TVERBERG'S THEOREM FOR CELL COMPLEXES

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1. Introduction

This is a survey of the paper [13].

Radon's theorem on configurations of points in the Euclidean space states that given and $d+2$ points in $\mathbb{R}^d$, we can partition these points into two subsets whose convex hulls have a point in common. It is natural to consider more points and more subsets, and Tverberg [17] generalized Radon's theorem along this direction: any given $(d+1)(r-1)+1$ points in $\mathbb{R}^d$ can be a partitioned into r disjoint subsets whose convex hulls have a point in common. It is useful to translate Tverberg's theorem in terms of an affine map: given any affine map $f\colon \Delta^{(d+1)(r-1)} \to \mathbb{R}^d$, there are pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta^{(d+1)(r-1)}$ such that $f(\sigma_1), \ldots, f(\sigma_r)$ have a point in common. This theorem has been of great interest in combinatorics for over 50 years, and a variety of its generalization have been obtained. See comprehensive surveys [1, 3, 6] for history and developments around Tverberg's theorem. Now we consider a topological generalization.

Question 1.1. What happens if a map $\Delta^{(d+1)(r-1)} \to \mathbb{R}^d$ is not affine but only continuous?

Here is an answer to this question, which is now called the topological Tverberg theorem.

Theorem 1.2. *If r is a prime power, then for any continuous map $f\colon \Delta^{(d+1)(r-1)} \to \mathbb{R}^d$, there are pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta^{(d+1)(r-1)}$ such that $f(\sigma_1), \ldots, f(\sigma_r)$ have a point in common.*

Remarks on the topological Tverberg theorem are in order. The topological Tverberg theorem was proved by Bárány, Shlosman and Szűcs [4] when r is a prime, and by Özaydin [16] and Volovoikov [18] when r is a prime power. As long as we look at the proof the condition for r being a prime power seems quite technical. But Frick [9] proved that the condition that r is a prime power is necessary.

Let us consider a generalization of the topological Tverberg theorem. In [10], Tverberg asked whether or not it is possible to generalize the topological Tverberg theorem to continuous maps from $(d+1)(r-1)$-polytopes into $\mathbb{R}^d$. The answer is positive because the boundary of a convex n-polytope is a refinement of the boundary of an n-simplex as in [11, p. 200] and the result follows from the topological Tverberg theorem. Then Tverberg's question does not contribute to a proper generalization of the topological Tverberg theorem, and so we further ask:

Question 1.3. For which CW complexes can we generalize the topological Tverberg theorem to continuous maps from them into Euclidean spaces?

Recently, Bárány, Kalai and Meshulam [2] and Blagojević, Haase and Ziegler [5] constructed affirmative examples of matroid complexes for Question 1.3 in a purely

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combinatorial way. In this survey, we give a new affirmative class of regular CW complexes from a topological point of view.

To state the main theorem, we set notation and terminology. Let X be a regular CW complex. A *face* of X means its closed cell. For faces $\sigma_1,\ldots,\sigma_k$ of X, let $X(\sigma_1,\ldots,\sigma_k)$ denote the subcomplex of X consisting of faces which do not intersect with $\sigma_1,\ldots,\sigma_k$. Recall that a space Y is called *n-acyclic* if $\widetilde{H}_*(Y)=0$ for $*\le n$. For convenience, a non-empty space will be called (-1)-acyclic, so that any n-acyclic space for $n\ge 0$ will be assumed non-empty. We define a regular CW complex that we are going to consider in this paper.

Definition 1.4. We say that a regular CW complex X is *k-complementary n-acyclic* if $X(\sigma_1,\ldots,\sigma_i)$ is $(n-\dim\sigma_1-\cdots-\dim\sigma_i)$-acyclic for any pairwise disjoint faces $\sigma_1,\ldots,\sigma_i$ of X such that $\dim\sigma_1+\cdots+\dim\sigma_i\le n+1$ and $0\le i\le k$.

Now we state the main theorem.

Theorem 1.5. *If X is an $(r-1)$-complementary $(d(r-1)-1)$-acyclic regular CW complex and r is a prime power, then for any continuous map $f\colon X\to\mathbb{R}^d$, there are pairwise disjoint faces $\sigma_1,\ldots,\sigma_r$ of X such that $f(\sigma_1),\ldots,f(\sigma_r)$ have a point in common.*

Since a $(d+1)$-simplex is k-complementary $(d-k)$-acyclic for $1\le k\le d+1$, the topological Tverberg theorem is recovered by Theorem 1.5. Moreover, we can prove:

Proposition 1.6. *Every simplicial d-sphere is k-complementary $(d-k)$-acyclic for $1\le k\le d+1$.*

Then we get:

Corollary 1.7. *If S is a simplicial $((d+1)(r-1)-1)$-sphere and r is a prime power, then for any continuous map $f\colon S\to\mathbb{R}^d$, there are pairwise disjoint faces $\sigma_1,\ldots,\sigma_r$ of S such that $f(\sigma_1),\ldots,f(\sigma_r)$ have a point in common.*

Grünbaum and Sreedharan [12] constructed a simplicial 3-sphere which is not polytopal. Moreover, Kalai [14] proved that for d large, "most" simplicial d-spheres are not polytopal. Then Corollary 1.7, hence Theorem 1.5 is a substantial generalization of the topological Tverberg theorem. We began with a property of a configuration of points in the Euclidean space and ended up with a property of a simplicial sphere.

2. Sketch of the proof

Let X be a regular CW complex. The *discretized configuration space*

$$\mathtt{Conf}_r(X)$$

is defined as the subcomplex of the direct product X^r consisting of faces $\sigma_1\times\cdots\times\sigma_r$ such that $\sigma_1,\ldots,\sigma_r$ are pairwise disjoint faces of X. The discretized configuration space is often called the deleted product in combinatorics, alternatively. Let $\Delta=\{(x_1,\ldots,x_r)\in(\mathbb{R}^d)^r\mid x_1=\cdots=x_r\}$. There is a homotopy equivalence

$$(\mathbb{R}^d)^r-\Delta\simeq S^{d(r-1)-1}. \tag{2.1}$$

Note that the symmetric group Σ_r acts on $\mathtt{Conf}_r(X)$ and $(\mathbb{R}^d)^r-\Delta$ by permuting of entries. The following lemma is proved in [6, Theorem 3.9].

Lemma 2.1. *Let X be a regular CW complex. If there is a continuous map $X\to\mathbb{R}^d$ such that $f(\sigma_1)\cap\cdots\cap f(\sigma_r)=\emptyset$ for all pairwise disjoint faces $\sigma_1,\ldots,\sigma_r$ of X, then there is a Σ_r-map*

$$\mathtt{Conf}_r(X)\to(\mathbb{R}^d)^r-\Delta.$$

If r is a prime, then the actions of $\mathbb{Z}/r \subset \Sigma_r$ on $\texttt{Conf}_r(X)$ and $(\mathbb{R}^d)^r - \Delta$ are free. So we can apply the Borsuk-Ulam theorem to Lemma 2.1. If r is a prime power, then we can also apply a generalization of the Borsuk-Ulam theorem in [18] (cf. [7]). More precisely, we get:

Proposition 2.2. *Let X be a regular CW complex such that $\texttt{Conf}_r(X)$ is $(d(r-1)-1)$-acyclic. If r is a prime power, then for any continuous map $f\colon X \to \mathbb{R}^d$, there are pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of X such that $f(\sigma_1), \dots, f(\sigma_r)$ have a point in common.*

In order to compute the homology of $\texttt{Conf}_r(X)$, we shall give its homotopy decomposition. Let P be a poset. Hereafter, we understand P as a category such that objects are elements of P and there is a unique morphism $x \to y$ for $x > y \in P$. For $x \in P$, let $P_{\le x} = \{y \in P \mid y \le x\}$. The order complex $\Delta(P)$ is the geometric realization of an abstract simplicial complex whose simplices are finite chains $x_0 < x_1 < \cdots < x_n$ in P. Let $F\colon P \to \mathbf{Top}$ be a functor. We define two maps

$$f, g\colon \coprod_{x<y\in P} \Delta(P_{\le x}) \times F(y) \to \coprod_{x\in P} \Delta(P_{\le x}) \times F(x)$$

by

$$f = \coprod_{x<y\in P} 1_{\Delta(P_{\le x})} \times F(y > x) \quad \text{and} \quad g = \coprod_{x<y\in P} \iota_{x,y} \times 1_{F(y)},$$

where $\iota_{x,y}\colon \Delta(P_{\le x}) \to \Delta(P_{\le y})$ denotes the inclusion for $x < y$. As in [19], the homotopy colimit $\operatorname{hocolim} F$ is defined to be the coequalizer of f and g. By definition, there is a natural projection

$$\pi\colon \operatorname{hocolim} F \to \Delta(P). \tag{2.2}$$

We recall a property of regular CW complexes that we are going to use. For a CW complex X, let $P(X)$ denote its face poset. The following lemma is proved in [15, Theorem 1.6, Chapter III].

Lemma 2.3. *Let X be a regular CW complex. Then there is a homeomorphism*

$$\Delta(P(X)) \xrightarrow{\cong} X$$

which restricts to a homeomorphism $\Delta(P(X)_{\le\sigma}) \xrightarrow{\cong} \sigma$ for each face σ.

Now we describe $\texttt{Conf}_r(X)$ in terms of a homotopy colimit. Similarly to the Fadell-Neuwirth fibration [8], we consider the first projection $\pi\colon \texttt{Conf}_r(X) \to X$. Then for each face σ of X, we have

$$\pi^{-1}(\operatorname{Int}(\sigma)) = \texttt{Conf}_{r-1}(X(\sigma)).$$

Thus since $X(\sigma) \subset X(\tau)$ for $\sigma > \tau$, $\texttt{Conf}_r(X)$ is obtained by gluing $\sigma \times \texttt{Conf}_{r-1}(X(\sigma))$ along the inclusions

$$\sigma \times \texttt{Conf}_{r-1}(X(\sigma)) \leftarrow \tau \times \texttt{Conf}_{r-1}(X(\sigma)) \to \tau \times \texttt{Conf}_{r-1}(X(\tau))$$

for $\sigma > \tau$. In other words, $\texttt{Conf}_r(X)$ is homeomorphic to the coequalizer of two maps

$$f, g\colon \coprod_{\tau<\sigma\in P(X)} \tau \times \texttt{Conf}_{r-1}(X(\sigma)) \to \coprod_{\tau\in P(X)} \tau \times \texttt{Conf}_{r-1}(X(\tau))$$

defined by

$$f = \coprod_{\tau<\sigma\in P(X)} 1_\tau \times \theta_{\sigma,\tau} \quad \text{and} \quad g = \coprod_{\tau<\sigma\in P(X)} \iota_{\tau,\sigma} \times 1_{\texttt{Conf}_{r-1}(X(\sigma))},$$

where $\theta_{\sigma,\tau}\colon \mathtt{Conf}_{r-1}(X(\sigma)) \to \mathtt{Conf}_{r-1}(X(\tau))$ and $\iota_{\tau,\sigma}\colon \tau \to \sigma$ are inclusions for $\sigma > \tau$. Now we define a functor $F_r\colon P(X) \to \mathbf{Top}$ by

$$F_r(\sigma) = \mathtt{Conf}_{r-1}(X(\sigma)) \quad \text{and} \quad F(\sigma > \tau) = \theta_{\sigma,\tau}.$$

By Lemma 2.3, there is a natural homeomorphism $\Delta(P(X)_{\le\sigma}) \cong \sigma$ for each face σ of X. Then by the above observation, we get:

Theorem 2.4. *There is a homeomorphism*

$$\mathtt{Conf}_r(X) \cong \operatorname{hocolim} F_r.$$

Then we can apply the Bousfield-Kan spectral sequence to compute the homology of $\mathtt{Conf}_r(X)$. However, the E^1-term of the Bousfield-Kan spectral sequence includes a plenty of degenerate elements, and so we will apply the following variant of the Bousfield-Kan spectral sequence.

Proposition 2.5. *Let X be a regular CW complex, and let $F\colon P(X) \to \mathbf{Top}$ be a functor. Then there is a spectral sequence*

$$E^1_{p,q} \cong \bigoplus_{\substack{\sigma \in P(X)\\ \dim\sigma = p}} H_q(F(\sigma)) \quad \Longrightarrow \quad H_{p+q}(\operatorname{hocolim} F).$$

Now we get:

Lemma 2.6. *Let X be a regular CW complex, and let $F\colon P(X) \to \mathbf{Top}$ be a functor such that $F(\sigma)$ is $(n - \dim\sigma)$ acyclic for each $\sigma \in P(X)$ with $\dim\sigma \le n+1$. Then there is an isomorphism for $* \le n$*

$$H_*(\operatorname{hocolim} F) \cong H_*(X)$$

Thus we obtain the following corollary which completes the proof of Theorem 1.5 by Proposition 2.2.

Corollary 2.7. *If X is an $(r-1)$-complementary n-acyclic regular CW complex, then $\mathtt{Conf}_r(X)$ is n-acyclic.*

3. Atomicity

Theorem 1.5 shows that the Tverberg property is possessed not only by a simplex but also by a variety of CW complexes. But the Tverberg property of some CW complexes can be deduced from the that of other complexes. For example, as mentioned in Section 1, the Tverberg property of a polytopal sphere is deduced from a simplex. This section studies CW complexes having the Tverberg property that is not induced from other CW complexes.

We say that a regular CW complex X is (d,r)-*Tverberg* if for any continuous map $f\colon X \to \mathbb{R}^d$, there are pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of X such that $f(\sigma_1), \dots, f(\sigma_r)$ have a point in common. For example, by Theorem 1.5, $(r-1)$-complementary $(d(r-1)-1)$-acyclic regular CW complexes are (d,r)-Tverberg. Let X be a (d,r)-Tverberg regular CW complex. Observe that a regular CW complex Y is (d,r)-Tverberg if either of the following conditions is satisfied:

(1) X is a subcomplex of Y;
(2) Y is a refinement of X, that is, $X \cong Y$ and each face of X is the union of faces of Y.

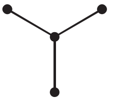
This observation leads us to:

Definition 3.1. A (d,r)-Tverberg regular CW complex is called *atomic* if it does not include a proper subcomplex which is (d,r)-Tverberg or it is not a refinement of a proper (d,r)-Tverberg complex.

Here is a fundamental problem on (d, r)-Tverberg complexes.

Problem 3.2. Given d, r and n, are there only finitely many atomic (d, r)-Tverberg finite complexes of dimension n?

First, we consider 1-dimensional $(1, 2)$-Tverberg finite complexes. Let C_n denote the cycle graph with n vertices for $n \geq 3$. Then by Corollary 1.7, C_n is $(1, 2)$-Tverberg. Let Y be the Y-shaped graph depicted below. Then by the intermediate value theorem, we can see that Y is $(1, 2)$-Tverberg too.



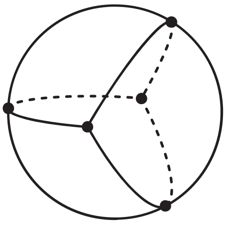
Proposition 3.3. *The only atomic 1-dimensional* $(1, 2)$*-Tverberg finite complexes are* C_3 *and* Y*.*

Remark 3.4. If we remove its center vertex of Y, then it becomes disconnected. Hence Y is not 1-complementary 0-acyclic, so that we cannot apply Theorem 1.5 for $d = 1$ and $r = 2$ to deduce that Y is $(1, 2)$-Tverberg. However, we can directly see $\mathrm{wgt}_{\mathbb{Z}/2}(\mathtt{Conf}_2(Y)) = 1$, implying that Y is $(1, 2)$-Tverberg, because $\mathtt{Conf}_2(Y)$ is a hexagon so that Lemma **??** applies.

Next, we consider $(2, 2)$-Tverberg polyhedral 2-spheres. Let $\partial\Delta^n$ denote the boundary of an n-simplex.

Proposition 3.5. *The only atomic* $(2, 2)$*-Tverberg polyhedral 2-sphere is* $\partial\Delta^3$*.*

The following 2-sphere is an atomic $(2, 2)$-Tverberg complex, and so there may be other atomic $(2, 2)$-Tverberg 2-spheres which are not polyhedral.



Then we pose a problem much weaker than Problem 3.2 but still interesting.

Problem 3.6. Are there only finitely many atomic $(2, 2)$-Tverberg 2-spheres?

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