Disturbing Actions of a Shaft Governor.

By

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The present paper is devoted exclusively to a consideration of the disturbing actions of a shaft governor applied to a steam engine as its speed regulator. In the case of a conical pendulum governor, the force at the sleeve required for overcoming the resistance of the regulating gear, when a movement of the sleeve is to commence, may be regarded as constant in magnitude, and, if there be any unbalanced force transmitted from the regulating gear to the sleeve in the state of equilibrium, this force is comparatively small and might be considered an additional loading or unloading on the sleeve. The case is quite different in a shaft governor. Most shaft governors in practical use are attached to the fly wheel, the eccentric disc being held in position by the governor; and on this account a comparatively large amount of resistance of the valve gear continuously reacts on the governor. Fluctuating and reversing periodically according to the position of the crank, these forces cause the pendulum of the governor to vibrate about its position of equilibrium, and this vibratory motion may cause a false steam distribution. Also when the load on the engine is altered, the relative configration of the governor changes accordingly and the pendulum moves from one position of equilibrium to another. On account of the inertia of the pendulum and the parts connected therewith this change of configuration is sometimes accompanied by a number of bibratory motions which way cause a serious disturbance of steam distribution. The object of this paper is to report an investigation of the laws of these disturbing motions; but, before entering upon the subject proper, it is necessary to consider the statical conditions of the governor, i. e. the conditions of equilibrium under the action of several statical forces, and to give a method of designing it in connection therewith. In this part of the paper the

fluctuation of reacting forces will be neglected and a constant force, equal in magnitude to their mean value, will be taken as continuously acting on the oscillating system.

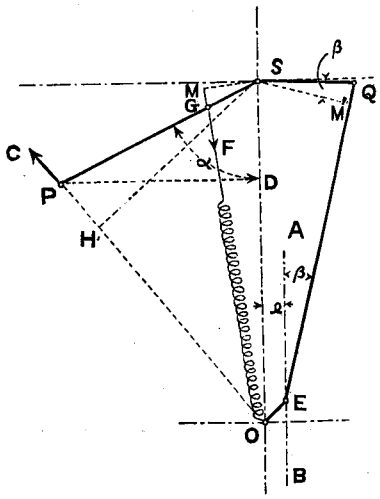


Fig. 1

Fig. 1 is a diagrammatic sketch of a shaft governor, showing about one half. It is attached to and rotates with the crank shaft O, as usual, and consists of a pair of pendulums arranged symmetrically with respect to O, each pendulum having its fulcrum at S and its centre of gravity at P. A spring F attached at G and, passing through O, connects the pendulums. The centrifugal force C of each pendulum tends to throw it out against the force of the spring F. The pendulum has a lever SQ on the opposite side of S; and the end Q of this lever is connected with the eccentric centre E by means of a rad QE. In order to keep the lead of the valve constant the eccentric centre is assumed to be guided so as to move along a straight line AB perpendicular to the centre line of the crank. If the engine is unloaded either suddenly or gradually, the speed of the engine increases, the pendulum flies out owing to the corresponding increase of centrifugal force and consequently the steam is cut off earlier, diminishing the supply so as to bring down the speed to its proper level. On the contrary, when the pendulum swings in, owing to a decrease of speed, the supply of steam is increased so as to bring the speed up again.

I. Equilibrium of the Pendulum.

If there were no vibratory motion of the governor, the pendulum would be at rest with respect to the fly wheel in a steady working condi-

tion of the engine; and, consequently, all moments of force acting at several points of the pendulum would be in equilibrium. For the equilibrium of the pendulum about its fulcrum S, the following three moments must form a balancing set: the moment of the centrifugal force M_z acting clock-wise, the moment of force of the spring M_f acting counter clock-wise and the moment of the mean reacting force of the valve gear M_{r_0} acting counter clock-wise. The first moment M_z depends on the angular velocity and the position of the pendulum, the second and third moments, M_f and M_{r_0}, on the position of the pendulum only.

1° Moment of the centrifugal force of the pendulum:

Let, (see Fig. 1).

ω = angular velocity of the governor shaft,

α = angle OSP,

ρ = distance $\overline{SP}$,

r = distance $\overline{PO}$,

a = distance $\overline{SO}$,

$d = \overline{PD}$ = perpendicular distance of P from SO,

m = mass of each pendulum.

Draw SH perpendicular to OP, then

$$M_z = \omega^2 mr \,.\, \overline{SH}.$$

But, twice the area of the triangle SPO is

$$r \,.\, \overline{SH} = ad = a\rho \sin\alpha.$$

Therefore we have

$$M_z = \omega^2 ma\rho \sin\alpha. \qquad (1)$$

2° Moment of force of the spring:

The spring is assumed to be attached to the pendulum at a point G in the line SP.

Let, l = half length of the spring = $\overline{OG}$,

l_0 = half its natural length,

$\varkappa$ = force of the spring caused by extension of unit length,

b = distance $\overline{SG}$,

$h = \overline{SM}$ = perpendicular distance of S from the centre line of the coil of the spring OG,

then we have

$$M_f = 2\varkappa(l-l_0)h = 2\varkappa lh - 2\varkappa l_0 h.$$

But since

$$lh = ab \sin \alpha$$

or

$$h = \frac{ab \sin \alpha}{l},$$

it follows that

$$M_f = 2\varkappa ab \sin\alpha - 2\varkappa l_0 \frac{ab \sin\alpha}{l}, \tag{2}$$

in which

$$\frac{1}{l} = (a^2+b^2-2ab\cos\alpha)^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{a^2+b^2}}\left[1-\frac{2ab\cos\alpha}{a^2+b^2}\right]^{-\frac{1}{2}}.$$

If $2b \leqq a$, the second term in the square bracket is smaller than unity for all values of α, and so the expression in the square bracket may be expanded into a convergent series by the binomial theorem. But in practice as b is very small in comparison with $\frac{a}{2}$, we may neglect those terms which contain higher powers of $\frac{2ab\cos\alpha}{a^2+b^2}$ in the series.

Thus we obtain

$$\frac{1}{l} \doteqdot \frac{1}{\sqrt{a^2+b^2}}\left(1+\frac{ab\cos\alpha}{a^2+b^2}\right) = \frac{1}{\sqrt{a^2+b^2}} + \frac{ab\cos\alpha}{(a^2+b^2)^{\frac{3}{2}}}.$$

Substituting this in (2)

$$M_f = 2\varkappa ab \sin\alpha - 2\varkappa l_0 \frac{ab\sin\alpha}{\sqrt{a^2+b^2}}$$

$$-\varkappa l_0 \frac{a^2b^2 \sin 2\alpha}{(a^2+b^2)^{\frac{3}{2}}}. \tag{3}$$

3° Moment of reacting forces of the valve gear:

These forces may be reduced to the eccentric centre and consist of the following:

(i) frictional resistance to the reciprocating motion of the valve gear, chiefly that in the stuffing box,

(ii) force due to inertia of reciprocating parts,

(iii) unbalanced steam pressure on the valve spindle,

(iv) weight of valve, valve spindle, eccentric and eccentric rod, and

(v) centrifugal force of eccentric disc and strap.

In a vertical engine the first four forces act in a direction parallel to the centre line of the engine, if the angularity of the eccentric rod is neglected, and their components in the direction of the line AB in Fig. 1, in which the movement of the eccentric centre takes place, vary periodically. In Fig. 2, P denotes one of those forces, which is taken as positive when it acts in the direction shown by the arrow head, θ the angular displacement of the crank measured from its inner dead position C_0O— the cylinder being supposed to be at the left hand side of the figure —, δ the angular advance of the eccentric; and AB the direction of motion of E which is perpendicular to CO as mentioned above; then the components F' of P in the direction of AB, which is assumed positive when it is directed outwards, is expressed by

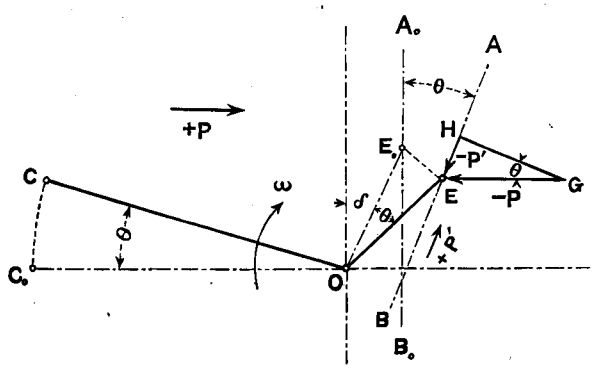


Fig. 2

$$P' = +P\sin\theta. \tag{4}$$

(i) Denote the frictional resistance to the reciprocating motion of the valve gear by $\pm P_1$, whose absolute value may be taken constant; then the component P_1' of this force along AB is obtained from equation (4)

$$P_1' = \pm P_1 \sin\theta, \tag{5}$$

where upper sign is to be taken for the value of θ from $\left(\frac{\pi}{2}-\delta\right)$ to

$\left(\frac{3\pi}{2}-\delta\right)$ and lower sign for the value of θ from $\left(\frac{3\pi}{2}-\delta\right)$ to $\left(\frac{\pi}{2}-\delta\right)$.

(ii) If we neglect the angularity of the eccentric rod, the force of inertia P_2 of the reciprocating parts is

$$P_2 = m_v x \omega^2,$$

where m_v = total mass of the reciprocating system,

x = its displacement from the mid position, takiug it positive when it is to the right

$= r\sin(\delta+\theta)$

$= e\,.\,\frac{\sin(\delta+\theta)}{\sin\delta}$ (see Fig. 3).

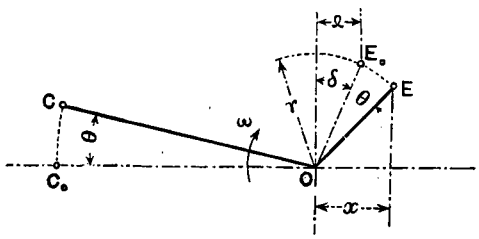


Fig. 3

Hence, from equation (4) the component of P_2 is

$$P_2' = P_2\sin\theta$$

$$= m_v\omega^2 e\sin(\delta+\theta)\,.\,\frac{\sin\theta}{\sin\delta} \qquad (6)$$

(iii) & (iv) The forces P_3 and P_4 (mentioned in iii & iv) are both constant, the first acting always in the direction of the axis of the cylinder, while the latter has its whole effect only when the engine is of the vertical type. The components of P_3 and P_4 are

$$P_3'+P_4' = (P_3+P_4)\sin\theta \qquad (7)$$

for a vertical engine. For a horizontal engine, if $P_{4'}$ is the weight of eccentric and the portion of the weight of the eccentric rod which has effect on the eccentric centre, they are

$$P_3'+P_4' = P_3\sin\theta - P_{4'}\cos\theta. \qquad (8)$$

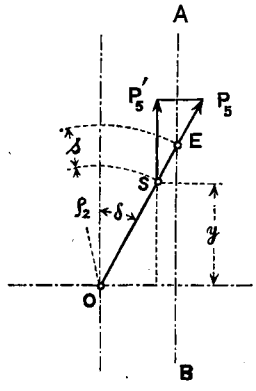


Fig. 4

(v) If m_e is the mass of the rotating parts of the valve gear, and ρ_2 the distance of its centre of gravity S from the axis of rotation O (see Fig. 4), the centrifugal force P_5 is

$$P_5 = m_e\omega^2\rho_2;$$

and then the component of P_5 in the direction of AB is

$$P_5' = P_5 \cos\delta = m_e\omega^2\rho_2 \cos\delta$$

or $$= m_e\omega^2 y$$

or $$= m_e\omega^2(e \cot\delta - s \cos\delta). \qquad (9)$$

The sum of these reacting forces is

$$\Sigma P' = P_1' + P_2' + P_3' + P_4' + P_5'.$$

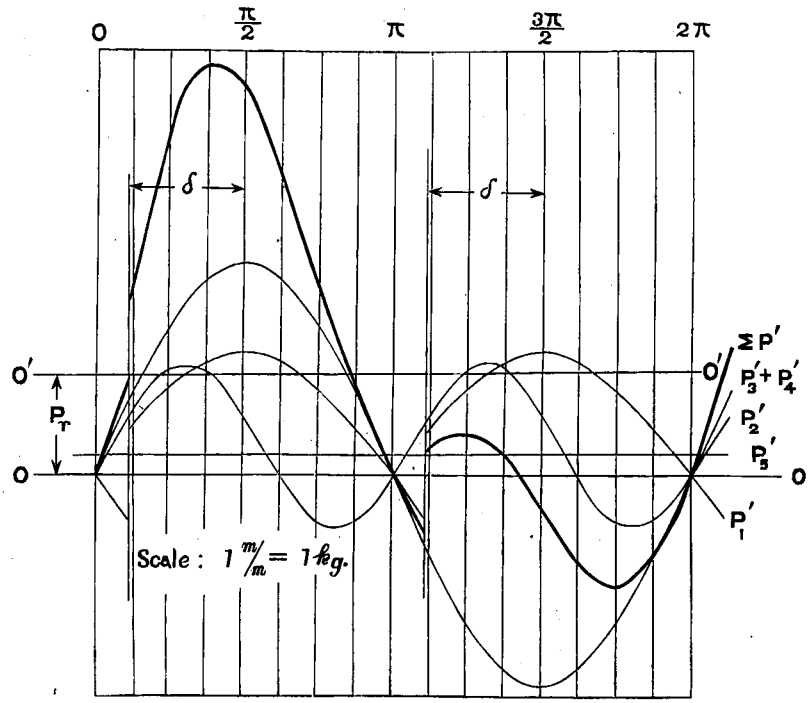


Fig. 5

The diagram shown in Fig. 5 gives an example of the fluctuation of these forces (P' 's) and their resultant during one revolution of the crank. As may be seen from the diagram, the ordinate denotes the force and the obscissa the angular displacement or time, since the angular velocity is supposed to be constant, and the area may be supposed to represent impulse—force into time—; consequently it also gives the change of momentum of the pendulum and all other oscillating parts. But as the mass of these moving parts is constant, the area of the diagram between any two ordinates gives also their change of velocity in that interval of

time. When the load on the engine is constant the mean configuration of the governor must remain unaltered and, therefore, the motion of the pendulum, produced by such forces, must be a periodic oscillation about the mean configuration, i.e. the configuration determined by conditions of equilibrium for that load; and so its velocity must assume the same value after each complete oscillation. This condition is satisfied when the areas of the diagram above and below the line O′O′, drawn at a height above OO representing the mean value of the forces, are equal to each other. The moment of this mean force P_r must be in equilibrium with the resultant of the other two moments M_z and M_f.

Therefore we get an equation of statical equilibrium of the governor:

$$M_f - M_z + M_{r_0} = 0, \tag{10}$$

in which M_z and M_f are given in equations (1) and (3); the moment M_{r_0} of P_r is obtained as follows:—

The area of the curve $\Sigma P'$ may be easily obtained by finding the area of each of the P' curves:

(i) the area of the curve P_1' is, from equation (5),

$$F_1 = \int_0^{2\pi} P_1' d\theta = \int_{\frac{\pi}{2}-\delta}^{\frac{3\pi}{2}-\delta} P_1 \sin\theta d\theta + \int_{\frac{3\pi}{2}-\delta}^{\frac{\pi}{2}-\delta} -P_1 \sin\theta d\theta$$

$$= 4P_1 \sin\delta;$$

(ii) the area of the curve P_2' is, from equation (6),

$$F_2 = \int_0^{2\pi} P_2' d\theta = \int_0^{2\pi} \frac{m_v \omega^2 e}{\sin\delta} (\cos\delta \sin^2\theta + \sin\delta \sin\theta \cos\theta) d\theta$$

$$= \frac{m_v \omega^2 e}{\sin\delta} \cdot \pi \cos\delta;$$

(iii) & (iv) the area of the curve $P_3' + P_4'$ is, from equation (7),

$$F_3 + F_4 = \int_0^{2\pi} (P_3' + P_4') d\theta = \int_0^{2\pi} (P_3 + P_4) \sin\theta d\theta = 0:$$

(v) the area of the curve P_5' is, from equation (8),

$$F_5 = 2\pi m_e \omega^2 y\,;$$

and hence the resultant area of $\Sigma P'$ is

$$\Sigma F = F_1 + F_2 + F_3 + F_4 + F_5$$

$$= 4P_1 \sin\delta + \frac{m_v \omega^2 e}{\tan\delta} \,.\, \pi + 2\pi m_e \omega^2 y.$$

The mean force P_r is

$$P_r = \frac{\Sigma F}{2\pi}$$

$$= \frac{2P_1}{\pi} \sin\delta + \frac{m_v \omega^2 e}{2\tan\delta} + m_e \omega^2 (e \cot\delta - s \cos\delta).$$

As there are two pendulums in the governor, one half of P_r is transmitted to each pendulum.

Let, q = length of the lever SQ,

β = angle AEQ (see Fig. 1),

then the moment of P_r may be taken approximately to be

$$M_{r_0} = \frac{P_r}{2\cos\beta} \,.\, \overline{SM'}$$

$$= \frac{P_r}{2\cos\beta} \,.\, q\cos\beta$$

$$= \frac{q}{2} P_r$$

$$= \frac{q}{2}\left[\frac{2P_1}{\pi} \sin\delta + \frac{m_v \omega^2 e}{2\tan\delta} + m_e \omega^2 (e \cot\delta - s \cos\delta)\right], \qquad (11)$$

where $\overline{SM'}$ is the perpendicular distance of S from QE.

Therefore equation (10) becomes

$$2\varkappa ab\left(1 - \frac{l_0}{\sqrt{a^2+b^2}}\right) \sin\alpha - \varkappa l_0 \frac{a^2 b^2}{(a^2+b^2)^{\frac{3}{2}}} \sin 2\alpha$$

$$- \omega^2 am\rho \sin\alpha + \frac{q}{2}\left[\frac{2P_1}{\pi} \sin\delta + \frac{m_v \omega^2 e}{2\tan\delta} + m_e \omega^2 (e \cot\delta - s \cos\delta)\right]$$

$$= 0, \qquad (12)$$

which is an equation expressing the state of equilibrium of the governor. From this equation we can find the values of the moment of centrifugal

force of the pendulum at different positions, and draw the characteristic curve of the governor. The irregularity of the governor can be found from the curve. Conversely, assuming a value of the permissible irregularity for a governor, we can determine the dimensions of the governor spring from the equation.

Example 1.—Take a vertical single cylinder condensing engine with the cylinder 25 cm. in diameter by 25 cm. in stroke, fitted with a piston valve 11 cm. in diameter, running at 240 revolutions per minute under a

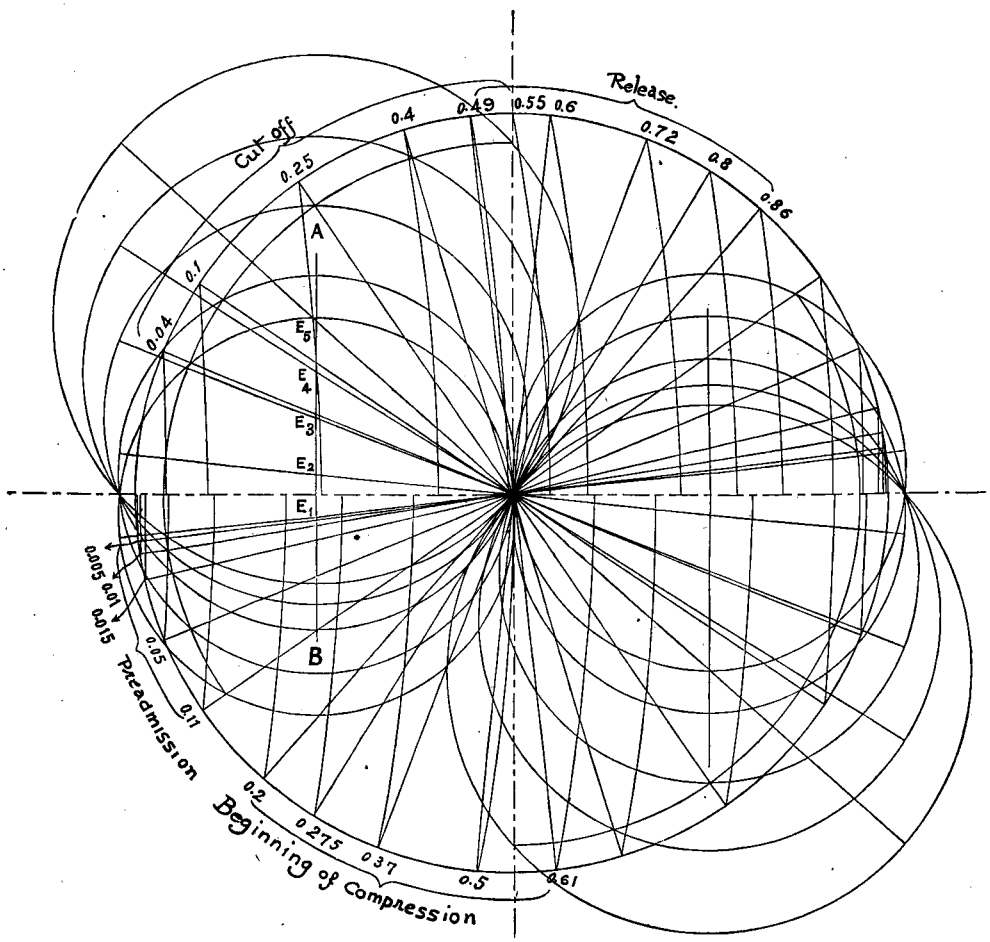


Fig. 6

steam pressure of 6 atm. absolute. The engine is to be regulated by a shaft governor mounted on the crank shaft. The dimensions of the governor are as follows:—

$a=18$ cm., $\rho=12$ cm., $b=3$ cm., $q=5$ cm. and $e=2.6$ cm.

Determine the dimensions of the spring of the governor.

Fig. 6 gives the valve diagram for various positions of the eccentric centre:

$$\delta=45°-30', 58°, 69°, 84° \text{ and } 96°,$$

corresponding to the configurations of the governor:

$$a=50°, 57°-20', 64°-40', 73°-30' \text{ and } 80° \text{ respectively.}$$

We must first find P_r and then M_{r0}.

Taking: $P_1=16$ kg., $P_3=12$ kg. $P_4=16$ kg.,

$$m_v = \frac{12}{981} = 0.01223 \text{ and } m_e = \frac{6}{981} = 0.006116;$$

$$\frac{2P_1}{\pi} \cdot \frac{q}{2} = \frac{16 \times 5}{\pi} = 25.5 \text{ kg. cm.},$$

$$\frac{m_v\omega^2 e}{2} \cdot \frac{q}{2} = \frac{0.01223 \times 630 \times 2.6 \times 5}{4} = 25 \text{ kg. cm.},$$

$$m_e\omega^2 e \cdot \frac{q}{2} = \frac{0.006116 \times 630 \times 2.6 \times 5}{2} = 25 \text{ kg. cm.},$$

$$m_e\omega^2 s \cdot \frac{q}{2} = \frac{0.006116 \times 630 \times 0.6 \times 5}{2} = 5.8 \text{ kg. cm.}$$

Therefore from equation (11), at the different positions of the governor we have

δ	$48°-30'$	$58°$	$69°$	$84°$	$96°$
$\frac{2P_1}{\pi} \cdot \frac{q}{2} \sin\delta$	19.1	21.6	23.8	25.4	25.4
$\frac{m_v\omega^2 e \cot\delta}{2} \cdot \frac{q}{2}$	22.1	15.6	9.6	2.6	−2.6
$m_e\omega^2 e \cot\delta \cdot \frac{q}{2}$	22.1	15.6	9.6	2.6	−2.6
$m_e\omega^2 s \cos\delta \cdot \frac{q}{2}$	3.9	3.07	2.08	0.61	−0.61
M_{r0} in kg. cm.	59.4	49.7	40.9	30.0	20.8

The curve of M_{r_0} is given in Fig. 7.

For determining the dimensions of the springs we have still to know the values of δ_g, the permissible irregularity and W, the weight of the pendulum. Let us now see the influences of these values upon the dimensions of the spring required.

For this purpose assume different values of δ_g and W and make the calculations for the spring for each set of values. In general let:

ω_i and ω_a = angular velocities of the fly wheel at the innermost and the outermost positions of the governor respectively,

M_{z_i} and M_{z_a} = corresponding moments of centrifugal forces,

M_{f_i} and M_{f_a} = corresponding moments of forces of the spring,

F_i and F_a = corresponding forces of the spring,

d = diameter of wire of the spring in centimeters,

r = mean radius of coil in centimeters,

n = number of turns,

k_s = allowable shearing stress of spring steel
= 4000 kg. per sq. cm.,

G = Modulus of shearing elasticity
= 850000 kg. per sq. cm.

(a) $\delta_g = 0.12$, $W = 12$ kg.; then

$$\omega_i^2 = 630 \times \left(1 - \frac{0.12}{2}\right)^2 = 557,$$

$$\omega_a^2 = 630 \times \left(1 + \frac{0.12}{2}\right)^2 = 707,$$

$$M_{z_i} = 0.01223 \times 557 \times 18 \times 12 \sin 50^\circ \fallingdotseq 1127 \text{ kg. cm.},$$

$$M_{z_a} = 0.01223 \times 707 \times 18 \times 12 \sin 80^\circ \fallingdotseq 1839 \text{ kg. cm.},$$

$$M_{f_i} = 1127 - 59.4 = 1068 \text{ kg. cm.},$$

$$M_{f_a} = 1839 - 20.8 = 1818 \text{ kg. cm.},$$

$$F_i = \frac{1068}{2.5} = 427 \text{ kg.},$$

$$F_a = \frac{1818}{3} = 606 \text{ kg.}.$$

Therefore the value of x required is

$$\varkappa = \frac{606-427}{2\times 1.5} = 59.7 \text{ kg. per cm. of deformation.}$$

Taking r at 6 cm.,

$$d = \sqrt[3]{\frac{F_a \times r}{k_s} \cdot \frac{16}{\pi}} = \sqrt[3]{\frac{606\times 6}{4000} \cdot \frac{16}{\pi}} \doteqdot 1.7 \text{ am.}$$

and

$$n = \frac{G\times d^4}{64\times r^3 \times \varkappa} = \frac{850000\times 1.7^4}{64\times 6^3\times 59.7} \doteqdot 8.5.$$

If we take: $d = 1.7$ cm., $r = 6$ cm. and

$n = 8$ turns; then

$$\varkappa = \frac{850000\times 17^4}{64\times 6^3\times 8} \doteqdot 64.2 \text{ kg. per cm. of deformation.}$$

In the further calculations the actual value of $\varkappa$ thus found will be taken. If we take the normal cut off at 0.25, which corresponds to the configuration of the governor: $\alpha = 64°-40'$,

$$M_{z_n} = 0.01223\times 630\times 18\times 12 \sin(64°-40')$$
$$= 1506 \text{ kg. cm.}$$
$$M_{f_n} = 1506-40.9 = 1465 \text{ kg. cm.,}$$
$$F_n = \frac{1465}{2\times 1.44} = 508 \text{ kg.,}$$

	$2(l-l_n)$	$2\varkappa(l-l_n)$	$F\times h = M_f$		
i	−1.36 cm.	− 87 kg.	420.7	2.5	1051 kg. cm.
	−0.67 „	− 43 „	465	2.72	1264 „ „
n	0 „	0	508	2.88	1465 „ „
	+0.9 „	+ 57.7 „	565.7	2.97	1679 „ „
a	+1.6 „	+102.7 „	610.7	3.00	1829 „ „

$$M_{z_i} = 1051+59.4 = 1110 \text{ kg. cm.,}$$
$$M_{z_a} = 1829+20.8 = 1850 \text{ kg. cm.,}$$
$$\omega_i^2 = \frac{1110}{0.01223\times 18\times 12 \sin 50°} \doteqdot 548,$$
$$\omega_a^2 = \frac{1850}{0.01223\times 18\times 12 \sin 80°} \doteqdot ,700,$$

$$\omega_i = \sqrt{548} = 23.4,$$

$$\omega_a = \sqrt{700} = 26.4,$$

Therefore the irregularity of the governor is

$$\delta_g = \frac{26.4-23.4}{25.1} = 0.119,$$

a result practically the same as assumed.

(b) $\delta_g=0.12$, $W=10$ kg.; then $d=1.6$ cm., $r=6$ cm., $n=8$ turns and $\varkappa = 50.4$ kg. per cm. of deformation.

(c) $\delta_g=0.104$, $W=8$ kg.; then $d=1.5$ cm., $r=6$ cm., $n=8$ turns and $\varkappa = 38.9$ kg. per cm. of deformation.

(d) $\delta_g=0.056$, $W=6$ kg.; then $d=1.3$ cm., $r=6$ cm., $n=8$ turns and $\varkappa = 21.95$ kg. per cm. of deformation.

(d′) $\delta_g=0.094$, $W=6$ kg.; then $d=1.3$ cm., $r=6$ cm., $n=6$ turns and $\varkappa = 29.3$ kg. per cm. of deformation.

The characteristic curves of the governors considered in the example are shown in Fig. 7.

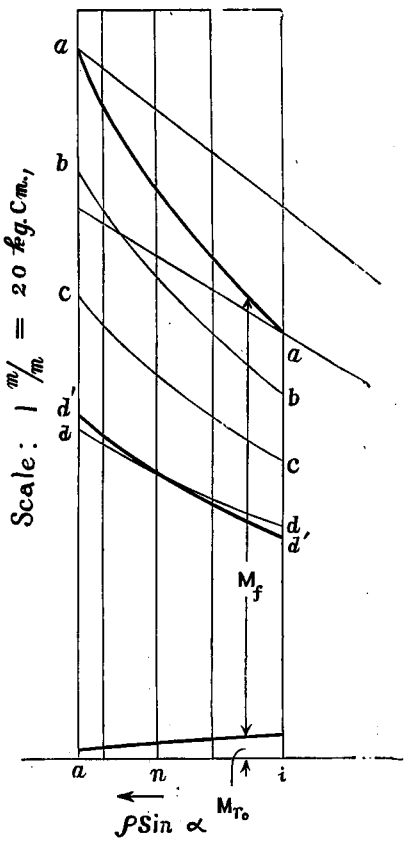


Fig. 7

II. Vibration of the Pendulum caused by Periodic Changes of the Reacting Forces of the Valve Gear.

P_1' in equation (5), expanded as an infinite series of trigonometrical functions by Fourier's theorem, becomes:

$$\begin{aligned} P_1' = \frac{2P_1}{\pi}\Big[&\sin\delta-(\sin\delta+\tfrac{1}{3}\sin3\delta)\cos2\theta \\ &-(\cos\delta+\tfrac{1}{3}\cos3\delta)\sin2\theta \\ &+(\tfrac{1}{3}\sin3\delta+\tfrac{1}{5}\sin5\delta)\cos4\theta \\ &+(\tfrac{1}{3}\cos3\delta+\tfrac{1}{5}\cos5\delta)\sin4\theta+\ldots\ldots\Big]. \end{aligned}$$

The other terms are:

$$\begin{aligned} P_2' &= \frac{m_v\omega^2 e}{\sin\delta}\sin(\delta+\theta)\sin\theta \\ &= \frac{m_v\omega^2 e}{2\sin\delta}\Big[\cos\delta-\cos\delta\cos2\theta+\sin\delta\sin2\theta\Big], \\ P_3'+P_4' &= (P_3+P_4)\sin\theta, \\ P_5' &= m_e\omega^2 y. \end{aligned}$$

Summing up these forces

$$\begin{aligned} \Sigma P' = &\Big[\frac{2P_1}{\pi}\sin\delta+\frac{m_v\omega^2 e}{2\tan\delta}+m_e\omega^2 y\Big]+[P_3+P_4]\sin\theta \\ &-\Big[\Big(\frac{2P_1}{\pi}(\sin\delta+\tfrac{1}{3}\sin3\delta)+\frac{m_v\omega^2 e}{2\tan\delta}\Big)\cos2\theta \\ &+\Big(\frac{2P_1}{\pi}(\cos\delta+\tfrac{1}{3}\cos3\delta)-\frac{m_v\omega^2 e}{2}\Big)\sin2\theta\Big] \\ &+\Big[\frac{2P_1}{\pi}(\tfrac{1}{3}\sin3\delta+\tfrac{1}{5}\sin5\delta)\cos4\theta \\ &+\frac{2P_1}{\pi}(\tfrac{1}{3}\cos3\delta+\tfrac{1}{5}\cos5\delta)\sin4\theta\Big]+\ldots\ldots. \end{aligned} \tag{13}$$

The moment of these forces acting on each pendulum is in the form:

$$
\begin{aligned}
M_r &= \frac{q}{2}\,\Sigma P' \\
&= A_0 + B_1 \sin\theta - [A_2 \cos 2\theta + B_2 \sin 2\theta] \\
&\quad + [A_4 \cos 4\theta + B_4 \sin 4\theta] + \ldots\ldots, \qquad (14)
\end{aligned}
$$

where

$$
\left.
\begin{aligned}
A_0 &= \left[\frac{2P_1}{\pi}\sin\delta + \frac{m_v\omega^2 e}{2\tan\delta} + m_e\omega^2 y\right]\frac{q}{2}, \\
B_1 &= (P_3 + P_4)\frac{q}{2}, \\
A_2 &= \left[\frac{2P_1}{\pi}(\sin\delta + \tfrac{1}{3}\sin 3\delta) + \frac{m_v\omega^2 e}{2\tan\delta}\right]\frac{q}{2}, \\
B_2 &= \left[\frac{2P_1}{\pi}(\cos\delta + \tfrac{1}{3}\cos 3\delta) - \frac{m_v\omega^2 e}{2}\right]\frac{q}{2}, \\
A_4 &= \frac{2P_1}{\pi}\left(\tfrac{1}{3}\sin 3\delta + \tfrac{1}{5}\sin 5\delta\right)\frac{q}{2}, \\
B_4 &= \frac{2P_1}{\pi}\left(\tfrac{1}{3}\cos 3\delta + \tfrac{1}{5}\cos 5\delta\right)\frac{q}{2}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
\right\} \qquad (15)
$$

In the above equations A_0 is the same as M_{r_0} in equation (11), and consequently it must be in statical equilibrium with M_f and M_z. The other terms are those moments which put the pendulum in periodic oscillation about its neutral position.

The equation for such oscillation of the pendulum is, neglecting the friction of pins:

$$M_i + M_f - M_z + M_r = 0, \qquad (16)$$

where M_i = moment of force of inertia of the pendulum about its centre of suspension

$$= \frac{d^2\alpha}{dt^2}\rho^2 m.$$

Substituting the values of M_f, M_z and M_r from equations (3), (1) and (14), in the above equation, we have:

$$\frac{d^2\alpha}{dt^2}\rho^2 m + 2\varkappa ab\left(1 - \frac{l_0}{\sqrt{a^2+b^2}}\right)\sin\alpha - \varkappa l_0 \frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\sin 2\alpha$$

$$-\omega^2 am\rho \sin\alpha + A_0 + B_1 \sin\theta$$
$$-[A_2 \cos 2\theta + B_2 \sin 2\theta] + [A_4 \cos 4\theta + B_4 \sin 4\theta]$$
$$+ \ldots\ldots\ldots\ldots \quad = 0. \tag{17}$$

Put, $$\alpha = \alpha_0 + \xi,$$

where α_0 is the value of α when the pendulum is in its mean position of oscillation and ξ is the angle of oscillation about the mean position. Since ξ should be very small, we may take approximately:

$$\sin\xi = \xi, \quad \cos\xi = 1, \quad \sin 2\xi = 2\xi \quad \text{and} \quad \cos 2\xi = 1;$$

then equation (17) becomes:

$$\frac{d^2\xi}{dt^2}\rho^2 m + \xi \,.\, 2\varkappa ab\left(1 - \frac{l_0}{\sqrt{a^2+b^2}}\right)\cos\alpha_0 + 2\varkappa ab\left(1 - \frac{l_0}{\sqrt{a^2+b^2}}\right)\sin\alpha_0$$
$$-\xi \,.\, 2\varkappa l_0 \frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\cos 2\alpha_0 - \varkappa l_0 \frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\sin 2\alpha_0$$
$$-\xi \,.\, \omega^2 am\rho \cos\alpha_0 - \omega^2 am\rho \sin\alpha_0 + A_0 + B_1 \sin\theta$$
$$-[A_2 \cos 2\theta + B_2 \sin 2\theta] + [A_4 \cos 4\theta + B_4 \sin 4\theta]$$
$$+ \ldots\ldots\ldots\ldots \quad = 0. \tag{18}$$

Now the angular velocity ω can not be constant unless the mass of the fly wheel be infinitely great: it fluctuate periodically between two limits $\omega_0\left(1 + \frac{\delta_f}{2}\right)$ and $\omega_0\left(1 - \frac{\delta_f}{2}\right)$, δ_f denoting the unsteadiness of the fly wheel.

Put $$\omega = \omega_0 + \eta,$$

where ω_0 is the mean angular velocity of the fly wheel and is a constant, and η is a small change of angular velocity. And

$$\theta = \int \omega dt = \int \omega_0 dt + \int \eta dt = \omega_0 t + \int \eta dt.$$

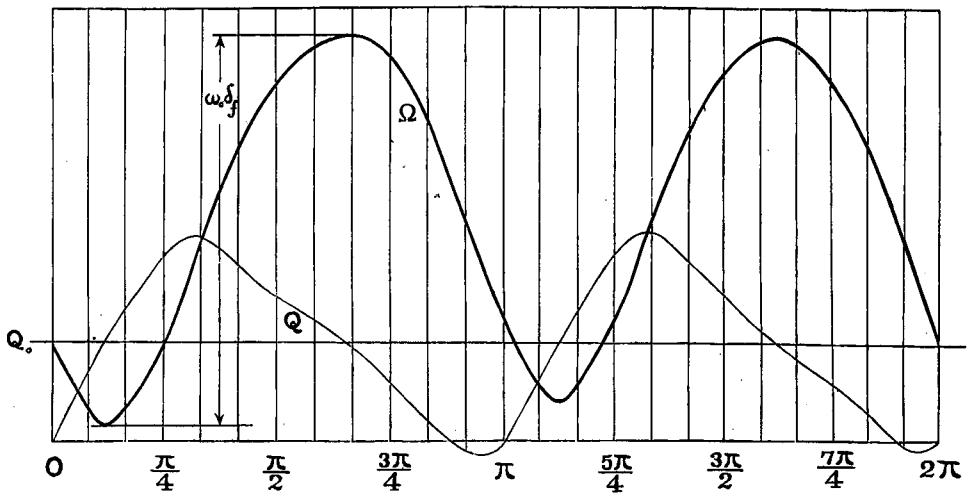


Fig. 8

Referring to Fig. 8 let Q be the crank effort diagram drawn for a complete revolution of the shaft, i. e. from $\theta = 0$ to $\theta = 2\pi$. Draw a horizontal straight line Q_0 at a height equal to the mean height of the curve Q. Construct another curve Ω such that its ordinate represents the area included between the curves Q and Q_0, then the ordinate of the curve Ω also represents the variation of the angular velocity; and, consequently, η is a periodic function whose law of variation is represented by this curve, and so may be expanded as a series of trigonometrical functions of ωt by Fourier's theorem. Take, for the sake of simplicity, the first five terms of this series, then it may be written in the form:

$$\eta = E_0 + E_1 \cos\omega t + F_1 \sin\omega t + E_2 \cos 2\omega t + F_2 \sin 2\omega t,$$

in which E_1, F_1, E_2 and F_2 must be certain fractions of $\omega_0\delta_f$. If we take the datum line of the curve Ω at its mean height, the constant term E_0 vanishes. Also, in the case of a single cylinder engine, E_1 and F_1 become very small compared with E_2 and F_2, so that they may be neglected and then the above equation may be put approximately:

$$\eta = E_2 \cos 2\omega t + F_2 \sin 2\omega t.$$

Since η itself is a very small quantity we may put in the above equation

$$\omega = \omega_0 = \text{constant}.$$

Thus ω and θ may be expressed by the equation:

$$\left.\begin{aligned} \omega &= \omega_0 + E_2 \cos 2\omega_0 t + F_2 \sin 2\omega_0 t, \\ \theta &= \omega_0 t + \frac{E_2}{2\omega_0} \sin 2\omega_0 t - \frac{F_2}{2\omega_0} \cos 2\omega_0 t. \end{aligned}\right\} \qquad (19)$$

Substituting these values in equation (18), and neglecting terms of powers of ξ and η higher than the second inclusive, it becomes:

$$\begin{aligned} &\frac{d^2\xi}{dt^2}\rho^2 m + \left[2\varkappa ab\left(1 - \frac{l_0}{\sqrt{a^2+b^2}}\right)\cos\alpha_0 - 2\varkappa l_0 \frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\cos 2\alpha_0\right. \\ &\left. - \omega_0^2 am\rho \cos\alpha_0\right]\xi + \left[2\varkappa ab\left(1 - \frac{l_0}{\sqrt{a^2+b^2}}\right)\sin\alpha_0\right. \\ &\left. - \varkappa l_0 \frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\sin 2\alpha_0 - \omega_0^2 am\rho \sin\alpha_0 + A_0\right] \\ &- 2\omega_0 am\rho \sin\alpha_0 (E_2 \cos 2\omega_0 t + F_2 \sin 2\omega_0 t) \\ = &- B_1 \sin\left(\omega_0 t + \frac{E_2}{2\omega_0}\sin 2\omega_0 t - \frac{F_2}{2\omega_0}\cos 2\omega_0 t\right) \\ &+ A_2 \cos 2\left(\omega_0 t + \frac{E_2}{2\omega_0}\sin 2\omega_0 t - \frac{F_2}{2\omega_0}\cos 2\omega_0 t\right) \\ &+ B_2 \sin 2\left(\omega_0 t + \frac{E_2}{2\omega_0}\sin 2\omega_0 t - \frac{F_2}{2\omega_0}\cos 2\omega_0 t\right) \\ &- A_4 \cos 4\left(\omega_0 t + \frac{E_2}{2\omega_0}\sin 2\omega_0 t - \frac{F_2}{2\omega_0}\cos 2\omega_0 t\right) \\ &- B_4 \sin 4\left(\omega_0 t + \frac{E_2}{2\omega_0}\sin 2\omega_0 t - \frac{F_2}{2\omega_0}\cos 2\omega_0 t\right) \\ &+ \ldots\ldots\ldots\ldots \end{aligned}$$

In this equation the third term in the left hand side vanishes by equation (12); and the coefficient of ξ in the second term becomes:

$$\begin{aligned} &2\varkappa ab\left(1 - \frac{l_0}{\sqrt{a^2+b^2}}\right)\cos\alpha_0 - 2\varkappa l_0 \frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\cos 2\alpha_0 \\ &\quad - \omega_0^2 am\rho \cos\alpha_0 \end{aligned}$$

$$= \cot\alpha_0\left[2\varkappa ab\left(1-\frac{l_0}{\sqrt{a^2+b^2}}\right)\sin\alpha_0-\varkappa l_0\frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\sin 2\alpha_0\right.$$

$$\left.+2\varkappa l_0\frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\cdot\frac{\sin^3\alpha_0}{\cos\alpha_0}-\omega_0^2 am\rho\sin\alpha_0\right]$$

$$= \cot\alpha_0\left[2\varkappa l_0\frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\cdot\frac{\sin^3\alpha_0}{\cos\alpha_0}-A_0\right]$$

$$= 2\varkappa l_0\frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\sin^2\alpha_0-A_0\cot\alpha_0.$$

As $\frac{E_2}{2\omega_0}$ and $\frac{E_2}{2\omega_0}$ are both certain fractions of δ_f, the right hand side is transformed as follows:

$$-B_1\sin(\omega_0 t+\frac{E_2}{2\omega_0}\sin 2\omega_0 t-\frac{F_2}{2\omega_0}\cos 2\omega_0 t)$$

$$+A_2\cos 2(\omega_0 t+\frac{E_2}{2\omega_0}\sin 2\omega_0 t-\frac{F_2}{2\omega_0}\cos 2\omega_0 t)$$

$$+B_2\sin 2(\omega_0 t+\frac{E_2}{2\omega_0}\sin 2\omega_0 t-\frac{F_2}{2\omega_0}\cos 2\omega_0 t)$$

$$-A_4\cos 4(\omega_0 t+\frac{E_2}{2\omega_0}\sin 2\omega_0 t-\frac{F_2}{2\omega_0}\cos 2\omega_0 t)$$

$$-B_4\sin 4(\omega_0 t+\frac{E_2}{2\omega_0}\sin 2\omega_0 t-\frac{F_2}{2\omega_0}\cos 2\omega_0 t)$$

$$+A_6\cos 6(\omega_0 t+\frac{E_2}{2\omega_0}\sin 2\omega_0 t-\frac{F_2}{2\omega_0}\cos 2\omega_0 t)$$

$$+B_6\sin 6(\omega_0 t+\frac{E_2}{2\omega_0}\sin 2\omega_0 t-\frac{F_2}{2\omega_0}\cos 2\omega_0 t)$$

$$+\ldots\ldots\ldots\ldots$$

$$=-B_1\left[\sin\omega_0 t+\left(\frac{E_2}{2\omega_0}\sin 2\omega_0 t-\frac{F_2}{2\omega_0}\cos 2\omega_0 t\right)\cos\omega_0 t\right]$$

$$+A_2\left[\cos 2\omega_0 t-2\left(\frac{E_2}{2\omega_0}\sin 2\omega_0 t-\frac{F_2}{2\omega_0}\cos 2\omega_0 t\right)\sin 2\omega_0 t\right]$$

$$+B_2\left[\sin 2\omega_0 t+2\left(\frac{E_2}{2\omega_0}\sin 2\omega_0 t-\frac{F_2}{2\omega_0}\cos 2\omega_0 t\right)\cos 2\omega_0 t\right]$$

$$- A_4\left[\cos 4\,\omega_0 t - 4\left(\frac{E_2}{2\omega_0}\sin 2\omega_0 t - \frac{F_2}{2\omega_0}\cos 2\omega_0 t\right)\sin 4\omega_0 t\right]$$

$$- B_4\left[\sin 4\,\omega_0 t + 4\left(\frac{E_2}{2\omega_0}\sin 2\omega_0 t - \frac{F_2}{2\omega_0}\cos 2\omega_0 t\right)\cos 4\omega_0 t\right]$$

$$+ A_6\left[\cos 6\,\omega_0 t - 6\left(\frac{E_2}{2\omega_0}\sin 2\omega_0 t - \frac{F_2}{2\omega_0}\cos 2\omega_0 t\right)\sin 6\omega_0 t\right]$$

$$+ B_6\left[\sin 6\,\omega_0 t + 6\left(\frac{E_2}{2\omega_0}\sin 2\omega_0 t - \frac{F_2}{2\omega_2}\cos 2\omega_0 t\right)\cos 6\omega_0 t\right]$$

$$+ \ldots\ldots\ldots\ldots$$

$$= - B_1 \sin\omega_0 t + [A_2 \cos 2\omega_0 t + B_2 \sin 2\omega_0 t]$$

$$- [A_4 \cos 4\omega_0 t + B_4 \sin 4\omega_0 t] + [A_6 \cos 6\omega_0 t + B_6 \sin 6\omega_0 t]$$

$$+ \ldots\ldots\ldots\ldots$$

$$+ \frac{E_2}{2\omega_0}\,[-\tfrac{1}{2}B_1(\sin\omega_0 t + \sin 3\omega_0 t) - A_2(1 - \cos 4\omega_0 t)$$

$$+ B_2 \sin 4\omega_0 t + 2A_4(\cos 2\omega_0 t - \cos 6\omega_0 t)$$

$$+ 2B_4(\sin 2\omega_0 t - \sin 6\omega_0 t) - 3A_6(\cos 4\omega_0 t - \cos 8\omega_0 t)$$

$$- 3B_6(\sin 4\omega_0 t - \sin 8\omega_0 t) + \ldots\ldots\ldots\ldots]$$

$$+ \frac{F_2}{2\omega_0}\,[\tfrac{1}{2}B_1(\cos\omega_0 t + \cos 3\omega_0 t) - A_2 \sin 4\omega_0 t$$

$$- B_2(1 + \cos 4\omega_0 t) - 2A_4(\sin 2\omega_0 t + \sin 6\omega_0 t)$$

$$+ 2B_4(\cos 2\omega_0 t + \cos 6\omega_0 t) + 3A_6(\sin 4\omega_0 t + \sin 8\omega_0 t)$$

$$- 3B_6(\cos 4\omega_0 t + \cos 8\omega_0 t) + \ldots\ldots\ldots\ldots]$$

$$= -\left(\frac{E_2}{2\omega_0}A_2 + \frac{F_2}{2\omega_0}B_2\right)$$

$$+ \tfrac{1}{2}\cdot\frac{F_2}{2\omega_0}B_1 \cos\omega_0 t - \left(1 + \tfrac{1}{2}\cdot\frac{E_2}{2\omega_0}\right)B_1 \sin\omega_0 t$$

$$+ \left(A_2 + 2\frac{E_2}{2\omega_0}A_4 + 2\frac{F_2}{2\omega_0}B_4\right)\cos 2\omega_0 t$$

$$+ \left(B_2 + 2\frac{E_2}{2\omega_0}B_4 - 2\frac{F_2}{2\omega_0}\mathrm{A}_4\right)\sin 2\omega_0 t$$

$$+ \tfrac{1}{2}\cdot\frac{F_2}{2\omega_0}B_1 \cos 3\omega_0 t - \tfrac{1}{2}\cdot\frac{E_2}{2\omega_0}B_1 \sin 3\omega_0 t$$

$$+\left(-A_4+\frac{E_2}{2\omega_0}A_2-\frac{E_2}{2\omega_0}3A_6-\frac{F_2}{2\omega_0}-\frac{F_2}{2\omega_0}3B_6\right)\cos 4\omega_0 t$$

$$+\left(-B_4+\frac{E_2}{2\omega_0}B_2-\frac{E_2}{2\omega_0}3B_6-\frac{F_2}{2\omega_0}A_2-\frac{F_2}{2\omega_0}3A_6\right)\sin 4\omega_0 t$$

$$+\cdots\cdots\cdots\cdots$$

Therefore the above differential equation may be put into the form:

$$\begin{aligned}\frac{d^2\xi}{dt^2}+k^2\xi+L=&[M_1\cos\omega_0 t+N_1\sin\omega_0 t]\\&+[M_2\,\mathrm{ces}2\omega_0 t+N_2\sin 2\omega_0 t]\\&+[M_3\cos 3\omega_0 t+N_3\sin 3\omega_0 t]\\&+[M_4\cos 4\omega_0 t+N_4\sin 4\omega_0 t]\\&+\cdots\cdots\cdots\cdots,\end{aligned}\tag{20}$$

in which

$$\left.\begin{aligned}
k_2&=\frac{1}{\rho^2 m}\left[2\varkappa l_0\frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\sin^2 a_0-A_0\cot a_0\right],\\
L&=\frac{1}{\rho^2 m}\left[\frac{E_2}{2\omega_0}A_2+\frac{F_2}{2\omega_0}B_2\right],\\
M_1&=\frac{1}{\rho^2 m}\cdot\tfrac{1}{2}\cdot\frac{F_2}{2\omega_0}B_1,\\
N_1&=-\frac{1}{\rho^2 m}\left[1+\tfrac{1}{2}\cdot\frac{E_2}{2\omega_0}\right]B_1,\\
M_2&=\frac{1}{\rho^2 m}\left[\frac{E_2}{2\omega_0}4\omega_0^2 am\rho\sin a_0+A_2+2\frac{E_2}{2\omega_0}A_4+2\frac{F_2}{2\omega_0}B_4\right],\\
N_2&=\frac{1}{\rho^2 m}\left[\frac{F_2}{2\omega_0}4\omega_0^2 am\rho\sin a_0+B_2+2\frac{E_2}{2\omega_0}B_4-2\frac{F_2}{2\omega_0}A_4\right],\\
M_3&=\frac{1}{\rho^2 m}\cdot\tfrac{1}{2}\cdot\frac{F_2}{2\omega_0}B_1,\\
N_3&=-\frac{1}{\rho^2 m}\cdot\tfrac{1}{2}\cdot\frac{E_2}{2\omega_0}B_1,\\
M_4&=\frac{1}{\rho^2 m}\left[-A_4+\frac{E_2}{2\omega_0}A_2-3\frac{E_2}{2\omega_0}A_6-\frac{F_2}{2\omega_0}B_2-3\frac{F_2}{2\omega_0}B_6\right],\\
N_4&=\frac{1}{\rho_2 m}\left[-B_4+\frac{E_2}{2\omega_0}B_2-3\frac{E_2}{2\omega_0}B_6-\frac{F_2}{2\omega_0}A_2-3\frac{F_2}{2\omega_0}B_6\right],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\}\tag{21}$$

The solution of equation (20) is

$$\xi + \frac{L}{k^2} = C_1 \cos kt + C_2 \sin kt$$
$$+ \left[\frac{M_1}{k^2 - \omega_0^2} \cos\omega_0 t + \frac{N_1}{k^2 - \omega_0^2} \sin\omega_0 t \right]$$
$$+ \left[\frac{M_2}{k^2 - 4\omega_0^2} \cos 2\omega_0 t + \frac{N_2}{k^2 - 4\omega_0^2} \sin 2\omega_0 t \right]$$
$$+ \left[\frac{M_3}{k^2 - 9\omega_0^2} \cos 3\omega_0 t + \frac{N_3}{k^2 - 9\omega_0^2} \sin 3\omega_0 t \right]$$
$$+ \left[\frac{M_4}{k^2 - 16\omega_0^2} \cos 4\omega_0 t + \frac{N_4}{k^2 - 16\omega_0^2} \sin 4\omega_2 t \right]$$
$$+ \ldots\ldots\ldots\ldots, \qquad (22)$$

in which C_1 and C_2 are both integration constants.

The influence of the value of k^2 upon the forced oscillation expressed by this equation must be considered, as it plays the most important part in the motion.

Case 1.—When $k^2 > 0$ and $k^2 < \omega_0^2$:

The oscillation expressed by the first two terms in equation (22) has a longer period than that of the third, and accordingly it has also a longer period than that of the combined oscillations of all the other terms. The amplitude of this oscillation is determined by integration constants C_1 and C_2, which are difficult to find; but we may suppose that the oscillation is originated by the action of the spring and would not take a considerable part in the total oscillation. The series following the first two terms in the right hand side of equation (22) has finite coefficients in all terms and converges rapidly.

Case 2.—When $k^2 > 0$ and $k^2 > \omega_0^2$:

The series converges slowly; and, moreover, there is a possibility of the denominator of the constant coefficient of some of the terms becoming very small, i. e. a danger of the occurrence of a resonance of the pendulum. A governor of such quality is useless; and, therefore, k^2 should always be made smaller than ω_0^2. Even a governor designed with this precaution,

as ω increases gradually from zero when the engine starts, passes such a dangerous point quickly before reaching its normal speed, and therefore without injury.

Case 3.—When $k^2 < 0$:

The first two terms become

$$C_1' e^{kt} + C_2' e^{-kt},$$

and the motion represented by these terms is no more an oscillation but an ordinary accelerating motion. Furthermore, as the first term in the expression increases with t without limit, the pendulum never comes to rest but is displaced indefinitely. The governor of such an unstable equilibrium is clearly useless. Therefore the governor must be designed to fulfill the condition, $\omega_0^2 > k^2 > 0$, for all its configurations.

Example 2.—Required to establish the laws of vibration of the governors of example 1, the crank effort diagram of the engine being as shown in Fig. 8. Assume

$$\delta_f = \frac{1}{150}.$$

From Fig. 8 it can be found that

$$E_2 \fallingdotseq -0.642 \cdot \frac{\omega_0 \delta_f}{2}$$

and

$$F_2 \fallingdotseq -0.658 \cdot \frac{\omega^0 \delta_f}{2}.$$

The values of k^2 and the coefficients of terms, except the first two in the right hand side of equation (22), when the pendulum is in equilibrium at its normal configuration, are found by calculation and the results given in the table below:

	(a)	(b)	(c)	(d)	(d′)
Weight of pend. in kgs.	12	10	8	6	6
$\varkappa$ of spring in kg. per cm.	64.2	50.4	38.9	21.95	29.3
δ_g	0.12	0.12	0.104	0.056	0.094
k^2	360	330	310	200	233
$\frac{M_1}{k^2-\omega_0^2}$ in degrees.	0.00235	0.00247	0.00291	0.00291	0.00316
$\frac{N_1}{k^2-\omega_0^2}$,,	8.45	9.05	10.4	10.4	11.4
$\frac{M_2}{k^2-4\omega_0^2}$,,	−0.395	−0.473	−0.604	−0.783	−0.79
$\frac{N_2}{k^2-4\omega_0^2}$,,	−0.35	−0.422	0.542	−0.7	−0.707
$\frac{M_3}{k^2-9\omega_0^2}$,,	0.000118	0.000138	0.000174	0.000228	0.00023
$\frac{N_3}{k^2-9\omega_0^2}$,,	0.000112	0.000134	0.000168	0.000219	0.00022
$\frac{M_4}{k^2-16\omega_0^2}$,,	−0.0174	−0.0207	−0.0258	−0.034	−0.0342
$\frac{N_4}{k^2-16\omega_0^2}$,,	−0.009	−0.0108	−0.0134	−0.0177	−0.0178

The curve of the motion expressed by each of the terms of equation (22) can be drawn separately. The coefficients in all terms, except those of $\sin\omega_0 t$, $\cos 2\omega_0 t$ and $\sin 2\omega_0 t$, are so small as to have no appreciable effect on the result if neglected altogether. Fig. 9, Pl. I, shows these curves. In the figure the curves marked N_1, M_2 and N_2 represent terms:

$$\frac{N_1}{k^2-\omega_0^2}\sin\omega_0 t, \quad \frac{M_2}{k^2-4\omega_0^2}\cos 2\omega_0 t$$

and

$$\frac{F_2}{k^2-\omega 4_0^2}\sin 2\omega_0 t$$

respectively; and the curve $\Sigma M+N$ their resultant.

It is seen from the results of calculation, that the vibration of the pendulum caused by the reacting forces of the valve gear rises to a degree which is inadmissible in a regulator. If we try, as with a conical governor, to make the "energy" so great that reacting forces may be inappreciable,

a disproportionately heavy governor is necessitated. Therefore, we must remain content with a governor accompanied by vibrations, which are damped in practice by frictional resistance at pins or at knife-edges which was not taken into account in the deduction of the equation of motion; or also we must perfect some device whereby to suppress these vibrations. In a conical governor, also, vibrations may be caused by the varying reacting forces of the regulating gear; but the chief source of vibration is the unsteadiness due to the finite mass of the fly wheel; hence they may be removed by fulfilling the condition that the insensibility of the governor should not be less than the unsteadiness of the fly wheel. But in the case of a shaft governor, which is a pure spring governor, the vibrations are caused by the reacting forces as well as by the influence of the unsteadiness of the fly wheel; and meeting the above condition would not sufficiently remedy the evil. To obviate this a shaft governor is often provided with a dash pot or an oil brake. Because of the vibration of the pendulum, the pins or knife edges in the governor suffer an incessant rubbing unnecessarily; and to stand the severe wear they must be made of ample size so as not to shorten the life of the governor. Apparently it is impossible fully to annihilate the vibrations, and it is believed by some that on account of its vibrations the governor get its absolute sensitiveness, i. e. the least variation of angular velocity of the governor-shaft causes immediately a movement of the pendulum (Z. d. V. d. Ing. 1899 S. 913: J. Isachsen, Das Regulieren von Kraftmachinen).

III. Motion of the Pendulum when the Load on the Engine Changes.

If we do not take into account the vibration of the governor considered in II, but put A_0 in place of M_r, assuming the reacting forces of the valve gear to be steady and constant for all configurations of the

governor, and if we assume the dampiug effect to be in the form $j\frac{d\alpha}{dt}$, equation (16) becomes

$$\frac{d^2\alpha}{dt^2}\rho^2 m + j\frac{d\alpha}{dt} + 2\varkappa ab\left(1-\frac{l_0}{\sqrt{a^2+b^2}}\right)\sin\alpha$$

$$-\varkappa l_0\frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\sin 2\alpha - \omega^2 am\rho\sin\alpha + A_0 = 0. \qquad (23)$$

Put $$\alpha = \alpha_0 + \xi',$$

where α_0 denotes the value of α in the new configuration of equilibrium and ξ' denotes the small angle of deflection about α_0.

Taking $\sin\xi' = \xi'$, $\cos\xi' = 1$, $\sin 2\xi' = 2\xi'$

and $\cos 2\xi' = 1$, the above equation becomes

$$\frac{d^2\xi'}{dt^2}\rho^2 m + j\frac{d\xi'}{dt} + \xi'.2ab\left(1-\frac{l_0}{\sqrt{a^2+b^2}}\right)\cos\alpha_0$$

$$+2\varkappa ab\left(1-\frac{l_0}{\sqrt{a^2+b^2}}\right)\sin\alpha_0 - \xi'\varkappa l_0\frac{2a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\cos 2\alpha_0$$

$$-\varkappa l_0\frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\sin 2\alpha_0 - \xi'\omega^2 am\rho\cos\alpha_0$$

$$-\omega^2 am\rho\sin\alpha_0 + A_0 = 0.$$

Again put $$\omega = \omega_0 + \eta',$$

where ω_0 is the value of ω corresponding to the new configuration α_0 and η' denotes the fluctuation of ω. Since η' is a small quantity of the same order as ξ', we may neglect terms of powers of ξ' and η' higher than the second inclusive, and have

$$\frac{d^2\xi'}{dt^2}\rho^2 m + j\frac{d\xi'}{dt} + \xi'.2\varkappa ab\left(1-\frac{l_0}{\sqrt{a^2+b^2}}\right)\cos\alpha_0$$

$$+2\varkappa ab\left(1-\frac{l_0}{\sqrt{a^2+b^2}}\right)\sin\alpha_0 - \xi'\varkappa l_0\frac{2a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\cos 2\alpha_0$$

$$-\varkappa l_0\frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\sin 2\alpha^0 - \xi'\omega_0^2 am\rho\cos\alpha_0$$

$$-\eta'.2\omega_0 am\rho\sin\alpha_0 - \omega_0^2 am\rho\sin\alpha_0 + A = 0.$$

From the equation of statical equilibrium (12) the above equation becomes

$$\frac{d^2\xi'}{dt^2}\rho^2 m + j\frac{d\xi'}{dt} + \xi'\left[2\varkappa l_0 \frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}\sin^2 a_0 - A_0 \cot a_0\right]$$
$$-\eta' \,.\, 2\omega_0\, am\rho \sin a_0 = 0. \tag{24}$$

Let, T = mean turning moment of the engine,

ΔT = variation of T corresponding to small fluctuation of a or ξ'.

T depends upon the point of cut off and this latter is, in its turn, governed by the position of the pendulum. Therefore T must be a certain function of a, so that we may write

$$T = f(a). \tag{25}$$

If we draw a series of probable indicator diagrams for different points of cut off corresponding to different configurations of the governor, we shall get a curve of T referred to a. The curve in Fig. 10 is the one drawn according to our example. It is always concave towards the axis of abscissa, but the curvature is very slight.

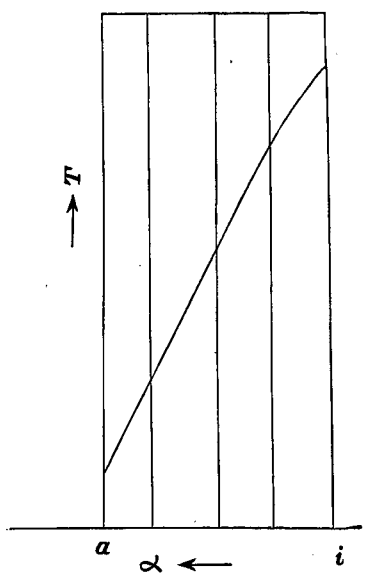


Fig. 10

For ΔT we have

$$\Delta T = \xi'\left(\frac{df}{da}\right)_0,$$

where $\left(\frac{df}{da}\right)_0$ is a value of $\frac{df}{da}$ at the new configuration of equilibrium, and is represented by the tangent of the angle between the tangent of T curve at a_0 and the axis of abscissa.

Also

$$\Delta T = I\frac{d\omega}{dt} = I\frac{d\eta'}{dt}$$

where I denotes the moment of inertia of the fly wheel.

Therefore we have

$$I\frac{d\eta'}{dt} = \xi'\left(\frac{df}{da}\right)_0$$

or

$$\frac{d\eta'}{dt} = \xi'\frac{\left(\frac{df}{da}\right)_0}{I}. \tag{26}$$

$\left(\frac{df}{da}\right)_0$ has always a negative value, because T decreases when a increases for ordinary governors of stable equilibrium; so if we put

$$\frac{\left(\frac{df}{da}\right)_0}{I} = -R, \tag{27}$$

R is a positive quantity depending on a. Since the T curve in Fig. 10 is very nearly a straight line we may assume R to be a constant throughout the whole oscillation. Elliminating η' between the two equations (24) and (26), we have

$$\left.\begin{aligned}
&a_0\frac{d^3\xi'}{dt^3}+3a_1\frac{d^2\xi'}{dt^2}+3a_2\frac{d\xi'}{dt}+a_3\xi' = 0,\\
\text{where}\quad &a_0 = \rho^2 m,\\
&a_1 = \tfrac{1}{3}j,\\
&a_2 = \tfrac{1}{3}\left[2x'_0\frac{a^2b^2}{(a^2+b^3)^{\frac{3}{2}}}\sin^2\alpha_0 - A_0\cot\alpha_0\right],\\
&a_3 = R\,.\,2\omega_0\,am\rho\,\sin\alpha_0.
\end{aligned}\right\} \tag{28}$$

The characteristic equation of the above differential equation is

$$a_0x^3+3a_1x^2+3a_2x+a_3 = 0. \tag{29}$$

If the three roots of this equation are α', β' and γ', all of which may be real, or one real and two complex, the general solution of equation (28) becomes

$$\xi' = Ae^{\alpha' t}+Be^{\beta' t}+Ce^{\gamma' t}; \tag{30}$$

or, in the latter case, if the pair of complex roots be $p+ig$ and $p-ig$, the above equation may be expressed in the form:

$$\xi' = Ae^{\alpha t} + e^{pt}\,(D\cos qt + \mathrm{E}\sin qt), \tag{31}$$

in which A, B, C, D and E are integration constants.

In equations (30) and (31) the term with positive exponent increases with t without limit, while that with the negative exponent approaches to zero, as it increases.

Therefore all real parts of the roots of the cubic equation (29) should be negative in our case, for otherwise ξ' would increase without limit with t and a governor of such quality would be of unstable equilibrium and useless. To meet this condition, the constant coefficients of the terms of equation (29) must all be positive. But a_0 and a_3 are always positive for ordinary governors; and for a_2 this condition coincides with that in II. Furthermore there is a condition

$$\begin{vmatrix} 3a_1 & a_3 \\ a_0 & 3a_2 \end{vmatrix} > 0$$

or

$$9a_1a_2 > a_0a_3,$$

after Hurwitz's law (Mathematische Annalen 1895), which includes the condition $a_1 > 0$. Therefore we have a new condition for our requirement:

$$\left.\begin{aligned} & a_1 > \frac{a_0a_3}{9a^2} \\ \text{or} \quad & j > \frac{R\,.\,2\omega_0 am\rho\sin\alpha}{k_2}. \end{aligned}\right\} \tag{32}$$

When condition (32) is fulfilled, if the three roots are all real, there is no oscillation in the movement of the governor, as can be seen from equation (30); and the pendulum approaches to the new configuration of equilibrium gradually; but in case of one real and two complex roots, equation (31) shows that the pendulum oscillates with diminishing amplitude about its centre of oscillation, which displaces itself gradually to its new configuration. The roots of equation (29) have now to be investigated. In equation (29) substitute

$$y = x + \frac{a_1}{a_0}, \tag{33}$$

and we have

$$\left.\begin{aligned} & y^3+\frac{3H}{a_0^2}y+\frac{G}{a_0^3}=0, \\ \text{where} \quad & H=a_0a_2-a_1^2, \\ & G=a_0^2a_3-3a_0a_1a_2+2a_1^3. \end{aligned}\right\} \tag{34}$$

If the three roots of this equation are α, β and γ, there must be the relations:

$$\left.\begin{aligned} \alpha' &= \alpha-\frac{a_1}{a_0}, \\ \beta' &= \beta-\frac{a_1}{a_0}, \\ \gamma' &= \gamma-\frac{a_1}{a_0}. \end{aligned}\right\} \tag{35}$$

If we form a cubic equation

$$Z^3+\frac{18H}{a_0^2}Z^2+\frac{81H^2}{a_0^4}Z+\frac{27(G^2+4H^3)}{a_0^6}=0, \tag{36}$$

whose three roots Z_1, Z_2 and Z_3 are

$$\begin{aligned} Z_1 &= (\alpha-\beta)^2, \\ Z_2 &= (\beta-\gamma)^2, \\ Z_3 &= (\gamma-\alpha)^2; \end{aligned}$$

then, from equation (35),

$$\begin{aligned} Z_1 &= (\alpha'-\beta')^2, \\ Z_2 &= (\beta'-\gamma')^2, \\ Z_3 &= (\gamma'-\alpha')^2. \end{aligned}$$

Denoting

$$-\frac{27}{a_0^6}(G^2+4H^3)=(\alpha'-\beta')^2(\beta'-\gamma')^2(\gamma'-\alpha')^2=Q,$$

we have three cases:

(a).— $\qquad Q>0, \qquad$ i. e. $\quad G^2+4H^3<0$

Then $\qquad H<0,$

and the second and third terms in equation (36) become negative, hence it has no negative root and consequently equation (29) has three distinct

real roots, because if it had any complex root, there would of necessity be another conjugate to it, and therefore the square of their difference, or a root of equation (36) would be negative.

(b).— $$Q = 0$$

In the case at least one root of equation (36) must be zero; and,

(i) if $$H \neq 0,$$

two roots of equation (29) are equal and,

(ii) if $$H = 0,$$

the three roots are all equal.

(c).— $$Q < 0, \quad \text{i. e.} \quad G^2 + 4H^3 > 0$$

Here either one root only or all the three roots of equation (36) must be negative; and then equation (29) has a pair of complex roots.

Since case (b) may be regarded as a particular case of (a), the cubic equation (29) has or has not a pair of complex roots according as

$$\left.\begin{array}{l} G^2 + 4H^3 > 0 \\ \text{or} \quad G^2 + 4H^3 \not> 0. \end{array}\right\} \tag{37}$$

The arbitrary constants must now be determined.

A. Case in which there is no oscillation and the motion is expressed by equation (30),

$$\xi' = Ae^{\alpha' t} + Be^{\beta' t} + Ce^{\gamma' t}.$$

Suppose that the engine be unloaded suddenly when it is running steadily, and the governor is in a certain configuration α_1; then the fly wheel will be accelerated, the angular velocity ω increased, and the governor put in motion for its new configuration α_0 which is determined by equation (25). If we take this instant as the starting point of the motion, then at this instant $\frac{d\xi'}{dt}$ must be zero and the value of ξ' may be taken as $\alpha_1 - \alpha_0$, which is negative in the case of unloading. The value of η' also may be taken as the difference between ω_0, which corresponds to the new configuration α_0, and ω_1 which the fly wheel possessed at the very moment of starting in motion. ω_0 and ω_1 are both found by putting

$a=a_0$ and $a=a_1$ in equation (12). In practice, as the governor possesses some insensibility, though small, the pendulum cannot start in motion at $\omega=\omega_1$; but at some higher value of ω:

$$\omega_1' = \omega_1\left(1+\frac{\varepsilon}{2}\right),$$

where ε is the insensibility of the governor. Conversely, when the load of the engine increases, the pendulum begins to move when the angular velocity has assumed a somewhat lower value than the original. But if the governor is absolutely sensitive, we have $\omega_1'=\omega_1$.

Putting $$\xi_0' = a_1-a_0$$

and $$\eta_0' = \omega_1'-\omega_0,$$

we get the three following relations:

$$\xi_0' = A+B+C$$

and $$0 = A\alpha'+B\beta'+C\gamma'$$

from equation (30), and

$$\frac{a_3}{R}\eta_0' = a_0(A\alpha'^2+B\beta'^2+c\gamma'^2)+3a_2\xi_0'$$

or $$\frac{1}{a_0}\left(\frac{a_3}{R}\eta_0'-3a_2\xi_0'\right) = A\alpha'^2+B\beta'^2+C\gamma'^2$$

from equation (24).

Solving the above equations we have

$$\left.\begin{aligned}
A &= \frac{\left\{\frac{1}{a_0}\left(\frac{a_3}{R}\eta_0'-3a_2\xi_0'\right)+\xi_0'\beta'\gamma'\right\}(\gamma'-\beta')}{\beta'\gamma'(\gamma'-\beta')+\gamma'\alpha'(\alpha'-\gamma')+\alpha'\beta'(\beta'-\alpha')},\\
B &= \frac{\left\{\frac{1}{a_0}\left(\frac{a_3}{R}\eta_0'-3a_2\xi_0'\right)+\xi_0'\gamma'\alpha'\right\}(\alpha'-\gamma')}{\beta'\gamma'(\gamma'-\beta')+\gamma'\alpha'(\alpha'-\gamma')+\alpha'\beta'(\beta'-\alpha')},\\
C &= \frac{\left\{\frac{1}{a_0}\left(\frac{a_3}{R}\eta_0'-3a_2\xi_0'\right)+\xi_0'\alpha'\beta'\right\}(\beta'-\alpha')}{\beta'\gamma'(\gamma'-\beta')+\gamma'\alpha'(\alpha'-\gamma')+\alpha'\beta'(\beta'-\alpha')};
\end{aligned}\right\} \qquad (38)$$

in which α', β' and γ', the roots of equation (29), are found as follows:

Since in this case $H < 0$, as is seen from (37), we may write equation (34) in the form,

$$y^3 - uy \pm v = 0,$$

where u and v are positive and their values are

$$u = \left| \frac{3H}{a_0^2} \right| \quad \text{and} \quad v = \left| \frac{G}{a_0^3} \right|;$$

then its three roots are

$$\alpha = \mp 2\sqrt{\frac{u}{3}} \cos\tfrac{1}{3}\varphi,$$

$$\beta = \pm 2\sqrt{\frac{u}{3}} \cos\left(\frac{\pi}{3} - \tfrac{1}{3}\varphi\right),$$

$$\gamma = \pm 2\sqrt{\frac{u}{3}} \cos\left(\frac{\pi}{3} + \tfrac{1}{3}\varphi\right);$$

and therefore

$$\left.\begin{aligned} \alpha' &= \mp 2\sqrt{\frac{u}{3}} \cos\tfrac{1}{3}\varphi - \frac{a_1}{a_0}, \\ \beta' &= \pm 2\sqrt{\frac{u}{3}} \cos\left(\frac{\pi}{3} - \tfrac{1}{3}\varphi\right) - \frac{a_1}{a_0}, \\ \gamma' &= \pm 2\sqrt{\frac{u}{3}} \cos\left(\frac{\pi}{3} + \tfrac{1}{3}\varphi\right) - \frac{a_1}{a_0}; \end{aligned}\right\} \tag{39}$$

where φ is determined by the relation

$$\cos\varphi = \frac{\frac{v}{2}}{\left(\frac{u}{3}\right)^{\frac{3}{2}}}.$$

B. Case in which the oscillation takes place and the motion is expressed by equation (31),

$$\xi' = Ae^{\alpha' t} + e^{pt}(D \cos qt + E \sin qt).$$

In the same way as before, we get the three relations:

$$\xi_0' = A + D,$$

$$0 = A\alpha' + Dp + Eq,$$

$$\frac{1}{a_0}\left(\frac{a_3}{R}\eta_0' - 3a_2\xi_0'\right) = A\alpha'^2 + D(p^2 - q^2) + E\,.\,2pq.$$

Therefore we obtain

$$\left.\begin{aligned}
A &= \frac{\xi_0'(p^2+q^2) + \frac{1}{a_0}\left(\frac{a_3}{R}\eta_0' - 3a_2\xi_0'\right)}{p^2+q^2+\alpha'(\alpha'-2p)}, \\
D &= \frac{\xi_0'\alpha'(\alpha'-2p) - \frac{1}{a_0}\left(\frac{a_3}{R}\eta_0' - 3a_2\xi_0'\right)}{p^2+q^2+\alpha'(\alpha'-2p)}, \\
E &= \frac{\xi_0'\alpha'(p^2-q^2-p\alpha') + (p-\alpha')\frac{1}{a_0}\left(\frac{a_3}{R}\eta_0' - 3a_2\xi_0'\right)}{\{p^2+q^2+\alpha'(\alpha'-2p)\}q}.
\end{aligned}\right\} \qquad (40)$$

The roots of equation (34) are

$$\alpha = \sqrt[3]{-\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right) + \sqrt{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)^2 + \left(\frac{H}{a_0^2}\right)^3}}$$

$$- \sqrt[3]{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right) + \sqrt{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)^2 + \left(\frac{H}{a_0^2}\right)^3}},$$

$$\beta = p_1 + iq,$$

$$\gamma = p_1 - iq;$$

in which

$$p_1 = -\frac{\alpha}{2},$$

$$q = \left[\sqrt[3]{-\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right) + \sqrt{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)^2 + \left(\frac{H}{a_0^2}\right)^3}}\right.$$

$$\left. + \sqrt[3]{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right) + \sqrt{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)^2 + \left(\frac{H}{a_0^2}\right)^3}}\right]\frac{\sqrt{3}}{2};$$

and then the roots of equation (29) are

$$\alpha' = \alpha - \frac{a_1}{a_0},$$

$$\beta' = p' + iq - \frac{a_1}{a_0},$$

$$\gamma' = p_1 - iq - \frac{a_1}{a_0}.$$

Therefore

$$\left.\begin{aligned}
a' &= \sqrt[3]{-\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)+\sqrt{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)^2+\left(\frac{H}{a_0^2}\right)^3}} \\
&\quad -\sqrt[3]{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)+\sqrt{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)^2+\left(\frac{H}{a_0^2}\right)^3}} - \frac{a_1}{a_3}, \\
p &= -\tfrac{1}{2}\left[\sqrt[3]{-\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)+\sqrt{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)^2+\left(\frac{H}{a_0^2}\right)^3}}\right. \\
&\quad \left. -\sqrt[3]{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)+\sqrt{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)^2+\left(\frac{H}{a_0^2}\right)^3}}\right] - \frac{a_1}{a_0}, \\
q &= \left[\sqrt[3]{-\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)+\sqrt{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)^2+\left(\frac{H}{a_0^2}\right)^3}}\right. \\
&\quad \left. +\sqrt[3]{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)+\sqrt{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)^2+\left(\frac{H}{a_0^2}\right)^3}}\right]\frac{\sqrt{3}}{2}.
\end{aligned}\right\} \quad (41)$$

Example 3.—Required to establish the laws of motion of the pendulum of the governor for the engine in example 1 fitted with a fly wheel of $I = 43$ kg. m.^{2}; supposing that, when it is running in a steady condition corresponding to the configuration of the governor, $\alpha = 57° - 20'$, it is suddenly unloaded to such a degree that the new configuration of equilibrium of the governor shall be $\alpha = 64° - 40'$.

From the diagram in Fig. 10 it can be found that

$$\left(\frac{dT}{d\alpha}\right)_0 = 300 \text{ kg. m. per radian.}$$

and so

$$R = \frac{300}{43} \doteqdot 7.$$

(1). Governor designed under case (a).

Here the coefficients of equation (28) are:

$$a_0 = \rho^2 m = 1.76,$$

$$a_2 = \tfrac{1}{3}\left[2xl_0 \frac{a^2b^2}{(a^2+b^2)^{\frac{3}{2}}} \sin^2 a_0 - A_0 \cot a_0\right] = 211,$$

$$a_3 = 2R\omega_0 am\rho \sin a_0 = 840.$$

Therefore

$$\frac{a_0 a_3}{9a_2} = 0.78.$$

Take $$a_1 = 24,$$

this value of a_1 evidently satisfies condition (32), then

$$G^2 = (a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3)^2 \doteqdot 13000000$$

and $$4H^3 = 4(a_0 a_2 - a_1^2)^3 \doteqdot -34500000.$$

Therefore, according to unequality (37), the motion of the pendulum will be expressed by equation (30), in which the exponents of e, or the roots of equation (29), α', β' and γ' and the arbitrary constants A, B and C, may be obtained in the same way as in case A; thus in the equation

$$y^3 - ug \pm v = 0,$$

$$u = \left| \frac{3H}{a_0^2} \right| = 198,$$

$$v = \left| \frac{G}{a_0^3} \right| = 660;$$

and

$$\cos\varphi = \frac{\frac{v}{2}}{\left(\frac{u}{3}\right)^{\frac{3}{2}}} = 0.617,$$

or $$\varphi = 51° - 54'.$$

From equation (39),

$$\alpha' = -2\sqrt{\frac{u}{3}}\cos\tfrac{1}{3}\varphi - \frac{a_1}{a_0} = -29.1,$$

$$\beta' = +2\sqrt{\frac{u}{3}}\cos\left(\frac{\pi}{3} - \tfrac{1}{3}\varphi\right) - \frac{a_1}{a_0} = -1.7,$$

$$\gamma' = +2\sqrt{\frac{u}{3}}\cos\left(\frac{\pi}{3} + \tfrac{1}{3}\varphi\right) - \frac{a_1}{a_0} = -10.$$

Also $\xi_0' = a_1 - a_0 \quad = (57^\circ - 20') - (64^\circ - 40')$

$= 7^\circ - 20'$

or $= 0.128$ radian,

and $\eta_0' = \omega_1' - \omega_0 = \omega_1\left(1 + \frac{\varepsilon}{2}\right) - \omega_0$

$$= 24.3\left(1 + \frac{0.03}{2}\right) - 25.1 = -0.4 \text{ radian per sec.}$$

From equation (38),

$$A = 0.0318 \quad \text{radian},$$

$$B = -0.0814 \quad ,,$$

$$C = -0.0784 \quad ,, \quad .$$

Therefore the equation of motion of the pendulum is here

$$\xi' = 0.0318e^{-29.1t} - 0.0814e^{-1.7t} - 0.0784e^{-10t}$$

The curve of this equation is given by the thick line in Fig. 11a, Pl. II.

If we take $a_1 = 15,$

$$G = -7340, \qquad H = 146$$

and $\left(\frac{1}{2}\cdot\frac{G}{a_0^3}\right)^2 = 455000,$

$$\left(\frac{H}{a_0^2}\right)^3 = 104300.$$

Therefore the motion belongs to case B, and

$$\sqrt{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)^2+\left(\frac{H}{a_0^2}\right)^3}=747,$$

$$\sqrt[3]{-\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)+\sqrt{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)^2+\left(\frac{H}{a_0^2}\right)^3}}=11.23,$$

$$\sqrt[3]{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)+\sqrt{\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)^2-\left(\frac{H}{a_0^2}\right)^3}}=4.2.$$

Therefore, by equation (41),

$$\alpha' = -1.5$$
$$p = -12.043$$
$$q = 13.35\,;$$

and, from (40),

$$A = -0.0783,$$
$$D = -0.0497,$$
$$E = -0.028.$$

Therefore the equation of motion becomes

$$\xi' = -0.0783e^{-1.5t}-e^{-12.043t}(0.0497\cos 13.35t+0.028\sin 13.35t).$$

The motion is shown in Fig. 11_b, Pl. III.

(2). Governor designed under case (b).

$$a_0 = 1.47,$$
$$a_2 = 162,$$
$$a_3 = 700.$$

Therefore

$$\frac{a_0a_3}{9a_2} = 0.707.$$

Take again $\qquad a_1 = 24, \qquad$ then

$$\left(\tfrac{1}{2}\cdot\frac{G}{a_0^3}\right)^2 = 3650000,$$

$$\left(\frac{H}{a_0^2}\right)^3 = -3830000.$$

Therefore the motion belongs to case A; and from equation (39),

$$\alpha' = -41.2,$$
$$\beta' = -2.2,$$
$$\gamma' = -5.46;$$

and, from (38),

$$A = 0.01,$$
$$B = -0.108,$$
$$C = -0.03.$$

Therefore we get the equation of motion

$$\xi' = 0.01e^{-41.2t} - 0.108e^{-2.2t} - 0.03e^{-5.46t}$$

The curve of this equation is given in Fig. 12a, Pl. IV.

If we take $a_1 = 15,$

$$\left(\frac{1}{2}\cdot\frac{G}{a_0^3}\right)^2 = 148200,$$

$$\left(\frac{H}{a_0^2}\right)^3 = 218.$$

Therefore the motion belongs to case B; so, putting the values of α', p, q, A, D and E in equation (31), we get the equation of motion

$$\xi' = -0.083e^{-1.08t} - e^{-14.78t}(0.045\cos 7.94t + 0.0943\sin 7.94t),$$

which is represented by the curve in Fig. 12b, Pl. V.

(3). Governor designed under case (C).

$$a_0 = 1.17,$$
$$a_2 = 121,$$
$$a_3 = 560;$$

and therefore

$$\frac{a_0 a_3}{9a_2} = 0.6.$$

Take $$a_1 = 15,$$

$$\left(\tfrac{1}{2} \cdot \frac{G}{a_0^3}\right)^2 = 126000,$$

$$\left(\frac{H}{a_0^2}\right)^3 = -222000.$$

Therefore the motion belongs to case A, and so we get the equation of motion

$$\xi' = 0.034e^{-27.9t} - 0.067e^{-2.1t} - 0.095e^{-8.43t},$$

which is represented by the curve in Fig. 13_a, Pl. VI.

If we take $$a_1 = 5,$$

$$\left(\tfrac{1}{2} \cdot \frac{G}{a_0^3}\right)^2 = 120500,$$

$$\left(\frac{H}{a_0^2}\right)^3 = 625000.$$

Therefore the motion belongs to case B, and the equation of motion is

$$\xi' = -0.079e^{-1.64t} - e^{-9.6t}(-0.049 \cos 16.1t + 0.029 \sin 16.1t),$$

which is represented by the curve in Fig. 13_b, Pl. VII.

(4). Governor designed under case (d).

$$a_0 = 0.88,$$
$$a_2 = 58.7,$$
$$a_3 = 420;$$

and therefore

$$\frac{a_0 a_3}{9 a_2} = 0.7.$$

If we take $$a_1 = 30,$$

$$\left(\tfrac{1}{2} \cdot \frac{G}{a_0^3}\right)^2 = 1324000000,$$

$$\left(\frac{H}{a_0^2}\right)^3 = -1312000000.$$

Therefore the motion belongs to case B, and the equation of motion is

$$\xi' = 0.001e^{-100.2t} - e^{-1.1t}(0.129 \cos 1.73t + 0.008 \sin 1.73t),$$

which is represented by the curve in Fig. 14_{b_1}, Pl. VIII.

If we take $$a_1 = 15,$$

$$\left(\frac{1}{2} \cdot \frac{G}{a_0^3}\right)^2 = 12230000,$$

$$\left(\frac{H}{a_0^2}\right)^3 = -11100000.$$

Therefore the motion belongs to case B, and the equation of motion is

$$\xi' = 0.0052e^{-47t} - e^{-2t}(0.132 \cos 2.7t + 0.0078 \sin 2.7t),$$

which is represented by the curve in Fig. 14_{b_2}, Pl. IX.

If we take $a_1 = 5$, the equation of motion becomes

$$\xi' = -0.049e^{-3.04t} - e^{-4.36t}(0.079 \cos 12t + 0.0334 \sin 12t).$$

The motion is shown by the curve in Fig. 14_{b_3}, Pl. X.

(4′) Governor designed under case (d′).

$$a_0 = 0.88,$$

$$a_2 = 68.3,$$

$$a_3 = 420;$$

and so

$$\frac{a_0 a_3}{9a_2} = 0\ 6.$$

Take $$a_1 = 15, \quad \text{then}$$

$$\left(\frac{1}{2} \cdot \frac{G}{a_0^3}\right)^2 = 10240000,$$

$$\left(\frac{H}{a_0^2}\right)^3 = -9680000.$$

The motion belongs to case B, and the equation of motion becomes

$$\xi' = 0.0126e^{-36.28t} - e^{-2.36t}(0.1406\cos 2t - 0.092\sin 2t),$$

which is represented by the curve in Fig. 15_{b_1}, Pl. XI.

If we take $a_1 = 5$, the equation of motion is

$$\xi' = -0.0535e^{-2.4t} - e^{-4.05t}(0.0745\cos 13.81t + 0.0306\sin 13.81t),$$

which is represented by the curve in Fig. 15_{b_2}, Pl. XII.

The results in the above examples show the manner in which the governors approach to their new position of equilibrium. Governors whose motions are given in Fig. 11_a, Fig. 11_b, Fig. 12_a, Fig. 12_b, Fig. 13_a and Fig. 13_b come sooner or later to their new position. This approaching motion may be accompanied by some oscillation or not, but is without any over regulation at all. On the other hand, the governors in all other cases make over regulation in different degrees. But we can perceive a common fact that, at last, all of the governors come practically to rest in the new position of equilibrium after some interval of time.

If there is no friction and no special damping device, such as an oil brake, the differential equation (28) loses its second term; and thus either the real root or the real part of the complex roots of the characteristic equation becomes positive and the pendulum comes to rest no more, but passes to the extreme position through its entire range of path. Frictional resistance at pins and knife edges, in fact, act to damp the oscillation of the governor and in most cases they sustain the governor in a stable condition.

If we denote by M_0 the moment of these frictional forces about the fulcrum of the pendulum and assume it to be constant, equation (24) becomes

$$a_0\frac{d^2\xi'}{dt^2} + 3a_2\xi' - \eta'\frac{a_3}{R} \pm M_0 = 0, \qquad (42)$$

in which the constants a_0, a_2 and a_3 are the same as in (28), and the constant term M_0 is to have the upper sign for the outward motion of the pendulum and the lower sign for the reversed motion.

Elliminating η' between this equation and (26), we get

$$a_0\frac{d^3\xi'}{dt^3}+3a_2\frac{d\xi'}{dt}+a_3\xi' = 0. \tag{43}$$

The general solution of this equation is

$$\xi' = A'e^{\lambda t}+e^{p't}(D'\cos q't+E'\sin q't), \tag{44}$$

where

$$\lambda = \sqrt[3]{-\left(\frac{a_3}{2a_0}\right)+\sqrt{\left(\frac{a_3}{2a_0}\right)^2+\left(\frac{a_2}{a_0}\right)^3}}-\sqrt[3]{\left(\frac{a_3}{2a_0}\right)+\sqrt{\left(\frac{a_3}{2a_0}\right)^2+\left(\frac{a_2}{a_0}\right)^3}},$$

$$p' = -\frac{\lambda}{2},$$

$$q' = \left[\sqrt[3]{-\left(\frac{a_3}{2a_0}\right)+\sqrt{\left(\frac{a_3}{2a_0}\right)^2+\left(\frac{a_2}{a_0}\right)^3}}+\sqrt[3]{\left(\frac{a_3}{2a_0}\right)+\sqrt{\left(\frac{a_3}{2a_0}\right)^2+\left(\frac{a_2}{a_0}\right)^3}}\right]\frac{\sqrt{3}}{2}.$$

Equation (43) is the same as (28) except that in the former the second term of the latter is absent, and so it seems as if it were a special case of (28) when a_1 is taken equal to zero. But there are different points as regards the determination of the integration constants A', D' and E'. The most important difference is that the amplitude of the oscillation varies suddenly when it passes to the next stroke of the oscillation, but the state of the motion between both ends of one stroke is the same as when the constant friction is neglected. Therefore for the first stroke of the motion we may determine the arbitrary constant in the same way as before, but in the reduction of the third boundary condition we must take equation (42) instead of (24) with the upper sign of the constant term, if the case is unloading of the engine. We have, therefore, the three following relations containing three arbitrary constants A', D' and E':

$$\left.\begin{aligned} \xi_0' &= A'+D', \\ 0 &= \lambda A'+p'D'+q'E', \\ \frac{a_3}{R}\eta_0' &= a_0\{\lambda^2A'+(p'^2-q'^2)D'+2p'q'E'\}+a_3\xi_0'+M_0; \end{aligned}\right\} \tag{45}$$

from which A', D' and E' are determined. Again solving the equation

$$\frac{d\xi'}{dt} = 0 = \lambda A' e^{\lambda t} + e^{p't}(-q'D' \sin q't + q'E' \cos q't)$$
$$+ p' e^{p't}(D' \cos q't + E' \sin q't), \qquad (46)$$

we get the time t_0 for the first stroke of oscillation; and thus the motion of the pendulum in the first stroke is completely known; If the amplitude of the first stroke, which may be got by putting t_0 in equation (44), be greater than ξ'_0, that is if the pendulum passes through its new position of equilibrium to the other side to a greater distance than before, the governor becomes unstable. For the second stroke we can find the new constants in the same way by taking its initial point at the end of this first stroke, and so on.

As am example, if we take $a_1 = 0$

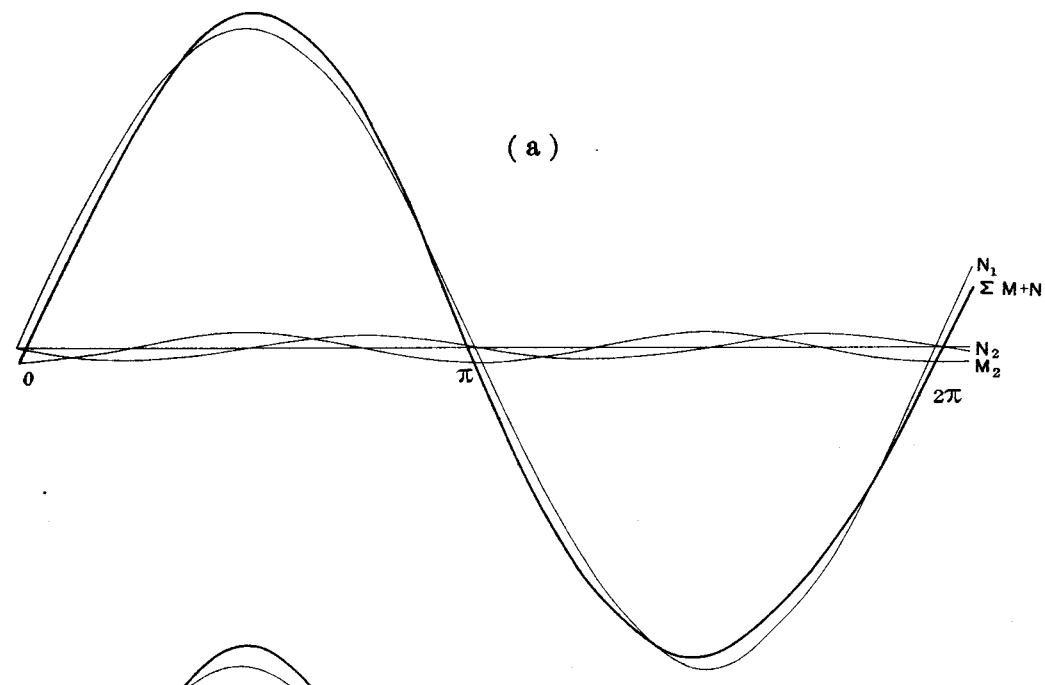
and $$M_0 = \frac{0.03}{2} \times M_z = 22.6 \text{ kg. cm.}$$

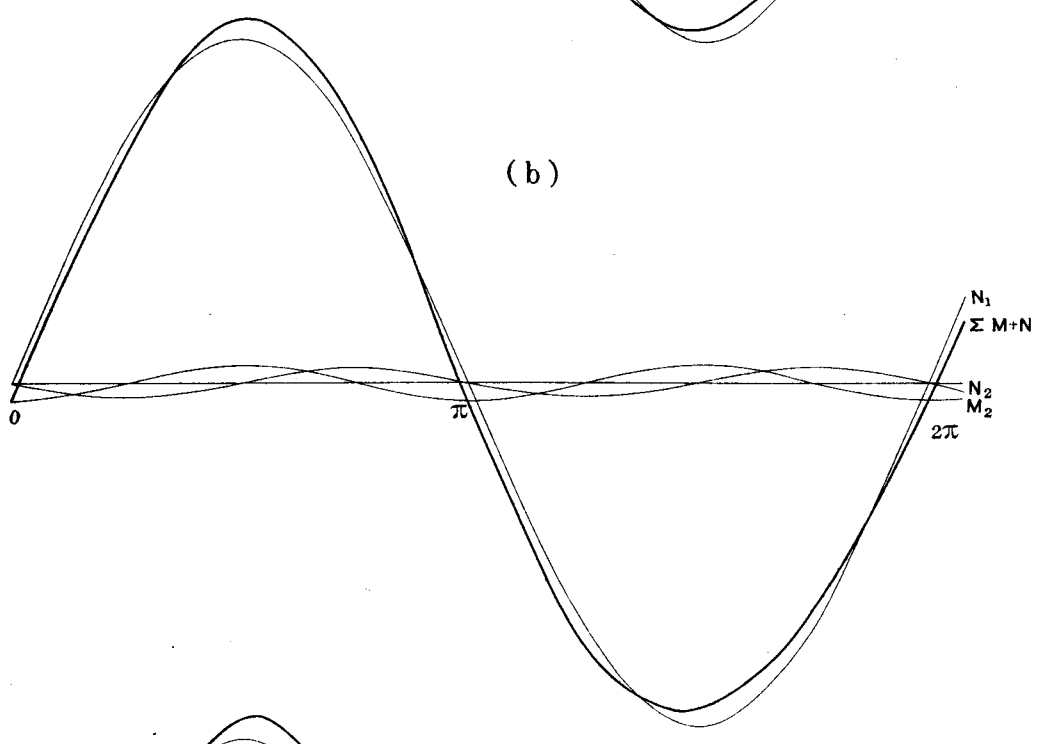
in the governor of the above example, we get, for the governor under case 1, the curve of motion for its first stroke as shown in Fig. 16, Pl. XIII. And for those under cases 4′ and 4, we get the curves of motion in Fig. 17 and Fig. 18, Pl. XIII. From these curves we may see that all these governors have no fear of over-regulation, the value of ξ' at the end of the first stroke being negative. Next assume so small a value as nearly one third for M_0, and the curve becomes as shown in Fig. 18_a, and even in this case over-regulation does not yet occur. If a positive value of ξ' is obtained at the end of the first stroke of oscillation, over-regulation clearly takes place. Furthermore when ξ' comes out greater than ξ_0' in its absolute magnitude, the initial distance of the pendulum from the new position of equilibrium in the second stroke is greater than that in the first; in the third stroke it becomes still greater, and so on; and thus the pendulum never comes to rest. But in this case, in which the frictional resistances are taken constant, it will be difficult to determine a definite limit like (32) in the previous case for the stable condition of the governor. When the load on the engine is altered, the governor commences its motion, shifting the eccentric to give a greater or less admission of steam

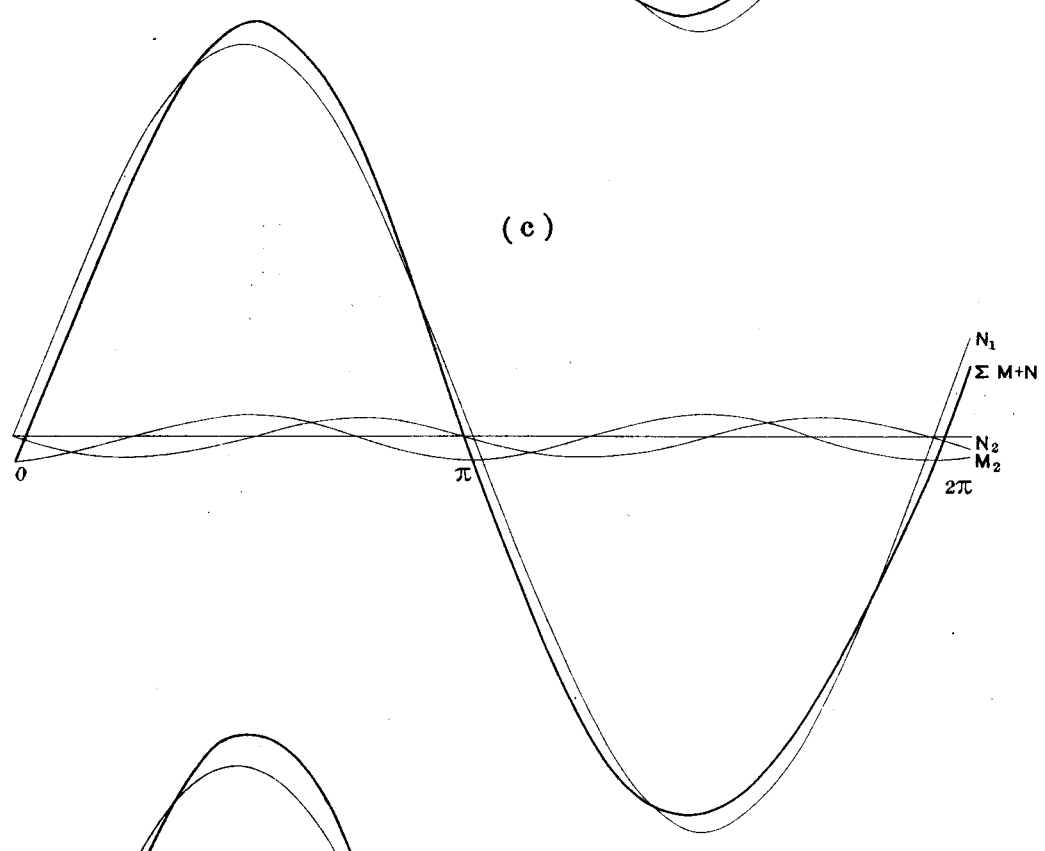
for a heavier or lighter load. The governor is desired to reach its new configuration as soon as possible, but without over-regulation. From the above investigations we may conclude that the governors which are ascertained to be stable by means of their characteristic curves, do not necessarily come to their new position of equilibrium; in other words a governor, not fulfilling the condition (32) and having no constant friction, is of an unsteady character. Also some govornors may be defective on account of over-regulation or hunting. Moreover, if a governor be constructed so as to obviate these defects, too steady a governor may result, requiring a long time for its displacement; and such a governor is again not fit for practical use. Therefore, if we have to consider the precise qualities necessary for a governor, its law of motion must be determined as above.

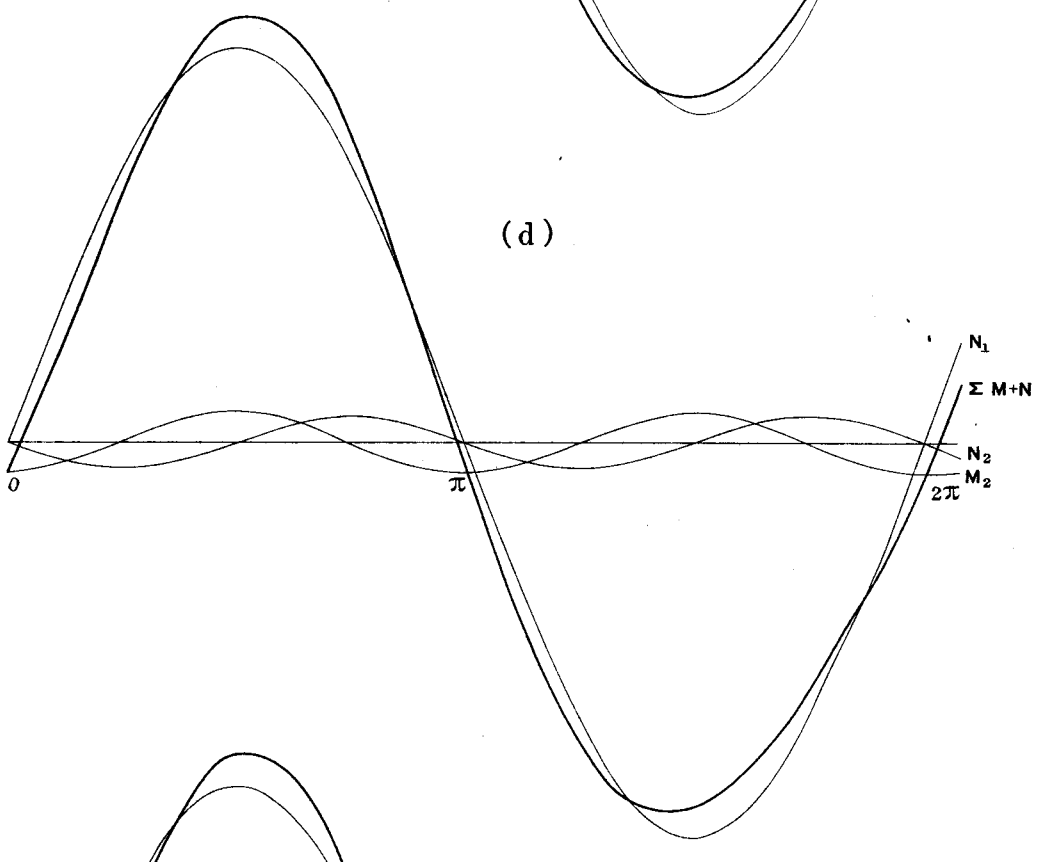
In this study it is assumed that the turning moment of the engine at any instant has an amount corresponding to the configuration of the governor at that instant. This assumption will be correct for a prime mover of continuous working, such as a steam or water turbine, but not for such as cut off their working fluid periodically, as a reciprocating engine. In the latter case the regulation can take place only once in a stroke and any displacement of the governor after the cut off has no effect on the turning moment of the engine till the next cut off. Therefore the curves drawn in the above examples do not represent the true motions of a governor attached to a steam engine. However, when the engine is of a high speed type, the governor requires a comparatively long time for its displacement; and so the forgoing conforms with tolerable accuracy to the actual facts.

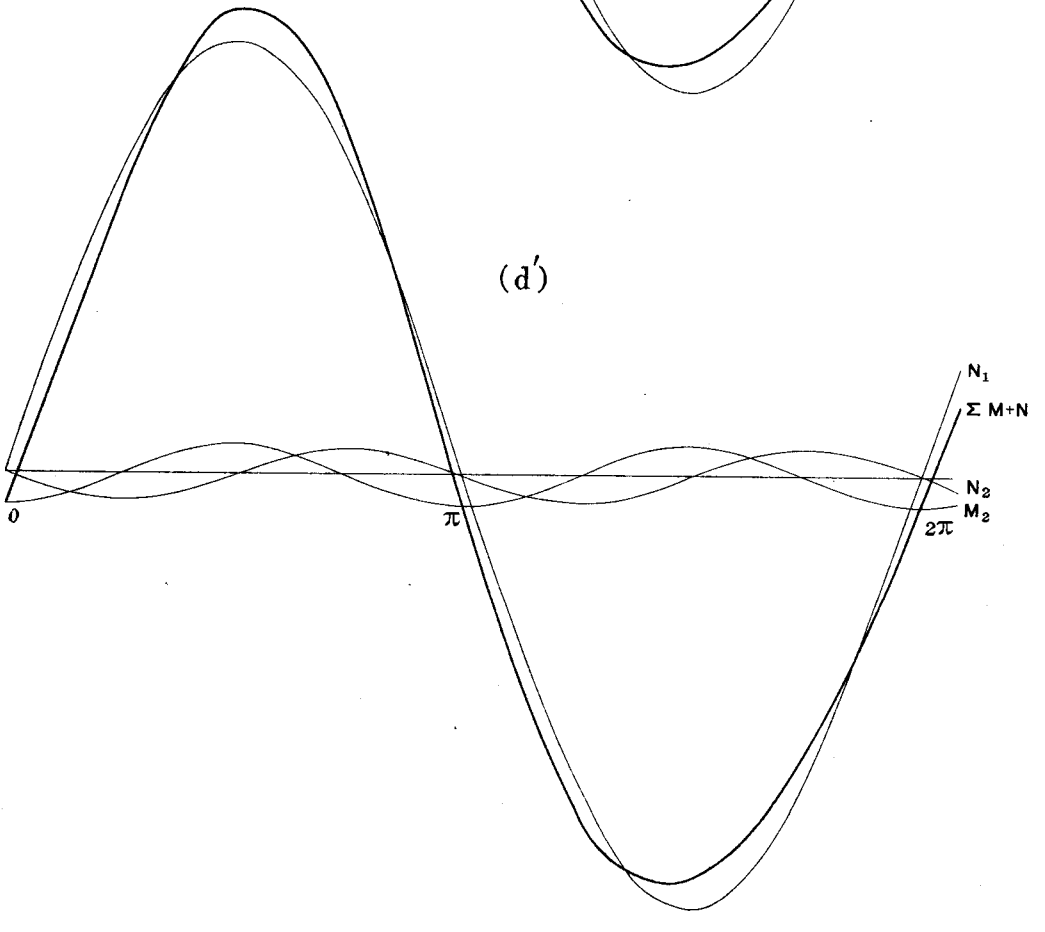
Fig. 9


(a)
N1
Σ M+N
N2
M2
0
π
2π


(b)
N1
Σ M+N
N2
M2
0
π
2π


(c)
N1
Σ M+N
N2
M2
0
π
2π


(d)
N1
Σ M+N
N2
M2
0
π
2π


(d′)
N1
Σ M+N
N2
M2
0
π
2π

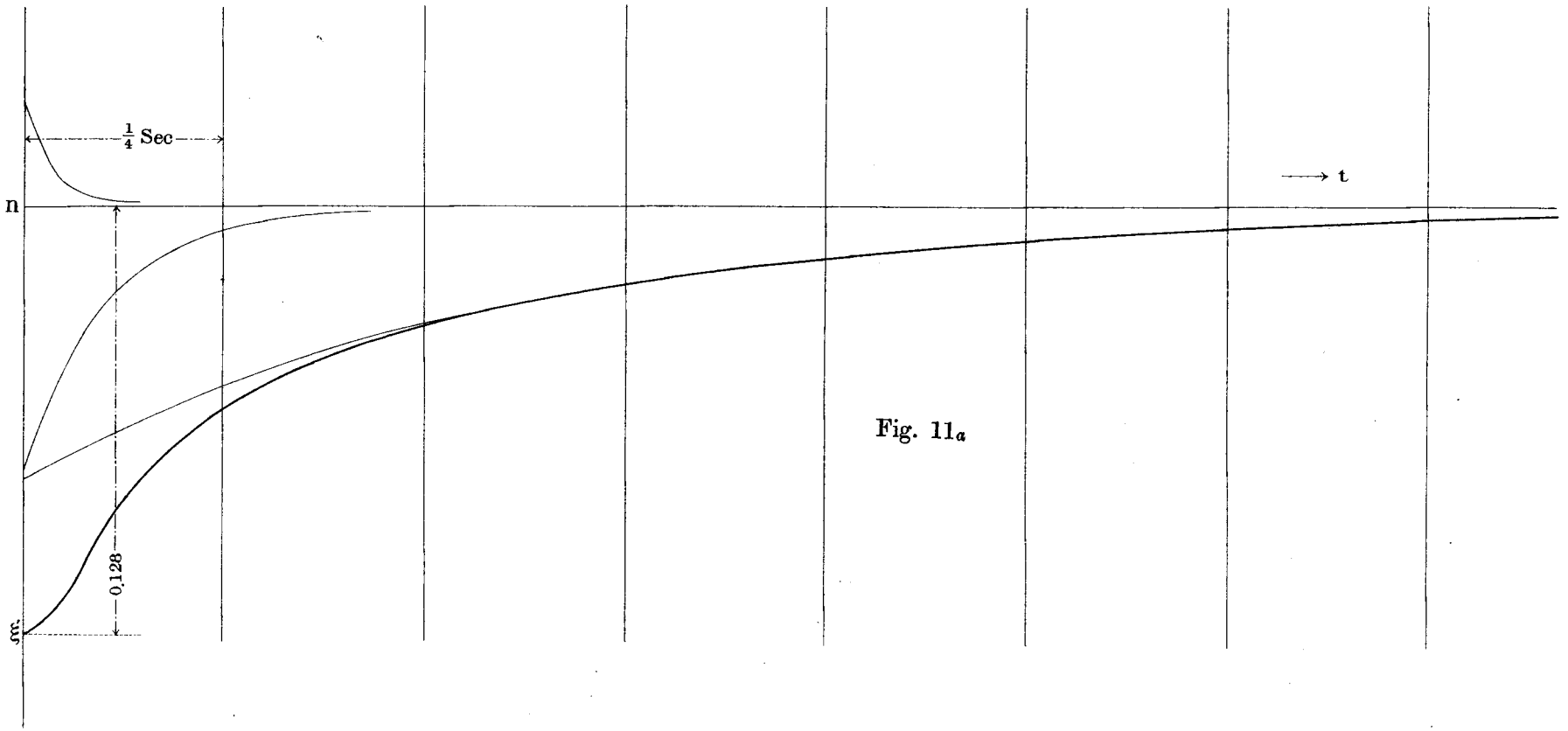


Fig. 11$_a$

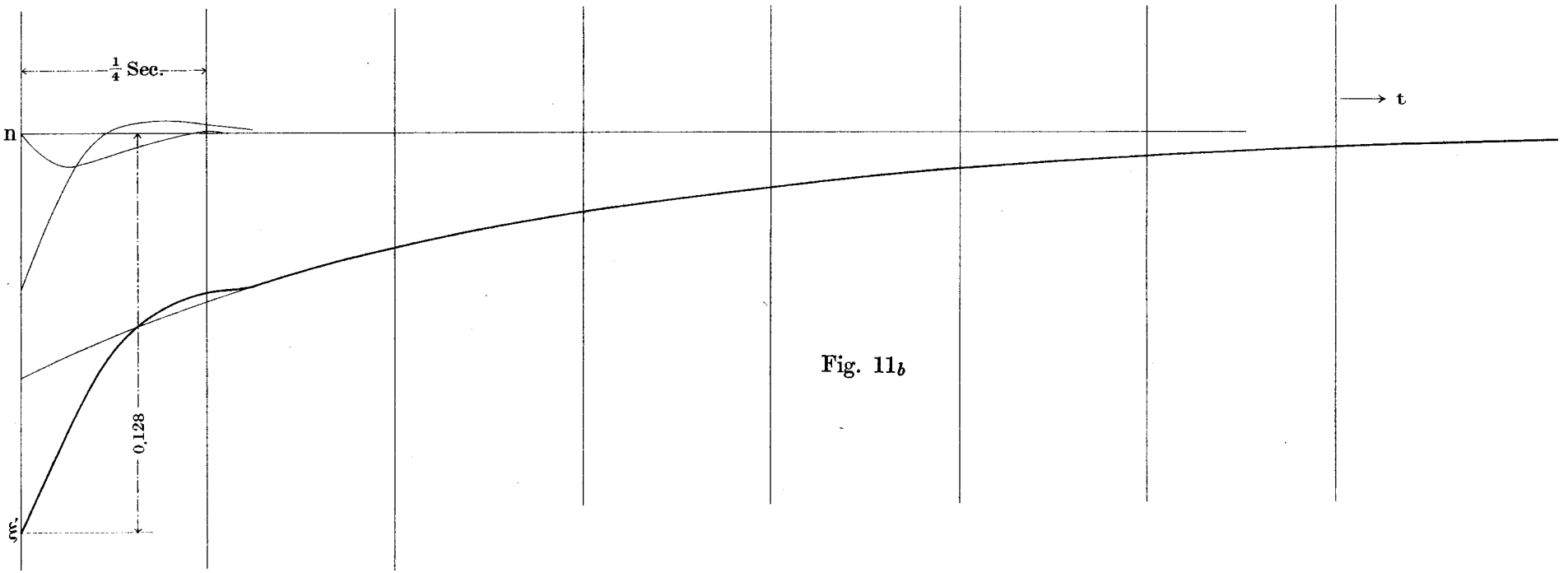


Fig. 11$_b$

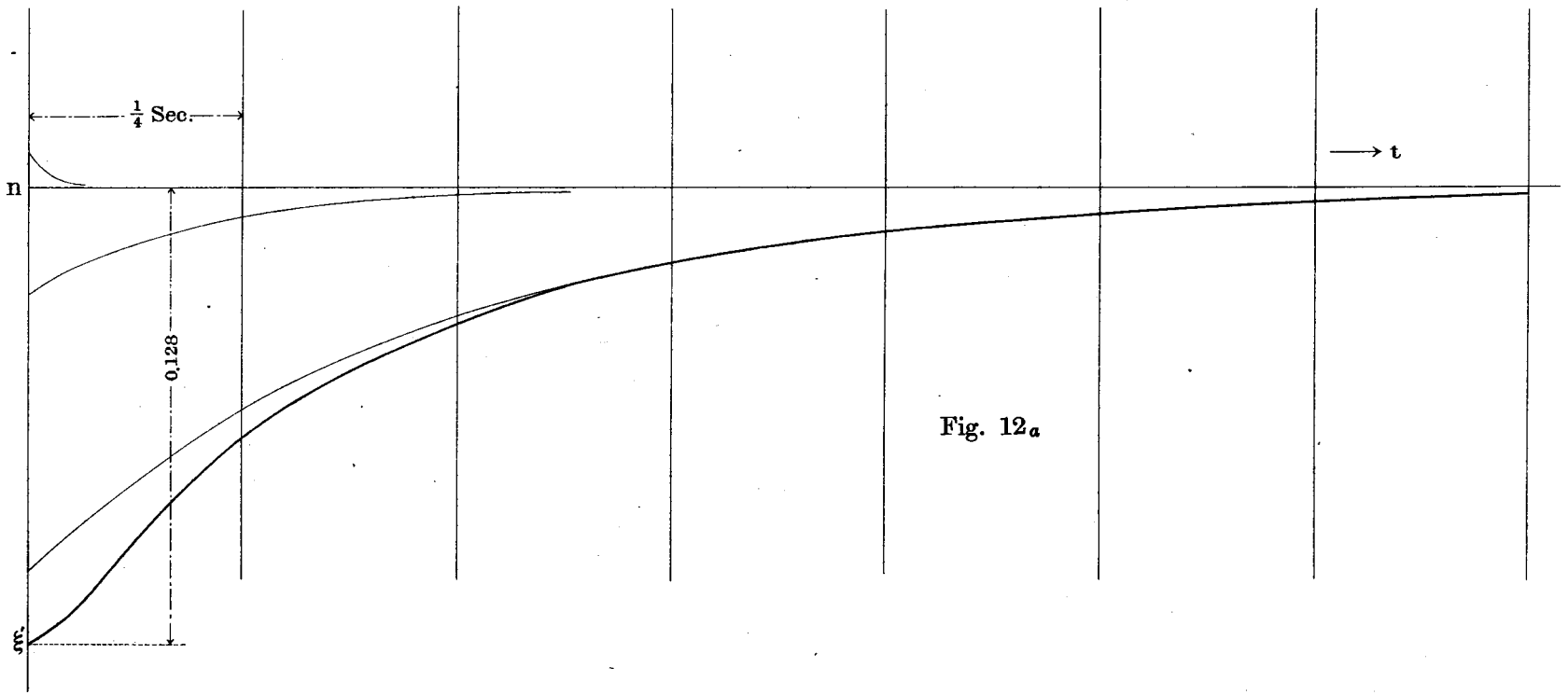


Fig. 12$_a$

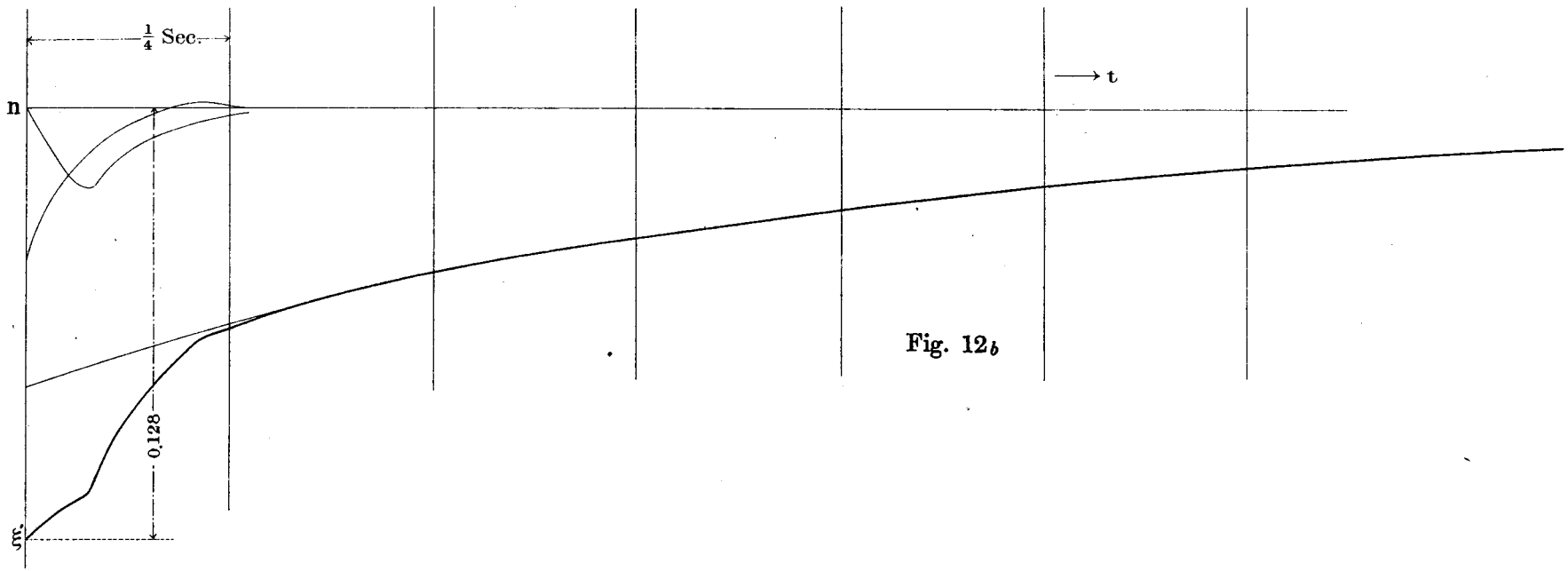


Fig. 12*b*

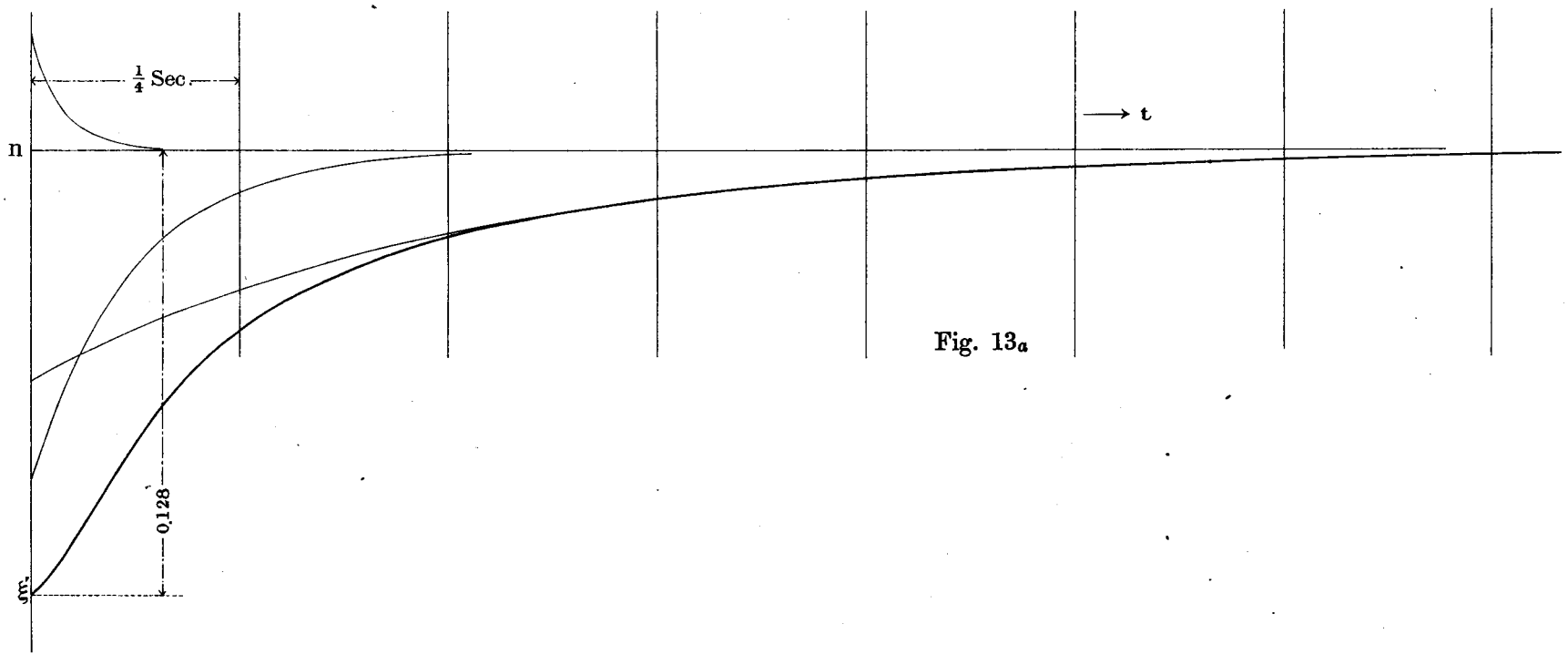


Fig. 13a

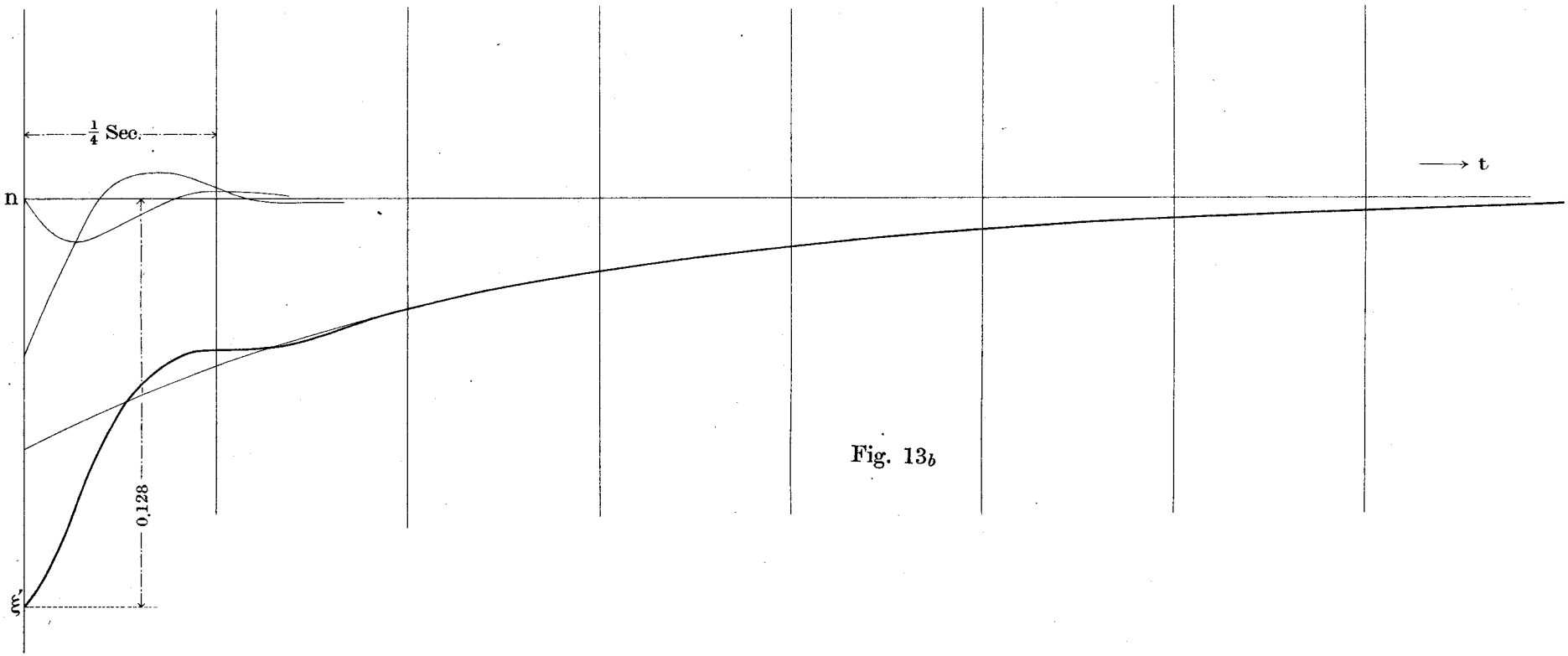


Fig. 13*b*

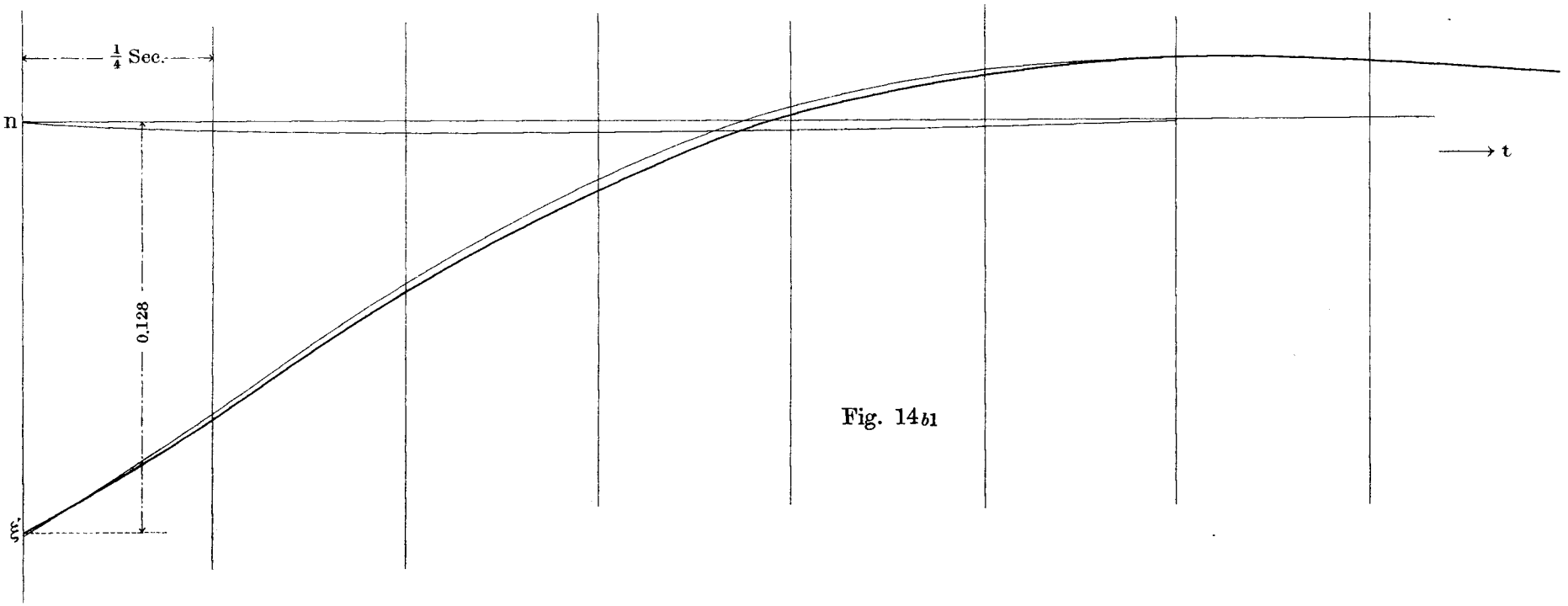


Fig. 14$_{b1}$

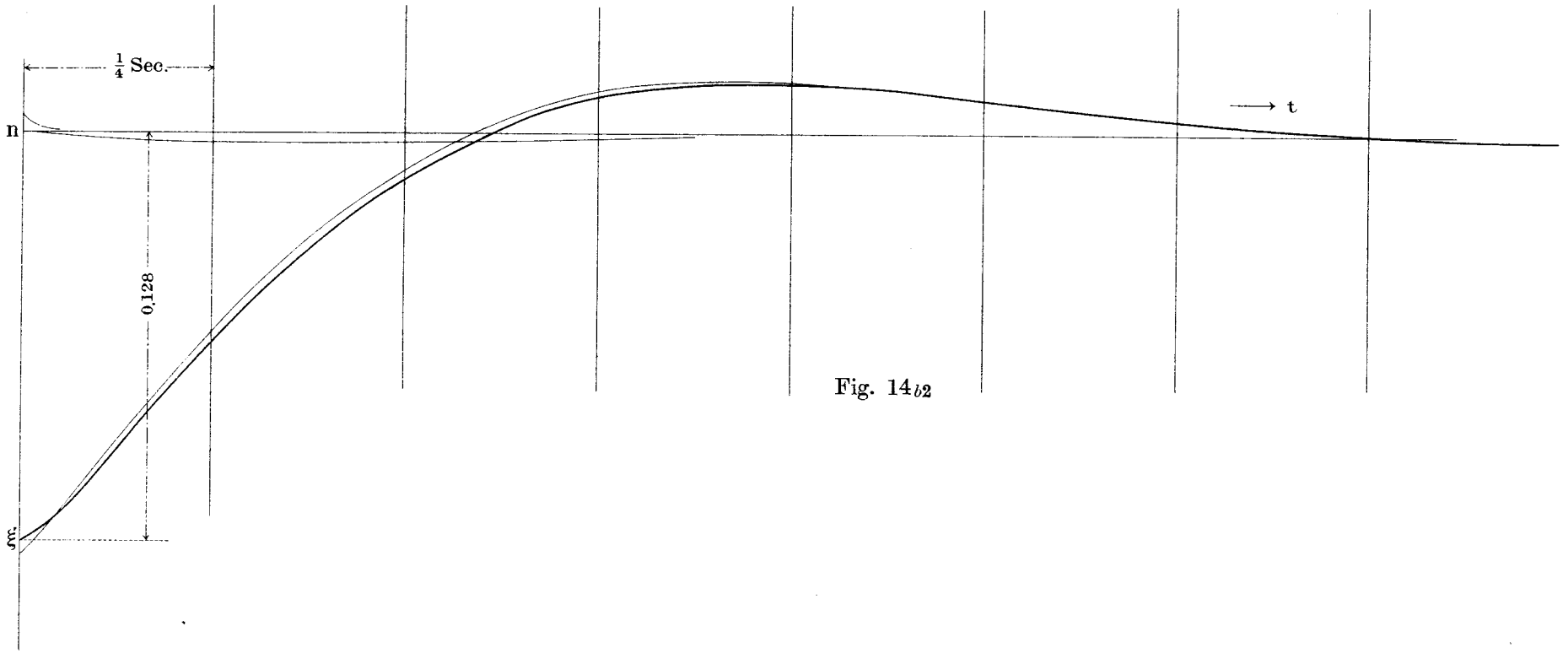


Fig. 14$_{b2}$

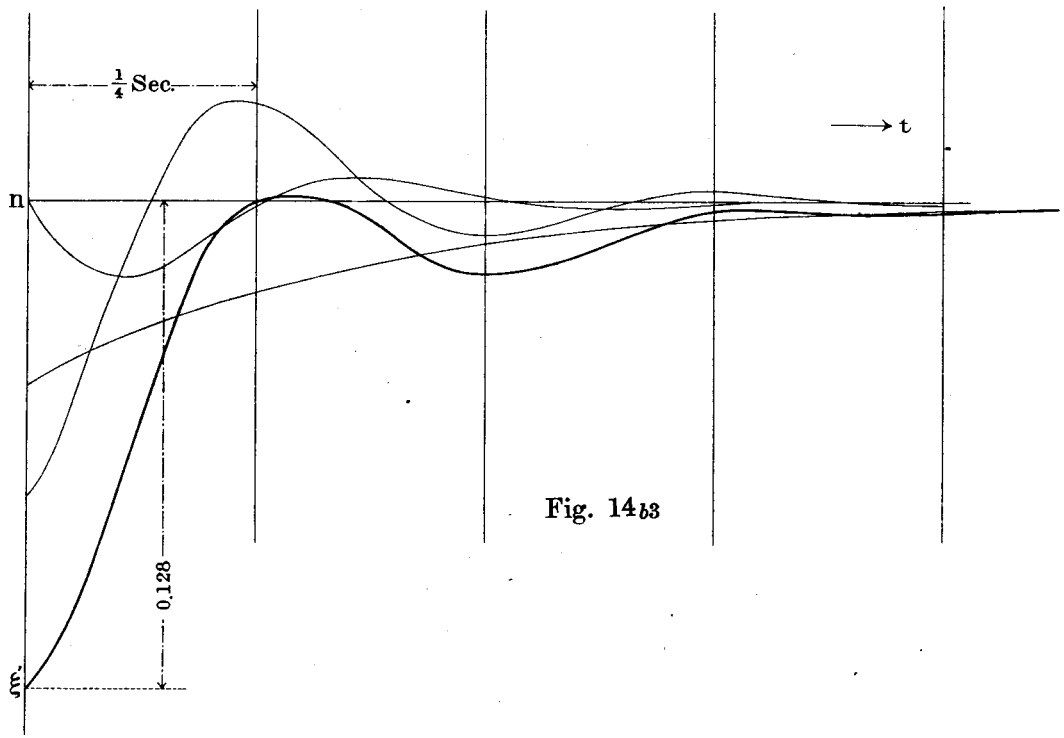


Fig. 14*b*3

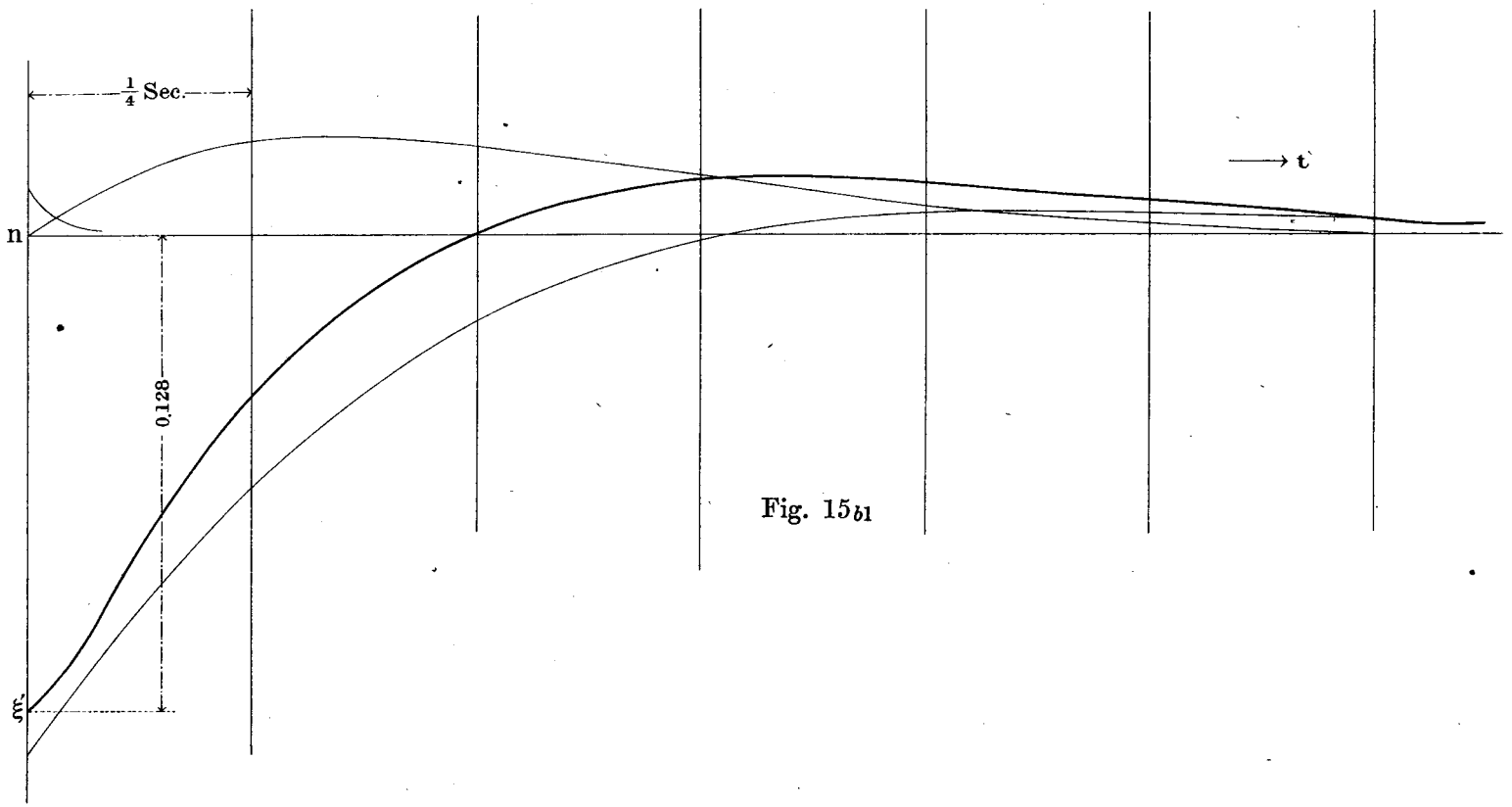


Fig. 15$_{b1}$

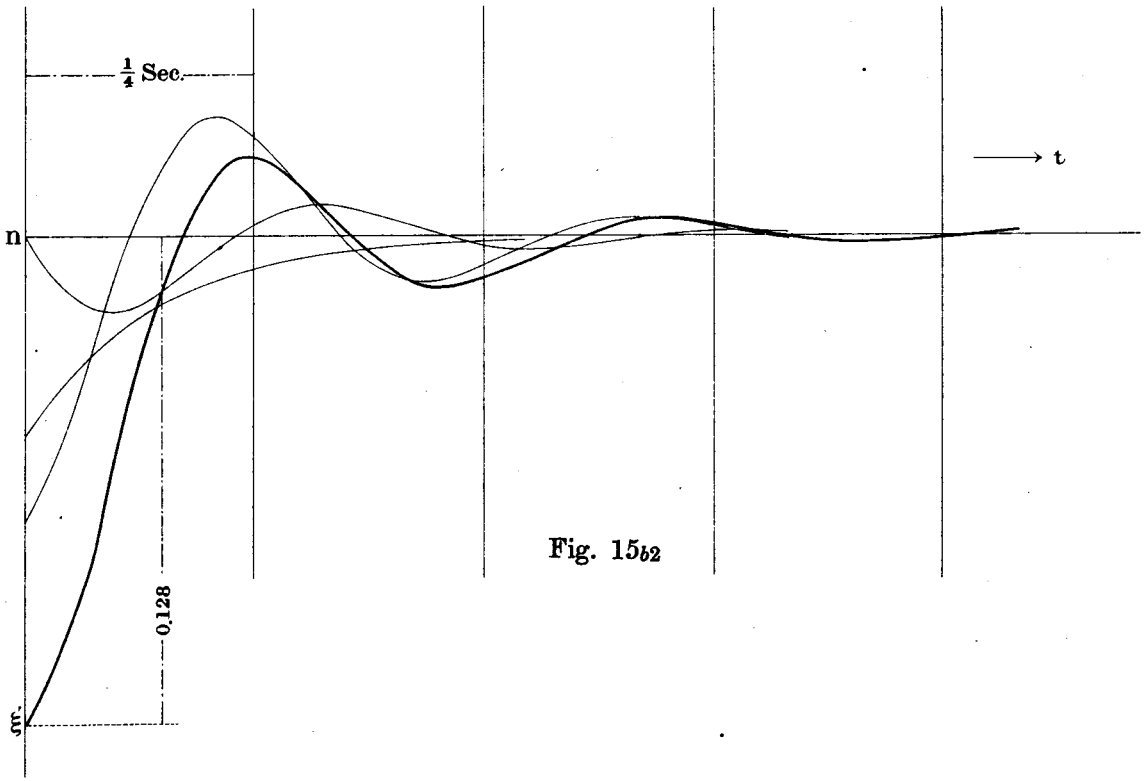


Fig. 15$_{b2}$

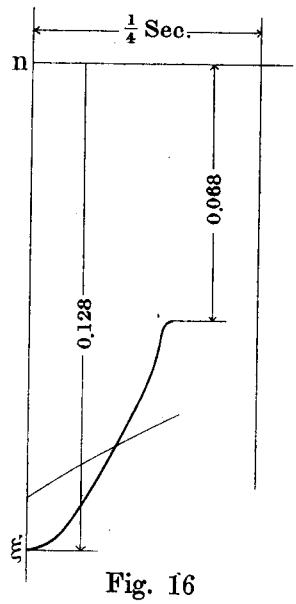


Fig. 16

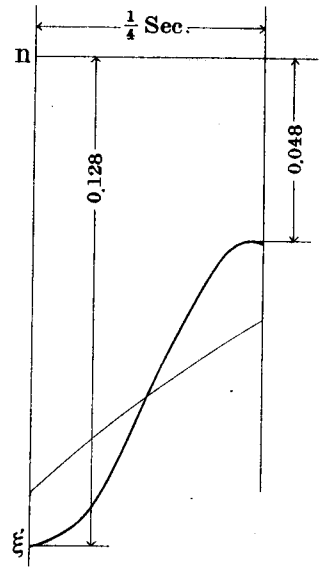


Fig. 17

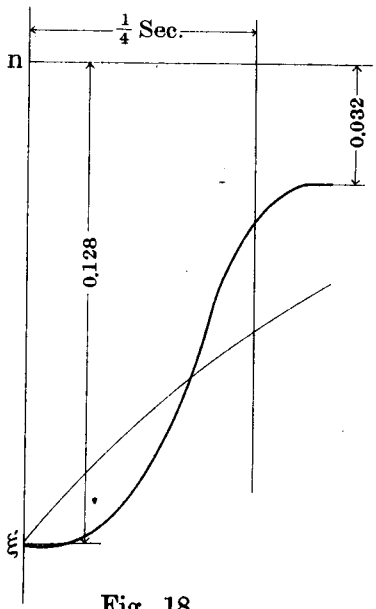


Fig. 18

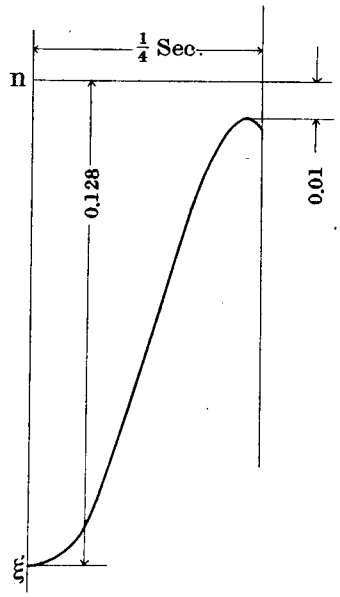


Fig. 18a