Second-Order Logic and Related Systems — a game-semantical perspective —

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Abstract

This is based on tutorial lectures on second-order logic in SAML 2022. Among others, we here discuss monadic second-order logic (MSO) from a game-theoretical view-point. Although the validity of MSO in terms of standard structures is not decidable (not axiomatizable), the MSO theory of full binary tree is decidable and modal μ-calculus can be viewed as a decidable fragment of MSO.

1 Game semantics

We start with an example of real analysis to explain the game semantics of first-order logic.

Example

"A real function $f(x)$ is continuous at $x = a$" can be expressed by first-order logic as follows.

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \Big(\underbrace{\mid x - a \mid < \delta}_{A} \rightarrow \underbrace{\mid f(x) - f(a) \mid < \varepsilon}_{B} \Big).$$

Consider a game where two players, Pro and Con, debate whether this formula is true or not. Pro asserts the truth of this formula, while Con tries to refute it. The game proceeds by evaluating the formula from left to right as follows:

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- The formula starts with $\forall \varepsilon > 0$. Then, Con chooses a value for ε with which the remaining subformula would be false.
- The remaining subformula starts with $\exists \delta > 0$, and so Pro determines a value of δ so that the subformula after $\exists \delta > 0$ holds with the chosen values of ε and δ.
- The rest subformula starts with $\forall x$, and Con selects a value for x.
- The left subformula is in the form $A \to B$, which can be rewriten as $(\neg A) \vee B$. Pro chooses either $\neg A$ or B.
- Finally, if Pro's last choice holds with the selected values, she wins the game. Otherwise, she loses.

If the last move should be carried out by Con and what he selects is true, Pro also wins. For the above formula in the example, if it is true, Pro will win as long as she plays correctly in each round, and vice versa. In other words, the truth of the logical formula can be determined by the existence of Pro's winning strategy.

Generally speaking, Pro and Con decompose the logical formula and select some elements from the domain according to some rules, and eventually Pro wins when a true atomic formula is chosen finally. A true proposition cannot be refuted if it is asserted correctly. The concept of game semantics can be applied not only to first-order logic, but also to various logics, and it is also an effective tool for examining the complexity of decision problems.

Next, we consider the game semantics of modal logic. The model we here deal with is a Kripke model $M = (W, R, v)$, where (W, R) is a transition system (directed graph) and $v(p)$ is the set of worlds (states, vertices) in W where an atomic proposition p holds. Two players, Pro and Con, proceed by moving a token on the game arena, consisting of the transition graph of a Kripke model and the subformulas of a given logical formula. For simplicity, we assume that the formulas are given in the negative normal form defined as follows.

$$\varphi ::= p \mid \neg p \mid \varphi \vee \varphi' \mid \varphi \wedge \varphi' \mid \Box \varphi \mid \Diamond \varphi.$$

Each position of the game arena is a pair (s, φ), where s is a vertex of the graph and φ a subformula of the given formula. The two players move the token on the arena obeying rules described in Table 1 below. In each position, the player who takes charge of this position has the right to select the next position and also move the token to it. Then, if one reaches a final position (s, p) (or $(s, \neg p)$) and

$s \in v(p)$ (or $s \notin v(p)$ respectively) holds, player Pro wins. Otherwise, if one cannot choose a next lawful move at a non-final position, this player will lose the game. This game is called a model checking game, and with a start position (s, φ), it is denoted as $\mathcal{E}(M, s, \varphi)$.

Table 1: Model checking rules for modal logic

Positions of Pro	possible choices for next position of Pro
$(s, \varphi \vee \varphi')$	$\{(s, \varphi), (s, \varphi')\}$
$(s, \Diamond\varphi)$	$\{(t, \varphi) \mid (s, t) \in R\}$
Positions of Con	**possible choices for next position of Con**
$(s, \varphi \wedge \varphi')$	$\{(s, \varphi), (s, \varphi')\}$
$(s, \Box\varphi)$	$\{(t, \varphi) \mid (s, t) \in R\}$

Then, we have the following theorem.

Theorem 1.1 (Adequacy Theorem of Modal Logic)**.** *The following are equivalent.*

- *Pro has a winning strategy in the model checking game $\mathcal{E}(M, s, \varphi)$.*
- *$M, s \models \varphi$.*

This can be easily proved by induction on the construction of φ. In fact, if φ is p, then Pro has a winning strategy in $\mathcal{E}(M, s, \varphi) \Leftrightarrow s \in v(p) \Leftrightarrow M, s \models \varphi$. Similarly for $\neg p$. If φ is $\psi \vee \psi'$, then

$$\begin{aligned}
&\text{Pro has a winning strategy in } \mathcal{E}(M, s, \varphi)\\
\Leftrightarrow\ &\text{Pro has a winning strategy in } \mathcal{E}(M, s, \psi) \text{ or } \mathcal{E}(M, s, \psi')\\
\Leftrightarrow\ &M, s \models \psi \text{ or } M, s \models \psi'\\
\Leftrightarrow\ &M, s \models \varphi.
\end{aligned}$$

If φ is $\Diamond\psi$, then

$$\begin{aligned}
&\text{Pro has a winning strategy in } \mathcal{E}(M, s, \varphi)\\
\Leftrightarrow\ &\text{Pro has a winning strategy in } \mathcal{E}(M, t, \psi) \text{ for some } t \in sR\\
\Leftrightarrow\ &M, t \models \psi \text{ for some } t \in sR\\
\Leftrightarrow\ &M, s \models \varphi.
\end{aligned}$$

Similarly we can prove the cases $\psi \wedge \psi'$ and $\Box\psi$.

2 Second-order logic

In **first-order logic** (FO), $\forall$ and $\exists$ quantify over the elements of a model, while in **second-order logic** (SO), they quantify over the relations and functions on a model. For simplicity, from now on, we deal only with quantification over relations, not functions.

Definition 2.1. *Consider the first-order language $\mathcal{L}$ and a n-ary relation symbol* R ($\notin \mathcal{L}$). *For a formula $\varphi(\mathrm{R}) \in \mathcal{L} \cup \{\mathrm{R}\}$, the truth of $\forall R\varphi(R)$ and $\exists R\varphi(R)$ in the structure $\mathcal{A}$ of $\mathcal{L}$ is defined as follows.*

$$\mathcal{A} \models \forall R\varphi(R) \Leftrightarrow \textit{for any } \dot{\mathrm{R}} \subseteq A^n, (\mathcal{A}, \dot{\mathrm{R}}) \models \varphi(\mathrm{R}) \textit{ holds.}$$
$$\mathcal{A} \models \exists R\varphi(R) \Leftrightarrow \textit{there exists } \dot{\mathrm{R}} \subseteq A^n \textit{such that } (\mathcal{A}, \dot{\mathrm{R}}) \models \varphi(\mathrm{R}).$$

In the following, we do not strictly distinguish among the relation variable R, relation $\dot{\mathrm{R}}$, and relation constant R. For simplicity, we often restrict second-order variables to unary relations, namely subsets of the domain. Such a logic is called **monadic second-order logic** (MSO), formalized in the language with first-order variables x, y, z, $\cdots$ ranging over the domain of a target structure, and second-order variables X, Y, Z, $\cdots$ ranging over the subsets of the first-order domain.

How to consider the domain of second-order variables? In the above definition, "any $\dot{\mathrm{R}} \subseteq A^n$" means that "all" subsets of A^n should be considered. A structure with such an interpretation is called a **standard structure** of second-order logic. However, second-order logic with standard structures cannot be formalized in an axiomatic system. In other words, we have the following.

Theorem 2.1 (Gödel)**.** *The validity of* (M)SO *in terms of standard structures is not axiomatizable, hence not decidable.*

Proof. Assume MSO were axiomatized. We can define second-order Peano Arithmetic PA_2 by adding arithmetic axioms to MSO. In any model of PA_2, since all subsets of the first-order domain M are in the second-order domain, then the minimum set containing 0 and closed under $+1$ exists in the second-order domain, which is isomorphic to $\mathbb{N}$. However, since induction holds in $\mathcal{M}$, the set containing 0 and closed under $+1$ must agree with the whole M. That is, $\mathcal{M}$ is isomorphic to $\mathbb{N}$. Therefore, the unique model for PA_2 is $\mathbb{N} \cup \mathcal{P}(\mathbb{N})$, which implies that there is no sentence independent from PA_2. This condradicts with Gödel's first incompleteness theorem. $\square$

Henkin instead considered a **general structure** of second-order logic, whose second-order part varies similarly to a first-order logic domain. In other words, such a logic can be regarded as two-sorted first-order logic. In particular, a general structure in MSO is defined as follows.

Definition 2.2. *A* **general structure** *of monadic second-order logic $\mathcal{B} = (\mathcal{A}, \mathcal{S})$ consists of first-order logic structure $\mathcal{A}$ and set $\mathcal{S} \subset \mathcal{P}(A)$. The set quantifiers run over $\mathcal{B}$ as follows.*

$$\mathcal{B} \models \forall X \varphi(X) \Leftrightarrow \textit{for any } S \in \mathcal{S}, \mathcal{B} \models \varphi(S) \textit{ holds},$$
$$\mathcal{B} \models \exists X \varphi(X) \Leftrightarrow \textit{there exists } S \in \mathcal{S} \textit{ such that } \mathcal{B} \models \varphi(S).$$

Henkin assumed that the general structure should satisfy certain amounts of comprehension axiom and axiom of choice. The comprehension axiom is an assertion that for a formula $\varphi(x)$ with no free occurrence of X, $\exists X \forall x (x \in X \leftrightarrow \varphi(x))$, that is, the set $\{x : \varphi(x)\}$ exists in the second-order domain.

Theorem 2.2 (Completeness theorem of MSO)**.** *An* MSO *formula is provable from appropriate comprehension and other axioms in two-sorted first-order system if and only if it is true in any general structure that satisfies those axioms.*

This theorem can be proved in the same way as in first-order logic. It can also be generalized to higher-order logics. In fact, Henkin's proof for the completeness theorem of first-order logic was made with such a generalization scheme.

Example: MSO is more expressive than FO

- FO cannot distinguish $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$. In MSO, it can express that "a bounded set $X (\neq \varnothing)$ has a least upper bound", and hence $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are distinguishable.

- MSO can express the sentence that determines the parity (even or odd) of the length of a finite linear order, which is not expressible by FO.

Example: SO is more expressive than MSO

The MSO theory of $(\mathbb{N}, x+1, 0)$ is decidable due to Büchi. But SO theory of $(\mathbb{N}, x+1, 0)$ is not, since addition $m + n = k$ is defined by

$$\forall R([R(0,m) \wedge \forall x, y(R(x,y) \to R(x+1, y+1))] \to R(n,k),$$

and multiplication can be defined in a similar way, which means that first-order arithmetic is embedded into the theory.

The relations between arithmetic theories are summarized as follows.

$$\begin{array}{ccccc} \mathsf{FO}(\mathbb{N}, S(x)) \subset & \mathsf{FO}(\mathbb{N}, S(x), +) & \subset & \mathsf{FO}(\mathbb{N}, S(x), +, \cdot) \\ & \Cap * & & \Cap \\ & \mathsf{MSO}(\mathbb{N}, S(x)) & \subset & \mathsf{MSO}(\mathbb{N}, S(x), +) \\ & & & \Cap \\ & & & \mathsf{SO}(\mathbb{N}, S(x)) \end{array}$$

Here, $S(x)$ denotes $x+1$, and $\mathsf{FO}(\mathbb{N}, S(x))$ is the FO theory of $(\mathbb{N}, S(x))$. Similarly for $\mathsf{MSO}(\mathbb{N}, S(x))$, etc. $A \subset B$ is the usual set inclusion, $A \Subset B$ a relation via a formula translation, $A \Subset^* B$ a formula translation with coding.

Finally, let us take a look at a brief history of MSO.

Decidability results

- $\mathsf{S1S} = \mathsf{MSO}(\mathbb{N}, S(x))$ is decidable.
  Büchi (1960)'s proof relied on ω-automata with a Büchi condition. That is, an infinite word is accepted if a final state appears infinitely many times when reading the input.
- $\mathsf{S2S} = \mathsf{MSO}(2^{<\omega}, x^\frown 0, x^\frown 1)$ is decidable.
  The proof utilized tree-automata (Rabin 1969), and later improved with the help of infinite games (Gurevich-Harrington 1982).

Definability results (Note that weak quantifiers range over finite sets only).

- For $(\mathbb{N}, S(x))$, weak monadic definability $=$ monadic definablity.
- For binary trees, weak monadic definability $=$ Büchi $\cap$ co-Büchi (Rabin 1970).
- For infinitely branching trees, weak monadic definability $\subsetneq$ Büchi $\cap$ co-Büchi, because WMSO, in which second-order quantifiers only range over finite sets, cannot distinguish an infinite path from infinitely many finite paths.

3 Logics

The essence of logic is the relation between sentences and models, "$\mathcal{A} \models_{\mathsf{S}} \varphi$". Now, by a **logic**, we mean a set S of sentences together with a function $\mathrm{Mod}_{\mathsf{S}}$ from S to the structures, satisfying certain conditions so that for each sentence $\varphi \in \mathsf{S}$, $\mathrm{Mod}_{\mathsf{S}}(\varphi)$ intends to represent $\{\mathcal{A} : \mathcal{A} \models_{\mathsf{S}} \varphi\}$. See [5] for details.

Logic S is said to be **weaker than** logic S' ($\mathsf{S} \leq \mathsf{S}'$) iff for any $\varphi \in \mathsf{S}$, there exists some $\varphi' \in \mathsf{S}'$ such that $\mathrm{Mod}_{\mathsf{S}}(\varphi) = \mathrm{Mod}_{S'}(\varphi')$. Obviously, $\mathsf{FO} \leq \mathsf{MSO} \leq \mathsf{SO}$.

We say that the (countable) **compactness theorem** holds for logic S iff for any countable $U \subset \mathsf{S}$, if $\bigcap\{\mathrm{Mod}_{\mathsf{S}}(\varphi) : \varphi \in U\} = \varnothing$, then there exists a finite $V \subset U$ such that $\bigcap\{\mathrm{Mod}_{\mathsf{S}}(\varphi) : \varphi \in V\} = \varnothing$.

We say that the (countable) **downward Löwenheim-Skolem theorem** (downward LS theorem) holds for logic S iff for any countable $U \subset \mathsf{S}$, if $\bigcap\{\mathrm{Mod}_{\mathsf{S}}(\varphi) : \varphi \in U\}$ contains an infinite structure $\mathcal{A}$, then it a countably infinite structure $\mathcal{B}$.

It is well-known that the compactness theorem and the downward LS theorem hold for FO, but they fail for MSO and SO. Surprisingly, Lindström has shown that FO is the strongest logic that satisfies both the compactness theorem and the downward LS theorem. We sketch the proof briefly in the following.

First, we consider a language of finitely many relational symbols and constants, without functional symbols (other than constants). Let $\mathcal{L}$ be $\{\mathrm{R}_0, \ldots, \mathrm{R}_{n-1}\}$, and consider its extensions by adding constants. The structure $\mathcal{A}$ in $\mathcal{L}$ can be expressed:

$$\mathcal{A} = (A, \mathrm{R}_0^{\mathcal{A}}, ..., \mathrm{R}_{n-1}^{\mathcal{A}}).$$

Then, for any $B \subset A$, we set

$$\mathcal{A}{\restriction}B = (B, \mathrm{R}_0^{\mathcal{A}} \cap B^{k_0}, \ldots, \mathrm{R}_{n-1}^{\mathcal{A}} \cap B^{k_{n-1}}).$$

Furthermore, for $\vec{a} = (a_1, \cdots, a_k)$ of A, naming them with constants $\vec{c}$, we define a structure $(\mathcal{A}, \vec{a})$ in $\mathcal{L} \cup \{\vec{c}\}$.

Definition 3.1 (Quantifier Rank)**.** *For a formula φ, the* **quantifier rank** *of φ, denoted as* $\mathrm{qr}(\varphi)$*, is defined recursively as follow,*

$$\mathrm{qr}(\textit{atomic formula}) = 0,$$
$$\mathrm{qr}(\neg\varphi) = \mathrm{qr}(\varphi), \qquad \mathrm{qr}(\varphi \wedge \psi) = \max\{\mathrm{qr}(\varphi), \mathrm{qr}(\psi)\},$$
$$\mathrm{qr}(\forall x\varphi) = \mathrm{qr}(\exists x\varphi) = \mathrm{qr}(\varphi) + 1.$$

Definition 3.2. *A set of sentences that hold in the structure $\mathcal{A}$ in $\mathcal{L}$ is called the* **theory** *of $\mathcal{A}$, represented by* $\mathrm{Th}(\mathcal{A})$*. Two structures with the same theory are said to be* **elementary equivalent***, denoted by $\mathcal{A} \equiv \mathcal{B}$. That is,*

$$\mathcal{A} \equiv \mathcal{B} \quad \Leftrightarrow \quad \mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B}) \quad \Leftrightarrow \quad \mathcal{B} \models \mathrm{Th}(\mathcal{A}).$$

Definition 3.3. *Let* $\mathrm{Th}_n(\mathcal{A})$ *be the subset of* $\mathrm{Th}(\mathcal{A})$ *consisting of sentences with rank n or below. For structures $\mathcal{A}, \mathcal{B}$ in the same language $\mathcal{L}$, a relation $\equiv_n$ between them is defined as follows.*

$$\mathcal{A} \equiv_n \mathcal{B} \Leftrightarrow \mathrm{Th}_n(\mathcal{A}) = \mathrm{Th}_n(\mathcal{B}).$$

Definition 3.4. *Let $\mathcal{A}, \mathcal{B}$ be structures in $\mathcal{L}$. A partial function $f : A \to B$ is a* **partial isomorphism** *if $\mathcal{A} \upharpoonright \mathrm{dom(f)}$ and $\mathcal{B} \upharpoonright \mathrm{range(f)}$ are isomorphic via f.*

If $\mathrm{dom}(f) = \vec{a}$, then the above definition is equivalent to

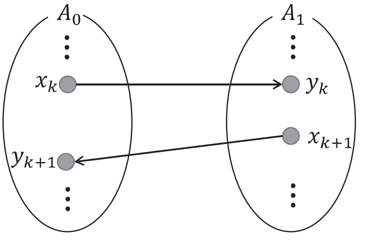
$$(\mathcal{A}, \vec{a}) \equiv_0 (\mathcal{B}, f(\vec{a})).$$

It is obvious that "if $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{A} \equiv \mathcal{B}$". Fraïssé showed a weak version of its reversal by using quantifier ranks. Ehrenfeucht reformulated Fraïssé's argument from a view of games. Now such a technique is referred to as Ehrenfeucht-Fraïssé game (EF game).

Definition 3.5. *Let $\mathcal{A}_0$, $\mathcal{A}_1$ be structures of $\mathcal{L}$ and n be a natural number. In an n-round* **EF game**, $\mathrm{EF}_n(\mathcal{A}_0, \mathcal{A}_1)$*,* player I (Spoiler) *and* player II (Duplicator) *alternately choose from A_i ($i = 0, 1$) following the rules described below, and the winner is determined according to the winning condition.*

- **Rules***: if* I *chooses $x_i \in A_j$ ($j = 0, 1$),* II *chooses $y_i \in A_{1-j}$.*

- **Winning conditions***: If the correspondence $x_i \leftrightarrow y_i$ chosen by the players up to n rounds determines a partial isomorphism of $\mathcal{A}_0$ and $\mathcal{A}_1$, then* II *wins.*



Example

Consider $\mathrm{EF}_3(\mathcal{A}, \mathcal{B})$ where $\mathcal{A} = (\mathbb{Z}, <), \mathcal{B} = (\mathbb{R}, <)$. In the following, $e \in \mathbb{R} \to 2 \in \mathbb{Z}$ represents that player I selects $e \in \mathbb{R}$ and then player II chooses $2 \in \mathbb{Z}$. For example, if $e \in \mathbb{R} \to 2 \in \mathbb{Z}$, $0 \in \mathbb{Z} \to 0 \in \mathbb{R}$ and $\pi \in \mathbb{R} \to 5 \in \mathbb{Z}$ are produced in the game, player II wins because $\{(0,0), (2,e), (5,\pi)\}$ is a partial isomorphism (order preserving).

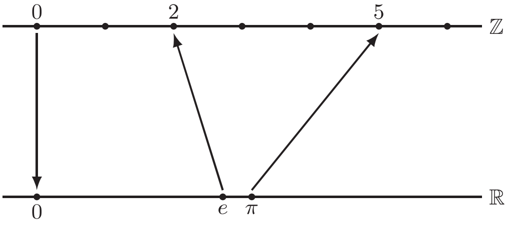


Figure: $\mathrm{EF}_3((\mathbb{Z}, <), (\mathbb{R}, <))$.

Definition 3.6. *$\mathcal{A} \simeq^n \mathcal{B}$ if player* II *has a winning strategy in* $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$ *.*

Note that if $\mathcal{A} \simeq^n \mathcal{B}$ then $\mathcal{B} \simeq^n \mathcal{A}$. We can easily show the following lemma.

Lemma 3.1.

$$\begin{aligned}
(\mathcal{A}, \vec{a}) \simeq^0 (\mathcal{B}, \vec{b}) &\Leftrightarrow \ \vec{a} \mapsto \vec{b} \text{ is partial isomorphism.} \\
&\Leftrightarrow \ (\mathcal{A}, \vec{a}) \equiv_0 (\mathcal{B}, \vec{b}). \\
(\mathcal{A}, \vec{a}) \simeq^{n+1} (\mathcal{B}, \vec{b}) &\Leftrightarrow \ \forall a \in A \exists b \in B (\mathcal{A}, \vec{a}a) \simeq^n (\mathcal{B}, \vec{b}b) \text{ and} \\
&\qquad \forall b \in B \exists a \in A (\mathcal{A}, \vec{a}a) \simeq^n (\mathcal{B}, \vec{b}b)
\end{aligned}$$

Then, essentially by induction, we can prove the following main theorem.

Theorem 3.2 (EF Theorem)**.** *For all $n \geq 0$, $(\mathcal{A}, \vec{a}) \simeq^n (\mathcal{B}, \vec{b}) \Leftrightarrow (\mathcal{A}, \vec{a}) \equiv_n (\mathcal{B}, \vec{b})$.*

There are some corollaries.

Corollary 3.3. *$\mathcal{A} \equiv \mathcal{B} \Leftrightarrow$ for any n, $\mathcal{A} \simeq^n \mathcal{B}$.*

It is natural to extend the play of the EF game to infinity (ω-round), denoted as $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$. We write $\mathcal{A} \simeq^\omega \mathcal{B}$ if player II has a winning strategy in $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$.

Corollary 3.4. *Suppose $\mathcal{A}, \mathcal{B}$ are countable. Then, $\mathcal{A} \simeq^\omega \mathcal{B} \ \Leftrightarrow \ \mathcal{A} \simeq \mathcal{B}$.*

Proof. $\Leftarrow$ is obvious because the isomorphism is a winning strategy for player II. To show $\Rightarrow$, let $A = \{a_0, a_1, \dots\}, B = \{b_0, b_1, \dots\}$. Player II follows the winning strategy, and I alternately chooses the smallest element that have not been selected from A and B, thus a bijection between $\mathcal{A}$ and $\mathcal{B}$ is produced, which is a desired isomorphism. □

Corollary 3.5. *Let K be a set of structures of $\mathcal{L}$. The following are equivalent.*

(1) *For any n, there exist $\mathcal{A} \in K$ and $\mathcal{B} \notin K$ such that $\mathcal{A} \equiv_n \mathcal{B}$.*

(2) *K is not an elementary class (K cannot be defined by a first-order formula).*

Proof. (1) $\Rightarrow$(2). By way of contradiction, assume K is defined by a first-order sentence φ. Let n be the rank of φ. If $\mathcal{A} \in K$ and $\mathcal{B} \notin K$ then $\mathcal{A} \not\equiv_n \mathcal{B}$.

(2) $\Rightarrow$(1). Assume the contrary that for some n, if $\mathcal{A} \equiv_n \mathcal{B}$ then $\mathcal{A} \in K \Leftrightarrow \mathcal{B} \in K$. Since the language consists of finitely many relational symbols, there is a first-order (Scott-Hintikka) sentence $\varphi^n_\mathcal{A}$ of rank n such that $\mathcal{A} \equiv_n \mathcal{C} \Leftrightarrow \mathcal{C} \models \varphi^n_\mathcal{A}$. Thus, K is defined by $\varphi^n_\mathcal{A}$. □

Now, we are ready to show the following theorem.

Theorem 3.6 (Lindström's theorem)**.** *For logic* S *such that* $\mathsf{FO} \leq \mathsf{S}$*, the following are equivalent.*

(1) *Compactness theorem and downward* LS *theorem holds for* S*.*

(2) $\mathsf{S} \leq \mathsf{FO}$.

Proof. (2) $\Rightarrow$ (1) is obvious.

To show (1) $\Rightarrow$ (2), assume $\mathsf{S} \leq \mathsf{FO}$ does not hold. There exists some $\varphi \in \mathsf{S}$ such that $\mathrm{Mod}_{\mathsf{S}}(\varphi)$ is not defined by a first-order sentence. That is, for any $n \in \omega$, there exist $\mathcal{A} \in \mathrm{Mod}_{\mathsf{S}}(\varphi)$ and $\mathcal{B} \in \mathrm{Mod}_{\mathsf{S}}(\neg\varphi)$ such that $\mathcal{A} \equiv_n \mathcal{B}$, or equivalently $\mathcal{A} \simeq^n \mathcal{B}$ by the EF theorem. We express this condition as a logical expression θ_n of S for each n (so that $\theta_{n+1} \to \theta_n$). Namely, $(\mathcal{A}, \mathcal{B}, \sigma) \models_{\mathsf{S}} \theta_n$ means that "$\mathcal{A} \models_{\mathsf{S}} \varphi$ and $\mathcal{B} \models_{\mathsf{S}} \neg\varphi$ and σ is player II's winning strategy in $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$".

Since this holds for all $n \in \omega$, by the compactness theorem, $(\mathcal{A}, \mathcal{B}, \sigma) \models_{\mathsf{S}} \{\theta_n : n \in \omega\}$ holds, and thus σ is a winning strategy in $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$. Moreover, $(\mathcal{A}, \mathcal{B}, \sigma)$ can be selected countable by downward LS theorem. Therefore, $\mathcal{A}, \mathcal{B}$ are isomorphic, which contradicts with $\mathcal{A} \in \mathrm{Mod}_{\mathsf{S}}(\varphi)$ and $\mathcal{B} \in \mathrm{Mod}_{\mathsf{S}}(\neg\varphi)$. Thus $\mathsf{S} \leq \mathsf{FO}$. $\square$

Examples of logic

Infinite logic $\mathcal{L}_{\omega_1,\omega}$: allowing countable disjunctions and conjunctions, but quantifying only over a finite number of variables.

$\mathsf{FO}(Q_1)$: adding the quantifier Q_1 to the first-order logic. $Q_1 x\ \varphi(x)$ means "there are uncountably many x that satisfy $\varphi(x)$".

WMSO: Second-order quantifiers range over finite sets only.

Table 2: The compactness and downward LS property for various logic

Logic	Compactness	Downward LS property
FO	○	○
WMSO	×	○
MSO, SO	×	×
$\mathsf{FO}(Q_1)$	○	×
$\mathcal{L}_{\omega_1,\omega}$	×	○

4 Modal logic: a bisimulation-invariant of FO

Modal logic is an extension of propositional logic with modal operators such as

- $\Box\, p$ expresses "$\forall$ next move (world), p holds."
- $\Diamond\, p$ expresses "$\exists$ next move (world), p holds."

A Kripke model $M = (W, R, v)$ consists of a directed graph (W, R) and a valuation $v : \{p_i : i < n\} \to \mathcal{P}(W)$ such that $v(p_i)$ is the set of worlds (states) in which an atomic proposition p_i holds. Denoting $P_i := v(p_i)$, M can be treated as a first-order relational structure $M' = (W, R, P_0, P_1, \dots)$. Then, a modal formula φ on M, s can be translated into a first-order formula $ST_s(\varphi)$ on M' as follows.

Definition 4.1. *For a modal formula φ, its* **standard translation** $ST_x(\varphi)$ *is defined as follows:*

$$ST_x(p_i) := P_i(x), \quad ST_x(\neg\varphi) := \neg ST_x(\varphi),$$
$$ST_x(\varphi \vee \psi) := ST_x(\varphi) \vee ST_x(\psi), \quad ST_x(\varphi \wedge \psi) := ST_x(\varphi) \wedge ST_x(\psi),$$
$$ST_x(\Box\varphi) := \forall y(R(x, y) \to ST_y(\varphi)), \quad ST_x(\Diamond\varphi) := \exists y(R(x, y) \wedge ST_y(\varphi)).$$

It is easy to show the following by induction on the formula:

(1) $M, s \models \varphi \Leftrightarrow M' \models ST_s(\varphi)$,

(2) $M \models \varphi \Leftrightarrow M' \models \forall x ST_x(\varphi)$.

Then, we can also translate many results on first-order logic to those on modal logic such as compactness theorem and downward Löwenheim-Skolem theorem.

There are many variations of modal logic, which can be also translated into first-order logic. For instance, multi-modal logic can be translated into first-order logic with many relations $R_i(x, y)$ almost in the same way. Modal predicate logic (with constant domain) can be translated into two-sorted first-order logic.

The directed graph $F = (W, R)$ under a Kripke model (W, R, v) is often called its **frame**. The validity of a formula φ in a frame F is defined in monadic second-order as follows:

$$F \models \varphi \Leftrightarrow \forall v\ (F, v) \models \varphi \Leftrightarrow\ F \models \forall \vec{P} \forall x ST_x(\varphi).$$

Since this can not be defined in first-order logic, the class of frames for some modal logic (e.g. GL) does not satisfy the compactness.

In first-order logic, Ehrenfeucht-Fraïssé game connects the concepts of elementary equivalence to isomorphism. In modal logic, this corresponds to the idea of "bisimulation".

Definition 4.2 (Bisimulation)**.** *Let $M = (W, R, v), M' = (W', R', v')$ be Kripke models. $Z \subset W \times W'$ is a* **bisimulation** *between M and M' if the following holds.*

(0) *$Z \neq \emptyset$.*

(1) *sZs', then for any $p \in P$, $M, s \models p \Leftrightarrow M', s' \models p$.*

(2) *If sZs' and sRt, then there is t' such that $s'R't'$ and tZt' (forth condition).*

(3) *If sZs' and $s'R't'$, then there is t such that sRt and tZt' (back condition).*

If there exists a bisimulation Z between M and M' such that sZs', we write $M, s \underline{\leftrightarrow} M', s'$.

The following is an example of (maximum) bisimulation Z between M and M'.

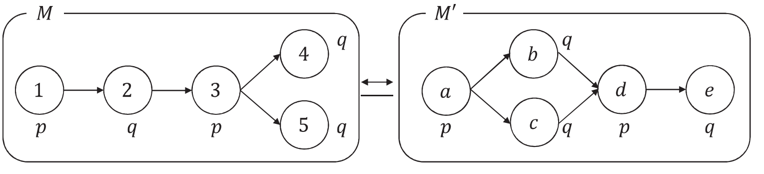


Figure 1: $Z = \{(1, a), (2, b), (2, c), (3, d), (4, e), (5, e)\}$

Definition 4.3 (Modally equivalence)**.** *Let M and M' be Kripke models. M, s and M', s' are* **modally equivalent**, *denoted $M, s \equiv M', s'$, if for all modal formulas φ, $M, s \models \varphi \Leftrightarrow M', s' \models \varphi$.*

Theorem 4.1 (Bisimulation invariant theorem)**.** *If $M, s \underline{\leftrightarrow} M', s'$, then $M, s \equiv M', s'$.*

Proof. We assume $M, s \underline{\leftrightarrow} M', s'$, and then want to show $M, s \equiv M', s'$, *i.e.*, for all formula φ, $M, s \models \varphi \Leftrightarrow M', s' \models \varphi$. We prove this by induction on the construction of modal formula φ. The case $\varphi = \Box\psi$ is only essential to treat. Suppose $M, s \models \Box\psi$, and we want to show $M', s' \models \Box\psi$. So, we will show that for any $t' \in s'R'$, $M', s' \models \psi$. $M, s \underline{\leftrightarrow} M', s'$ gives sZs', so by the backward condition, there is t such that sRt and tZt'. By $M, s \models \Box\psi$ and sRt, we have $M, t \models \psi$. Since tZt', $M, t \underline{\leftrightarrow} M', t'$, it follows that $M', t' \models \psi$ from the induction hypothesis. $\Box$

The converse of the above theorem does not hold in general. However, there are some special classes of Kripke models where the converse of the theorem also hold, which is called the **Hennessy-Milner property**.

Now, we say that $M = (W, R, v)$ is a finite branching model if sR is a finite set for any $s \in W$.

Theorem 4.2. *The class of finite branching models has the Hennessy-Milner property.*

Proof. Assume M, M' are finite branching and $M, s \equiv M', s'$. Let Z be the set of pairs (w, w') such that $M, w \equiv M', w'$. It is obvious that $Z \neq \varnothing$. Condition (1) of Definition 4.2 can be obtained from $M, s \equiv M', s'$. To prove (2) of Definition 4.2, suppose sZs' and sRt. Since $M, s \models \neg\Box\bot$ by sRt, we have $M', s' \models \neg\Box\bot$ and so there is t' such that $s'R't'$. Since there are only a finite number of such t' due to the finite branch property, and so we list them as $t'_1, t'_2, \ldots, t'_n$.

Suppose to the contrary that for all $i \leq n$, not tZt'_i. Then for each $i \leq n$, there is ψ_i such that $M, t \models \psi_i$ and $M', t'_i \models \neg\psi_i$. So $M, t \models \bigwedge_i \psi_i$ and $M', s' \models \Box \bigvee_i \neg\psi_i$. Then, $M, s \models \Box \bigvee_i \neg\psi_i$ by sZs'. So, $M, t \models \bigvee_i \neg\psi_i$ by sRt, which contradicts $M, t \models \bigwedge_i \psi_i$. Therefore, for some $i \leq n$, we have tZt'_i. This complete the proof for (2). Similarly, we can prove for (3). $\Box$

To generalize the above theorem, we introduce the concept of "modal saturation" as follows.

Definition 4.4 (Modal saturation)**.** *Let $M = (W, R, v)$ be a Kripke model. We say that M is* **modally saturated** *if for any set Σ of modal formulas, any $s \in W$ and any finite subset $\Sigma' \subset \Sigma$ satisfying $M, s \models \Diamond \bigwedge \Sigma'$, then there exists some $t \in sR$ such that $M, t \models \Sigma$ (i.e., $M, t \models \varphi$ for all $\varphi \in \Sigma$).*

Lemma 4.3. *Let $M = (W, R, v)$ be a countable Kripke model. There exists a modally saturated elementary extension $M^* \supset M$.*

Proof. Consider M as countable model of first-order logic. Let M^* be the ultrapower $M^I/\mathcal{U}$ by non-principle ultrafilter $\mathcal{U}$ on infinite set I. M^* is an elementary extension of M and satisfies countable saturation. $\Box$

Lemma 4.4. *The class of modal saturation models has the Hennessy-Milner properties.*

Proof. Suppose $M, s \equiv M', s'$ and M, M' are modally saturated. Let Z be the set of pairs (w, w') such that $M, w \equiv M', w'$. From the assumption, $Z \neq \varnothing$ is clear. Also, (1) of Definition 4.2 can be obtained immediately from the modal equivalence. To show (2), assume sZs' and sRt. Let $\Sigma := \{\varphi : M, t \models \varphi\}$. Then, for any finite subset $\Sigma' \subset \Sigma$, $M, s \models \Diamond \bigwedge \Sigma'$. By $M, s \equiv M', s'$, $M', s' \models \Diamond \bigwedge \Sigma'$.

Since M' is modally saturated, there exists some $t' \in s'R'$ such that $M', t' \models \Sigma$. Therefore, $M, t \equiv M', t'$, so $(t, t') \in Z$, which completes the proof for (2). (3) of Definition 4.2 can be proved similarly. $\square$

Theorem 4.5 (Modal invariant theorem)**.** *Over Kripke models, for a first-order formula $\varphi(x)$, the following are equivalent.*

(1) *It is equivalent to the standard translation of a modal formula.*
(2) *It is invariant with respect to bisimulation.*

Proof. (1) $\Rightarrow$ (2) can be proved by Theorem 4.1.

To show (2) $\Rightarrow$(1), we assume that $\varphi(x)$ is invariant with respect to bisimulation. For simplicity, let $\psi(x)$ denote (the meta-variable of) a first-order formula which is the standard translation of a modal formula. Let MC (φ) denote the set of all $\psi(x)$ such that $\forall M \forall s (M, s \models \varphi(x) \Rightarrow M, s \models \psi(x))$ (MC stands for Modal Consequence).

Assuming $M, s \models \mathrm{MC}(\varphi)$, we show $M, s \models \varphi$. By the compactness of first-order logic, there exists a finite subset S of $\mathrm{MC}(\varphi)$ such that $\bigwedge S \rightarrow \varphi$ holds and $\varphi \rightarrow \bigwedge S$ is clearer from the definition.

By the compactness theorem of first-order logic, there exists N, t for a countable model of $T(x) \cup \{\varphi(x)\}$. Now, since N, t satisfies $T(x)$, they are modally equivalent to M, s. By the Lemma 4.3, for M, s and N, t, we define elementary extensions M^*, s and N^*, t which are modally saturated, respectively. Since M, s and N, t are modally equivalent, so are M^*, s and N^*, t. By Lemma 4.4, there is a bisimulation between M^*, s and N^*, t.

Now, $N, t \models \varphi(x)$, and N^*, t is an elementary extension of N, t, so $N^*, t \models \varphi(x)$. Then, by (2), $M^*, s \models \varphi(x)$ and M^*, s is an elementary extension of M, s, and thus $M, s \models \varphi(x)$, which completes for the proof. $\square$

Finally, we mention the following theorem with no explanation. To state the theorem properly, we need to define "abstract modal logic" extending the basic modal logic. For details, see [1] (Section 25.3) and [2] (Section 7.6).

Theorem 4.6 (Modal Lindström theorem)**.** *Modal logic is the strongest logic in which the compactness theorem and the bisimulation invariant theorem hold.*

Now it is natural to consider what is the bisimulation-invariant MSO logic. The answer is modal μ-calculus, which we will discussed in the next section.

5 Modal μ-calculus

In modal logic, "φ holds after n steps" can be expressed as $\Diamond^n\varphi$ by n copies of $\Diamond$, but to express "at some point φ holds" an infinite formula $\varphi \vee \Diamond\varphi \vee \Diamond^2\varphi \vee \cdots$ might be needed. In modal μ-calculus, we define this infinite-long expression as the least fixed point of $x \leftrightarrow \varphi \vee \Diamond x$, which is represented by $\mu x.\varphi \vee \Diamond x$. In addition, modal μ-calculus has the largest fixed point operator νx, too.

In a Kripke model $M = (W, R, v)$, v is a mapping from the atomic propositions to the power set of W. This function v can be extended as a mapping from the general modal proposition φ to the power set of W such that $V(\varphi) = \{s \in W : M, s \models \varphi\}$. Now, consider $V(\mu x.\varphi \vee \Diamond x)$. As if we regard x as an atomic proposition and $v(x)$ is given, $V(\varphi \vee \Diamond x)$ is obtained. For instance, if $v(x) = \varnothing$, then $V(\varphi \vee \Diamond x) = V(\varphi)$. Then, let $v(x) = V(\varphi)$ and we obtain $V(\varphi \vee \Diamond x) = V(\varphi \vee \Diamond\varphi)$. Moreover, if $v(x) = V(\varphi \vee \Diamond\varphi)$, then $V(\varphi \vee \Diamond x) = V(\varphi \vee \Diamond\varphi \vee \Diamond\Diamond\varphi)$. Hence, if $v(x)$ is expanded as

$$\varnothing \subseteq V(\varphi) \subseteq V(\varphi \vee \Diamond\varphi) \subseteq \cdots,$$

then $V(\varphi \vee \Diamond x)$ is expanded as

$$V(\varphi) \subseteq V(\varphi \vee \Diamond\varphi) \subseteq V(\varphi \vee \Diamond\varphi \vee \Diamond\Diamond\varphi) \subseteq \cdots.$$

Since the limits of the two infinite sequences coincide, we obtain a fixed point of $x \leftrightarrow \varphi \vee \Diamond x$, which is defined as $V(\mu x.\varphi \vee \Diamond x)$.

To guarantee that such a fixed point exists, we assume that the variable x associated with μx must appear positively within its scope. Therefore, the formulas of modal μ-calculus L_μ are defined as follows:

$$\varphi ::= p \mid \neg p \mid x \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \mu x.\varphi \mid \nu x.\varphi,$$

where p is an atomic proposition and x is a variable. The negation $\neg$ is only attached to atomic propositions. But for convenience, we use the following negation rules to expand the L_μ formulas.

$$\neg\neg\varphi \equiv \varphi, \quad \neg\Box\varphi \equiv \Diamond\neg\varphi, \quad \neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi), \ \neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi),$$
$$\neg\mu x.\varphi \equiv \nu x.\neg\varphi[\neg x/x].$$

Notice that $\varphi[\neg x/x]$ is obtained by replacing all free occurrences of x in φ with $\neg x$. For example, $\neg\mu x.(p \vee \Diamond x) \equiv \nu x.\neg(p \vee \Diamond\neg x) \equiv \nu x.(\neg p \wedge \Box x)$. A general L_μ formula φ should be identified as a strict L_μ formula which is equivalent to φ by the negation rules.

The truth value function $V(\varphi) = \{s : M, s \models \varphi\}$ for an L_μ formula φ is defined by induction on the construction of φ as usual. We only treat the cases that φ is of the form $\mu x.\theta(x)$ and $\nu x.\theta(x)$. In the following, we also write $||\varphi||^M$ for $V(\varphi)$.

If we regard x as an atomic proposition and put $V(x) = X(\subset W)$, then we obtain $V(\theta(x))$ as a monotonic increasing function $\Psi(X)$ of X. We also denote $V(\theta(x))$ as $||\theta(x)||^M_{x:=X}$. Since $\Psi(X)$ is a monotonic increasing function of X, the least fixed point $||\mu x.\theta(x)||^M := \bigcap\{X : \Psi(X) \subseteq X\}$ and the largest fixed point $||\nu x.\theta(x)||^M := \bigcup\{X : \Psi(X) \supseteq X\}$ exist.

An L_μ formula that holds in any state of any relational structure is said to be **valid**. The formal system of modal μ-calculus was introduced by D. Kosen in1983 and the completeness theorem was proved by I. Walkiewicz in 1995 (the paper published in 2000). However, the proof is very difficult, and lots of attempts have been made to improve the understanding since then. In the following, we will briefly introduce a **game-semantical** view.

A model checking game $\mathcal{E}(M, s, \varphi)$ for modal μ-calculus is a simple extension of game semantics for modal logic in Section 1. Here, we only treat the fixed point operators. For convenience, we distinguish different formuls $\eta x.\varphi$ and $\eta x'.\varphi'$ with different bound variables x, x' ($\eta = \mu$ or ν). In addition to the rules in Table 1, the following rules are also considered.

Table 3: Extra model checking rules for modal μ-calculus in addition to Table 1

Positions	possible choices for next position
$(s, \mu x.\theta)$	(s, θ)
$(s, \nu x.\theta)$	(s, θ)
(t, x) where x is a μ or ν bounded variable	$(t, \mu x.\theta)$ or $(t', \mu x.\theta)$

A play ρ is a sequence of positions in the game. Then the winning conditions for model checking games of modal μ-calculus are given as follows.

Table 4: Winning conditions

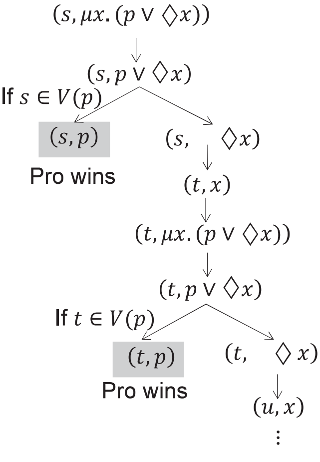
	Pro wins	Con wins
ρ is finite	the same as modal logic in Section 1	the same as modal logic in Section 1
ρ is infinite	the outermost subformula visited infinite many times is of the form $\nu x.\varphi$	the outermost subformula visited infinite many times is of the form $\mu x.\varphi$

Example

The following statements are equivalent.
(a) $M, s \models \mu x.p \vee \Diamond x$.
(b) In the graph of M, if we start from s, we will eventually reach a state where p holds.
(c) Pro has a winning strategy in the game $\mathcal{E}(M, s, \mu x.p \vee \Diamond x)$.

To show $(a) \Leftrightarrow (b)$.
Since $\mu x.p \vee \Diamond x$ is the least fixed point of $x \leftrightarrow p \vee \Diamond x$, $V(\mu x.p \vee \Diamond x) = \bigcup_n V(\Diamond^{<n} p)$. Therefore, if $s \in V(\mu x.p \vee \Diamond x)$, the state of p can be reached from s, and vice versa.

Next, look at $(b) \Rightarrow (c)$. The initial position is $(s, \mu x.p \vee \Diamond x)$, then it automatically moves to $(s, p \vee \Diamond x)$. It is Pro's turn to choose: if $s \in v(p)$, Pro wins by choosing (s, p). Otherwise, Pro selects $(s, \Diamond x)$. It is Pro's turn again to select t that is connected from s ($(s, t) \in R$), and the next position is (t, x).



From (t, x), the game automatically goes to $(t, \mu x.p \vee \Diamond x)$ and $(t, p \vee \Diamond x)$. At this time, if $t \in v(p)$, Pro wins by choosing (t, p). Otherwise, she selects u such that $(t, u) \in R$. Since it is Pro who chooses the successor of t, it is possible to reach some state where p holds by the assumption (b), and this is a Pro's winning strategy.

Finally, we show $(c) \Rightarrow (b)$. Assume that Pro has a winning strategy and she plchoose each position following this strategy. If the game does not stop, $\mu x.p \vee \Diamond x$ appears infinitely many times, then Con wins, which conflicts the assumption that Pro follows his winning strategy. Therefore, the play reaches p in finite steps.

In general, we have the adequacy theorem.

Theorem 5.1 (Adequacy Theorem of modal μ-calculus)**.** *The following are equivalent.*

- *Pro has a winning strategy in the model checking game* $\mathcal{E}(M, s, \varphi)$.
- $M, s \models \varphi$.

To show the adequacy theorem, the following facts are usefull.

(1) If $M, s \models \varphi$ then Pro has a **memoryless** winning strategy in the model checking game $\mathcal{E}(M, s, \varphi)$.

(2) If $M, s \not\models \varphi$ then Con has a **memoryless** winning strategy in the model checking game $\mathcal{E}(M, s, \varphi)$.

In a memoryless winning strategy, the player's next move only depends on the current position, regardless of the history of the game. The model checking game can be represented as a "parity game", so that (if M is computable) it is decidable whether one of the two players has a winning strategy. Furthermore, since the determinacy of this game can be expressed by monadic second-order logic $\mathsf{S2S}$, we can derive the decidability of modal μ-calculus from Rabin's theorem. For a proof of Rabin's theorem and the role of parity games, see [10]. The modal μ-calculus has finite model property, which also gives decidability, but the usual filter method does not apply to prove it. See references [3], [4], [6] for overview of the research. Kashima [9] is a textbook containing a chapter on modal μ-calculus.

The standard translation of modal μ-calculus (into MSO) is obtained from that of modal logic (into FO) by adding

$$ST_s(\mu x.\varphi) := \forall X(\forall y((ST_s(\varphi) \to y \in X) \to y \in X)),$$

then modal μ-calculus can be expressed by MSO. Then, we have

Theorem 5.2. *Modal μ-calculus is bisimulation-invariant* MSO *logic.*

The proof is similar to that for modal logic (Theorem 4.5) with induction on the number of μ/ν-operations. For example, if $M, s \models \mu x.\varphi \vee \Diamond x$, then for some n, $M, s \models \Diamond^n \varphi$. Then, from the induction hypothesis, we get $M', s' \models \Diamond^n \varphi$ and $M', s' \models \mu x.\varphi \vee \Diamond x$. Therefore, modal μ-calculus is bisimulation-invariant MSO logic.

The converse is also true, but the proof of Theorem 4.5 cannot be applied. Because it strongly depends on the compactness theorem of FO, and the compactness theorem fails in MSO (Table 2). Janin-Walukiewicz (1996) cleverly managed this by constructing automata. However, it is undecidable whether a given MSO formula is bisimulation-invariant, that is, whether it can be written by modal μ-calculus.

Pacheco et al. [11] and subsequent papers [12, 13] investigate the collapse of the μ alternation hierarchy in modal μ-calculus for some class of restricted relational structures (for example, weakly transitive frames).

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