Tangent spaces and a metric on geodesic spaces
測地距離空間上の接空間と計量

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Abstract

In this work, we introduce a notion of tangent spaces to geodesic spaces with curvature bounded above. We first consider functions like inner products and angles with the functions. Further, we consider relations between newly proposed angle and the Alexandrov angle. We finally define the tangent space and a metirc on geodesic spaces.

1 Introduction

In Hilbert spaces, a monotone operator and its zero points play an important role for convex optimisations. Rockafellar [8] showed a weak convergence theorem with the proximal point algorithm for a maximal monotone operator. Since the subdifferential of a proper lower semicontinuous convex function is a maximal monotone operator and the set of zero points of the operator coincides with the minimisers of the function, we can apply the zero points approximation theorem to finding minimisers of the function.

On the other hand, Berg and Nikolaev [2] proposed the concept called quasilinearisation. It satisfies some properties like inner products in Hilbert spaces. After that, Khatibzadeh and Ranjbar [6] introduced a notion of maximal monotone operators to Hadamard spaces, and investigated some properties using dual spaces proposed by Ahmadi Kakavandi and Amini [1]. The dual space in the sense of [1] is known to generalise the usual dual spaces when the considered space is a linear space. However, it is not known what relations there are between the dual space of a Hadamard manifold and the Riemannian tangent space.

In 2021, Chaipunya, Kohsaka and Kumam [4] introduced a notion of the tangent space to Hadamard spaces. Tangent spaces on a given Hadamard space were introduced earlier in [3]. In [4], for the technical convenience, they make a slight modification on the definition of tangent spaces.

In this paper, we adopt the similar methods of [4] and consider the tangent spaces of a CAT(κ) space. We first propose a function like inner products and a notion of angles. Further, we consider relations between newly proposed angle and the Alexandrov angle. After that, we define the tangent spaces and a metric on CAT(κ) spaces. In discussions about the tangent spaces, we mention the relation between the Euclidean cones in the sense of [3] and the tangent space proposed in this article.

2 Preliminaries

Let (X, d) be a metric space and let $D \in]0, \infty]$. For $x, y \in X$, we call an isometric mapping γ_{xy} from $[0, d(x, y)]$ into X a geodesic from x to y if $\gamma_{xy}(0) = x$ and $\gamma_{xy}(d(x, y)) = y$. X is said to be uniquely D-geodesic if for each $x, y \in X$ with $d(x, y) < D$, there is a unique geodesic. In a uniquely D-geodesic space, for $x, y \in X$ with $d(x, y) < D$, $\gamma_{xy}([0, d(x, y)]) \subset X$ is called a geodesic segment joining x and y, and we denote it by $[x, y]$. We denote a geodesic triangle with vertices $x, y, z \in X$ by $\triangle(x, y, z) = [y, z] \cup [z, x] \cup [x, z]$.

To define a CAT(κ) space, we use the following notation called a model space. Let $n \in \mathbb{N}$. For $\kappa = 0$, the n-dimensional model space $M_\kappa^n = M_0^n$ is the n-dimensional Euclidean space $\mathbb{E}^n$. For $\kappa > 0$, M_κ^n is the n-dimensional sphere $(1/\sqrt{\kappa})\,\mathbb{S}^n$ whose metric is a length of a minimal great arc joining each two points. For $\kappa < 0$, M_κ^n is the n-dimensional hyperbolic space $\left(1/\sqrt{-\kappa}\right)\mathbb{H}^n$ with the metric defined by a usual hyperbolic distance. The diameter of M_κ^n is denoted by D_κ, and is defined by $D_\kappa = \infty$ if $\kappa \leq 0$ and $D_\kappa = \pi/\sqrt{\kappa}$ if $\kappa > 0$. M_κ^n is a complete uniquely D_κ-geodesic space for each $\kappa \in \mathbb{R}$.

Let $\kappa \in \mathbb{R}$. For a geodesic triangle $\triangle(x, y, z)$ satisfying that $d(y, z) + d(z, x) + d(x, y) < 2D_\kappa$ in a uniquely D_κ-geodesic space X, there are points $\overline{x}, \overline{y}, \overline{z} \in M_\kappa^2$ such that $d(y, z) = d_{M_\kappa^2}(\overline{y}, \overline{z})$, $d(z, x) = d_{M_\kappa^2}(\overline{z}, \overline{x})$ and $d(x, y) = d_{M_\kappa^2}(\overline{x}, \overline{y})$. We call the triangle having vertices $\overline{x}, \overline{y}, \overline{z} \in M_\kappa^2$ a comparison triangle of $\triangle(x, y, z)$. Notice that it is unique up to an isometry of M_κ^2. For a specific choice of comparison triangles, we denote it by $\triangle(\overline{x}, \overline{y}, \overline{z})$. A point $\overline{p} \in [\overline{x}, \overline{y}]$ is called a comparison point for $p \in [x, y]$ if $d(x, p) = d_{M_\kappa^2}(\overline{x}, \overline{p})$.

Let $\kappa \in \mathbb{R}$ and X a uniquely D_κ-geodesic space. If for any $x, y, z \in X$ with $d(y, z) + d(z, x) + d(x, y) < 2D_\kappa$, for any $p, q \in \triangle(x, y, z)$, and for the comparison points $\overline{p}, \overline{q} \in \triangle(\overline{x}, \overline{y}, \overline{z})$ of M_κ^2, the CAT(κ) inequality

$$d(p, q) \leq d_{M_\kappa^2}(\overline{p}, \overline{q})$$

holds, then we call X a CAT(κ) space. For any $n \in \mathbb{N}$ and $\kappa \in \mathbb{R}$, the n-dimensional model space M_κ^n is a CAT(κ) space.

Let X be a CAT(κ) space for $\kappa \in \mathbb{R}$ such that $d(v, w) + d(w, u) + d(u, v) < 2D_\kappa$ for all $u, v, w \in X$. Note that then $d(x, y) < D_\kappa$ for all $x, y \in X$. For $x, y \in X$ and $t \in [0, 1]$, we denote $\gamma_{xy}((1 - t)d(x, y))$ by $tx \oplus (1 - t)y$. Note that $tx \oplus (1 - t)y$ tends to y as $t \searrow 0$. A subset C of X is said to be convex if $tx \oplus (1 - t)y \in C$ for each $x, y \in C$ and $t \in [0, 1]$. A nonempty convex subset of X is also CAT(κ) space.

We define a function c_κ from $\mathbb{R}$ into $[0,\infty[$ by

$$c_\kappa(a) = \frac{1}{2}a^2 + \sum_{n=2}^{\infty} \frac{(-\kappa)^{n-1}a^{2n}}{(2n)!} = \begin{cases} \frac{1}{\kappa}\left(1-\cos\left(\sqrt{\kappa}a\right)\right) & (\kappa > 0); \\ \frac{1}{2}a^2 & (\kappa = 0); \\ \frac{1}{-\kappa}\left(\cosh\left(\sqrt{-\kappa}a\right)-1\right) & (\kappa < 0) \end{cases}$$

for $a \in \mathbb{R}$. Then, we know

$$c'_\kappa(a) = \begin{cases} \frac{\sin\left(\sqrt{\kappa}a\right)}{\sqrt{\kappa}} & (\kappa > 0); \\ a & (\kappa = 0); \\ \frac{\sinh\left(\sqrt{-\kappa}a\right)}{\sqrt{-\kappa}} & (\kappa < 0) \end{cases} \quad \text{and} \quad c''_\kappa(a) = \begin{cases} \cos\left(\sqrt{\kappa}a\right) & (\kappa > 0); \\ 1 & (\kappa = 0); \\ \cosh\left(\sqrt{-\kappa}a\right) & (\kappa < 0) \end{cases}$$

for $a \in \mathbb{R}$. We know the following properties of c_κ, c'_κ and c''_κ:

- c_κ, c'_κ and c''_κ is continuous;
- $c_\kappa \colon [0, D_\kappa[\to [0,\infty[$ is strictly increasing;
- $c_\kappa(0) = c'_\kappa(0) = 0$ and $c''_\kappa(0) = 1$;
- $c'_\kappa(-a) = -c'_\kappa(a)$ and $c''_\kappa(-a) = c_\kappa(a)$ for all $a \in \mathbb{R}$.

Moreover, for $a, b \in \mathbb{R}$, we get the following equations:

$$\begin{aligned} 1 &= c''_\kappa(a) + \kappa c_\kappa(a); \\ 1 &= c''_\kappa(a)^2 + \kappa c'_\kappa(a)^2; \\ c'_\kappa(a+b) &= c'_\kappa(a)c''_\kappa(b) + c'_\kappa(b)c''_\kappa(a); \\ c''_\kappa(a+b) &= c''_\kappa(a)c''_\kappa(b) - \kappa c'_\kappa(a)c'_\kappa(b); \\ c''_\kappa(a) - c''_\kappa(b) &= -2\kappa c'_\kappa\left(\frac{a+b}{2}\right)c'_\kappa\left(\frac{a-b}{2}\right). \end{aligned}$$

Note that it holds from l'Hospital's rule that

$$\lim_{t\to 0} \frac{c'_\kappa(t)}{t} = \lim_{t\to 0} c''_\kappa(t) = 1.$$

For more details about the function c_κ, see [5].

For a metric space (X,d), we define a function ϕ_κ from X^2 into $\mathbb{R}$ by

$$\phi_\kappa(x,y) = c_\kappa(d(x,y))$$

for each $x, y \in X$. We get the following properties of ϕ_κ:

- $\phi_\kappa(x,y) \geq 0$ for all $x, y \in X$;
- $\phi_\kappa(x,y) = 0$ if and only if $x = y$, where $d(x,y) < 2D_\kappa$;
- $\phi_\kappa(x,y) = \phi_\kappa(y,x)$ for all $x, y \in X$.

For more details about ϕ_κ, refer to [7].

3 Angles

In this section, we consider angles on CAT(κ) spaces. We first introduce a function like inner products to metric spaces.

Let $\kappa \in \mathbb{R}$ and X a metric space such that $d(u,v) < D_\kappa$ for all $u,v \in X$. For $x,y,u,v \in X$, we define a product by

$$\langle x \ominus y, u \ominus v \rangle_\kappa = c''_\kappa(d(x,y))\phi_\kappa(x,v) + \phi_\kappa(y,u) - c''_\kappa(d(x,y))\phi_\kappa(x,u) - \phi_\kappa(y,v).$$

If $\kappa = 0$, then this product is quasilinearisation in the sense of [2].

Lemma 3.1. *Let $\kappa \in \mathbb{R}$ and X a metric space such that $d(u,v) < D_\kappa$ for all $u,v \in X$. Then, the following hold:*

(i) $\langle x \ominus y, u \ominus v \rangle_\kappa = -\langle x \ominus y, v \ominus u \rangle_\kappa$ *for each $u,v,x,y \in X$;*
(ii) $\langle x \ominus x, u \ominus v \rangle_\kappa = 0$ *for each $u,x,y \in X$;*
(iii) $\langle x \ominus y, u \ominus u \rangle_\kappa = 0$ *for each $u,v,x \in X$;*
(iv) $\langle x \ominus y, x \ominus y \rangle_\kappa = c'_\kappa(d(x,y))^2$ *for all $x,y \in X$;*
(v) $\langle p \ominus x, p \ominus y \rangle_\kappa = \langle p \ominus y, p \ominus x \rangle_\kappa$ *for all $p,x,y \in X$.*

Proof. Let $u,v,x,y \in X$ and set $d = d(x,y)$. Then, we easily get (i), (ii) and (iii). We show (iv). We know

$$\begin{aligned}\langle x \ominus y, x \ominus y \rangle_\kappa &= c''_\kappa(d)\phi_\kappa(x,y) + \phi_\kappa(y,x) - c''_\kappa(d)\phi_\kappa(x,x) - \phi_\kappa(y,y)\\ &= \left(c''_\kappa(d) + 1\right)\phi_\kappa(x,y).\end{aligned}$$

If $\kappa = 0$, then

$$\langle x \ominus y, x \ominus y \rangle_\kappa = \left(c''_0(d) + 1\right)\phi_0(x,y) = d(x,y)^2 = c'_0(d(x,y))^2.$$

Suppose $\kappa \neq 0$. Then, we have

$$\langle x \ominus y, x \ominus y \rangle_\kappa = \left(c''_\kappa(d) + 1\right)\phi_\kappa(x,y) = \frac{1}{\kappa}\left(c''_\kappa(d) + 1\right)\left(1 - c''_\kappa(d)\right) = \frac{1}{\kappa}\left(1 - c''_\kappa(d)^2\right).$$

Since $c''_\kappa(d)^2 + \kappa c'_\kappa(d)^2 = 1$, we obtain

$$\langle x \ominus y, x \ominus y \rangle_\kappa = c'_\kappa(d(x,y))^2.$$

We finally show (v). Let $p,x,y \in X$. Then,

$$\begin{aligned}\langle p \ominus x, p \ominus y \rangle_\kappa &= c''_\kappa(d(p,x))\phi_\kappa(p,y) + \phi_\kappa(x,p) - \phi_\kappa(x,y)\\ &= \phi_\kappa(p,y) - \left(1 - c''_\kappa(d(p,x))\right)\phi_\kappa(p,y) + \phi_\kappa(x,p) - \phi_\kappa(x,y)\\ &= \phi_\kappa(p,y) - \left(1 - c''_\kappa(d(p,y))\right)\phi_\kappa(p,x) + \phi_\kappa(x,p) - \phi_\kappa(x,y)\\ &= c''_\kappa(d(p,y))\phi_\kappa(p,x) + \phi_\kappa(y,p) - \phi_\kappa(y,x) = \langle p \ominus y, p \ominus x \rangle_\kappa.\end{aligned}$$

This is the desired result and it completes the proof. □

Let $n \in \mathbb{N}$ and $(M_\kappa^n, d_{M_\kappa^n})$ the n-dimensional model space for $\kappa \neq 0$. For $p, x, y \in M_\kappa^n$, the angle $\measuredangle_p^\kappa(x, y)$ of x and y at p is defined by

$$\measuredangle_p^\kappa(x, y) = \arccos\left(\frac{c_\kappa''(d_{M_\kappa^n}(p, x))c_\kappa''(d_{M_\kappa^n}(p, y)) - c_\kappa''(d_{M_\kappa^n}(x, y))}{\kappa c_\kappa'(d_{M_\kappa^n}(p, x))c_\kappa'(d_{M_\kappa^n}(p, y))}\right)$$

if $p \neq x$ and $p \neq y$; $\measuredangle_p^\kappa(p, y) = \measuredangle_p^\kappa(x, p) = \pi/2$; $\measuredangle_p^\kappa(p, p) = 0$. We also define $\measuredangle_p^0(x, y)$ for $p, x, y \in \mathbb{E}^n$ by the usual angle on the Euclidian space. For more details, refer to "The Law of Cosines in M_κ^n 2.13" in [3, Chapter I.2].

Let X be a CAT(κ) space for $\kappa \in \mathbb{R}$ and $p, x, y \in X$ with $d(x, y) + d(y, p) + d(p, x) < 2D_\kappa$. We define the κ-angle $\angle_p^\kappa(x, y)$ of x and y at p by

$$\angle_p^\kappa(x, y) = \measuredangle_{\overline{p}}^\kappa(\overline{x}, \overline{y}),$$

where $\triangle(\overline{p}, \overline{x}, \overline{y}) \subset M_\kappa^2$ is the comparison triangle of $\triangle(p, x, y)$ and $\measuredangle_{\overline{p}}^\kappa$ is the angle at $\overline{p}$ with respect to M_κ^2.

Now, we can prove the following:

Theorem 3.2. *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$ such that $d(v, w) + d(w, u) + d(u, v) < 2D_\kappa$ for all $u, v, w \in X$. Then,*

$$\frac{\langle p \ominus x, p \ominus y\rangle_\kappa}{c_\kappa'(d(p, x))c_\kappa'(d(p, y))} = \cos \angle_p^\kappa(x, y)$$

for all $p, x, y \in X$ with $p \neq x$ and $p \neq y$, where $\angle_p^\kappa(x, y)$ is the κ-angle of x and y at p.

Proof. Let $p, x, y \in X$ with $p \neq x$ and $p \neq y$. Then, we can take their comparison triangle $\triangle(\overline{p}, \overline{x}, \overline{y})$ of the two-dimensional model space M_κ^2. Note that $d(x, y) = d_{M_\kappa^2}(\overline{x}, \overline{y})$, $d(y, p) = d_{M_\kappa^2}(\overline{y}, \overline{p})$ and $d(p, x) = d_{M_\kappa^2}(\overline{p}, \overline{x})$. We first suppose that $\kappa = 0$. From the law of cosines with respect to $\mathbb{E}^2$, we have

$$\begin{aligned}\cos \angle_p^0(x, y) = \cos \measuredangle_{\overline{p}}^0(\overline{x}, \overline{y}) &= \frac{d_{\mathbb{E}^2}(\overline{p}, \overline{x})^2 + d_{\mathbb{E}^2}(\overline{p}, \overline{y})^2 - d_{\mathbb{E}^2}(\overline{x}, \overline{y})^2}{2d_{\mathbb{E}^2}(\overline{p}, \overline{x})d_{\mathbb{E}^2}(\overline{p}, \overline{y})} \\ &= \frac{d(p, x)^2 + d(p, y)^2 - d(x, y)^2}{2d(p, x)d(p, y)} = \frac{\langle p \ominus x, p \ominus y\rangle_0}{d(p, x)d(p, y)}.\end{aligned}$$

We next assume that $\kappa \neq 0$. Then, from the definition of angles on M_κ^2,

$$\begin{aligned}\cos \angle_p^\kappa(x, y) = \cos \measuredangle_{\overline{p}}^\kappa(\overline{x}, \overline{y}) &= \frac{c_\kappa''(d_{M_\kappa^2}(\overline{x}, \overline{y})) - c_\kappa''(d_{M_\kappa^2}(\overline{p}, \overline{x}))c_\kappa''(d_{M_\kappa^2}(\overline{p}, \overline{y}))}{\kappa c_\kappa'(d_{M_\kappa^2}(\overline{p}, \overline{x}))c_\kappa'(d_{M_\kappa^2}(\overline{p}, \overline{y}))} \\ &= \frac{c_\kappa''(d(x, y)) - c_\kappa''(d(p, x))c_\kappa''(d(p, y))}{\kappa c_\kappa'(d(p, x))c_\kappa'(d(p, y))} \\ &= \frac{\langle p \ominus x, p \ominus y\rangle_\kappa}{c_\kappa'(d(p, x))c_\kappa'(d(p, y))}.\end{aligned}$$

This is the desired result. $\square$

Let X be a CAT(κ) space for $\kappa \in \mathbb{R}$ such that $d(v,w)+d(w,u)+d(u,v) < 2D_\kappa$ for all $u,v,w \in X$, and let $p,x,y \in X$. From Theorem 3.2, we can redefine the κ-angle as follows:

$$\angle_p^\kappa(x,y) = \arccos\left(\frac{\langle p \ominus x, p \ominus y\rangle_\kappa}{c'_\kappa(d(p,x))c'_\kappa(d(p,y))}\right)$$

if $p \neq x$ and $p \neq y$; $\angle_p^\kappa(p,y) = \angle_p^\kappa(x,p) = \pi/2$; $\angle_p^\kappa(p,p) = 0$. Moreover, we define the Alexandrov angle $A_p(x,y)$ of x and y at p by

$$A_p(x,y) = \lim_{t\to 0} \angle_p^\kappa(\gamma_{px}(t), \gamma_{py}(t)) \in [0,\pi].$$

Here, γ_{px} and γ_{py} are geodesics from p to x and y, respectively. Note that $A_p(p,y) = A_p(x,p) = \pi/2$ and $A_p(p,p) = 0$. From the definition, we get $A_p(x,y) = A_p(y,x)$ and $A_p(x,x) = 0$. Further, the Alexandrov angle has the following property:

$$A_p(x,z) \leq A_p(x,y) + A_p(y,z)$$

for any $p,x,y,z \in X$. For more details about the Alexandrov angles, for instance, refer to [3, Proposition 1.14 in Chapter I.1 and Proposition 3.1 in Chapter II.3].

Theorem 3.3 (Bridson–Haefliger [3, Proposition 2.9 in Chapter I.2])**.** *Let X be a nonempty convex subset such that $d_{M_\kappa^n}(v,w) + d_{M_\kappa^n}(w,u) + d_{M_\kappa^n}(u,v) < 2D_\kappa$ for all $u,v,w \in X$ of the n-dimensional model space M_κ^n for $n \in \mathbb{N}$ and $\kappa \in \mathbb{R}$. Then,*

$$A_p(x,y) = \measuredangle_p^\kappa(x,y)$$

for each $p,x,y \in X$, where $\measuredangle_p^\kappa$ is the angle at p with respect to M_κ^n.

Theorem 3.4 (Bridson–Haefliger [3, Proposition 1.7 in Chapter II.1])**.** *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$ such that $d(v,w)+d(w,u)+d(u,v) < 2D_\kappa$ for all $u,v,w \in X$. Then,*

$$A_p(x,y) \leq \measuredangle_{\overline{p}}^\kappa(\overline{x},\overline{y})$$

for each $p,x,y \in X$ and its comparison triangle $\triangle(\overline{p},\overline{x},\overline{y}) \subset M_\kappa^2$, where $\measuredangle_{\overline{p}}^\kappa$ is the angle at $\overline{p}$ with respect to M_κ^2.

As a direct consequence of this theorem, we obtain the following lemma:

Lemma 3.5. *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$ such that $d(v,w)+d(w,u)+d(u,v) < 2D_\kappa$ for all $u,v,w \in X$. Then,*

$$A_p(x,y) \leq \angle_p^\kappa(x,y)$$

for each $p,x,y \in X$, where $\angle_p^\kappa$ is the κ-angle at p.

Further, the following theorem called the first variation formula holds:

Theorem 3.6 (Bridson–Haefliger [3, Corollary 3.6 in Chapter II.3])**.** *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$ such that $d(v,w)+d(w,u)+d(u,v) < 2D_\kappa$ for all $u,v,w \in X$. Then,*

$$\lim_{t \searrow 0} \frac{d(p,y) - d(\gamma_{px}(t), y)}{t} = \cos A_p(x,y)$$

for each $p,x,y \in X$ with $p \neq x$.

In what follows, we introduce a metric space with the Alexandrov angles.

Lemma 3.7. *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$ such that $d(v,w)+d(w,u)+d(u,v) < 2D_\kappa$ for all $u,v,w \in X$, and let $p \in X$. We define a binary relation $\sim_p$ on X by $x \sim_p y$ if $A_p(x,y) = 0$, where A_p is the Alexandrov angle at p. Then, $\sim_p$ is an equivalence relation on X.*

Proof. $\sim_p$ is obviously reflexive and symmetric. We show it has transitivity. We suppose $x \sim_p y$ and $y \sim_p z$. Then,

$$0 \leq A_p(x,z) \leq A_p(x,y) + A_p(y,z) = 0$$

and thus $x \sim_p z$. Therefore, $\sim_p$ is an equivalence relation on X. □

Let X be a CAT(κ) space for $\kappa \in \mathbb{R}$ such that $d(v,w)+d(w,u)+d(u,v) < 2D_\kappa$ for all $u,v,w \in X$, and let $p \in X$. For $x \in X$, we define an equivalence class of x by $[x]_p = \{z \in X \mid x \sim_p z\}$. Further, put

$$D_pX = X/{\sim_p} = \{[x]_p \mid x \in X\}.$$

Since $A_p(x,p) = \pi/2$ for all $x \in X$ and $A_p(p,p) = 0$, $[p]_p = \{p\}$.

Lemma 3.8. *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$ such that $d(v,w)+d(w,u)+d(u,v) < 2D_\kappa$ for all $u,v,w \in X$, and let $p \in X$. Then, (D_pX, A_p) is a metric space, where the distance function A_p is defined by $A_p([x]_p,[y]_p) = A_p(x,y)$ for $[x]_p,[y]_p \in D_pX$.*

Proof. We first show $A_p(x_1,y_1) = A_p(x_2,y_2)$ for each $x_1,x_2 \in [x]_p \in D_pX$ and $y_1,y_2 \in [y]_p \in D_pX$. Since $A_p(x_1,x_2) = A_p(y_1,y_2) = 0$, we have

$$\begin{aligned}
|A_p(x_1,y_1) - A_p(x_2,y_2)| &= |A_p(x_1,y_1) - A_p(x_1,y_2) + A_p(x_1,y_2) - A_p(x_2,y_2)| \\
&\leq |A_p(x_1,y_1) - A_p(x_1,y_2)| + |A_p(x_1,y_2) - A_p(x_2,y_2)| \\
&\leq A_p(y_1,y_2) + A_p(x_1,x_2) = 0.
\end{aligned}$$

Consequently, we can define a value $A_p([x]_p,[y]_p)$ by $A_p(x,y)$ for $[x]_p,[y]_p \in D_pX$.

Let $[x]_p,[y]_p,[z]_p \in D_pX$. Then, $A_p([x]_p,[y]_p) \geq 0$, $A_p([x]_p,[y]_p) = A_p([y]_p,[x]_p)$ and

$$A_p([x]_p,[z]_p) \leq A_p([x]_p,[y]_p) + A_p([y]_p,[z]_p).$$

Further, $A_p([x]_p,[y]_p) = 0$ if and only if $[x]_p = [y]_p$. Therefore, (D_pX, A_p) is a metric space. □

4 Tangent spaces

In this section, we introduce the tangent spaces on a CAT(κ) space with similar methods in [4].

Let X be a CAT(κ) space for $\kappa \in \mathbb{R}$ such that $d(v,w)+d(w,u)+d(u,v) < 2D_\kappa$ for all $u,v,w \in X$, and let $p \in X$. We define a function ζ_p from D_pX into $\{0,1\}$ by

$$\zeta_p([x]_p) = \begin{cases} 0 & ([x]_p = [p]_p); \\ 1 & ([x]_p \neq [p]_p) \end{cases}$$

for each $[x]_p \in D_pX$. We define a binary relation $\simeq_p$ on $[0,\infty[\times D_pX$ by $(\lambda_1,[x]_p) \simeq_p (\lambda_2,[y]_p)$ if one of the following conditions is satisfied:

- $\lambda_1\zeta_p([x]_p) = \lambda_2\zeta_p([y]_p) = 0$;
- $\lambda_1\zeta_p([x]_p) = \lambda_2\zeta_p([y]_p) > 0$ and $[x]_p = [y]_p$.

Then, we get the following:

Lemma 4.1. *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$ such that $d(v,w)+d(w,u)+d(u,v) < 2D_\kappa$ for all $u,v,w \in X$, and let $p \in X$. Then, $\simeq_p$ is an equivalence relation on $[0,\infty[\times D_pX$.*

Proof. $\simeq_p$ is obviously reflexive and symmetric. We show it has transitivity. Suppose that $(\lambda_1,[x]_p) \simeq_p (\lambda_2,[y]_p)$ and $(\lambda_2,[y]_p) \simeq_p (\lambda_3,[z]_p)$. If $\lambda_1\zeta_p([x]_p) = \lambda_2\zeta_p([y]_p) = 0$, then we obtain $\lambda_3\zeta_p([z]_p) = \lambda_2\zeta_p([y]_p) = 0$. It implies that $(\lambda_1,[x]_p) \simeq_p (\lambda_3,[z]_p)$. If $\lambda_1\zeta_p([x]_p) = \lambda_2\zeta_p([y]_p) > 0$ and $[x]_p = [y]_p$, then we get $\lambda_3\zeta_p([z]_p) = \lambda_2\zeta_p([y]_p) > 0$. Since $(\lambda_2,[y]_p) \simeq_p (\lambda_3,[z]_p)$, we have $[y]_p = [z]_p$, which implies that $\lambda_1\zeta_p([x]_p) = \lambda_3\zeta_p([z]_p) > 0$ and $[x]_p = [z]_p$, and hence $(\lambda_1,[x]_p) \simeq_p (\lambda_3,[z]_p)$. Therefore, $\simeq_p$ is an equivalence relation on $[0,\infty[\times D_pX$. ☐

Let X be a CAT(κ) space for $\kappa \in \mathbb{R}$ such that $d(v,w)+d(w,u)+d(u,v) < 2D_\kappa$ for all $u,v,w \in X$, and let $p \in X$. We define a set T_pX by

$$T_pX = ([0,\infty[\times D_pX)/{\simeq_p}.$$

Let us write $\lambda[x]_p$ for $[(\lambda,[x]_p)]_{\simeq_p} \in T_pX$, where $[(\lambda,[x]_p)]_{\simeq_p}$ is an equivalent class of $(\lambda,[x]_p) \in [0,\infty[\times D_pX$.

Lemma 4.2. *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$ such that $d(v,w)+d(w,u)+d(u,v) < 2D_\kappa$ for all $u,v,w \in X$, and let $p \in X$. Define a bifunction d_p on T_pX by*

$$d_p(\lambda[x]_p, \mu[y]_p) = \sqrt{\lambda^2\zeta_p([x]_p) + \mu^2\zeta_p([y]_p) - 2\lambda\mu\zeta_p([x]_p)\zeta_p([y]_p)\cos A_p([x]_p,[y]_p)}$$

for each $\lambda[x]_p, \mu[y]_p \in T_pX$. Then, (T_pX, d_p) is a semimetric space.

Proof. We first show

$$d_p((\lambda_1,[x_1]_p),(\mu_1,[y_1]_p)) = d_p((\lambda_2,[x_2]_p),(\mu_2,[y_2]_p))$$

for each $(\lambda_1, [x_1]_p), (\lambda_2, [x_2]_p) \in \lambda[x]_p$ and $(\mu_1, [y_1]_p), (\mu_2, [y_2]_p) \in \mu[y]_p$. Since

$$(\lambda_1, [x_1]_p) \simeq_p (\lambda_2, [x_2]_p) \text{ and } (\mu_1, [y_1]_p) \simeq_p (\mu_2, [y_2]_p),$$

we have $\lambda_1\zeta_p([x_1]_p) = \lambda_2\zeta_p([x_2]_p)$ and $\mu_1\zeta_p([y_1]_p) = \mu_2\zeta_p([y_2]_p)$. If $\lambda_1\zeta_p([x_1]_p) = \lambda_2\zeta_p([x_2]_p) = 0$ or $\mu_1\zeta_p([y_1]_p) = \mu_2\zeta_p([y_2]_p) = 0$, we easily get

$$d_p((\lambda_1, [x_1]_p), (\mu_1, [y_1]_p)) = d_p((\lambda_2, [x_2]_p), (\mu_2, [y_2]_p)).$$

We assume that $\lambda_1\zeta_p([x_1]_p) = \lambda_2\zeta_p([x_2]_p) > 0$ and $\mu_1\zeta_p([y_1]_p) = \mu_2\zeta_p([y_2]_p) > 0$. Then, since $[x_1]_p = [x_2]_p$ and $[y_1]_p = [y_2]_p$, we have

$$\begin{aligned} &d_p((\lambda_1, [x_1]_p), (\mu_1, [y_1]_p)) \\ &= \sqrt{\lambda_1^2\zeta_p([x_1]_p) + \mu_1^2\zeta_p([y_1]_p) - 2\lambda_1\mu_1\zeta_p([x_1]_p)\zeta_p([y_1]_p)\cos A_p([x_1]_p, [y_1]_p)} \\ &= \sqrt{\lambda_2^2\zeta_p([x_2]_p) + \mu_2^2\zeta_p([y_2]_p) - 2\lambda_2\mu_2\zeta_p([x_2]_p)\zeta_p([y_2]_p)\cos A_p([x_2]_p, [y_2]_p)} \\ &= d_p((\lambda_2, [x_2]_p), (\mu_2, [y_2]_p)). \end{aligned}$$

We next show (T_pX, d_p) is a semimetric space. Let $\lambda[x]_p, \mu[y]_p \in T_pX$. Note that

$$|\lambda\zeta_p([x]_p) - \mu\zeta_p([y_p])| \leq d_p(\lambda[x]_p, \mu[y]_p) \leq \lambda\zeta_p([x]_p) + \mu\zeta_p([y]_p).$$

Then, $d_p(\lambda[x]_p, \mu[y]_p) = d_p(\lambda[y]_p, \mu[x]_p)$, $d_p(\lambda[x]_p, \mu[y]_p) \geq 0$ and $d_p(\lambda[x]_p, \lambda[x]_p) = 0$. Assume that $d_p(\lambda[x]_p, \mu[y]_p) = 0$. Then, we get $\lambda\zeta_p([x]_p) = \mu\zeta_p([y_p])$. If $\lambda\zeta_p([x]_p) = \mu\zeta_p([y_p]) = 0$, then we have $\lambda[x]_p = \mu[y]_p$. If $\lambda\zeta_p([x]_p) = \mu\zeta_p([y_p]) > 0$, then

$$0 = d_p(\lambda[x]_p, \mu[y]_p)^2 = 2\lambda^2\zeta_p([x]_p) - 2\lambda^2\zeta_p([x]_p)\cos A_p([x]_p, [y]_p)$$

and thus $\cos A_p([x]_p, [y]_p) = 1$. It means that $A_p([x]_p, [y]_p) = 0$ and hence $[x]_p = [y]_p$. Therefore, $\lambda[x]_p = \mu[y]_p$. Consequently, (T_pX, d_p) is a semimetric space. □

Let X be a CAT(κ) space for $\kappa \in \mathbb{R}$ such that $d(v, w) + d(w, u) + d(u, v) < 2D_\kappa$ for all $u, v, w \in X$, and let $p \in X$. Put $S_pX = D_pX \setminus \{[p]_p\}$. Note that (S_pX, A_p) is a metric space. Then, we denote the Euclidean cone C_0S_pX in the sense of [3], and define

$$C_0S_pX = ([0, \infty[\times S_pX)/\simeq_p,$$

where $\simeq_p$ is the same equivalence relation adopted in T_pX. For more details about the Euclidean cones, see [3, Definition 5.6 in Chapter I.5]. From the definition of the Euclidean cone C_0S_pX, we get $C_0S_pX \subset T_pX$, and C_0S_pX can adopt the same semimetric of (T_pX, d_p). Further, the following holds:

Theorem 4.3 (Bridson–Haefliger [3, Proposition 5.9 in Chapter I.5])**.** *Let X be a CAT(κ) space for $\kappa \in \mathbb{R}$ such that $d(v, w) + d(w, u) + d(u, v) < 2D_\kappa$ for all $u, v, w \in X$, and let $p \in X$. Then, (C_0S_pX, d_p) is a metric space. Namely, d_p satisfies the triangle inequality on C_0S_pX.*

As a direct consequence of this theorem, we obtain the following:

Theorem 4.4. *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$ such that $d(v,w) + d(w,u) + d(u,v) < 2D_\kappa$ for all $u, v, w \in X$. Then, (T_pX, d_p) is a metric space for each $p \in X$.*

Proof. We show d_p satisfies the triangle inequality. Take $\lambda[x]_p, \mu[y]_p, \nu[z]_p \in T_pX$. If $[x]_p = [p]_p$, then

$$\begin{aligned} d_p(\lambda[x]_p, \nu[z]_p) = \nu\zeta_p([z]_p) &= \mu\zeta_p([y]_p) + \nu\zeta_p([z]_p) - \mu\zeta_p([y]_p) \\ &\leq d_p(\lambda[x]_p, \mu[y]_p) + d_p(\mu[y]_p, \nu[z]_p). \end{aligned}$$

In the same fashion, we obtain the inequality if $[z]_p = [p]_p$. Further, in the case $[y]_p = [p]_p$, we obtain

$$d_p(\lambda[x]_p, \nu[z]_p) \leq \lambda\zeta_p([x]_p) + \nu\zeta_p([z]_p) = d_p(\lambda[x]_p, \mu[y]_p) + d_p(\mu[y]_p, \nu[z]_p).$$

Therefore, if one of the three elements $[x]_p, [y]_p, [z]_p \in S_pX$ coincides with $[p]_p$, then the triangle inequality holds. We assume $[x]_p \neq [p]_p, [y]_p \neq [p]_p$ and $[z]_p \neq [p]_p$. Then, $\lambda[x]_p, \mu[y]_p, \nu[z]_p \in C_0S_pX$, where $C_0S_pX \subset T_pX$ is the Euclidean cone of $S_pX = D_pX \setminus \{[p]_p\}$. Since (C_0S_pX, d_p) is a metric space, we get

$$d_p(\lambda[x]_p, \nu[z]_p) \leq d_p(\lambda[x]_p, \mu[y]_p) + d_p(\mu[y]_p, \nu[z]_p).$$

Consequently, (T_pX, d_p) is a metric space. □

Theorem 4.5. *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$ such that $d(v,w) + d(w,u) + d(u,v) < 2D_\kappa$ for all $u, v, w \in X$. Then, (C_0S_pX, d_p) and (T_pX, d_p) are isometric.*

Proof. We show that there is a surjective isometric mapping from T_pX to C_0S_pX. We define a mapping ι from T_pX to C_0S_pX by

$$\iota(\lambda[x]_p) = \begin{cases} \lambda[x]_p & \text{if } [x]_p \neq [p]_p; \\ [(0, [x]_p)]_{\simeq_p} & \text{if } [x]_p = [p]_p \end{cases}$$

for $\lambda[x]_p \in T_pX$. Then, the mapping ι preserves the distance, namely, it is isometric. We show that it is surjective. Let $\mu[y]_p \in C_0S_pX$. From the definition of C_0S_pX, we have $[y]_p \neq [p]_p$. Then, $\mu[y]_p = \iota(\mu[y]_p)$ if $\mu > 0$. Furthermore, $0[y]_p = \iota(0[p]_p)$. Hence, (C_0S_pX, d_p) and (T_pX, d_p) are isometric. □

If X is a nonempty convex subset of the n-dimensional model space M_κ^n for $n \in \mathbb{N}$ and $\kappa \in \mathbb{R}$ such that $d_{M_\kappa^n}(v,w) + d_{M_\kappa^n}(w,u) + d_{M_\kappa^n}(u,v) < 2D_\kappa$ for all $u, v, w \in X$, then C_0S_pX is the usual Riemannian tangent space at $p \in X$. Namely, T_pX is also the Riemannian tangent space at $p \in X$. For details, see [3, "The Space of Directions" in Chapter II.3].

Let X be a CAT(κ) space for $\kappa \in \mathbb{R}$ such that $d(v,w)+d(w,u)+d(u,v) < 2D_\kappa$ for all $u, v, w \in X$, and let $p \in X$. We call (T_pX, d_p) the tangent space over X at p. We write $TX = \bigcup_{p \in X} T_pX$ and call it the tangent bundle of X. For $x^* = \lambda[x]_p \in T_pX$, we denote $\nu\lambda[x]_p$ by νx^*. For each $x \in X$, we denote a normalised vector $c'_\kappa(d(p,x))[x]_p \in T_pX$ by x_p. Similarly, for $x \in X$, we denote $d(p,x)[x]_p \in T_pX$ by $\widehat{x}_p$. Note that

$$\frac{d(p,x)}{c'_\kappa(d(p,x))} x_p = \widehat{x}_p$$

for each $x \in X$ with $p \neq x$. Further, set $0_p = p_p$. Note that $0_p = \lambda p_p = 0x_p$ for each $\lambda > 0$ and $x \in X$.

5 A metric on CAT(κ) spaces

Let X be a CAT(κ) space for $\kappa \in \mathbb{R}$ such that $d(v,w) + d(w,u) + d(u,v) < 2D_\kappa$ for all $u, v, w \in X$. For each $p \in X$, we define a function $g_p \colon T_pX \times T_pX \to \mathbb{R}$ by

$$g_p(\lambda[x]_p, \mu[y]_p) = \langle 0_p \ominus \lambda[x]_p, 0_p \ominus \mu[y]_p \rangle_0 = \lambda\mu\zeta_p([x]_p)\zeta_p([y]_p)\cos A_p([x]_p, [y]_p)$$

for each $\lambda[x]_p, \mu[y]_p \in T_pX$. Note that

$$g_p(x_p, y_p) = c'_\kappa(d(p,x))c'_\kappa(d(p,y))\cos A_p(x,y)$$

for each $x, y \in X$. We call a family of the functions $\{g_p \mid p \in X\}$ a metric on X. Note that the following hold:

- $g_p(x^*, x^*) \geq 0$ for all $p \in X$ and $x^* \in T_pX$;
- $g_p(x^*, y^*) = g_p(y^*, x^*)$ for all $p \in X$ and $x^*, y^* \in T_pX$;
- $\lambda g_p(x^*, y^*) = g_p(\lambda x^*, y^*)$ for all $p \in X$, $x^*, y^* \in T_pX$ and $\lambda \geq 0$;
- $g_p(x^*, 0_p) = 0$ for all $p \in X$ and $x^* \in T_pX$.

If X is a nonempty convex subset of the n-dimensional model space M^n_κ for $n \in \mathbb{N}$ and $\kappa \in \mathbb{R}$ such that $d_{M^n_\kappa}(v,w) + d_{M^n_\kappa}(w,u) + d_{M^n_\kappa}(u,v) < 2D_\kappa$ for all $u, v, w \in X$, then $\{g_p \mid p \in X\}$ is the usual Riemannian metric. That is, g_p is an inner product on the Riemannian tangent space at $p \in X$.

At the end of this article, we prove the following theorems:

Theorem 5.1. *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$ such that $d(v,w) + d(w,u) + d(u,v) < 2D_\kappa$ for all $u, v, w \in X$, and let $p \in X$. Then, for each $x, y \in X$,*

$$g_p(x_p, y_p) \geq \langle p \ominus x, p \ominus y \rangle_\kappa .$$

Proof. Let $x, y \in X$. From Lemma 3.5, since $A_p(x,y) \leq \angle^\kappa_p(x,y)$,

$$\begin{aligned} g_p(x_p, y_p) &= c'_\kappa(d(p,x))c'_\kappa(d(p,y))\cos A_p(x,y) \\ &\geq c'_\kappa(d(p,x))c'_\kappa(d(p,y))\cos \angle^\kappa_p(x,y) = \langle p \ominus x, p \ominus y \rangle_\kappa \end{aligned}$$

and this is the desired result. □

Theorem 5.2. *Let X be a nonempty convex subset of the n-dimensional model space M^n_κ for $n \in \mathbb{N}$ and $\kappa \in \mathbb{R}$ such that $d_{M^n_\kappa}(v,w) + d_{M^n_\kappa}(w,u) + d_{M^n_\kappa}(u,v) < 2D_\kappa$ for all $u, v, w \in X$, and let $p \in X$. Then, for each $x, y \in X$,*

$$g_p(x_p, y_p) = \langle p \ominus x, p \ominus y \rangle_\kappa .$$

Theorem 5.3. *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$ such that $d(v,w) + d(w,u) + d(u,v) < 2D_\kappa$ for all $u, v, w \in X$, and let $p \in X$. Then, for each $x, y \in X$ with $p \neq x$,*

$$\lim_{t \searrow 0} \frac{\phi_\kappa(p,y) - \phi_\kappa(tx \oplus (1-t)p, y)}{t} = \frac{d(p,x)}{c'_\kappa(d(p,x))} g_p(x_p, y_p).$$

Proof. Let $p, x, y \in X$ with $p \neq x$. We denote $z_t = tx \oplus (1-t)p$ and

$$f(t) = \frac{\phi_\kappa(p,y) - \phi_\kappa(z_t,y)}{t}$$

for $t \in]0,1]$. Note that $z_t = \gamma_{px}(td(p,x))$. If $\kappa = 0$, then

$$f(t) = \frac{d(p,y)^2 - d(z_t,y)^2}{2t} = \frac{d(p,x)(d(p,y)+d(z_t,y))}{2} \cdot \frac{d(p,y) - d(\gamma_{px}(td(p,x)),y)}{td(p,x)}.$$

Therefore, from the first variation formula,

$$\lim_{t\searrow 0} \frac{d(p,y) - d(\gamma_{px}(td(p,x)),y)}{td(p,x)} = \cos A_p(x,y)$$

and thus

$$\lim_{t\searrow 0} f(t) = d(p,x)d(p,y)\cos A_p(x,y) = g_p(x_p, y_p).$$

Assume that $\kappa \neq 0$. Put $D_t = (d(p,y) - d(z_t,y))/2$. Then, we get

$$f(t) = \frac{c''_\kappa(d(z_t,y)) - c''_\kappa(d(p,y))}{\kappa t} = \frac{2}{t} c'_\kappa \left(\frac{d(z_t,y)+d(p,y)}{2} \right) c'_\kappa(D_t)$$

and therefore

$$f(t) = c'_\kappa \left(\frac{d(z_t,y)+d(p,y)}{2} \right) \frac{c'_\kappa(D_t)}{D_t} \cdot \frac{2D_t}{t}.$$

Note that $D_t \to 0$ as $t \searrow 0$ and hence $c'_\kappa(D_t)/D_t$ tends to 1 as $t \searrow 0$. Further, it holds from the first variation formula that

$$\lim_{t\searrow 0} \frac{2D_t}{t} = \lim_{t\searrow 0} \frac{d(p,y) - d(\gamma_{px}(td(p,x)),y)}{t} = d(p,x)\cos A_p(x,y).$$

Therefore, we have

$$\lim_{t\searrow 0} f(t) = d(p,x)c'_\kappa(d(p,y))\cos A_p(x,y) = \frac{d(p,x)}{c'_\kappa(d(p,x))} g_p(x_p,y_p).$$

Consequently, we obtain the desired result. □

Corollary 5.4. *Let X be a* CAT(κ) *space for $\kappa \in \mathbb{R}$ such that $d(v,w) + d(w,u) + d(u,v) < 2D_\kappa$ for all $u, v, w \in X$, and let $p \in X$. Then, for each $x, y \in X$,*

$$\lim_{t\searrow 0} \frac{\phi_\kappa(p,y) - \phi_\kappa(tx \oplus (1-t)p, y)}{t} = g_p(\widehat{x}_p, y_p).$$

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