Execution performance under specific price models*

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Abstract

In this paper we give a discussion of how off-exchange transaction fees change due to price reversion, taking into account two pricing models. Institutional investors can easily profit by manipulating prices on the exchange and then trading against them on the off-exchange because they influence prices through large executions. From this point, we obtain the transaction fee for the off-exchange trading so that the profit is zero by using off-exchange trading. Specifically, the transaction fee for the off-exchange will be defined as the profit generated by round-trip trading, which takes into account the use of optimal execution strategies for exchange trades and the off-exchange trade.

1 Introduction

In recent years, we have to pay more attention to fraudulent trades, due to the diversification of securities trading venues and flourishing of automated trading. By using various trading venues for the security execution, it is possible in principle to manipulate the price in one trading venue and to execute at an advantageous price in the other trading venue. Such trading, often called gaming, is unintentionally prohibited by the law, but the complex execution process blurs the lines. Therefore, rather than prohibiting it by law, it is considered more effective to remove the opportunity to earn profits and eliminate the incentive to engage in illegal tradings.

This paper considers how execution costs change due to the resilience of how prices that have changed due to large-scale execution by institutional investors will return. As for the price impact, we assume a linear, that is, block-shaped LOB, and consider a linear (constant) decay price model and an exponential decay price model based on [11]. However, the specific parameters of both price model are adjusted under the TWAP cost equivalent condition based on [8] where the costs in both models are the same when TWAP trading strategy is used. Then, we numerically obtain the execution cost when using the optimal execution strategy for round-trip trade in both models, and compare the execution performance.

The remainder of this paper is organized as follows. Section 2 gives two pricing models, the permanent model and the transient model, and their optimal execution strategies. In Section 3, we give definitions of the cost and show price manipulation. There are various definitions of price manipulation, but here we define *Pure-Price Manipulation* based on [5] Then we show the relationship between the two price models based on [8]. In Section 4, we use each pricing model

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with numerical examples to construct the cost of round-trip trading and the commission level when using off-exchange trading. Section5 concludes the paper.

2 Price models and optimal executions

Suppose that p_t^i is the price of a single risky asset at time t, q_t is the large trader's execution volume. Q_t is the number of shares which the large trader remains to purchase, if $Q_t > 0$(or liquidate, if $Q_t < 0$). That is,

$$Q_{t+1} = Q_t - q_t. \tag{2.1}$$

Moreover, w_t^i is a wealth at time t. The large trader submits a large amount of her market order q_t at time t just after she has recognized the price p_t^i at that time. Although the order is executed immediately, the execution price may not be equal to p_t^i. The executed price $\hat{p}_t^i$ will be instantly lifted upward from p_t^i because of the temporary imbalance of supply and demand. Assume that λ_t denote to the price change per share (called price impact), the dynamics of w_t^i and $\hat{p}_t^i$ are,

$$w_{t+1}^i = w_t^i - \hat{p}_t^i q_t, \tag{2.2}$$

$$\hat{p}_t^i = p_t^i + \lambda_t q_t. \tag{2.3}$$

More generally, by [4],

$$p_t^i = p_0^i + \int_0^t \mu_s ds + \int_0^t f^i(\dot{x}_s) G^i(t-s) ds + \int_0^t \sigma dB_s, \tag{2.4}$$

where f is the price impact function at time s, G is the decay factor, and i represents permanent price model ($i = pe$) and transient price model ($i = tr$). Moreover, $\int_0^t \mu_s ds$ represents the drift term and $\int_0^t \sigma dB_s$ represents the random term. The changed price (execution price) by the large order reverts to previous price level to a certain extent since temporary imbalance of supply and demand on the order book moves to new equilibrium with progress of the time. We introduce following two price models regarding the way how to revert the execution price. One is the permanent price model which was mainly introduced by [6], the other is the transient price model considered by [4] and [11].

2.1 Price dynamics

Permanent price model: In the permanent price model, the execution price is diminished instantly to the permanent impact level and the expected price is maintained until the next trading time. That is,

$$p_{t+1}^{pe} = \alpha_t p_t^{pe} + (1-\alpha_t)\hat{p}_t^{pe} + \epsilon_{t+1}. \tag{2.5}$$

Using equation (2.3) and (2.4),

$$p_{t+1}^{pe} = p_t^{pe} + (1-\alpha_t)\lambda_t + \epsilon_{t+1}, \tag{2.6}$$

where α_t represents the deterministic reversion rate of price and follows $0 \leq \alpha_t \leq 1$. ϵ_{t+1} represents the public news effect to the fundamental price between time t and $t+1$ and is recognized by the large trader at time $t+1$. Further, $\{\epsilon_t\}_t$ are $i.i.d$ random variables on a probability space $(\Omega, \mathcal{F}, P)$ as follows,

$$\epsilon_t \sim N(0, \sigma_\epsilon^2). \tag{2.7}$$

All information available to the large trader before her trading at time t are,

$$\mathcal{F}_t := \sigma\{(\epsilon_{s+1}) : s = 1, \ldots, t-1\}. \tag{2.8}$$

In the permanent price model, the price impact, the temporary impact and the permanent impact are represented respectively λ_t, $(1-\alpha_t)\lambda_t$ and $\alpha_t\lambda_t$. In this model, the price reversion is constant. Applying the general model,

$$f^{pe}(\dot{x}_s) = \lambda \dot{x}_s, \tag{2.9}$$
$$G^{pe}(t-s) = \alpha_{t-s}\lambda_{t-s} = \alpha\lambda. \tag{2.10}$$

Transient price model: In the transient price model, on the other hand, it is the same as the permanent price model until the submitted order is executed. However the price recurrence is not immediate but gradual to permanent level. We set the time independent rate ρ as the resilience speed. Then,

$$p_t^{tr} = p_t^0 + \sum_{k=1}^{t-1} \lambda_k e^{-\rho(t-k)} q_k, \tag{2.11}$$

where p^0 denotes the fundamental price and $p_{t+1}^0 - p_t^0 =: \epsilon_{t+1}$, the same as (2.6) and (2.7). Furthermore, by the equation(2.8),

$$p_{t+1}^{tr} = p_t^{tr} + \lambda_t e^{-\rho} q_t - S_t + \epsilon_{t+1}. \tag{2.12}$$

Here, we define S as,

$$S_t := e^{-\rho t}\left(1 - e^{-\rho}\right) \sum_{k=1}^{t-1} \lambda_k e^{-\rho(t-k)} q_k = l_{t-1} q_{t-1} + e^{-\rho} S_{t-1}, \tag{2.13}$$

where,

$$l_t := \lambda_t (1 - e^{-\rho}) e^{-\rho}. \tag{2.14}$$

In this transient price model, price impact and transient impact are λ_t and $\lambda_t e^{-\rho(t-k)}$. On the other hand, temporary and permanent impact are both 0. See [8] for more details on the economic interpretation of S_t. Similarly to the permanent price model, when applied more generally to the [4] model,

$$f^{tr}(\dot{x}_s) = \lambda \dot{x}_s, \tag{2.15}$$
$$G^{tr}(t-s) = e^{-\rho(t-s)}. \tag{2.16}$$

2.2 optimal execution

In both two price models, a deterministic execution strategy becomes optimal by [8]. It can be seen that the transient price model leaves execution volume in later periods because of price reversion.

Permanent price model: When we use the permanent price model, the optimal execution volume at time t denoting $q_t^{'*}$ is represented as an affine function of the remaining execution volume Q_t at the time. Then the optimal execution volume and the optimal value function are,

$$q_t^{'*} = \frac{D_t^{'} Q_t}{2C_t^{'}} \quad (= \beta_t^{'} Q_t), \tag{2.17}$$

where,

$$\left(\begin{array}{l} C_t^{'} := \alpha_t \lambda_t + \frac{R\sigma_\epsilon^2}{2} + A_{t+1}^{'} \\ D_t^{'} := -(1-\alpha_t)\lambda_t + R\sigma_\epsilon^2 + 2A_{t+1}^{'}, \end{array} \right. \tag{2.18}$$

and

$$A_t^{'} := A_{t+1}^{'} + \frac{R\sigma_\epsilon^2}{2} - \frac{D_t^{'2}}{4C_t^{'}}. \tag{2.19}$$

Transient price model: When we use the transient price model, the optimal execution volume of large trader at time t denoting q_t^{*tr} is represented as the function of the remaining execution volume Q_t and the cumulative effect of past executions S_t at the time. Then the optimal execution volume and the corresponding optimal value function are,

$$q_t^* = \frac{D_t Q_t - L_t S_t}{2C_t} \tag{2.20}$$

where,

$$\left(\begin{array}{l} C_t := l_t e^{\rho} + \frac{R\sigma_\epsilon^2}{2} + A_{t+1} - B_{t+1} l_t - K_{t+1} l_t^2 \\ D_t := -\lambda_t e^{-\rho} + R\sigma_\epsilon^2 + 2A_{t+1} - B_{t+1} l_t \\ L_t := 1 - B_{t+1} e^{-\rho} - 2K_{t+1} l_t e^{-\rho}, \end{array} \right. \tag{2.21}$$

and,

$$\left(\begin{array}{l} A_t := A_{t+1} + \frac{R\sigma_\epsilon^2}{2} - \frac{D_t^2}{4C_t} \\ B_t := B_{t+1} e^{-\rho} - 1 + \frac{D_t L_t}{2C_t} \\ K_t := K_{t+1} e^{-2\rho} + \frac{L_t^2}{4C_t}. \end{array} \right. \tag{2.22}$$

Figure 1 shows the optimal execution strategy for the transient price model and the permanent price model. Here, $Q_0 = 50000, R = 0.001$, and $\lambda = 0.0005$. The intrinsic factors are set to $\rho = 0.5$ and $\alpha = 0.739$, respectively. The reason for setting α in this way will be discussed in Section 3 below.

3 Execution cost and price manipulation

3.1 Execution cost

Let the set of static execution strategies be $\Pi = \{x_t; t \in [0,T]\}$, and the expected execution cost $C[\Pi]$ by the IS method is

$$C[\Pi] = E\left[\int_0^T \dot{x}_t (P_t - P_0) dt\right] \tag{3.1}$$

$$= \int_0^T \int_0^t \dot{x}_t f(\dot{x}_s) G(t-s) ds dt. \tag{3.2}$$

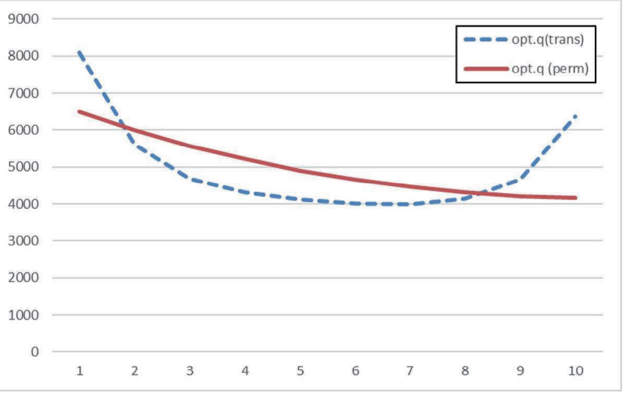

Figure 1: Optimal execution volume for both two price models

Also, for discrete T times executions, we get

$$C[\Pi] = \sum_{t=1}^{T} q_t \sum_{s=1}^{t} f(q_s)G(t-s). \tag{3.3}$$

Since the impact of all past executions is reflected in the price, it is not easy to derive a general solution, and verification by numerical calculation is often performed.

3.2 Price Manipulation

Here, we introduce the definition of price manipulation.

Definition 1 (Round-Trip Trading Strategy) *For the trading strategy* $\{x_t; t \in [0,T]\}$, *a trading strategy which satisfies the following equation is called a round-trip trade,*

$$\int_0^T \dot{x}_t dt = 0, \tag{3.4}$$

where $\dot{x}$ *represents the trade rate, which is the time derivative. More simply, for discrete execution volumes* ${}^T\boldsymbol{Q} = (q_1, q_2, \cdots, q_T), t = 1, 2, \cdots, T$,

$${}^T\boldsymbol{1Q} = \boldsymbol{0}, \tag{3.5}$$

where $\boldsymbol{1}$ *represents the unit vector, and* ${}^T\boldsymbol{1}$ *represents the transposition of the unit vector.*

A round-trip trade is a execution strategy in which the sum of all executed trade volumes during the execution period is 0.

Definition 2 (Pure Price manipulation [5]**)** *For a round-trip trading strategy* $x_t; t \in [0,T]$ *such that,*

$$E\left[\sum_{t=1}^{T} \hat{p}_t q_t\right] < 0. \tag{3.6}$$

is called pure price manipulation strategy.

The above definition of price manipulation is not for the act of execution itself (market manipulation), but for what appears as a result of execution. In [5], they showed if the permanent impact is linear in terms of the execution volume, then the market is absent of price manipulation in the risk neutral sense. Our control for the risk averse large trader describes that when we apply the round trip trade, 0 trade is always optimal.

3.3 relationship between two intrinsic parameters

For the two price models, if the expected costs derived from these two price models respectively with the same execution volume at the same intervals are different from each other, an arbitrage opportunity may occur. We need to give α and ρ a proper relationship so that there is no arbitrage opportunity. Therefore, in [8], when the TWAP strategy is used for each price model, the relationship of the intrinsic parameters is defined as follows.

Definition 3 (TWAP Cost Equivalent) *If $E[C_{pe}] = E[C_{tr}]$, then we say the market is TWAP cost equivalent.*

The total expected costs for transient price model and permanent price model using the TWAP strategy are respectively,

$$\begin{aligned} E[C_{tr}] &= E\left[\sum_{t=1}^{T} \hat{p}_t^{tr} q\right] && (3.7)\\ &= E\left[\sum_{t=1}^{T}\left(p_1^0 + \sum_{i=1}^{t-1}\sum_{k=1}^{i-1} S_k + \sum_{k=2}^{t}\epsilon_k + \lambda_q\right) q\right] && (3.8)\\ &= Tp_1^0 q + T\lambda q^2 + \frac{T\lambda q^2 e^{-\rho}}{1-e^{-\rho}} - \frac{\lambda q^2 e^{-\rho} - e^{-\rho(T+1)})}{(1-e^{-\rho})^2}, && (3.9) \end{aligned}$$

$$\begin{aligned} E[C_{pe}] &= E\left[\sum_{t=1}^{T} \hat{p}_t^{pe} q\right] && (3.10)\\ &= E\left[\sum_{t=1}^{T}\left(p_1^0 + (t-1)(1-\alpha)\lambda + \sum_{k=2}^{t}\epsilon_k + \lambda q\right) q\right] && (3.11)\\ &= Tp_1^0 q + T\lambda q^2 + \lambda q^2(1-\alpha)\frac{T(T-1)}{2}. && (3.12) \end{aligned}$$

Then if the market is TWAP cost equivalent, then the following condition holds:

$$\alpha = 1 - \frac{2e^{-\rho}}{(T-1)(1-e^{-\rho})} + \frac{2(e^{-\rho} - e^{-\rho(T+1)})}{T(T-1)(1-e^{-\rho})^2}. \tag{3.13}$$

4 Execution cost under round-trip trade

In this section, we consider two types of round-trip trading strategies using the optimal execution strategy for the pricing model defined in the previous section. Then, through round-trip trading

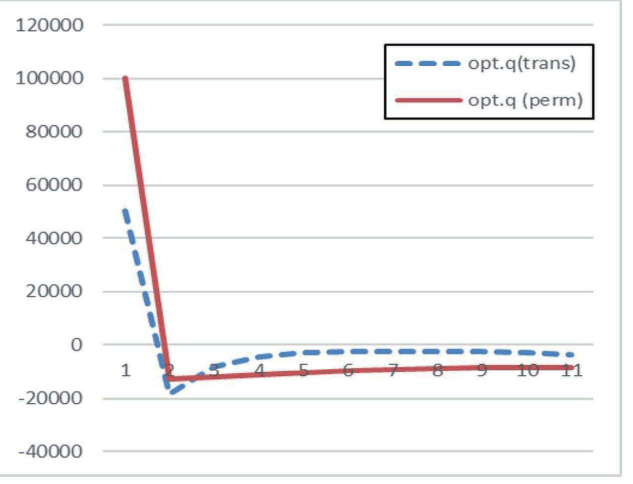

Figure 2: Optimal round-trip trade

that takes off-exchange trades into account, we present a numerical example of transaction fee on off-exchange trading.

4.1 Round-trip trade only on a single exchange

In [9] they consider a round-trip trade in which they first buy a large quantity on the exchange and then sell their purchases in small equal amounts, that is TWAP round trip strategy. If we buy everything first, and then trade TWAP round-trip trade using only in the exchange, the cost per execution volume increases monotonically and does not have a negative expected cost of execution. In other words, price manipulation becomes impossible in such executions. We consider buying all X shares first and selling an equal amount k times. Then,

$$C[\Pi^{TWAP}] = (p_0 + f(X))X - \left(p_0 + f(X)G(1) - f\left(\frac{X}{k}\right)G(0)\right)\frac{X}{k} \tag{4.1}$$

$$- \left(p_0 + f(X)G(2) - f\left(\frac{X}{k}\right)G(1) - f\left(\frac{X}{k}\right)G(0)\right)\frac{X}{k} - \cdots \tag{4.2}$$

$$- \left(p_0 + f(X)G(k) - f\left(\frac{X}{k}\right)G(k-1) - \cdots - f\left(\frac{X}{k}\right)G(0)\right)\frac{X}{k} \tag{4.3}$$

$$= X \ln X - \sum_{j=1}^{k}\left(\frac{X}{k}f(X)G(j) + \frac{X}{k}f\left(\frac{X}{k}\right)G(j-1)(k-j+1)\right). \tag{4.4}$$

For the optimal execution strategy,

$$C[\Pi^{opt}] = (p_0 + f(X))X - (p_0 + f(X)G(1) - f(q_1^*)G(0))\, q_1^* \tag{4.5}$$

$$- (p_0 + f(X)G(2) - f(q_1^*)G(1) - f(q_2^*)G(0))\, q_2^* - \cdots \tag{4.6}$$

$$- (p_0 + f(X)G(k) - f(q_1^*)G(k-1) - \cdots - f(q_k^*)G(0))\, q_k^*. \tag{4.7}$$

Figure 2 illustrates the amount of execution when 50,000 shares are initially purchased and then sold using the optimal execution strategy. Same as Figure 1, $R = 0.001$, $\lambda = 0.0005$, $\rho = 0.5$, and $\alpha = 0.739$. Moreover, initial price is 1,000. The costs in each pricing model are respectively,

$$C[\Pi_{pe}^{opt}] = (1,000 + 25)50,000 - \sum_{j=1}^{10} \hat{p}_j^{pe} q_j^{'*} = 1,201,498, \quad (4.8)$$

$$C[\Pi_{tr}^{opt}] = (1,000 + 25)50,000 - \sum_{j=1}^{10} \hat{p}_j^{pe} q_j^{'*} = 1,215,580. \quad (4.9)$$

The costs are equal when the TWAP strategy is used, but when the optimal execution strategy is used, the transient price model is more costly due to imperfect price reversion at the time of delayed execution. If an institutional trader buys 50000 shares using the optimal execution strategy over 10 periods and then sells them all in the 11th period,

$$C[\Pi_{pe}^{opt}] = \sum_{j=1}^{10} \hat{p}_j^{pe} q_j^{'*} - 981.40 \times 50,000 = 1,204,060, \quad (4.10)$$

$$C[\Pi_{tr}^{opt}] = \sum_{j=1}^{10} \hat{p}_j^{pe} q_j^{'*} - 979.78 \times 50,000 = 1,279,444. \quad (4.11)$$

In this case, the transient price model is also more costly.

4.2 Round-trip trade only on a single exchange

We consider that making a contract with a sell side broker that manages off-exchanges to refer to the price before the 11th period on the exchange in off-exchange trades and sell at that fixed price. The institutional trader then pay the fee and sells. Then,

$$C[\Pi_{pe}^{opt}] = \sum_{j=1}^{10} \hat{p}_j^{pe} q_j^{'*} - 1006.40 \times 50,000 = -45,939, \quad (4.12)$$

$$C[\Pi_{tr}^{opt}] = \sum_{j=1}^{10} \hat{p}_j^{pe} q_j^{'*} - 1004.79 \times 50,000 = 29,445. \quad (4.13)$$

The cost of using the permanent price model is negative. This means that we can make profit by taking the optimal execution strategy on the exchange and then selling off-exchange. By setting this profit of 45,939 as a fee in this trading environment, the incentive for gaming is eliminated.

5 Conclusion

In this paper, we considered the execution cost when using the optimal execution strategy under a round-trip trading. In particular, when taking off-exchange trading into consideration, we give estimated trading fee for the off-exchange based on two price models, when so-called gaming activities are not conducted. A permanent price model immediately returns to the permanent level, but no price reversion can be expected thereafter, while a transient price model delays execution as the decay of execution information is also taken into account in later periods. As a result, since the transient model increases the degree of freedom of action, so it seems that the execution cost is smaller for the transient model than that when the cost is the same as the permanent model in TWAP trading. In reality, however, the transient model tends to have slower price returns, indicating higher costs under the condition that costs are equal in TWAP.

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