EXTENDED ABSTRACT FOR "GEOMETRIC VERTEX DECOMPOSITION, GRÖBNER BASES, AND FROBENIUS SPLITTINGS FOR REGULAR NILPOTENT HESSENBERG VARIETIES"

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Hessenberg varieties are subvarieties of the full flag variety $\mathrm{Flags}(\mathbb{C}^n)$ and the investigation of their properties lies in the fruitful intersection of algebraic geometry, representation theory, and combinatorics, among other research areas.[1] First introduced to the algebraic geometry community by De Mari, Procesi, and Shayman [6], they have recently garnered attention due in part to their connection to the well-known and unresolved Stanley-Stembridge conjecture in combinatorics (see e.g. [9] for a leisurely account of some of the history). However, there are many other reasons aside from the Stanley-Stembridge conjecture that Hessenberg varieties are of interest; for instance, they arise in the study of quantum cohomology of flag varieties, they are generalizations of the Springer fibers which arise in geometric representation theory, total spaces of families of suitable Hessenberg varieties support interesting integrable systems [1], and the study of some of their cohomology rings suggests that there is a rich Hessenberg analogue to the theory of Schubert calculus on $\mathrm{Flags}(\mathbb{C}^n)$ [10].

Motivated by Schubert calculus, we study **local patches of Hessenberg varieties** - i.e. intersections of the Hessenberg subvariety with certain choices of affine Zariski-open subsets of $\mathrm{Flags}(\mathbb{C}^n)$. The analogous study of local patches of Schubert varieties is a classical topic and a great deal is known about the corresponding (local defining) ideals, from which properties of Schubert varieties can be deduced.

Another motivation for our results is to introduce the theory of geometric vertex decompositions to the study of the geometry of Hessenberg varieties. Geometric vertex decompositions were first defined and studied by Knutson, Miller, and Yong in their influential work [18], where they used their new theory to study Schubert determinantal ideals. More recently, the theory of geometric vertex decomposition, the definition of which is inherently inductive (recursive), has been linked to liaison theory. Our arguments show, first, that our local Hessenberg patch ideals $I_{w_0,h}$ are geometrically vertex decomposable, and from this it follows that a certain set of generators (those found in [2]) is a Gröbner basis for an appropriately chosen monomial order.

We now establish the terminology and notation to state our results. Proofs are contained in [5]. We begin with the flag variety. More specifically, the full flag variety $\mathrm{Flags}(\mathbb{C}^n)$ is the set of nested sequences of subspaces

$$\mathrm{Flags}(\mathbb{C}^n) := \{V_\bullet = (0 \subsetneq V_1 \subseteq V_2 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}}(V_i) = i\}$$

in $\mathbb{C}^n$. By representing $V_\bullet$ by an $n \times n$ matrix (whose leftmost i many columns span V_i), we may identify $\mathrm{Flags}(\mathbb{C}^n)$ as the homogeneous space $GL_n(\mathbb{C})/B$. Here, B is the Borel subgroup of $GL_n(\mathbb{C})$ consisting of upper-triangular invertible matrices. Let U^- denote the subgroup in $GL_n(\mathbb{C})$ consisting of lower-triangular matrices with 1's along the diagonal. Then $U^-B \subset GL_n(\mathbb{C})/B$ is the set of left cosets uB with $u \in U^-$. This is an open dense subset of $GL_n(\mathbb{C})/B \cong \mathrm{Flags}(\mathbb{C}^n)$ and can be profitably viewed as a "coordinate chart" on $GL_n(\mathbb{C})/B$.

Let S_n denote the symmetric group on n letters and $w \in S_n$ a permutation. We can identify S_n with the set of permutation flags in $\mathrm{Flags}(\mathbb{C}^n)$ and view it as a subgroup of $GL_n(\mathbb{C})$ by taking w to the associated permutation matrix. By abuse of notation we will often denote by the same w the element in S_n, its associated flag, and its associated permutation matrix. Translating the coordinate chart U^-B by multiplication by w on the left, we can define

$$\mathcal{N}_w := wU^-B \subseteq GL_n(\mathbb{C}^n)/B \tag{1}$$

[1]in this abstract, we restrict to the case of Lie type A, i.e., when the flag variety corresponds to the group $GL(n, \mathbb{C})$ (or $SL(n, \mathbb{C})$). Much can be said about other Lie types, but we do not delve into that here.

which is an open cell (i.e., a coordinate chart) in $GL_n(\mathbb{C}^n)/B$ containing the permutation flag w. It is well-known that $\mathrm{Flags}(\mathbb{C}^n) \cong GL_n(\mathbb{C}^n)/B$ can be covered by these $n!$ many coordinate charts, each centered around a permutation flag w.

In fact, each $\mathcal{N}_w$ is isomorphic to a complex affine space of dimension $\frac{n(n-1)}{2}$. To see this, let

$$M := \begin{bmatrix} 1 & & & & \\ \star & 1 & & & \\ \vdots & \vdots & \ddots & & \\ \star & \star & \cdots & 1 & \\ \star & \star & \cdots & \star & 1 \end{bmatrix} \tag{2}$$

denote an element in U^- where the $\star$'s represent complex numbers, and consider the map $M \mapsto wMB \in GL_n(\mathbb{C})/B$. It is not difficult to check that this defines an embedding $U^- \cong \mathbb{A}^{n(n-1)/2} \overset{\cong}{\to} \mathcal{N}_w \subset \mathrm{Flags}(\mathbb{C}^n)$ parametrizing the coordinate chart $\mathcal{N}_w$. A point in $\mathcal{N}_w$ can be uniquely identified with the w-translate of an element M in U^-, and thus a point in $\mathcal{N}_w$ is uniquely determined by a matrix $wM = (x_{i,j})$ satisfying

$$x_{w(j),j} = 1 \text{ for } j \in [n] \text{ and } x_{w(i),j} = 0 \text{ for } i, j \in [n], j > i.$$

Thus the coordinate ring of $\mathcal{N}_w$, which we denote by $\mathbb{C}[\mathbf{x}_w]$, is isomorphic to the polynomial ring in the $n(n-1)/2$ variables not specified by the above relations.

For instance, let w_0 be the Bruhat-longest element in S_n, so in one-line notation,

$$w_0 = [n \; n-1 \; n-2 \cdots 2 \; 1].$$

Then for any positive integer n, the coordinate chart $\mathcal{N}_{w_0}$ can be parametrized by matrices of the form

$$w_0 M = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n-2} & x_{1,n-1} & 1 \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n-2} & 1 & 0 \\ \vdots & & & & \vdots & \vdots \\ x_{n-1,1} & 1 & \cdots & & 0 & 0 \\ 1 & 0 & \cdots & & 0 & 0 \end{bmatrix}. \tag{3}$$

where we think of the variable $x_{i,j}$ in the matrix above as indeterminates (i.e., coordinates), taking values in $\mathbb{C}$. For a different choice of permutation w, these indeterminates will be located at different places within the matrix, but the idea is similar. We now define the regular nilpotent Hessenberg varieties which are the focus of our results. We call a function $h : [n] := \{1, 2, \cdots, n\} \to [n] := \{1, 2, \cdots, n\}$ a **Hessenberg function** if it satisfies the conditions $h(i) \geq i$ for all i and $h(i+1) \geq h(i)$ for $1 \leq i \leq n-1$. We say that a Hessenberg function is **indecomposable** if $h(i) \geq i+1$ for all $1 \leq i \leq n-1$. Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator and let $h : [n] \to [n]$ be an indecomposable Hessenberg function. Then we define the **Hessenberg variety associated to A and h** to be the subvariety of $\mathrm{Flags}(\mathbb{C}^n)$ given by

$$\mathrm{Hess}(A, h) := \{V_\bullet = (V_i) \in \mathrm{Flags}(\mathbb{C}^n) \mid AV_i \subseteq V_{h(i)}, \forall i\} \subset \mathrm{Flags}(\mathbb{C}^n). \tag{4}$$

In this abstract, we focus on the special case when A is a regular nilpotent operator. Specifically, define

$$\mathsf{N} := \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & 0 & & 0 & 0 \end{bmatrix} \tag{5}$$

to be the matrix with 0's everywhere except the 1's immediately above the diagonal entries. (In other words, N has a single Jordan block with eigenvalue 0.) The Hessenberg varieties $\mathrm{Hess}(\mathsf{N}, h)$ defined as in (4) with $A = \mathsf{N}$ are called **regular nilpotent Hessenberg varieties**.

We now describe "local defining equations" for $\mathrm{Hess}(\mathsf{N}, h)$ following the method of [2]. By "local" we mean that for each choice of permutation $w \in S_n$ we focus on the local coordinate chart $\mathcal{N}_w \subseteq \mathrm{Flags}(\mathbb{C}^n)$ centered at w and ask for the defining equations for $\mathcal{N}_w \cap \mathrm{Hess}(\mathsf{N}, h)$ in the affine space $\mathcal{N}_w$. The method for deriving these equations is explained in detail in [2, Section 3], to which we refer the reader; here we will only briefly recall the results therein. Following [2, Definition 3.3] we define certain polynomials $f^w_{k,\ell}$ in $\mathbb{C}[\mathbf{x}_w]$ as follows.

Definition 6. Let $w \in S_n$ and let $k, \ell \in [n]$ with $k > h(\ell)$. We define the polynomial $f^w_{k,\ell} \in \mathbb{C}[\mathbf{x}_w]$ by

$$f^w_{k,\ell} := ((wM)^{-1}\mathsf{N}(wM))_{k,\ell}.$$

where some matrix entries of the matrix wM are viewed as variables, as described above.

We also define, using the polynomials $f^w_{k,\ell}$ defined above, the following ideals

$$I_{w,h} := \langle f^w_{k,\ell} \mid k > h(\ell) \rangle \subseteq \mathbb{C}[\mathbf{x}_w] \tag{7}$$

which we call **Hessenberg patch ideals**. In other words, $I_{w,h}$ is the ideal generated by the (k,ℓ)-th matrix entries of $((wM)^{-1}\mathsf{N}(wM))$ where $k > h(\ell)$. Examples of $I_{w,h}$ are computed in [2, Section 3]. We also have the following result from [2].

Lemma 8. *The ideal $I_{w,h}$ is the defining ideal of the affine variety* $\mathrm{Hess}(\mathsf{N}, h) \cap \mathcal{N}_w$. *In particular, it is radical.*

There are inductive formulas for the polynomials $f^w_{k,\ell}$ which generate the ideals $I_{w,h}$. For more details see [5]. It may be helpful to see one example computation of these $f^w_{k,\ell}$ here. We restrict to the simplest case, namely, the $w = w_0$ chart.

Example 9. Let $n = 5$ and $w_0 = [5\,4\,3\,2\,1]$. We can compute the matrix $(w_0M)^{-1}\mathsf{N}(w_0M)$ to obtain

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ f^{w_0}_{3,1} & 1 & 0 & 0 & 0 \\ f^{w_0}_{4,1} & f^{w_0}_{4,2} & 1 & 0 & 0 \\ f^{w_0}_{5,1} & f^{w_0}_{5,2} & f^{w_0}_{5,3} & 1 & 0 \end{bmatrix}$$

where the $f_{k,\ell} \in \mathbb{C}[\mathbf{x}_{w_0}]$ are defined by the following formulas:

$$\begin{aligned} f^{w_0}_{5,1} &= -x_{1,2} + x_{1,3}(x_{3,2} - x_{4,1}) + x_{1,4}(x_{2,2} - x_{2,3}x_{3,2} + x_{2,3}x_{4,1} - x_{3,1}) + x_{2,1} \\ f^{w_0}_{5,2} &= -x_{1,3} + x_{1,4}(x_{2,3} - x_{3,2}) + x_{2,2} \\ f^{w_0}_{5,3} &= -x_{1,4} + x_{2,3} \\ f^{w_0}_{4,1} &= -x_{2,2} + x_{2,3}(x_{3,2} - x_{4,1}) + x_{3,1} \\ f^{w_0}_{4,2} &= -x_{2,3} + x_{3,2} \\ f^{w_0}_{3,1} &= -x_{3,2} + x_{4,1}. \end{aligned}$$

Therefore, if $h_1 = (2,3,4,5,5)$ and $h_2 = (3,4,4,5,5)$, then we have

$$I_{w_0,h_1} = \langle f^{w_0}_{3,1}, f^{w_0}_{4,1}, f^{w_0}_{4,2}, f^{w_0}_{5,1}, f^{w_0}_{5,2}, f^{w_0}_{5,3} \rangle$$

and

$$I_{w_0,h_2} = \langle f^{w_0}_{4,1}, f^{w_0}_{5,1}, f^{w_0}_{5,2}, f^{w_0}_{5,3} \rangle.$$

It is also useful to introduce a set of variables $y_{i,j}$ in addition to the $x_{i,j}$ in order to express the polynomials $f^{w_0}_{k,\ell}$. Once again we restrict ourselves to giving an example and refer the reader to [5] for more details.

Example 10. Let $n = 4$ and $h = (3,3,4,4)$. The longest element of S_4 is the permutation $w_0 = [4\,3\,2\,1]$. The coordinate ring of $\mathcal{N}_{w_0}$ is

$$\mathbb{C}[\mathbf{x}_{w_0}] \cong \mathbb{C}[x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{3,1}],$$

and a point in $\mathcal{N}_{w_0}$ is determined by a matrix

$$w_0M = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & 1 \\ x_{2,1} & x_{2,2} & 1 & 0 \\ x_{3,1} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The inverse must then have the form

$$(w_0 M)^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & y_{3,1} \\ 0 & 1 & y_{2,2} & y_{2,1} \\ 1 & y_{1,3} & y_{1,2} & y_{1,1} \end{pmatrix} \tag{11}$$

for some $y_{i,j}$. Note that the indexing is such that $y_{i,j}$ is the $(n+1-i, n+1-j)$-th entry in the inverse matrix. It is possible to obtain expressions for the $y_{i,j}$ in terms of the $x_{i,j}$ by starting from the matrix equality $(w_0M)^{-1}(w_0M) = \mathbf{1}_{4\times 4}$ (the 4×4 identity matrix) and comparing entries. For example,

$$\begin{aligned} y_{1,3} &= -x_{1,3}, \\ y_{1,2} &= -x_{1,2} - y_{1,3}x_{2,2} = -x_{1,2} + x_{1,3}x_{2,2}. \end{aligned}$$

Alternatively, the $y_{i,j}$ can also be expressed using the standard adjoint formula for inverses of matrices, and thus can be computed using certain minors of the original matrix w_0M. We will mainly stick to the latter point of view in the arguments that follow.

In fact, the above discussion for the case $n = 4$ readily generalizes to all n. Indeed, we have for general n that

$$(w_0 M)^{-1} = \begin{pmatrix} & & & & 1 \\ & & & 1 & y_{n-1,1} \\ & & & \vdots & \vdots \\ & 1 & \dots & y_{2,2} & y_{2,1} \\ 1 & y_{1,n-1} & \dots & y_{1,2} & y_{1,1} \end{pmatrix} \tag{12}$$

where again the $y_{i,j}$ are polynomials in the $\mathbf{x}_{w_0}$ variables. There is an inductive procedure to compute the $y_{i,j}$ and some useful facts about the $y_{i,j}$ which we won't recount in detail here, referring the reader to [5].

We also need some of the basic terminology of the theory of geometric vertex decomposition. We let R denote a polynomial ring over $\mathbb{C}$ with a finite and fixed set of indeterminates $\mathbf{x}$. (In our setting of the local defining ideals $I_{w_0,h}$ of Hessenberg varieties as above, the set of indeterminates will be the "$\mathbf{x}_{w_0}$ variables" as in the example above.) Now let I be an ideal in R. Suppose $y \in \mathbf{x}$ is one of the indeterminates in $\mathbf{x}$. The **initial y-form** $\text{in}_y f$ of $f \in R$ is the sum of all terms of f having the highest power of y. In particular, if y does not divide any term of f, then $\text{in}_y(f) = f$. We say a monomial order $<$ on R is **y-compatible** if it satisfies $\text{in}_<(f) = \text{in}_<(\text{in}_y(f))$ for every $f \in R$. With respect to such a y-compatible monomial order $<$, suppose $\mathcal{G} = \{y^{d_i}q_i + r_i \mid 1 \le i \le m\}$ is a Gröbner basis for I, where y does not divide any q_i and $\text{in}_y(y^{d_i}q_i + r_i) = y^{d_i}q_i$. In this situation it is straightforward to see that $\text{in}_y(I) = \langle y^{d_i}q_i \mid 1 \le i \le m\rangle$. We have the following.

Definition 13. ([14, Definition 2.3]) In the setting above, define $C_{y,I} := \langle q_i \mid 1 \le i \le m\rangle$ and $N_{y,I} := \langle q_i \mid d_i = 0\rangle$. If $\text{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y\rangle)$, then we call this decomposition a **geometric vertex decomposition of I with respect to** y. A geometric vertex decomposition is **degenerate** if $C_{y,I} = N_{y,I}$ or if $C_{y,I} = \langle 1\rangle$, and **non-degenerate** otherwise.

For further motivation and history surrounding these ideas see [14]. For the purposes of our work, it is important to have an inductive framework for GVDs, in the sense that the ideals $N_{y,I}$ and $C_{y,I}$ can also be equipped with such decompositions. This idea is made precise in Definition 14 below. Recall that I is said to be **unmixed** if $\dim(R/P) = \dim(R/I)$ for all $P \in \text{Ass}(I)$.

Definition 14. ([14, Definition 2.6]) Let I be an ideal in R. We say I is **geometrically vertex decomposable** if I is unmixed and if

(1) $I = \langle 1\rangle$ or I is generated by a (possibly empty) list of indeterminates, or,
(2) for some fixed indeterminate y of R, $\text{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y\rangle)$ is a geometric vertex decomposition and the contractions of $N_{y,I}$ and $C_{y,I}$ to $\mathbb{C}[\mathbf{x} \setminus y]$ are geometrically vertex decomposable.

We can now state our results from [5]. For precise statements and proofs, see [5].

Theorem 15. *Let n be a positive integer with $n \geq 3$. Let $h : [n] \to [n]$ be a Hessenberg function satisfying $h(i) \geq i+1$ for all $1 \leq i \leq n-1$. Then the Hessenberg patch ideal $I_{w_0,h}$ of $\mathrm{Hess}(\mathsf{N}, h)$ in the w_0-coordinate chart is geometrically vertex decomposable. Moreover, the set of polynomials $\{f_{k,\ell}^{w_0}\}$ form a Gröbner basis with respect to an appropriately chosen monomial order, and its initial ideal is an ideal of indeterminates.*

We also apply our result above to initiate a study of Frobenius splittings in the context of Hessenberg varieties. It is known that there exists a Frobenius splitting of the flag variety $\mathrm{Flags}(\mathbb{C}^n)$ which is compatible in a suitable sense with all Schubert and opposite Schubert varieties [3]; this has a geometric interpretation in terms of the anticanonical divisor class of $\mathrm{Flags}(\mathbb{C}^n)$. Thus, it is natural to ask whether there is a Hessenberg analogue of this theory, namely, we may ask whether there exists a Frobenius splitting of $\mathrm{Flags}(\mathbb{C}^n)$ which simultaneously compatibly splits all regular nilpotent Hessenberg varieties for (indecomposable) Hessenberg functions. It is known that a Frobenius splitting on an ambient variety restricts to a Frobenius splitting on an open dense affine coordinate chart, so if such a statement were true, then it must also hold true on a coordinate chart. In [5] we show that for a specific and explicit choice of Frobenius splitting on the w_0-coordinate chart, this necessary condition holds. For precise statements and and proofs, see [5].

Theorem 16. *Let $p > 0$ be a prime. There is an explicit Frobenius splitting φ of the coordinate ring of the w_0-chart of $\mathrm{Flags}(\mathbb{C}^n)$ with respect to which the local Hessenberg patch ideal $I_{w_0,h,p}$ is compatibly split. In particular, there is a partially ordered set (ordered by inclusion) of ideals $\{I_{w_0,h,p}\}$, indexed by the set of (indecomposable) Hessenberg functions h, which are simultaneously compatibly split with respect to φ.*

Much of the discussion above has focused exclusively on the w_0-chart. It is natural to ask what happens to the other coordinate charts for $w \neq w_0$. We have some computational evidence that suggests that, for $w \neq w_0$, the restriction of the local Hessenberg patch ideals $I_{w,h}$ to the coordinates corresponding to the Schubert cell has computationally convenient properties. We also have preliminary evidence suggesting that there are conditions on h and w (and an appopriate choice of monomial order) such that the initial ideal of $I_{w,h}$ possesses a square-free monomial degeneration. We expect to explore these questions further in future work.

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