - 2 - 2\tau)L \\ &= -22H_i^2 + (88 + 24 - 2 - 48)L = -22H_i^2 + 72L, \\ H_iE_i^2 &= H_i(-22H_i^2 + 72L) = -22H_i^3 + 72H_iL = -22 \cdot 22 = -484, \qquad \text{and} \\ E_i^3 &= E_i(-22H_i^2 + 72L) = -22E_iH_i^2 + 72E_iL = -72. \end{aligned}$$

Consequently, we have the following table of the multiplication of the intersection forms on $H^{2*}(Y_i, \mathbb{Z})$:

	H_i^2	L	$H^4(Y_i,\mathbb{Z})$
H_i	22	0	
E_i	0	-1	
$H^2(Y_i,\mathbb{Z})$			

	H_i	E_i	$H^2(Y_i,\mathbb{Z})$
H_i	H_i^2	$22L$	
E_i	$22L$	$-22H_i^2 + 72L$	
$H^2(Y_i,\mathbb{Z})$			

Substituting these values into the cubic products, we see that

$$\begin{aligned} e_1^3 &= (H_1, H_2)^3 = H_1^3 + H_2^3 = 44, \\ e_1^2e_2 &= (H_1, H_2)^2(H_1 - E_1, 0) = H_1^3 - H_1^2E_1 = 22, \\ e_1e_2^2 &= (H_1, H_2)(H_1 - E_1, 0)^2 = H_1^3 - 2H_1^2E_1 + H_1E_1^2 = -462, \\ e_2^3 &= (H_1 - E_1, 0)^3 = H_1^3 - 3H_1^2E_1 + 3H_1E_1^2 - E_1^3 = -1358. \end{aligned}$$

Now, we compute the λ-invariant. As we have seen in Section 3.1, the first Chern class of V is given by $c_1(V) = H$. In order to calculate the second Chern class of V, we use (3.2) for $n = 3$ and $p = 0$. Then we obtain

$$h^{0,0} - h^{0,1} + h^{0,2} - h^{0,3} = \frac{1}{24}\int_V c_1(V)c_2(V). \tag{3.5}$$

Suppose that $c_2(V) = aH^2$ for $a \in \mathbb{Q}$. Since the left hand side of (3.5) is 1 by (3.4), we see that

$$\frac{1}{24}\int_V aH^3 = 1 \quad \Leftrightarrow \quad a = \frac{12}{11}$$

where we used $\int_V H^3 = (-K_V^3) = 22$. Thus, we find $c_2(V) = \frac{12}{11}H^2$. By (2.1), the second Chern classes of Y_i are

$$\begin{aligned} c_2(Y_i) &= \pi_i^*(c_2(V) + \eta_C) - \pi_i^*(c_1(V)) \cdot E_i \\ &= \pi_i^*\left(\frac{12}{11}H^2 + H^2\right) - H_iE_i = \frac{23}{11}H_i^2 - H_iE_i. \end{aligned}$$

Then the products of $c_2(M)$ and e_i are

$$\begin{aligned} e_1 \cdot c_2(M) &= \frac{23}{11}H_1^3 - H_1^2E_1 + \frac{23}{11}H_2^3 - H_2^2E_2 = 92 = 2 \cdot 46, \\ e_2 \cdot c_2(M) &= (H_1 - E_1)c_2(Y_1) = (H_1 - E_1)\left(\frac{23}{11}H_1^2 - H_1E_1\right) \\ &= \frac{23}{11}H_1^3 + H_1E_1^2 = \frac{23}{11} \cdot 22 + (-484) = -438 = -2 \cdot 219. \end{aligned}$$

Since the subgroup $\{\, e \in \langle e_1, e_2 \rangle \mid e \cdot c_2(M) = 0 \,\}$ of $H^2(M, \mathbb{Z})$ is generated by $219e_1 + 46e_2$, we see that

$$\lambda(M) = |(219e_1 + 46e_2)^3| = |219^3e_1^3 + 3 \cdot 219^2 \cdot 46 \cdot e_1^2e_2 + 3 \cdot 219 \cdot 46^2e_1e_2^2 + 46^3e_2^3| = 122507896.$$

4. $(h^{1,1}(M), h^{2,1}(M)) = (2, 58)$ CASE

Now we consider the case where the doubling Calabi-Yau 3-folds have the same Hodge numbers $(h^{1,1}(M), h^{2,1}(M)) = (2, 58)$, that is, the underlying Fano 3-folds are (a) ID 1-4, (b) ID 1-12 and (c) 1-14. These Fano 3-folds are described as follows:

(a) a complete intersection of three quadrics in $\mathbb{C}P^6$; $V(2,2,2) \subset \mathbb{C}P^6$,
(b) a hypersurface of degree 4 in the weighted projective space $\mathbb{C}P(1,1,1,1,2)$; $V(4) \subset \mathbb{C}P^4(1^4, 2)$, and
(c) a complete intersection of two quadrics in $\mathbb{C}P^5$; $V(2,2) \subset \mathbb{C}P^5$.

4.1. **ID 1-14: $V(2,2) \subset \mathbb{C}P^5$ case.** Let V be a smooth complete intersection of 3 quadrics in $\mathbb{C}P^5$, which is the Fano 3-fold with $-K_V^3 = 32$ and

$$h^{p,q}(V) = \begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 2 & & 2 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix} .$$

By the adjunction formula, we see that

$$K_{V(2)} \cong (K_{\mathbb{C}P^5} + [V(2)])\big|_{V(2)} = -4H, \quad \text{and}$$
$$K_V \cong \big(K_{V(2)} + [V]\big)\big|_V = (-4+2)H = -2H$$

where $H \in H(V, \mathbb{Z})$ is the ample generator and $V(2) \subset \mathbb{C}P^5$ is a smooth quadric hypersurface in $\mathbb{C}P^5$. Let $D = 2H \in |-K_V|$ be an anticanonical divisor and $C \in |\mathcal{O}_D(2)|$ a smooth curve in D representing the intersection class of $D \cdot D$. For $i = 1, 2$, we take the blow-ups $Y_i = \mathrm{Bl}_C(V)$ which have the cohomology rings $H^2(Y_i) = \mathbb{C}\langle H_i, E_i\rangle$. Then the proper transforms D_i of D in Y_i are $2H_i - E_i$. Thus we set δ by $\langle -D_1, D_2\rangle = \langle E_1 - 2H_1, 2H_2 - E_2\rangle$. We observe that any element in $H^2(Y_1, \mathbb{Z}) \times H^2(Y_2, \mathbb{Z})$ is written as

$$(aH_1 + bE_1, cE_2 + (a + 2b - 2c)H_2) = (a + 2b)(H_1, H_2) - (b + c)(2H_1 - E_1, 0) - c\delta.$$

Consequently, we find that

$$H^2(M, \mathbb{Z}) \cong \langle (H_1, H_2), (2H_1 - E_1, 0)\rangle$$

up to torsion. This implies that two generators of $H^2(M, \mathbb{Z})$ can be taken as $e_1 = (H_1, H_2)$ and $e_2 = (2H_1 - E_1, 0)$.

In order to compute the cubic forms in $H^6(M, \mathbb{Z})$, we first see that the Fano genus g of V is

$$g = \frac{-K_V^3}{2} + 1 = \frac{32}{2} + 1 = 17.$$

Then the straightforward computation shows that $H_i^3 = 32$, $H_i L = 0$ and $E_i L = -1$ where L is a fiber over a point on C under the blow-up. Furthermore, for $d = \deg C$, we have

$$H_i E_i = dL = (2g-2)L = 32L \qquad \text{and}$$
$$H_i^2 E_i = H_i(H_i E_i) = 32 H_i L = 0.$$

In the same manner as in Section 3, let us denote the number of branches of the double curve $\widetilde{C}$ by τ. Then we find that

$$E_i^2 = -dH_i^2 + (4d + 2g - 2 - 2\tau)L = -32H_i^2 + (128 + 34 - 2 - 68)L = -32H_i^2 + 92L,$$
$$H_i E_i^2 = H_i(-32H_i^2 + 92L) = -32H_i^3 + 92H_i L = -32 \cdot 32 = -1024, \qquad \text{and}$$
$$E_i^3 = E_i(-32H_i^2 + 92L) = -32E_i H_i^2 + 92E_i L = -92.$$

In the following table, we summarize the values of the multiplication of the intersection forms on $H^{2*}(Y_i,\mathbb{Z})$:

	H_i^2	L	$H^4(Y_i,\mathbb{Z})$
H_i	32	0	
E_i	0	-1	
$H^2(Y_i,\mathbb{Z})$			

	H_i	E_i	$H^2(Y_i,\mathbb{Z})$
H_i	H_i^2	$32L$	
E_i	$32L$	$-32H_i^2+92L$	
$H^2(Y_i,\mathbb{Z})$			

Substituting these values into the cubic forms, we find that

$$\begin{aligned} e_1^3 &= (H_1,H_2)^3 = H_1^3+H_2^3 = 64,\\ e_1^2e_2 &= (H_1,H_2)^2(2H_1-E_1,0) = 2H_1^3 - H_1^2E_1 = 64,\\ e_1e_2^2 &= (H_1,H_2)(2H_1-E_1,0)^2 = 4H_1^3-4H_1^2E_1+H_1E_1^2 = 4\cdot 32 - 1024 = -896,\\ e_2^3 &= (2H_1-E_1,0)^3 = 8H_1^3-12H_1^2E_1+6H_1E_1^2-E_1^3 = 8\cdot 32+6\cdot(-1024)-(-92) = -5796. \end{aligned}$$

Next we compute the λ-invariant. Since V is a complete intersection of two quadrics in $\mathbb{C}P^5$, the total Chern classes of V are given by the formula

$$\begin{aligned} \frac{(1+H)^6}{(1+2H)^2} &= \left(1+6H+\binom{6}{2}H^2\right)\left(1+2H\right)^{-2}+O(H^3)\\ &= (1+6H+15H^2)(1-4H+12H^2)+O(H^3) = 1+2H+3H^2+O(H^3). \end{aligned}$$

Hence the second Chern classes of Y_i are computed as

$$\begin{aligned} c_2(Y_i) &= \pi_i^*(c_2(V)+\eta_C) - \pi_i^*(c_1(V))\cdot E_i\\ &= \pi_i^*(3H^2+4H^2) - 2H_iE_i = 7H_i^2 - 2H_iE_i. \end{aligned}$$

Then the products of $c_2(M)$ and e_i are given by

$$\begin{aligned} e_1\cdot c_2(M) &= 7H_1^3 - 2H_1^2E_1 + 7H_2^3 - 2H_2^2E_2 = 448 = 2^6\cdot 7,\\ e_2\cdot c_2(M) &= (2H_1-E_1)(7H_1^2-2H_1E_1)\\ &= 14H_1^3 - 4H_1^2E_1 - 7H_1^2E_1 + 2H_1E_1^2\\ &= 14\cdot 32 - 2\cdot 2^{10} = -1600 = 2^6\cdot(-25). \end{aligned}$$

Since the subgroup $\{\, e\in\langle e_1,e_2\rangle \mid e\cdot c_2(M)=0 \,\}$ of $H^2(M,\mathbb{Z})$ is generated by a single element $25e_1+7e_2$, the λ-invariant of M is

$$\begin{aligned} \lambda(M) &= |(25e_1+7e_2)^3| = |25^3e_1^3 + 3\cdot 25^2\cdot 7e_1^2e_2 + 3\cdot 25\cdot 7^2e_1e_2^2 + 7^3e_2^3|\\ &= |25^3\cdot 64 + 3\cdot 25^2\cdot 7\cdot 64 + 3\cdot 25\cdot 7^2\cdot(-896) + 7^3\cdot(-5796)| = 3440828. \end{aligned}$$

4.2. **ID 1-12: $V(4)\subset\mathbb{C}P(1^4,2)$ case.** Let V be a smooth hypersurface of degree 4 in the weighted projective space $\mathbb{C}P^4(1^4,2)$, which is the Fano 3-fold with $-K_V^3=16$ and

$$h^{p,q}(V) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 10 & & 10 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}.$$

By the adjunction formula, we find that

$$K_V \cong (K_{\mathbb{P}}+[V])\big|_V = (\mathcal{O}_{\mathbb{P}}(-6)+\mathcal{O}_{\mathbb{P}}(4))\big|_V = \mathcal{O}_{\mathbb{P}}(-2)\big|_V = \mathcal{O}_V(-2)$$

where we denote the weighted projective space $\mathbb{C}P^4(1^4,2)$ by $\mathbb{P}$. Let $D=2H\in|-K_V|$ be a smooth anticanonical divisor and $C\in|\mathcal{O}_D(2)|$ a smooth curve in D. Let $Y_i=\mathrm{Bl}_C(V)$ be the blow-ups of V along C and $H^2(Y_i) = \mathbb{C}\langle H_i, E_i\rangle$ the cohomology rings of Y_i for $i = 1, 2$. For the proper transforms $D_i = 2H_i - E_i$ of D in Y_i, we set δ by $\langle -D_1, D_2\rangle = \langle E_1 - 2H_1, 2H_2 - E_2\rangle$. Repeating the same computation in Section 4.1, we see that two generators of $H^2(M,\mathbb{Z})$ are $e_1 = (H_1, H_2)$ and $e_2 = (2H_1 - E_1, 0)$.

Now we compute the cubic products of e_i in $H^6(M,\mathbb{Z})$. Firstly, the genus of the Fano 3-fold V is given by

$$g = \frac{-K_V^3}{2} + 1 = \frac{16}{2} + 1 = 9.$$

Secondly, we readily see that

$$\begin{aligned} H_i^3 &= 16, \qquad H_iL = 0, \qquad E_iL = -1 \\ H_iE_i &= dL = (2g-2)L = 16L, \qquad \text{and} \\ H_i^2E_i &= H_i(H_iE_i) = 16H_iL = 0. \end{aligned}$$

Let $\tau = 2g$ be the number of branches of the double curve $\widetilde{C}$. Then we find that

$$\begin{aligned} E_i^2 &= -dH_i^2 + (4d + 2g - 2 - 2\tau)L = -16H_i^2 + (64 + 18 - 2 - 36)L = -16H_i^2 + 44L, \\ H_iE_i^2 &= H_i(-16H_i^2 + 44L) = -16H_i^3 + 44H_iL = -16 \cdot 16 = -256, \qquad \text{and} \\ E_i^3 &= E_i(-16H_i^2 + 44L) = -16E_iH_i^2 + 44E_iL = -44. \end{aligned}$$

The following table collects the values of the multiplication of the intersection forms on $H^{2*}(Y_i,\mathbb{Z})$:

	H_i^2	L	$H^4(Y_i,\mathbb{Z})$
H_i	16	0	
E_i	0	-1	
$H^2(Y_i,\mathbb{Z})$			

	H_i	E_i	$H^2(Y_i,\mathbb{Z})$
H_i	H_i^2	$16L$	
E_i	$16L$	$-16H_i^2 + 44L$	
$H^2(Y_i,\mathbb{Z})$			

Substituting these values into the cubic forms, we find that

$$\begin{aligned} e_1^3 &= (H_1, H_2)^3 = H_1^3 + H_2^3 = 32, \\ e_1^2e_2 &= (H_1, H_2)^2(2H_1 - E_1, 0) = 2H_1^3 - H_1^2E_1 = 32, \\ e_1e_2^2 &= (H_1, H_2)(2H_1 - E_1, 0)^2 = 4H_1^3 - 4H_1^2E_1 + H_1E_1^2 = 4 \cdot 16 - 256 = -192, \\ e_2^3 &= (2H_1 - E_1, 0)^3 = 8H_1^3 - 12H_1^2E_1 + 6H_1E_1^2 - E_1^3 = 8 \cdot 16 + 6 \cdot (-256) - (-44) = -1364. \end{aligned}$$

Let us compute the λ-invariant. Since V is a hypersurface of degree 4 in the weighted projective space $\mathbb{C}P^4(1^4, 2)$, the total Chern classes of V are given by

$$\begin{aligned} \frac{(1+H)^4(1+2H)}{(1+4H)} &= \left(1 + 4H + \binom{4}{2} H^2\right)\left(1+2H\right)\left(1+4H\right)^{-1} + O(H^3) \\ &= (1 + 4H + 6H^2)(1 + 2H)(1 - 4H + 16H^2) + O(H^3) \\ &= 1 + 2H + 6H^2 + O(H^3). \end{aligned}$$

Thus the second Chern classes of Y_i are

$$c_2(Y_i) = \pi_i^*(6H^2 + 4H^2) - 2H_iE_i = 10H_i^2 - 2H_iE_i.$$

Then we see that the products of $c_2(M)$ and e_i are

$$\begin{aligned} e_1 \cdot c_2(M) &= 10H_1^3 - 2H_1^2E_1 + 10H_2^3 - 2H_2^2E_2 = 320 = 2^6 \cdot 5, \\ e_2 \cdot c_2(M) &= (2H_1 - E_1)(10H_1^2 - 2H_1E_1) \\ &= 20H_1^3 - 4H_1^2E_1 - 10H_1^2E_1 + 2H_1E_1^2 \\ &= 20 \cdot 16 + 2 \cdot (-256) = -192 = 2^6 \cdot (-3). \end{aligned}$$

Since the subgroup $\{ e \in \langle e_1, e_2 \rangle \mid e \cdot c_2(M) = 0 \}$ of $H^2(M, \mathbb{Z})$ is generated by a single element $3e_1 + 5e_2$, the λ-invariant of M is

$$\begin{aligned}\lambda(M) &= |(3e_1 + 5e_2)^3| = |3^3 e_1^3 + 3 \cdot 3^2 \cdot 5 e_1^2 e_2 + 3 \cdot 3 \cdot 5^2 e_1 e_2^2 + 5^3 e_2^3| \\ &= |27 \cdot 32 + 3 \cdot 27 \cdot 5 \cdot 32 + 9 \cdot 25 \cdot (-192) + 125 \cdot (-1364)| = 208516.\end{aligned}$$

4.3. **ID** 1**-**4**:** $V(2,2,2) \subset \mathbb{C}P^6$ **case.** We refer the reader to [Y21] for the detailed computation of this example. This subsection collects the minimum amount of calculation necessary to see the values of the cubic forms and the λ-invariants.

Let $V = V(2,2,2) \subset \mathbb{C}P^6$ be a complete intersection of three quadrics in $\mathbb{C}P^6$. As usual, we set $D \in |\mathcal{O}_V(1)|$, $C \in |\mathcal{O}_D(1)|$ and $\pi_i : Y_i = \mathrm{Bl}_C(V) \dashrightarrow V$ for $i = 1, 2$. Then we see that the proper transform D_i of D in Y_i is $H_i - E_i$ and $H^2(Y_i) = \mathbb{C} \langle H_i, E_i \rangle$ for each i. Thus any element in $H^2(Y_1, \mathbb{Z}) \times H^2(Y_2, \mathbb{Z})$ can be written as

$$(a+b)(H_1, H_2) - (b+c)(H_1 - E_1, 0) - c\delta, \qquad \delta := \langle E_1 - H_1, H_2 - E_2 \rangle.$$

This implies that

$$H^2(M, \mathbb{Z}) \cong \langle (H_1, H_2), (H_1 - E_1, 0) \rangle$$

up to torsion. Setting $e_1 = (H_1, H_2)$ and $e_2 = (H_1 - E_1, 0)$ as generators of $H^i(M, \mathbb{Z})$, we find that

$$\begin{aligned} e_1^3 &= (H_1, H_2)^3 = H_1^3 + H_2^3 = 16, \\ e_1^2 e_2 &= (H_1, H_2)^2 (H_1 - E_1, 0) = H_1^3 - H_1^2 E_1 = 8, \\ e_1 e_2^2 &= (H_1, H_2)(H_1 - E_1, 0)^2 = H_1^3 - 2H_1^2 E_1 + H_1 E_1^2 = -56, \\ e_2^3 &= (H_1 - E_1, 0)^3 = H_1^3 - 3H_1^2 E_1 + 3H_1 E_1^2 - E_1^3 = -164. \end{aligned}$$

In the same manner as the previous calculation in Section 4.1, the second Chern class of Y_i is $c_2(Y_i) = 4H_i^2 - H_i E_i$ for each i. Consequently, the subgroup $\{ e \in \langle e_1, e_2 \rangle \mid e \cdot c_2(M) = 0 \}$ of $H^2(M, \mathbb{Z})$ is generated by $e_1 + 2e_2$. Hence we conclude that the λ-invariant is $\lambda(M) = |(e_1 + 2e_2)^3| = 1920$.

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