A generalization of fuzzy-set relations for intuitionistic fuzzy sets*

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Dedicated to the memory of Professor Kazimierz Goebel

1 Introduction

A membership function of a classical fuzzy set assigns to each element of the universe of discourse a number from the unit interval to indicate the degree of belongingness to the set under consideration. The necessity to deal with imprecision in real world problems has been a long term research challenge that has originated different extensions of fuzzy sets. Atanassov [1] introduced the concept of an intuitionistic fuzzy set (IFS) which is characterized by two functions expressing degree of belongingness and the degree of nonbelongingness, respectively. Intuitionistic fuzzy sets can be useful to deal with situations where the classical fuzzy tools are not so efficient. In 2016, Nayagam, Jeevaraj, and Sivaraman [6] introduced a complete ranking of intuitionistic fuzzy numbers.

Ranking fuzzy numbers and intuitionistic fuzzy numbers (IFN) have started long back, but till date there is no common method available to rank any two given IFN. The fuzzy max order for fuzzy numbers has been primarily defined by Ramik and Rimanek in 1985. Since then, several researchers have extended this order for fuzzy vectors and then for fuzzy sets which are closed, convex, normal, and support bounded.

General fuzzy sets seem to be much more suitable for modeling real world-problems rather than fuzzy numbers or fuzzy vectors. In 2014, Kon [4] extended the fuzzy max order for fuzzy number to orders for fuzzy sets based on orderings of level sets of fuzzy sets. In [3], Ike and Tanaka generalized this idea and applied

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it to the six types of set relations proposed by Kuroiwa, Tanaka, and Ha [5] using a vector ordering by a convex cone. Ike [2] presented further characterizations of possibility-theoretical indices in fuzzy optimization in 2020.

In this research, a partial ordering to the level sets or cuts of intuitionistic fuzzy sets is firstly introduced. Next, some basic properties are established. Finally, we propose various intuitionistic fuzzy set relations, and their difference evaluation functions through generalizing the notion of fuzzy set relations introduced by Ike and Tanaka.

2 Preliminaries

Let Z be a real topological vector space unless otherwise specified. Let $\mathcal{P}(Z)$ denote the set of all nonempty subsets of Z. The topological interior, topological closure, convex hull, and complement of a set A are denoted by $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{co} A$, and A^c, respectively.

A set $C \in \mathcal{P}(Z)$ is a cone if $tz \in C$ for all $z \in C$ and $t > 0$. The transitive relation $\leq_C$ is induced by a convex cone C as follows: for $z, z' \in Z$, $z \leq_C z'$ if $z' - z \in C$. If the zero vector θ_Z in Z belongs to C, then $\leq_C$ is reflexive and hence a preorder. If not, then $\leq_C$ is irreflexive and hence a strict order.

Definition 1. Let $C \subset Z$ be a convex cone. The eight types of set relations are defined by

$$
\begin{aligned}
A \leq_C^{(1)} B &\overset{\text{def}}{\iff} \forall a \in A, \forall b \in B, a \leq_C b \iff A \subset \bigcap_{b \in B} (b - C);\\
A \leq_C^{(2L)} B &\overset{\text{def}}{\iff} \exists a \in A \,\text{s.t.}\, \forall b \in B, a \leq_C b \iff A \cap \left(\bigcap_{b \in B} (b - C) \right) \neq \varnothing;\\
A \leq_C^{(2U)} B &\overset{\text{def}}{\iff} \exists b \in B \,\text{s.t.}\, \forall a \in A, a \leq_C b \iff \left(\bigcap_{a \in A} (a + C) \right) \cap B \neq \varnothing;\\
A \leq_C^{(2)} B &\overset{\text{def}}{\iff} A \leq_C^{(2L)} B \text{ and } A \leq_C^{(2U)} B;\\
A \leq_C^{(3L)} B &\overset{\text{def}}{\iff} \forall b \in B, \exists a \in A \,\text{s.t.}\, a \leq_C b \iff B \subset A + C;\\
A \leq_C^{(3U)} B &\overset{\text{def}}{\iff} \forall a \in A, \exists b \in B \,\text{s.t.}\, a \leq_C b \iff A \subset B - C;\\
A \leq_C^{(3)} B &\overset{\text{def}}{\iff} A \leq_C^{(3L)} B \text{ and } A \leq_C^{(3U)} B;\\
A \leq_C^{(4)} B &\overset{\text{def}}{\iff} \exists a \in A, \exists b \in B \,\text{s.t.}\, a \leq_C b \iff A \cap (B - C) \neq \varnothing.
\end{aligned}
$$

for $A, B \in \mathcal{P}(Z)$.

A pair $\widetilde{A} = (\mu_{\widetilde{A}}, \nu_{\widetilde{A}})$ is called an *intuitionistic fuzzy set* or *IFS* on Z, where $\mu_{\widetilde{A}} : Z \to [0, 1]$ and $\nu_{\widetilde{A}} : Z \to [0, 1]$ are the membership and non-membership functions, respectively, such that $\mu_{\widetilde{A}}(z) + \nu_{\widetilde{A}}(z) \leq 1$ for all $z \in Z$. When $\mu_{\widetilde{A}}(z) + \nu_{\widetilde{A}}(z) = 1$, $\widetilde{A}$ is called a *fuzzy set*.

Example 2. Let $b_1 \leq a_1 \leq b_2 \leq a_2 \leq a_3 \leq b_3 \leq a_4 \leq b_4$, for some a_1, a_2, a_3, a_4, b_1, b_2, b_3 , $b_4 \in \mathbb{R}$. Then $\widetilde{A}$ defined on $\mathbb{R}$ as follows is an intuitionistic fuzzy set.

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0; & x \leq a_1 \\ \dfrac{x-a_1}{a_2-a_1}; & a_1 \leq x \leq a_2 \\ 1; & a_2 \leq x \leq a_3 \\ \dfrac{x-a_4}{a_3-a_4}; & a_3 \leq x \leq a_4 \\ 0; & a_4 \leq x, \end{cases} \qquad \nu_{\widetilde{A}}(x) = \begin{cases} 1; & x \leq b_1 \\ \dfrac{x-b_2}{b_1-b_2}; & b_1 \leq x \leq b_2 \\ 0; & b_2 \leq x \leq b_3 \\ \dfrac{x-b_3}{b_3-b_4}; & b_3 \leq x \leq b_4 \\ 1; & b_4 \leq x. \end{cases}$$

◀

Let $I = \{(\alpha, \beta) \in [0,1]^2 \mid \alpha + \beta \leq 1\}$ be the the set which is used to give values for α, β in the (α, β)-*cut* of $\widetilde{A}$ defined as

$$\widetilde{A}_{(\alpha,\beta)} = \{z \in Z \mid \mu_{\widetilde{A}}(z) \geq \alpha \ \& \ \nu_{\widetilde{A}}(z) \leq \beta\}.$$

We define $\preceq$ as the partial order where $(\alpha_1, \beta_1) \preceq (\alpha_2, \beta_2)$ means $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$. Clearly, $\widetilde{A}_{(\alpha_2,\beta_2)} \subset \widetilde{A}_{(\alpha_1,\beta_1)}$ whenever $(\alpha_1, \beta_1) \preceq (\alpha_2, \beta_2)$, for any $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in I$. Throughout this paper, the minimum element $\min \Delta$ and maximum element $\max \Delta$ are considered as singleton sets provided that they exist with respect to the partial order $\preceq$.

Example 3. Let $U = \{x, y, z\}$ and $\widetilde{A}$ be an IFS on U defined as follows:

$$\widetilde{A}(x) = (0.7, 0.1),\ \widetilde{A}(y) = (0.1, 0.1),\ \widetilde{A}(z) = (0.4, 0.5).$$

Then $\widetilde{A}_{(0.1,0.7)} = \{x, y, z\}, \widetilde{A}_{(0.2,0.8)} = \{y, z\}, \widetilde{A}_{(0.4,0.2)} = \{x\}$. ◀

$\widetilde{A}$ is said to be normal if $\widetilde{A}_{(1,0)} \neq \varnothing$ (or equivalently, $\widetilde{A}_{(\alpha,\beta)} \neq \varnothing$ for all $(\alpha, \beta) \in I$). We denote by $\mathcal{F}_{\mathcal{N}}(V)$ the set of all normal intuitionistic fuzzy sets on Z.

Defining the translation $\widetilde{A} + z$ for $z \in Z$ by

$$\mu_{\widetilde{A}+z}(z') = \mu_{\widetilde{A}}(z' - z) \text{ and } \nu_{\widetilde{A}+z}(z') = \nu_{\widetilde{A}}(z' - z)$$

and the scalar multiplication $\lambda\widetilde{A}$ for $\lambda \neq 0$ by

$$\mu_{\lambda\widetilde{A}}(z') = \mu_{\widetilde{A}}\left(\tfrac{1}{\lambda}z'\right) \text{ and } \nu_{\lambda\widetilde{A}}(z') = \nu_{\widetilde{A}}\left(\tfrac{1}{\lambda}z'\right)$$

we have $(\widetilde{A} + z)_{(\alpha,\beta)} = \widetilde{A}_{(\alpha,\beta)} + z$ and $(\lambda\widetilde{A})_{(\alpha,\beta)} = \lambda\widetilde{A}_{(\alpha,\beta)}$, for $(\alpha, \beta) \in I$.

For IFS $\widetilde{A}$ and $\widetilde{B}$, their union and intersection are defined respectively as

$$(\mu_{\widetilde{A}\cup\widetilde{B}}, \nu_{\widetilde{A}\cup\widetilde{B}}) = (\max\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)\}, \min\{\nu_{\widetilde{A}}(x), \nu_{\widetilde{B}}(x)\})$$

and

$$(\mu_{\widetilde{A}\cap\widetilde{B}}, \nu_{\widetilde{A}\cap\widetilde{B}}) = (\min\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)\}, \max\{\nu_{\widetilde{A}}(x), \nu_{\widetilde{B}}(x)\}).$$

It can be seen that for any normal IFS $\widetilde{A}$ and $\widetilde{B}$, $(\widetilde{A} \cup \widetilde{B})_{(\alpha,\beta)} \supset \widetilde{A}_{(\alpha,\beta)} \cup \widetilde{B}_{(\alpha,\beta)}$ and $(\widetilde{A} \cap \widetilde{B})_{(\alpha,\beta)} = \widetilde{A}_{(\alpha,\beta)} \cap \widetilde{B}_{(\alpha,\beta)}$.

3 Intuitionistic fuzzy set relations

By considering the set relations between (α, β)-cuts of two IFS, the intuitionistic fuzzy set relations are defined as follows.

Definition 4. Let $C \subset Z$ be a convex cone and $\varnothing \neq \Delta \subset I$. For each $j = 1, 2L, 2U, 2, 3L, 3U, 3, 4$, the *intuitionistic fuzzy set relation (IFSR)* $\leq_C^{\Delta(j)}$ is defined by

$$\widetilde{A} \leq_C^{\Delta(j)} \widetilde{B} \Longleftrightarrow \forall(\alpha, \beta) \in \Delta, \widetilde{A}_{(\alpha,\beta)} \leq_C^{(j)} \widetilde{B}_{(\alpha,\beta)}$$

for normal IFS $\widetilde{A}$ and $\widetilde{B}$.

The set Δ is a collection of (α, β) values that are of concern in comparing intuitionistic fuzzy sets.

From the definition, we easily obtain the following implications:

$$\begin{aligned} \widetilde{A} \leq_C^{\Delta(1)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(2L)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(3L)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(4)} \widetilde{B}; \\ \widetilde{A} \leq_C^{\Delta(1)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(2U)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(3U)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(4)} \widetilde{B}; \\ \widetilde{A} \leq_C^{\Delta(1)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(2)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(3)} \widetilde{B} \Longrightarrow \widetilde{A} \leq_C^{\Delta(4)} \widetilde{B} \end{aligned} \quad (1)$$

for any IFS $\widetilde{A}, \widetilde{B}$.

If Δ has a minimum with respect to the order $\preceq$,

$$\widetilde{A} \leq_C^{\Delta(1)} \widetilde{B} \Longleftrightarrow \widetilde{A}_{\min\Delta} \leq_C^{(1)} \widetilde{B}_{\min\Delta}. \quad (2)$$

Also, if Δ has a maximum with respect to the order $\preceq$,

$$\widetilde{A} \leq_C^{\Delta(4)} \widetilde{B} \Longleftrightarrow \widetilde{A}_{\max\Delta} \leq_C^{(4)} \widetilde{B}_{\max\Delta}. \quad (3)$$

Proposition 5. *Let $C \subset Z$ be a convex cone, $\varnothing \neq \Delta \subset I$, and $\widetilde{A}, \widetilde{B}, \widetilde{S}$ normal IFS on Z. If $\widetilde{A} \leq_C^{\Delta(1)} \widetilde{B}$, then $\widetilde{A} \cap \widetilde{S} \leq_C^{\Delta(j)} \widetilde{B} \cap \widetilde{S}$ for each $j = 1, 2L, 2U, 2, 3L, 3U, 3, 4$.*

Proposition 6. Let $C \subset Y$ be a convex cone, $k \in C$, and $\varnothing \neq \Delta \subset I$, and $\widetilde{A}, \widetilde{B}$ normal IFS on Z. For each $j = 1, 2L, 2U, 2, 3L, 3U, 3, 4$,

(i) if $\widetilde{A} + \bar{s}k \leq_C^{(j)} B$, for some $\bar{s} \in \mathbb{R}$, then $\widetilde{A} + sk \leq_C^{(j)} \widetilde{B}$, $\forall s \in (-\infty, \bar{s}]$;

(ii) if $\widetilde{A} + \bar{s}k \not\leq_C^{(j)} B$, for some $\bar{s} \in \mathbb{R}$, then $\widetilde{A} + sk \not\leq_C^{(j)} \widetilde{B}$, $\forall s \in [\bar{s}, +\infty)$.

As evaluation measure of the difference between two IFS, the following functions called difference evaluation functions for IFS are defined.

Definition 7. Let $C \subset Z$ be a convex cone, $k \in \operatorname{int} C$, and $\varnothing \neq \Delta \subset I$. For each $j = 1, 2L, 2U, 2, 3L, 3U, 3, 4$, the *difference evaluation function* $D_{C,k}^{\Delta(j)} : \mathcal{F}_{\mathcal{N}}(Z) \times \mathcal{F}_{\mathcal{N}}(Z) \to \mathbb{R} \cup \{\pm\infty\}$ is defined by

$$D_{C,k}^{\Delta(j)}(\widetilde{A}, \widetilde{B}) = \sup \left\{ t \in \mathbb{R} : \widetilde{A} + tk \leq_C^{\Delta(j)} \widetilde{B} \right\},$$

for $\widetilde{A}, \widetilde{B} \in \mathcal{F}_{\mathcal{N}}(V)$.

Proposition 8. Let $C \subset Z$ be a convex cone, $k \in \operatorname{int} C$, $\varnothing \neq \Delta \subset I$, and $\widetilde{A}$, $\widetilde{B}$ normal IFS on Z. Then for each $j = 1, 2L, 2U, 2, 3L, 3U, 3, 4$,

$$D_{C,k}^{\Delta(j)}(\widetilde{A}, \widetilde{B}) = \inf_{(\alpha,\beta)\in\Delta} \sup \left\{ t \in \mathbb{R} : \widetilde{A}_{(\alpha,\beta)} + tk \leq_C^{(j)} \widetilde{B}_{(\alpha,\beta)} \right\}.$$

Proposition 9. Let $C \subset Z$ be a convex cone, $k \in \operatorname{int} C$, $\varnothing \neq \Delta \subset I$, and $\widetilde{A}, \widetilde{B}$ normal IFS on Z. Then

(i) if Δ has a minimum,

$$D_{C,k}^{\Delta(1)}(\widetilde{A}, \widetilde{B}) = \sup \left\{ t \in \mathbb{R} \ : \ \widetilde{A}_{\min \Delta} + tk \leq_C^{(1)} \widetilde{B}_{\min \Delta} \right\};$$

(ii) if Δ has a maximum,

$$D_{C,k}^{\Delta(4)}(\widetilde{A}, \widetilde{B}) = \sup \left\{ t \in \mathbb{R} \ : \ \widetilde{A}_{\max \Delta} + tk \leq_C^{(4)} \widetilde{B}_{\max \Delta} \right\};$$

(iii)

$$D_{C,k}^{\Delta(2)}(\widetilde{A}, \widetilde{B}) = \min \left\{ D_{C,k}^{\Delta(2L)}(\widetilde{A}, \widetilde{B}), D_{C,k}^{\Delta(2U)}(\widetilde{A}, \widetilde{B}) \right\};$$

(iv)

$$D_{C,k}^{\Delta(3)}(\widetilde{A}, \widetilde{B}) = \min \left\{ D_{C,k}^{\Delta(3L)}(\widetilde{A}, \widetilde{B}), D_{C,k}^{\Delta(3U)}(\widetilde{A}, \widetilde{B}) \right\}.$$

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