TORSION IN THE SPACE OF COMMUTING ELEMENTS IN A LIE GROUP

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Abstract. Let G be a compact connected Lie group, and let $\mathrm{Hom}(\mathbb{Z}^m, G)$ denote the space of homomorphisms from a free abelian group $\mathbb{Z}^m$ to G. We study the problem of which primes p $\mathrm{Hom}(\mathbb{Z}^m, G)$ has p-torsion in homology. We give a new homotopy decomposition of the space, and we prove that $\mathrm{Hom}(\mathbb{Z}^m, SU(n))$ for $m \geq 2$ has p-torsion in homology if and only if $p \leq n$. In this text we overview the proof and observe some examples.

1. Introduction

This text is based on the joint work with Daisuke Kishimoto "Torsion in the space of commuting elements in a Lie group"[14]. In this text, the focus will be on introducing the results of this joint work and observing examples.

Let G be a compact connected Lie group. Let $\mathrm{Hom}(\mathbb{Z}^m, G)$ denote the space of homomorphisms from a free abelian group $\mathbb{Z}^m$ to G. This space has induced topology of the space of continuous maps from $\mathbb{Z}^m$ to G. $\mathrm{Hom}(\mathbb{Z}^m, G)$ is homeomorphic to the subspace of the Cartesian product G^m consisting of $(g_1, \dots g_m) \in G^m$ such that $g_i g_j = g_j g_i$ for all i, j. So we call $\mathrm{Hom}(\mathbb{Z}^m, G)$ the space of commuting elements in G. We denote $\mathrm{Hom}(\mathbb{Z}^m, G)_1$ as the connected component of $\mathrm{Hom}(\mathbb{Z}^m, G)$ containing the trivial homomorphism.

Since $\mathrm{Hom}(\mathbb{Z}^m, G)$ is identified with the based moduli space of the flat bundle, $\mathrm{Hom}(\mathbb{Z}^m, G)$ is studied in geometry and mathematical physics, for example [1, 7, 12, 17, 18]. And there are many results about $\mathrm{Hom}(\mathbb{Z}^m, G)$ in topology, for example [2, 3, 4, 5, 6, 9, 10, 13, 15, 16].

In this text we denote T a maximal torus of G and W the Weyl group of G. Let $\mathbb{F}$ be a field of characteristic not dividing the order of W or 0. In [5] Baird described the cohomology of $\mathrm{Hom}(\mathbb{Z}^m, G)_1$ with coefficient $\mathbb{F}$ as a certain ring of invariants of W. Based on this result, Ramras and Stafa [15] proved that the Poincaré series of the cohomology of $\mathrm{Hom}(\mathbb{Z}^m, G)_1$ with the coefficient $\mathbb{F}$ is given by

$$\frac{\prod_{i=1}^{r}(1-t^{2d_i})}{|W|}\sum_{w\in W}\frac{\det(1+tw)^m}{\det(1-t^2w)},$$

where $d_1, \dots, d_r$ are the characteristic degrees of W. This formula doesn't depend on the characteristic of $\mathbb{F}$ as long as its characteristic does not divide the order of W or is zero. Thus we obtain the non-existence of torsion in homology.

Lemma 1.1. *The homology of* $\mathrm{Hom}(\mathbb{Z}^m, G)_1$ *doesn't have p-torsion in homology when p doesn't divide the order of W.*

On the other hand, there is few result about existence of torsion in the homology of $\mathrm{Hom}(\mathbb{Z}^m, G)_1$. Baird, Jeffrey and Selick [6] and Crabb [9] give the stable decomposition of $\mathrm{Hom}(\mathbb{Z}^m, SU(2))$. By this result, we can obtain that $\mathrm{Hom}(\mathbb{Z}^m, SU(2))$ has 2-torsion. By combining the result of the computation of fundamental groups by Adem, Gómez and Grischacher [4] and the computation of secomd homotopy groups by Gómez, Pettet and Souto [10], we obtain that $\mathrm{Hom}(\mathbb{Z}^m, Sp(n))$ has 2-torsion for $m \geq 3$. These are all result about existence of torsion in homology of $\mathrm{Hom}(\mathbb{Z}^m, G)$.

The main theorem in [14] is the following.

Theorem 1.2. *The homology of* $\mathrm{Hom}(\mathbb{Z}^m, SU(n))_1$ *for* $m \geq 2$ *has* p*-torsion if and only if* $p \leq n$*.*

To prove this theorem, we give a new homotopy decomposition of $\mathrm{Hom}(\mathbb{Z}^m, G)_1$. In this text, we overview the proof and observe some examples.

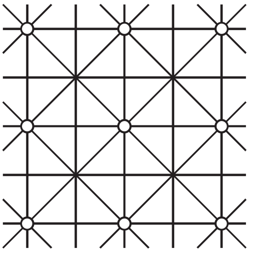
2. Triangulation of a maximal torus

In this section we briefly description a cell structure on T/W and a characterization of each cell. For more information on this section, please see Section 2 in [14].

Hereafter, let G denote a compact simply-connected simple Lie group with $\mathrm{rank}\, G = k$. Let $\mathfrak{t}$ be the Lie algebra of T, and let Φ be the set of roots of G. The Stiefel diagram is defined by

$$\bigcup_{\substack{\alpha \in \Phi \\ i \in \mathbb{Z}}} \alpha^{-1}(i) \subset \mathfrak{t}.$$

For example, the Stiefel diagram of $Sp(2)$ is given as follows.



We call each connected component of the complement of the Stiefel diagram a Weyl alcove. Since G is a compact simply-connected simple Lie group, the closure of any Weyl alcove is homeomorphic to k-simplex. Moreover a Weyl alcove is identified with the following a k-simplex

$$\Delta = \{x \in \mathfrak{t} \mid \alpha_1(x) \geq 0, \ldots, \alpha_k(x) \geq 0,\ \widetilde{\alpha}(x) \leq 1\},$$

where $\alpha_1, \ldots \alpha_k$ are simple roots, and $\widetilde{\alpha}$ is the highest root. Then the facets of Δ is corresponding to the one of the simple roots or the highest root. On the other hand, T/W is identified with the closure of a Weyl alcove. By combining the upper discussion, we obtain the next proposition.

Proposition 2.1. *The quotient space* T/W *is naturally identified with* Δ*.*

3. Homotopy colimit

In this section we recall the homotopy colimit. Let K be a simplicial complex and $P(K)$ be the face poset of K. We regard $P(K)$ as a category, and we take a functor $F \colon P(K) \to \mathrm{Top}$. Then the homotopy colimit of F, $\mathrm{hocolim}\, F$, is defined by

$$\mathrm{hocolim}\, F \cong \coprod_{\sigma \in K} F(\sigma) \times \sigma / \sim,$$

where the equivalence relation is generated by $(x, F(\iota)(y)) \sim (\iota(x), y)$ for $x \in \sigma$, $y \in F(\tau)$ and the inclusion $\iota\colon \sigma \hookrightarrow \tau$. Roughly, this is like a fiber space with different fibers on each cell.

To compute the homology of the homotopy colimit, we use the variant of the Bousfield-Kan spectral sequence constructed in [11]. In [8], the original Bousfield-Kan spectral sequence is explained.

Proposition 3.1. *Let $F\colon P(K) \to \mathbf{Top}$ be a functor, where $P(K)$ denotes the face poset of a simplicial complex K. Then there is a spectral sequence*

$$E^1_{p,q} = \bigoplus_{\sigma \in P_p(K)} H_q(F(\sigma)) \quad \Longrightarrow \quad H_{p+q}(\operatorname{hocolim} F),$$

where $P_p(K)$ denotes the set of p-simplices of K.

We can construct this spectral sequence by the similar way to construct the Serre spectral sequence.

4. Homotopy decomposition

This section constructs a new homotopy decomposition of $\operatorname{Hom}(\mathbb{Z}^m, G)_1$. The quotient space of G by the adjoint action of G is isomorphic to T/W, and by Proposition 2.1 it is isomorphic to Δ. We define a map π as the composition of the following maps

$$\pi\colon \operatorname{Hom}(\mathbb{Z}^m, G)_1 \to \operatorname{Hom}(\mathbb{Z}, G) \cong G \to \Delta,$$

where the first map is the m-th projection and the last map is the quotient map by the adjoint action of G. Then the following lemma hold.

Lemma 4.1. *If $x, y \in \Delta$ belong to the interior of a common face, then*

$$\pi^{-1}(x) \cong \pi^{-1}(y).$$

Sketch of proof. In [5] Baird induces the map

$$\phi\colon G/T \times T^m \to \operatorname{Hom}(\mathbb{Z}^m, G)_1 \quad (g, t_1, \ldots t_m) \mapsto (g^{-1}t_1 g, \ldots, g^{-1}t_m g),$$

for $g \in G/T, (t_1, \ldots t_m) \in T^m$ and proves this map is a surjection.

Suppose that $x, y \in \Delta^k$ are in the interior of a same face. Then for each $(t_1, \ldots, t_{m-1}) \in T^m$, the isotropy subgroups of $(t_1, \ldots, t_{m-1}, x)$ and $(t_1, \ldots, t_{m-1}, y)$ by the adjoint action of G are equal. And there are equivalences

$$(\phi \circ \pi)^{-1}(x) = G/T \times T^{m-1} \times W \cdot x \quad \text{and} \quad (\phi \circ \pi)^{-1}(y) = G/T \times T^{m-1} \times W \cdot y.$$

Thus by the definition of the map ϕ, we obtain $\pi^{-1}(x) \cong \pi^{-1}(y)$, as stated. □

Let σ_0 denote the barycenter of a face $\sigma \in P(K)$. Then we can obtain the following theorem.

Theorem 4.2. *Let G be a simple, simply connected, compact Lie group. Then there is a functor $F_m\colon P(\Delta) \to \mathrm{Top}$ with $F_m(\sigma) = \pi^{-1}(\sigma_0)$ such that there is a homeomorphism*

$$\operatorname{Hom}(\mathbb{Z}^m, G)_1 \cong \operatorname{hocolim} F_m.$$

We look at examples about this theorem.

Example 4.3. We look at $F_m(\sigma)$ for some σ. When σ is the top cell, there is a homeomorphism

$$F_m(\sigma) \cong G/T \times T^{m-1}.$$

When σ is the 0-cell of the center in G, there is a homeomorphism

$$F_m(\sigma) \cong \operatorname{Hom}(\mathbb{Z}^{m-1}, G).$$

Example 4.4. We consider the homotopy desomposition of $\mathrm{Hom}(\mathbb{Z}^m, SU(2))$. Since rank of $SU(2)$ is 1, Δ is a 1-simplex. Let v_0, v_1 be vertices of Δ, and let e be an edge of Δ. Since v_0 and v_1 correspond to the center, we have

$$F_m(v_i) \cong \mathrm{Hom}(\mathbb{Z}^{m-1}, SU(2))$$

for $i = 0, 1$. Then by Theorem 4.2 and Example 4.3, there is a homotopy pushout

$$\begin{array}{ccc} S^2 \times (S^1)^{m-1} & \longrightarrow & \mathrm{Hom}(\mathbb{Z}^{m-1}, SU(2)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\mathbb{Z}^{m-1}, SU(2)) & \longrightarrow & \mathrm{Hom}(\mathbb{Z}^{m}, SU(2)), \end{array}$$

where the map $S^2 \times (S^1)^{m-1} \to \mathrm{Hom}(\mathbb{Z}^{m-1}, SU(2))$ is equal to the map ϕ in the proof of Lemma 4.1. Especially when $m = 2$, there is a homotopy pushout

$$\begin{array}{ccc} S^2 \times S^1 & \longrightarrow & S^3 \\ \downarrow & & \downarrow \\ S^3 & \longrightarrow & \mathrm{Hom}(\mathbb{Z}^{2}, SU(2)), \end{array}$$

where the map $S^2 \times S^1 \to S^3$ is not a simple quotient map but the composition of the quotient map and the map of degree 2.

5. The functor $\widehat{F}_2$

Let $d = \dim(G)$. In the Bousfield-Kan spectral sequence of F_2, we call $E_{*,d}$ the top line. In this section to focus on this top line, we define a functor $\widehat{F}_2$ and a natural transformation ρ.

At first we observe some examples. For top cell $\sigma_{\mathrm{top}} \in P(\Delta)$ and the 0-cell $\sigma_0 \in P(\Delta)$ with corresponding to the center in G, the map $F_2(\sigma_{\mathrm{top}}) \to F_2(\sigma_0)$ is identified with the map

$$\phi\colon G/T \times T \to G \quad (g, t) \mapsto g^{-1}tg,$$

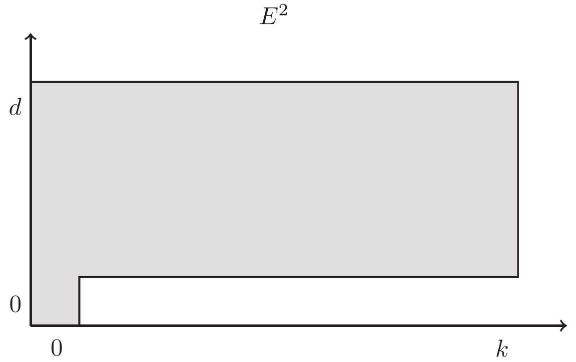
for $g \in G, t \in T$. It is well known that the induced map in top homology $\phi_*\colon H_d(G/T \times T) \to H_d(G)$ is the map of degree $|W|$. By considering the Bousfield-Kan spectral sequence of F_2, it seems that there may be p-torsion in the top line for prime number p that divides $|W|$. Moreover when $\sigma \in P(\Delta)$ is the top cell or a 0-cell with corresponding to the center in G, by Example 4.3 there is a quotient map $F_2(\sigma) \to S^d$ such that $H_d(F_2(\sigma)) \to H_d(S^d)$ is isomorphism. It seems that there may be the restriction to the top line. In fact we can construct such a natural transformation in general.

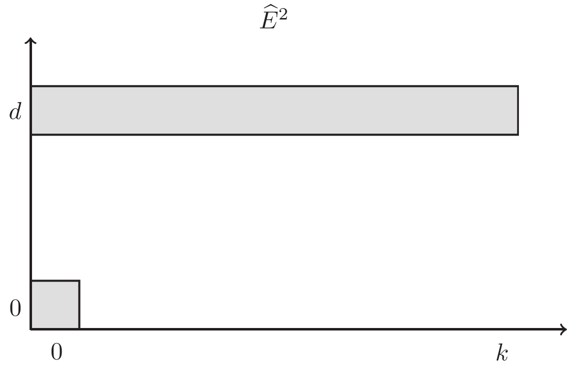
For $\sigma \in P(\Delta)$, let $W(\sigma) \subset W$ be the stabilizer of the barycenter of σ. In other words, $W(\sigma)$ is the group generated by the reflection corresponding to the root whose facet include σ. We define a functor $\widehat{F}_2\colon P(\Delta) \to \mathbf{Top}$ by $\widehat{F}_2(\sigma) = S^d$ such that the map $\widehat{F}_2(\sigma > \tau)\colon \widehat{F}_2(\sigma) \to \widehat{F}_2(\tau)$ is a map of degree $|W(\tau)|/|W(\sigma)|$. Then the following proposition holds.

Proposition 5.1. *There is a natural transformation $\rho\colon F_2 \to \widehat{F}_2$ such that the map $\rho_\sigma\colon H_d(F_2(\sigma)) \to H_d(\widehat{F}_2(\sigma))$ is an isomorphism for any $\sigma \in P(\Delta)$.*

About the construction of ρ, please see the section 4 in [14].

Let (E^r, d^r) and $(\widehat{E}^r, \widehat{d}^r)$ denote the spectral sequence of Proposition 3.1 for $\mathrm{hocolim}\, F_2$ and $\mathrm{hocolim}\, \widehat{F}_2$. Then the E^2 term of the (E^r, d^r) and $(\widehat{E}^r, \widehat{d}^r)$ are illustrated below, where possibly non-trivial parts are shaded.





Since the bottom lines of these spectral sequences correspond to the homology of Δ, the bottom lines are collard white except for $(0, 0)$. The natural transformation ρ induces the map between these spectral sequences that is isomorphic to the top line. Therefore by an canonical discussion, we obtain the next proposition.

Proposition 5.2. *$H_*(\mathrm{hocolim}\, \widehat{F}_2)$ is a direct summand of $H_*(\mathrm{Hom}(\mathbb{Z}^2, G)_1)$.*

By this proposition, if $\mathrm{hocolim}\, \widehat{F}_2$ has p-torsion in homology, then there exists p-torsion in $H_*(\mathrm{Hom}(\mathbb{Z}^2, G)_1)$. Moreover, since $\mathrm{Hom}(\mathbb{Z}^2, G)_1$ is a retract of $\mathrm{Hom}(\mathbb{Z}^m, G)_1$, the p-torsion in $\mathrm{hocolim}\, \widehat{F}_2$ induces the p-torsion in $\mathrm{Hom}(\mathbb{Z}^m, G)_1$ in homology.

Proposition 5.3. *If $\mathrm{hocolim}\, \widehat{F}_2$ has p-torsion in homology, then $\mathrm{Hom}(\mathbb{Z}^m, G)_1$ has p-torsion in homology.*

6. Computation of torsion in homology

This section computes some torsion in the homology of $\mathrm{Hom}(\mathbb{Z}^m, SU(n))_1$ for some small n. By Proposition 5.3, if we obtain torsion in the homology of $\mathrm{hocolim}\, \widehat{F}_2$, we can obtain torsion in the homology of $\mathrm{Hom}(\mathbb{Z}^m, SU(n))_1$. To obtain torsion in the homology of $\mathrm{hocolim}\, \widehat{F}_2$, we define a colored extended Dynkin diagram. A colored extended Dynkin diagram of G is an extended Dynkin diagram of G whose vertices are colored by black and white. For a colored extended Dynkin diagram Γ, let W_Γ denote the subgroup of W generated by the reflections corresponding to the roots with colored black in Γ. Then the next lemma follows from the definition.

Lemma 6.1. *There is a bijection*

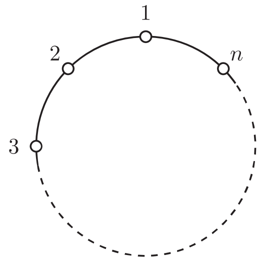
$$\Psi\colon P_i(\Delta) \xrightarrow{\cong} \{\textit{colored extended Dynkin diagrams with } k-i \textit{ black vertices}\}$$

which sends an i-face $\sigma \in P_i(\Delta)$ to a colored extended Dynkin diagram such that only $n-i$ vertices that correspond to the facets including σ are black-colored. Moreover there is an equation

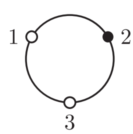
$$W_{\Psi(\sigma)} = W(\sigma).$$

We consider the chain complex of hocolim $\widehat{F}_2$. For a CW complex X, let $C_*(X)$ denote the cellular chain complex over $\mathbb{Z}$. The cell decomposition of S^d is given by $S^d = e_0 \cup e_d$, and let ι be the generator of $C_d(S^d)$ corresponding to the top cell e_d. Then $C_*(\text{hocolim}\,\widehat{F}_2)$ is spanned by $\sigma, \sigma \times \iota$ for $\sigma \in P(\Delta)$.

Let $1, 2, \ldots, n$ be vertices of the extended Dynkin diagram of $SU(n)$ as follows.



For $1 \geq i_1 < i_2 < \cdots < i_k \leq n$ we denote $\{i_1, \ldots i_k\}$ an $(k-1)$-face $\sigma \in P_{k-1}(\Delta)$ such that the white vertices of the extended Dynkin diagram $\Psi(\sigma)$ are $\{i_1, \ldots i_k\}$. For example, as for $G = SU(3)$, $\{1, 3\}$ corresponds the following colored extended Dynkin diagram.



Now we compute the homology for hocolim $\widehat{F}_2$ for $G = SU(3), SU(4)$. First we consider the case $G = SU(3)$. We compute the derivation, ∂, in $C_*(\text{hocolim}\,\widehat{F}_2)$. By the definition of $\widehat{F}_2$, the derivation on the basis corresponding to $\sigma \in P(\Delta)$ is equal to the derivation in $C_*(\Delta)$. And the derivation on the other basis is defined as follows.

$$\begin{aligned}
\partial(\{1,2,3\} \times \iota) &= 2\{2,3\} \times \iota - 2\{1,3\} \times \iota + 2\{1,2\} \times \iota \\
\partial(\{i_1, i_2\} \times \iota) &= 3\{i_2\} \times \iota - 3\{i_1\} \times \iota \\
\partial(\{i_1\} \times \iota) &= 0,
\end{aligned}$$

for $1 \leq i_1 < i_2 \leq 3$. Therefore when $G = SU(3)$ the homology of hocolim $\widehat{F}_2$ is

$$H_i(\text{hocolim}\,\widehat{F}_2) \cong \begin{cases} \mathbb{Z} & (i = 0) \\ \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 & (i = 8) \\ \mathbb{Z}/2 & (i = 9) \\ 0 & (\text{the others}). \end{cases}$$

Therefore we obtain that $\text{Hom}(\mathbb{Z}^m, SU(3))_1$ has p-torsion in homology for $p = 2, 3$.

Next we consider the case $G = SU(4)$. We compute the derivation, ∂, in $C_*(\mathrm{hocolim}\,\widehat{F}_2)$ by a similar way. The derivation is defined as follows.

$$\begin{aligned}
\partial(\{1,2,3,4\}\times\iota) &= 2\{2,3,4\}\times\iota - 2\{1,2,4\}\times\iota + 2\{1,3,4\}\times\iota - 2\{2,3,4\}\times\iota\\
\partial(\{1,2,3\}\times\iota) &= 3\{2,3\}\times\iota - 2\{1,3\}\times\iota + 3\{1,2\}\times\iota\\
\partial(\{1,2,4\}\times\iota) &= 2\{2,4\}\times\iota - 3\{1,4\}\times\iota + 3\{1,2\}\times\iota\\
\partial(\{1,3,4\}\times\iota) &= 3\{3,4\}\times\iota - 3\{1,4\}\times\iota + 2\{1,3\}\times\iota\\
\partial(\{2,3,4\}\times\iota) &= 3\{3,4\}\times\iota - 2\{2,4\}\times\iota + 3\{2,3\}\times\iota\\
\partial(\{i,i+1\}\times\iota) &= 4\{i+1\}\times\iota - 4\{i\}\times\iota\\
\partial(\{j,j+2\}\times\iota) &= 6\{j+2\}\times\iota - 6\{j\}\times\iota\\
\partial(\{k\}\times\iota) &= 0,
\end{aligned}$$

for $1 \le i \le 3$, $1 \le j \le 2$ and $1 \le k \le 4$. Therefore when $G = SU(4)$ the homology of $\mathrm{hocolim}\,\widehat{F}_2$ is

$$H_i(\mathrm{hocolim}\,\widehat{F}_2) \cong \begin{cases} \mathbb{Z} & (i=0)\\ \mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/4 & (i=15)\\ \mathbb{Z}/3 & (i=16)\\ \mathbb{Z}/2 & (i=17)\\ 0 & (\text{the others}).\end{cases}$$

Therefore we obtain that $\mathrm{Hom}(\mathbb{Z}^m, SU(4))_1$ has p-torsion in homology for $p = 2, 3$. In this case $\mathrm{Hom}(\mathbb{Z}^m, SU(4))_1$ has higher 2-torsion $\mathbb{Z}/4$, but we don't know when the torsion is higher torsion or not.

In our paper [14], by using an another property and prove the following theorem.

Theorem 6.2. *The homology of* $\mathrm{Hom}(\mathbb{Z}^m, SU(n+1))_1$ *for* $m \ge 2$ *has* p*-torsion in homology if and only if* $p \le n+1$.

By the upper calculation, we obtain this theorem for $n = 2, 3$.

7. Another results

In this section we see some results that we cannot write in the main part. By using the homotopy decomposition of $\mathrm{Hom}(\mathbb{Z}^m, G)$, we can compute the top term of the homology like the followings.

Theorem 7.1. *Let* G *be a compact simply-connected simple Lie group of rank* n*, and let*

$$d = \begin{cases} \dim G + n(m-1) - 1 & m \text{ is even}\\ \dim G + n(m-1) & m \text{ is odd.}\end{cases}$$

Then the top homology of $\mathrm{Hom}(\mathbb{Z}^m, G)_1$ *is given by*

$$H_d(\mathrm{Hom}(\mathbb{Z}^m, G)_1) \cong \begin{cases} \mathbb{Z}/2 & m \text{ is even}\\ \mathbb{Z} & m \text{ is odd.}\end{cases}$$

Corollary 7.2. *Let* G *be a compact simply-connected simple Lie group. Then* $\mathrm{Hom}(\mathbb{Z}^m, G)_1$ *for* $m \ge 2$ *has 2-torsion in homology.*

By the computation similar to the case $G = SU(n)$, we can obtain existence of p-torsion for some p.

Theorem 7.3. *If* $p \le n$ *and* $n \equiv 0, 1 \mod p$*, then for* $m \ge 2$*,* $\mathrm{Hom}(\mathbb{Z}^m, Spin(2n))_1$ *has* p*-torsion in homology.*

Theorem 7.4. *Let G be an exceptional Lie group. Then* $\mathrm{Hom}(\mathbb{Z}^m, G)_1$ *for* $m \geq 2$ *has p-torsion in homology if and only if p divides $|W|$, except possibly for* $(G, p) = (E_7, 5), (E_7, 7), (E_8, 7)$.

But in the other cases we can't obtain existence of p-torsion. Moreover we have proved that hocolim $\widehat{F}_2$ doesn't have p-torsion in almost all of the cases. We write this precisely in Section 8 of [14].

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