Convex combinations associated with the curvature of the space and their natures
曲率に対応して定義される凸結合の幾何的性質

東邦大学・理学部 木村泰紀
Yasunori Kimura
Department of Information Science
Toho University
東邦大学・理学研究科 佐々木和哉
Kazuya Sasaki
Department of Information Science
Toho University

Abstract

In this paper, we consider another type of convex combinations associated with the curvature, and investigate their natures.

1 Introduction

A convex combination is one of the basic notion for the convex analysis, and its definition is very simple. In a real vector space V, a convex combination of two points x and y with a ratio $\alpha \in [0,1]$, which is usually denoted by $\alpha x + (1-\alpha)y$, is a weighted average of x and y for weights α and $1-\alpha$. The concept of convex combination is defined not only for real vector spaces but also for geodesic spaces. A geodesic space X is a metric space that any two points on X have the shortest path joining these points. In a geodesic space X, a convex combination of two points x and y with a ratio $\alpha \in [0,1]$ is generally defined as a point z satisfying $d(x,z) = (1-\alpha)d(x,y)$ and $d(y,z) = \alpha d(x,y)$. We usually write that point z as $\alpha x \oplus (1-\alpha)y$.

In 2020, we defined a new breed of convex combination $\overset{1}{\oplus}$ and showed the following theorem in the context of fixed point approximation on a complete CAT(1) space:

Theorem 1.1 ([3]). *Let X be an admissible complete* CAT(1) *space such that* $\sup_{s,s' \in X} d(s,s') < \pi/2$. *Let $S,T\colon X \to X$ be strongly quasinonexpansive and Δ-demiclosed mappings such that S and T have a common fixed point. Let $\{\alpha_n\}, \{\gamma_n\} \subset]0,1[$ and suppose $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$, and $\gamma_n \to \gamma \in]0,1[$. Take $v, w, x_1 \in X$ and generate a iterative sequence $\{x_n\} \subset X$ by $s_n = \alpha_n v \overset{1}{\oplus} (1-\alpha_n)Sx_n$,*

$t_n = \alpha_n w \overset{1}{\oplus} (1-\alpha_n)Tx_n$, *and* $x_{n+1} = \gamma_n s_n \overset{1}{\oplus} (1-\gamma_n)t_n$ *for* $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges to a common fixed point of* S *and* T. *Moreover, its limit is a maximizer of the function* $g\colon F \to]0,1]$ *defined by* $g(x) = \gamma \cos d(v,x) + (1-\gamma)\cos d(w,x)$ *for* $x \in F$, *where* F *is the set of all common fixed points of* S *and* T.

In Theorem 1.1, we need to use a new convex combination $\overset{1}{\oplus}$ instead of the traditional convex combination $\oplus$ for the limit of the sequence $\{x_n\}$ to be the maximizer of the function g. Indeed, if we only use $\oplus$ instead of $\overset{1}{\oplus}$, then we can verify that the limit of $\{x_n\}$ may differ from the maximizer of g. This result suggests that the traditional convex combination is somewhat incompatible with CAT(1) spaces, and that $\overset{1}{\oplus}$ may be better adapted to CAT(1) spaces; note that the function g is well compatible with CAT(1) spaces. Particularly, since the model space of CAT(1) spaces is the unit sphere $\mathbb{S}^2$, it is expected that the new convex combination $\overset{1}{\oplus}$ is adapted to a geodesic space with the constant curvature 1. In this paper, we consider the natures of the new convex combination $\overset{1}{\oplus}$ and investigate its behavior on the unit sphere on Hilbert spaces, and its generalization $\overset{\kappa}{\oplus}$.

2 Preliminaries

Let A be a set and $f\colon A \to \mathbb{R}$. If f has the unique minimizer t_0, then we write t_0 by $\operatorname{argmin}_{t\in A} f(t)$. Similarly, $\operatorname{argmax}_{t\in A} f(t)$ denotes the unique maximizer of f.

Let X be a metric space. For $x, y \in X$, a mapping $\gamma\colon [0,1] \to X$ is called a *geodesic* joining x and y if $\gamma(0) = y$, $\gamma(1) = x$, and $d(\gamma(s), \gamma(t)) = |s-t|d(x,y)$ hold for any $s, t \in [0,1]$. For $D \in]0,\infty]$, X is called a *uniquely D-geodesic space* if a geodesic joining x and y exists uniquely for any two points $x, y \in X$ with $d(x,y) < D$. In particular, a uniquely ∞-geodesic space is simply called a *uniquely geodesic space*.

Let X be a uniquely D-geodesic space and let $x, y \in X$ such that $d(x,y) < D$. Then a point $tx \oplus (1-t)y := \gamma(t)$ is called a *convex combination* of x and y, where γ is the unique geodesic joining x and y. The set of all convex combinations of x and y is denoted by $[x,y]$, that is, $[x,y] = \{tx \oplus (1-t)y \mid x, y \in X,\ t \in [0,1]\}$. Then we get $[x,y] = [y,x]$ obviously. We call $[x,y]$ a *geodesic segment* (on X) joining x and y. Furthermore, a subset $C \subset X$ is said to be *convex* if $[x,y] \subset C$ for any $x, y \in C$.

Let M_κ be the complete simply connected 2-dimensional Riemannian manifold with constant sectional curvature $\kappa \in \mathbb{R}$ and a metric ρ. It is equal to $\frac{1}{\sqrt{\kappa}}\mathbb{S}^2$, $\mathbb{R}^2$, $\frac{1}{\sqrt{-\kappa}}\mathbb{H}^2$ if $\kappa > 0$, $\kappa = 0$, $\kappa < 0$, respectively, where $\mathbb{S}^2$ is the 2-dimensional unit sphere, and $\mathbb{H}^2$ is the 2-dimensional hyperbolic space. We define $D_\kappa \in]0,\infty]$ by $D_\kappa = \infty$ if $\kappa \le 0$, and $D_\kappa = \pi/\sqrt{\kappa}$ if $\kappa > 0$, which means a diameter of M_κ. M_κ is a uniquely D_κ-geodesic space. In what follows, $[u,v]_{M_\kappa}$ denotes a geodesic segment joining $u, v \in M_\kappa$.

For $\kappa \in \mathbb{R}$, let X be a uniquely D_κ-geodesic space. For each $x, y, z \in X$ with $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$, we define a *geodesic triangle* with vertices x, y, z by $[x,y] \cup [y,z] \cup [z,x]$, and write it by $\triangle(x,y,z)$. For each $\triangle(x,y,z)$, there exists three points $\overline{x}, \overline{y}, \overline{z} \in M_\kappa$ such that $d(x,y) = \rho(\overline{x}, \overline{y})$, $d(y,z) = \rho(\overline{y}, \overline{z})$, and $d(z,x) =$

$\rho(\overline{z},\overline{x})$. For these points $\overline{x},\overline{y},\overline{z}$, we define a *comparison triangle* $\overline{\triangle}(\overline{x},\overline{y},\overline{z})$ by $[\overline{x},\overline{y}]_{M_\kappa}\cup [\overline{y},\overline{z}]_{M_\kappa}\cup[\overline{z},\overline{x}]_{M_\kappa}$. For any $\triangle(x,y,z)$ and a point $p\in\triangle(x,y,z)$, there exists a point $\overline{p}\in\overline{\triangle}(\overline{x},\overline{y},\overline{z})$ such that the distances from two adjacent vertices are identical. That point $\overline{p}$ is called a *comparison point* of p.

Let $\kappa\in\mathbb{R}$. A uniquely D_κ-geodesic space X is called a CAT(κ) *space* if for any $\triangle := \triangle(x,y,z)$ and its comparison triangle $\overline{\triangle} := \overline{\triangle}(\overline{x},\overline{y},\overline{z})$, and for any two points $p,q\in\triangle$ and these comparison points $\overline{p},\overline{q}\in\overline{\triangle}$, the inequality $d(p,q)\le\rho(\overline{p},\overline{q})$ holds. A CAT(κ) space X is said to be *admissible* if $d(x,y)<D_\kappa/2$ for every $x,y\in X$. If $\kappa\le 0$, then every CAT(κ) space is admissible.

By the definition of CAT(κ) spaces, the unit sphere $\mathbb{S}^2$ embedded in a Euclidean space $\mathbb{R}^3$, a Hilbert space H, the hyperbolic space $\mathbb{H}^2$ are a CAT(1) space, a CAT(0) space, a CAT(-1) space, respectively. For more details, see [1].

3 κ-convex combination

In this section, we introduce the definition of new convex combination which is called the κ-convex combination, and we investigate its nature.

For each $\kappa\in\mathbb{R}$, define $c_\kappa\colon\mathbb{R}\to\mathbb{R}$ by

$$c_\kappa(d)=\begin{cases}\dfrac{1}{-\kappa}(\cosh(\sqrt{-\kappa}\,d)-1) & (\text{if } \kappa<0),\\ \dfrac{1}{2}d^2 & (\text{if } \kappa=0),\\ \dfrac{1}{\kappa}(1-\cos(\sqrt{\kappa}\,d)) & (\text{if } \kappa>0)\end{cases}$$

for $d\in\mathbb{R}$. In particular, $c_{-1}(d)=\cosh d-1$ and $c_1(d)=1-\cos d$. Note that c_κ is strictly convex and increasing on $[0,D_\kappa]$ for any $\kappa\in\mathbb{R}$.

The first definition of κ-convex combinations $\overset{\kappa}{\oplus}$ for $\kappa=-1$ and $\kappa=1$ were given by [2] and [3], respectively. Later, properties of the κ-convex combination for general $\kappa\in\mathbb{R}$ was shown in [4].

Let X be a uniquely D_κ-geodesic space. In [2], [3] and [4], the κ-convex combination of x and y is defined under the condition $d(x,y)<D_\kappa/2$. Actually, we can weaken the assumption to $d(x,y)<D_\kappa$ when define the κ-convex combination. In this paper, we use the condition $d(x,y)<D_\kappa$ to define the κ-convex combination.

Theorem 3.1. *Let $\kappa\in\mathbb{R}$ and X a uniquely D_κ-geodesic space. Take $x,y\in X$ with $d(x,y)<D_\kappa$ and $\alpha\in[0,1]$. Define $g_\kappa\colon X\to\mathbb{R}$ by*

$$g_\kappa(z)=\alpha c_\kappa(d(x,z))+(1-\alpha)c_\kappa(d(y,z))$$

for $z\in X$. Then the restriction $g_\kappa|_{[x,y]}$ has the unique minimizer, where $[x,y]$ is the geodesic segment joining x and y.

Proof. If $d(x,y)<D_\kappa/2$, then we obtain the conclusion, see [2], [3] and [4]. Furthermore, if $\kappa\le 0$, then we also have the conclusion, since $D_\kappa=\infty=D_\kappa/2$. Thus

we only show the case where $\kappa > 0$. It is sufficient to prove the case where $\kappa = 1$, henceforth we will assume $\kappa = 1$.

Let $x, y \in X$, $\alpha \in [0,1]$ and put $D = d(x,y)$. If $D = 0$, then we obtain the desired result obviously. Suppose that $0 < D < \pi$. Then we have

$$g_1(tx \oplus (1-t)y) = 1 - (\alpha \cos((1-t)D) + (1-\alpha)\cos tD)$$

for any $t \in [0,1]$. Define $f\colon [0,1] \to \mathbb{R}$ by $f(t) = \alpha\cos((1-t)D) + (1-\alpha)\cos tD$ for $t \in [0,1]$. Then $f'(t)/D = \alpha \sin((1-t)D) - (1-\alpha)\sin tD$ holds for each $t \in [0,1]$. Let $\tan^{-1}\colon \mathbb{R} \to [0,\pi[\setminus \{\pi/2\}$ be the inverse of the trigonometric tangent function. Then putting

$$t_0 = \frac{1}{D}\tan^{-1}\frac{\alpha \sin D}{1-\alpha+\alpha\cos D},$$

we get $t_0 \in [0,1]$ and $f'(t_0) = 0$. Take $t \in [0,1]$ arbitrarily. If $t < t_0$, then we obtain

$$f'(t)/D = \alpha\sin((1-t)D) - (1-\alpha)\sin tD > \alpha \sin((1-t_0)D) - (1-\alpha)\sin t_0 D = 0.$$

Similarly, if $t > t_0$, then $f'(t)/D < 0$. It concludes t_0 is the unique maximizer of f, and hence $t_0 x \oplus (1-t_0)y = \operatorname{argmin}_{z\in[x,y]} g_1(z)$. □

Let $\kappa \in \mathbb{R}$ and X a uniquely D_κ-geodesic space. Let $\alpha \in [0,1]$ and $x, y \in X$ such that $d(x,y) < D_\kappa$. Suppose that $d(x,y) < D_\kappa$. Then the unique minimizer of $g_\kappa|_{[x,y]}$ in Theorem 3.1 is called a *κ-convex combination* of x and y, and we write it by $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y$. That is, $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y = \operatorname{argmin}_{z\in[x,y]} g_\kappa(z)$. Note that $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y$ can be expressed by using a traditional convex combination $tx \oplus (1-t)y$. In fact, define $t \in [0,1]$ by

$$t = \begin{cases} \dfrac{1}{\sqrt{-\kappa}\,d(x,y)}\tanh^{-1}\dfrac{\alpha\sinh\left(\sqrt{-\kappa}\,d(x,y)\right)}{1-\alpha+\alpha\cosh\left(\sqrt{-\kappa}\,d(x,y)\right)} & (\text{if } \kappa < 0 \text{ and } x \neq y); \\ \alpha & (\text{if } \kappa = 0 \text{ or } x = y); \\ \dfrac{1}{\sqrt{\kappa}\,d(x,y)}\tan^{-1}\dfrac{\alpha\sin\left(\sqrt{\kappa}\,d(x,y)\right)}{1-\alpha+\alpha\cos\left(\sqrt{\kappa}\,d(x,y)\right)} & (\text{if } \kappa > 0 \text{ and } x \neq y). \end{cases}$$

Then we get

$$1-t = \begin{cases} \dfrac{1}{\sqrt{-\kappa}\,d(x,y)}\tanh^{-1}\dfrac{(1-\alpha)\sinh\left(\sqrt{-\kappa}\,d(x,y)\right)}{\alpha+(1-\alpha)\cosh\left(\sqrt{-\kappa}\,d(x,y)\right)} & (\text{if } \kappa < 0 \text{ and } x \neq y); \\ 1-\alpha & (\text{if } \kappa = 0 \text{ or } x = y); \\ \dfrac{1}{\sqrt{\kappa}\,d(x,y)}\tan^{-1}\dfrac{(1-\alpha)\sin\left(\sqrt{\kappa}\,d(x,y)\right)}{\alpha+(1-\alpha)\cos\left(\sqrt{\kappa}\,d(x,y)\right)} & (\text{if } \kappa > 0 \text{ and } x \neq y) \end{cases}$$

and $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y = tx \oplus (1-t)y$, where $\tanh^{-1}\colon [0,1[\to [0,\infty[$ is the inverse of the hyperbolic tangent function, and $\tan^{-1}\colon \mathbb{R} \to [0,\pi[\setminus \{\pi/2\}$ is the inverse of the trigonometric tangent function.

For $\kappa \in \mathbb{R}$, let X be a uniquely D_κ-geodesic space. Then, the following properties hold for any $\kappa \in \mathbb{R}$, $\alpha \in [0,1]$, and $x, y \in X$ with $d(x,y) < D_\kappa$.

(a) $1x \overset{\kappa}{\oplus} 0y = x$ and $0x \overset{\kappa}{\oplus} 1y = y$.
(b) $\alpha x \overset{\kappa}{\oplus} (1-\alpha)x = x$.
(c) $\frac{1}{2}x \overset{\kappa}{\oplus} \frac{1}{2}y = \frac{1}{2}x \oplus \frac{1}{2}y$.

These properties (a), (b) and (c) are obtained directly from the definition of κ-convex combination.

Theorem 3.2. *The 0-convex combination $\overset{0}{\oplus}$ is identical with the traditional convex combination $\oplus$.*

Proof. For $D \in]0, \infty]$, let X be a uniquely D-geodesic space and take $x, y \in X$ with $d(x,y) < D$. Then we show $\alpha x \overset{0}{\oplus} (1-\alpha)y = \alpha x \oplus (1-\alpha)y$ for any $\alpha \in [0,1]$. Since $\alpha x \overset{0}{\oplus} (1-\alpha)y \in [x,y]$, we get

$$\alpha x \overset{0}{\oplus} (1-\alpha)y = \operatorname*{argmin}_{z \in [x,y]} \left(\alpha d(x,z)^2 + (1-\alpha)d(y,z)^2\right) = \alpha' x \oplus (1-\alpha')y,$$

where

$$\alpha' = \operatorname*{argmin}_{t \in [0,1]} \left(\alpha((1-t)d(x,y))^2 + (1-\alpha)(td(x,y))^2\right) = \alpha.$$

Thus we get the conclusion. □

Lemma 3.3. *Let $\kappa \in \mathbb{R}$ and X a uniquely D_κ-geodesic space. Take $x, y \in X$ with $d(x,y) < D_\kappa$ and $\alpha \in [0,1]$. Define $g_\kappa \colon X \to \mathbb{R}$ by $g_\kappa(z) = \alpha c_\kappa(d(x,z)) + (1-\alpha)c_\kappa(d(y,z))$ for $z \in X$. Let C be a subset of X such that $d(u,v) < D_\kappa$ for any $u, v \in C$ and $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y \in C$. Then $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y = \operatorname{argmin}_{z\in C} g_\kappa(z)$.*

Proof. Put $v = \alpha x \overset{\kappa}{\oplus} (1-\alpha)\, y = \operatorname{argmin}_{z \in [x,y]} g_\kappa(z) \in C$. If $x = y$, then we obtain $v = x = \operatorname{argmin}_{z\in C} c_\kappa(d(x,z)) = \operatorname{argmin}_{z \in C} g_\kappa(z)$, which is the conclusion. Suppose that $x \neq y$ and take $w \in C \setminus \{v\}$ arbitrarily. Put $t = d(y,w)/(d(x,w)+d(y,w))$ and $v' = tx \oplus (1-t)\, y$. Then $d(x,v') : d(y,v') = d(x,w) : d(y,w)$. Moreover, we obtain $g_\kappa(v) \le g_\kappa(v')$, notably we get $g_\kappa(v) < g_\kappa(v')$ if $v \neq v'$.

Suppose that $v = v'$. Then we get $w \neq v'$ and hence $w \notin [x,y]$. Thus we have $d(x,v') + d(y,v') = d(x,y) < d(x,w) + d(y,w)$. It implies that $d(x,v') < d(x,w)$ and $d(y,v') < d(y,w)$. Therefore we get $g_\kappa(v') < g_\kappa(w)$ and it follows that $g_\kappa(v) < g_\kappa(w)$.

Next we assume $v \neq v'$. Then we have $d(x,v') \le d(x,w)$ and $d(y,v') \le d(y,w)$, and hence $g_\kappa(v') \le g_\kappa(w)$. It implies $g_\kappa(v) < g_\kappa(w)$ and thus we get the conclusion. □

Corollary 3.4. *Let $\kappa \in \mathbb{R}$ and X a uniquely geodesic space such that $d(u,v) < D_\kappa$ for any $u, v \in X$. Take $x, y \in X$, $\alpha \in [0,1]$ and define $g_\kappa \colon X \to \mathbb{R}$ by $g_\kappa(z) = \alpha c_\kappa(d(x,z)) + (1-\alpha)c_\kappa(d(y,z))$ for $z \in X$. Then $\alpha x \overset{\kappa}{\oplus} (1-\alpha)y = \operatorname{argmin}_{z\in X} g_\kappa(z)$.*

4 1-convex combination

The κ-convex combination is not defined only in geodesic manifolds with a curvature κ. For instance, we can define κ-convex combinations on an Euclidean space $\mathbb{R}^n$ for any $\kappa \in \mathbb{R}$. However, not all of κ-convex combinations have good properties on $\mathbb{R}^n$. In fact, it is obvious that the most useful κ-convex combination on $\mathbb{R}^n$ is the 0-convex combination. We consider that the κ-convex combination defined on a geodesic manifold with a curvature exactly κ should play a beneficial role, that is implied by previous studies [2, 3, 4].

In this section, we investigate properties of the 1-convex combination on geodesic spaces. Additionally, we confirm that the 1-convex combination has good behavior on the unit sphere in an Hilbert space, especially the 2-dimensional unit sphere $\mathbb{S}^2$.

4.1 1-convex combination on geodesic spaces

For $D \in \,]0, \pi]$, let X be a uniquely D-geodesic space. Then the 1-convex combination of $x, y \in X$ is defined by

$$\begin{aligned} \alpha x \overset{1}{\oplus} (1-\alpha) y &= \operatorname*{argmin}_{z \in X} (\alpha c_1(d(x,z)) + (1-\alpha) c_1(d(y,z))) \\ &= \operatorname*{argmax}_{z \in [x,y]} (\alpha \cos d(x,z) + (1-\alpha) \cos d(y,z)) \end{aligned}$$

for each $\alpha \in [0,1]$, where $d(x,y) < D$.

Lemma 4.1. *For $D \in \,]0, \pi]$, let X be a uniquely D-geodesic space. Let $x, y \in X$ such that $0 < d(x,y) < D$, and put $d_0 = d(x,y)$. Then for any $\alpha \in [0,1]$,*

$$\begin{aligned} &\alpha x \overset{1}{\oplus} (1-\alpha) y \\ &\quad = \left(\frac{1}{d_0} \tan^{-1} \frac{\alpha \sin d_0}{1 - \alpha + \alpha \cos d_0} \right) x \oplus \left(\frac{1}{d_0} \tan^{-1} \frac{(1-\alpha) \sin d_0}{\alpha + (1-\alpha) \cos d_0} \right) y. \end{aligned}$$

Proof. The proof of Theorem 3.1 exactly implies the conclusion. □

Let X be a CAT(1) space, and take $\triangle(x,y,z) \subset X$ and $\alpha \in [0,1]$ arbitrarily. Then

$$\cos d(\alpha x \oplus (1-\alpha) y, z) \sin D \geq \sin(\alpha D) \cos d(x,z) + \sin((1-\alpha) D) \cos d(y,z) \tag{i}$$

holds, where $D = d(x,y)$. This inequality is often called the parallelogram law on CAT(1) spaces. In an admissible subspace S of the unit sphere $\mathbb{S}^2$, the inequality (i) holds as the equation. On the other hand, for any $\triangle(x,y,z) \subset X$ and $\alpha \in [0,1]$,

$$\cos d(\alpha x \overset{1}{\oplus} (1-\alpha) y, z) \geq \frac{\alpha \cos d(x,z) + (1-\alpha) \cos d(y,z)}{\sqrt{\alpha^2 + 2\alpha(1-\alpha) \cos D + (1-\alpha)^2}} \tag{ii}$$

holds. Incidentally, we know that two inequalities are equivalent, which can be proved from Lemma 4.1, see [3]. Therefore, in S, the inequality (ii) also holds as the equation.

Lemma 4.2. *Let $d \in]0, \pi/2[$ and define $f\colon]0,1[\to \mathbb{R}$ by $f(t) = (\sin td)/t$ for $t \in]0,1[$. Then f is strictly decreasing.*

Lemma 4.3. *Let $d \in]0, \pi/2[$, $\alpha \in]0,1[$ and put*

$$\sigma = \frac{1}{d} \tan^{-1} \frac{\alpha \sin d}{1 - \alpha + \alpha \cos d} \in]0,1[.$$

Then the following hold:

- *If $\alpha < 1/2$, then $\alpha > \sigma$;*
- *if $\alpha = 1/2$, then $\alpha = \sigma$;*
- *if $\alpha > 1/2$, then $\alpha < \sigma$.*

Proof. The case where $\alpha = 1/2$ is obviously true. It is enough to prove only the case where $\alpha < 1/2$ by the symmetric property.

Suppose that $\alpha < 1/2$, and define a strictly concave function $g\colon [0,1] \to \mathbb{R}$ by $g(t) = \alpha \cos((1-t)d) + (1-\alpha)\cos td$ for $t \in [0,1]$. Then σ is a unique maximizer of g. In addition, we obtain

$$\begin{aligned} g'(\alpha) &= \alpha d \sin((1-\alpha)d) - (1-\alpha) d \sin \alpha d \\ &= \alpha(1-\alpha)d \cdot \left(\frac{\sin((1-\alpha)d)}{1-\alpha} - \frac{\sin \alpha d}{\alpha} \right) < 0 \end{aligned}$$

from Lemma 4.2. It implies $\alpha > \sigma$ and thus we get the conclusion. □

Corollary 4.4. *For $D \in]0, \pi]$, let X be a uniquely D-geodesic space, and take $x, y \in X$ such that $0 < d(x,y) < R$. Let $\alpha \in]0,1[$. Then $\alpha x \overset{1}{\oplus} (1-\alpha) y = \alpha x \oplus (1-\alpha) y$ holds if and only if $\alpha = 1/2$.*

Proof. Lemma 4.3 implies the conclusion. □

Corollary 4.5. *For $D \in]0, \pi]$, let X be a uniquely D-geodesic space, and take $x, y \in X$ such that $0 < d(x,y) < R$. Let $\alpha \in]0,1[\setminus \{1/2\}$. Then a point $u_1 = \alpha x \overset{1}{\oplus} (1-\alpha) y$ is farther from the midpoint $\frac{1}{2}x \oplus \frac{1}{2}y$ than $u_0 = \alpha x \oplus (1-\alpha) y$.*

Proof. Put $\sigma x \oplus (1-\sigma) y := u_1$. If $\alpha < 1/2$, then we have $1/2 > \alpha > \sigma$ by Lemma 4.3. Otherwise, we get $1/2 < \alpha < \sigma$. Therefore u_1 is farther from the midpoint $\frac{1}{2}x \oplus \frac{1}{2}y$ than u_0 in both cases. □

Lemma 4.6. *Let $d \in]0, \pi/2[$, and define a function $f\colon [0,1] \to [0,1]$ by*

$$f(\alpha) = \frac{1}{d} \tan^{-1} \frac{\alpha \sin d}{1 - \alpha + \alpha \cos d}$$

for $\alpha \in [0,1]$. Then f is continuous, strictly increasing, and bijective.

Proof. By basic calculations, we get $f'(\alpha) > 0$ for any $\alpha \in [0,1]$. Since $f(0) = 0$ and $f(1) = 1$, we get the conclusion. □

Corollary 4.7. *For $D \in \,]0, \pi]$, let X be a uniquely D-geodesic space, and take $x, y \in X$ such that $0 < d(x, y) < D$. Then $[x, y] = \{tx \overset{1}{\oplus} (1-t)y \mid t \in [0, 1]\}$.*

Proof. Define a function $f\colon [0, 1] \to [0, 1]$ by

$$f(\alpha) = \frac{1}{D} \tan^{-1} \frac{\alpha \sin D}{1 - \alpha + \alpha \cos D}$$

for $\alpha \in [0, 1]$. Then we have $\{tx \overset{1}{\oplus} (1-t)y \mid t \in [0, 1]\} = \{f(t)x \oplus (1-f(t))y \mid t \in [0, 1]\}$ by Lemma 4.1, thus we get the conclusion by bijectivity of f. □

Corollary 4.8. *For $D \in \,]0, \pi]$, let X be a uniquely D-geodesic space, and take $x, y \in X$ such that $0 < d(x, y) < D$. Put $d_0 = d(x, y)$. Then for any $\sigma \in [0, 1]$,*

$$\sigma x \oplus (1-\sigma)y = \frac{\sin(\sigma d_0)}{\sin(\sigma d_0) + \sin((1-\sigma)d_0)}\, x \overset{1}{\oplus} \frac{\sin((1-\sigma)d_0)}{\sin(\sigma d_0) + \sin((1-\sigma)d_0)}\, y.$$

Proof. Take $\sigma \in [0, 1]$. Then there exists $\alpha \in [0, 1]$ such that $\alpha x \overset{1}{\oplus} (1-\alpha)y = \sigma x \oplus (1-\sigma)y$ by Corollary 4.7. Thus, using Lemma 4.1, we obtain

$$\sigma = \frac{1}{d_0} \tan^{-1} \frac{\alpha \sin d_0}{1 - \alpha + \alpha \cos d_0},$$

which is equivalent to

$$\alpha = \frac{\sin(\sigma d_0)}{\sin(\sigma d_0) + \sin((1-\sigma)d_0)}.$$

Consequently we obtain the conclusion. □

Lemma 4.9. *For $a, b, c, d \in \mathbb{R}$,*

$$\begin{aligned}\sin((a+b)(c-d)) \sin((a-b)(c+d)) - \sin((a+b)(c+d)) \sin((a-b)(c-d)) \\ = -\sin 2ac \sin 2bd + \sin 2ad \sin 2bc.\end{aligned}$$

Lemma 4.10. *Let $k \in \,]0, 1[$ and define $f\colon]0, \pi[\,\to \mathbb{R}$ by $f(x) = (\sin kx)/\sin x$ for $x \in \,]0, \pi[$. Then f is strictly increasing.*

Theorem 4.11. *Let $\alpha \in \,]0, 1[$, and define a function $f\colon]0, \pi/2[\,\to \,]0, 1[$ by*

$$f(d) = \frac{1}{d} \tan^{-1} \frac{\alpha \sin d}{1 - \alpha + \alpha \cos d}$$

for $d \in \,]0, \pi/2[$. Then the following hold:

- $\lim_{d \to 0} f(d) = \alpha$;
- *if $\alpha < 1/2$, then f is strictly decreasing;*
- *if $\alpha > 1/2$, then f is strictly increasing.*

Proof. The equation $\lim_{d\to 0} f(d) = \alpha$ can be verified easily, thus we prove the other properties. It suffices to show the case where $\alpha < 1/2$. Let $\alpha \in \,]0, 1/2[$, $d_1, d_2 \in \,]0, \pi/2[$ and suppose $d_1 < d_2$. Put $\sigma_1 = f(d_1)$ and $\sigma_2 = f(d_2)$. Then we obtain $\sigma_1 < 1/2$ and $\sigma_2 < 1/2$ by Lemma 4.3. Moreover, using the equation $\sigma_2 = f(d_2)$, we get

$$\alpha = \frac{\sin(\sigma_2 d_2)}{\sin(\sigma_2 d_2) + \sin((1-\sigma_2)d_2)}. \tag{iii}$$

Define a strictly concave function $g\colon [0,1] \to \mathbb{R}$ by

$$g(t) = \alpha \cos((1-t)d_1) + (1-\alpha)\cos td_1$$

for $t \in [0,1]$. Then σ_1 is a unique maximizer of g. By the formula (iii), we obtain

$$g(t) = \frac{\sin(\sigma_2 d_2)\cos((1-t)d_1) + \sin((1-\sigma_2)d_2)\cos td_1}{\sin(\sigma_2 d_2) + \sin((1-\sigma_2)d_2)}$$

for any $t \in [0,1]$ and hence

$$g'(t) = \frac{d_1\left(\sin(\sigma_2 d_2)\sin((1-t)d_1) - \sin((1-\sigma_2)d_2)\sin td_1\right)}{\sin(\sigma_2 d_2) + \sin((1-\sigma_2)d_2)}$$

for any $t \in [0,1]$. Put

$$C = \frac{d_1}{\sin(\sigma_2 d_2) + \sin((1-\sigma_2)d_2)}.$$

Then we get $C > 0$ and

$$\frac{1}{C} g'(\sigma_2) = \sin(\sigma_2 d_2)\sin((1-\sigma_2)d_1) - \sin((1-\sigma_2)d_2)\sin(\sigma_2 d_1).$$

Put $p = (d_1 + d_2)/2$, $q = (d_2 - d_1)/2$, and $k = 1 - 2\sigma_2$. Then using Lemma 4.9, we have

$$\begin{aligned}
\frac{1}{C} g'(\sigma_2) &= \sin\left((p+q)\left(\frac{1}{2} - \frac{1}{2}k\right)\right)\sin\left((p-q)\left(\frac{1}{2} + \frac{1}{2}k\right)\right) \\
&\quad - \sin\left((p+q)\left(\frac{1}{2} + \frac{1}{2}k\right)\right)\sin\left((p-q)\left(\frac{1}{2} - \frac{1}{2}k\right)\right) \\
&= -\sin kp \sin q + \sin kq \sin p \\
&= \sin p \sin q \left(\frac{\sin kq}{\sin q} - \frac{\sin kp}{\sin p}\right).
\end{aligned}$$

Since $0 < q < p < \pi/2$ and $0 < k < 1$, we get $g'(\sigma_2) > 0$ from Lemma 4.10. Therefore we obtain $\sigma_1 > \sigma_2$ and it implies $f(d_1) > f(d_2)$. □

Theorem 4.11 implies that the greater the distance between two points x and y, the further the point $\alpha x \overset{1}{\oplus} (1-\alpha)y$ is from the midpoint of x and y as a ratio than the point $\alpha x \oplus (1-\alpha)y$.

4.2 1-convex combination on unit spheres

Next, we observe the nature of the 1-convex combination on a unit sphere of a Hilbert space to know a relation between $\oplus$ and $\overset{1}{\oplus}$. Hereafter, we consider S_H the unit sphere embedded in a Hilbert space H, that is, $S_H = \{x \in H \mid \|x\| = 1\}$. Suppose that a metric $d\colon S_H \to [0, \pi]$ is defined by $d(x, y) = \cos^{-1}\langle x, y\rangle$ for each $x, y \in S_H$, where $\cos^{-1}\colon [-1, 1] \to [0, \pi]$ is the inverse of the trigonometric cosine function. Then S_H is a complete CAT(1) space. If $H = \mathbb{R}^3$, then S_H becomes a model of the unit sphere $\mathbb{S}^2$, which has a constant curvature 1.

In what follows, $[x, y]$ denotes a geodesic segment on S_H joining $x, y \in S_H$, and $[x, y]_H$ denotes a geodesic segment on H joining $x, y \in H$. Furthermore, we write 0_H for the origin of H.

Theorem 4.12. *Let $x, y \in S_H$ such that $0 < d(x, y) < \pi$. Then a convex combination $tx \oplus (1-t)y \in S_H$ is expressed by*

$$tx \oplus (1-t)y = \frac{\sin(td(x, y))}{\sin d(x, y)}x + \frac{\sin((1-t)d(x, y))}{\sin d(x, y)}y$$

for any $t \in [0, 1]$.

Theorem 4.13. *Let $x, y \in S_H$ such that $d(x, y) < \pi$. Then a 1-convex combination $tx \overset{1}{\oplus} (1-t)y \in S_H$ is expressed by*

$$tx \overset{1}{\oplus} (1-t)y = \frac{tx + (1-t)y}{\|tx + (1-t)y\|}$$

for any $t \in [0, 1]$.

Proof. By the definition of 1-convex combination, we have

$$\begin{aligned} tx \overset{1}{\oplus} (1-t)y &= \operatorname*{argmax}_{z \in S_H} (t \cos d(x, z) + (1-t) \cos d(y, z)) \\ &= \operatorname*{argmax}_{z \in S_H} \langle tx + (1-t)y, z\rangle . \end{aligned}$$

Put $p = tx + (1-t)y$ and $w = p/\|p\|$. Then for any $z \in S_H$, we obtain

$$\langle tx + (1-t)y, w\rangle - \langle tx + (1-t)y, z\rangle = \|p\| - \langle p, z\rangle = \|p\|\|z\| - \langle p, z\rangle \geq 0.$$

Thus we get $tx \overset{1}{\oplus} (1-t)y = w$, which is the desired result. □

Corollary 4.14. *Take $x, y \in S_H$ with $d(x, y) < \pi$. For $\alpha \in [0, 1]$, take $u = \alpha x + (1-\alpha)y \in H$ and put $v = \alpha x \overset{1}{\oplus} (1-\alpha)y \in [x, y]$. Then three points u, v, and 0_H are on a straight line.*

Proof. Since $v = u/\|u\|$, we get the conclusion. □

Theorem 4.13 implies that $\alpha x \overset{1}{\oplus} (1-\alpha)y \in S_H$ is a projection of $\alpha x + (1-\alpha)y \in H$ into the unit sphere S_H.

Lemma 4.15. *Take $x, y \in S_H$ with $d(x,y) < \pi$. Let $k, l \in]0,1]$ and put $x' = kx$, $y' = ly$. Then the geodesic segment $[x,y] \subset S_H$ is expressed by*

$$[x,y] = \left\{ \frac{tx' + (1-t)y'}{\|tx' + (1-t)y'\|} \;\middle|\; t \in [0,1] \right\} = \left\{ \frac{p}{\|p\|} \;\middle|\; p \in [x', y']_H \right\}.$$

Proof. Take $u \in [x,y]$ arbitrarily. Then there exists $t \in [0,1]$ such that $u = tx \overset{1}{\oplus} (1-t)y$ by Corollary 4.7. Thus, putting $t' = tl/(tl + (1-t)k)$, we get

$$u = \frac{tx + (1-t)y}{\|tx + (1-t)y\|} = \frac{t'x' + (1-t')y'}{\|t'x' + (1-t')y'\|}.$$

On the other hand, take $s \in [0,1]$ and put $u' = (sx' + (1-s)y')/\|sx' + (1-s)y'\|$. Then putting $s' = sk/(sk + (1-s)l)$, we obtain

$$u' = \frac{sx' + (1-s)y'}{\|sx' + (1-s)y'\|} = \frac{s'x + (1-s')y}{\|s'x + (1-s')y\|} = s'x \overset{1}{\oplus} (1-s')y \in [x,y],$$

which implies the conclusion. □

Lemma 4.15 yields the following two corollaries.

Corollary 4.16. *Take $x, y \in S_H$ arbitrarily. Let $k, l \in]0,1]$ and put $x' = kx$, $y' = ly$. Then $v/\|v\| \in [x,y]$ holds for any $v \in [x', y']_H$.*

Corollary 4.17. *Take $x, y \in S_H$ arbitrarily. Let $k, l \in]0,1]$ and put $x' = kx$, $y' = ly$. Then for any $u \in [x,y]$, there exists $v \in [x', y']_H$ such that $u = v/\|v\|$.*

Fact 4.18 (Ceva's theorem in plane geometry)**.** *Let V be a real vector space and $x, y, z \in V$. For $\alpha, \beta, \gamma \in]0,1[$, take $p = (1-\alpha)x + \alpha y$, $q = (1-\beta)y + \beta z$ and $r = (1-\gamma)z + \gamma x$. Put $[u,v]_V = \{tu + (1-t)v \mid t \in [0,1]\}$ for each $u, v \in V$. Suppose that $[x,y]_V \cap [y,z]_V \cap [z,x]_V = \varnothing$. Then $[x,q]_V \cap [y,r]_V \cap [z,p]_V \neq \varnothing$ if and only if*

$$\frac{\alpha}{1-\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{\gamma}{1-\gamma} = 1.$$

Using the 1-convex combination and the fact above, we get the following theorem which can be said to be Ceva's theorem on the unit sphere.

Theorem 4.19. *Let S be a nonempty convex subspace of S_H such that $d(u,v) < \pi$ for any $u, v \in S$, and $\triangle(x,y,z)$ a geodesic triangle on S such that $[x,y] \cap [y,z] \cap [z,x] = \varnothing$. For $\alpha, \beta, \gamma \in]0,1[$, take $p = (1-\alpha)x \overset{1}{\oplus} \alpha y$, $q = (1-\beta)y \overset{1}{\oplus} \beta z$ and $r = (1-\gamma)z \overset{1}{\oplus} \gamma x$. Then $[x,q] \cap [y,r] \cap [z,p] \neq \varnothing$ if and only if*

$$\frac{\alpha}{1-\alpha} \cdot \frac{\beta}{1-\beta} \cdot \frac{\gamma}{1-\gamma} = 1.$$

Proof. Let $\triangle_H(x,y,z) = [x,y]_H \cup [y,z]_H \cup [z,x]_H$ be a geodesic triangle on H. Put $\overline{p} = (1-\alpha)x + \alpha y$, $\overline{q} = (1-\beta)y + \beta z$, and $\overline{r} = (1-\gamma)z + \gamma x$. Then we have $p = \overline{p}/\|\overline{p}\|$, $q = \overline{q}/\|\overline{q}\|$, $r = \overline{r}/\|\overline{r}\|$, and $\overline{p}, \overline{q}, \overline{r} \in \triangle_H(x,y,z)$. By Fact 4.18, we obtain $[x,\overline{q}]_H \cap [y,\overline{r}]_H \cap [z,\overline{p}]_H \neq \varnothing$ holds if and only if $\alpha\beta\gamma/((1-\alpha)(1-\beta)(1-\gamma)) = 1$. Furthermore, Corollaries 4.16 and 4.17 imply that $[x,\overline{q}]_H \cap [y,\overline{r}]_H \cap [z,\overline{p}]_H \neq \varnothing$ if and only if $[x,q] \cap [y,r] \cap [z,p] \neq \varnothing$. □

5 Balanced 1-convex combination

In a Hilbert space H, let $x_1, x_2, \ldots, x_m \in H$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \in [0,1]$ such that $\sum_{i=1}^m \alpha_i = 1$. Then

$$\sum_{i=1}^m \alpha_i x_i = \operatorname*{argmin}_{z \in H} \sum_{i=1}^m \alpha_i \|x_i - z\|^2$$

holds. Based on this fact, we generalize the 1-convex combination to be defined for a finite number of points. Let S be a nonempty convex subspace of S_H such that $d(u,v) < \pi$ for any $u, v \in S$. For $x_1, x_2, \ldots, x_m \in S$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \in [0,1]$ with $\sum_{i=1}^m \alpha_i = 1$, we define $B(\{x_1, \ldots, x_m\}, \{\alpha_1, \ldots, \alpha_m\}) \in S$ by

$$B(\{x_1, \ldots, x_m\}, \{\alpha_1, \ldots, \alpha_m\}) = \operatorname*{argmax}_{z \in S} \sum_{i=1}^m \alpha_i \cos d(x_i, z).$$

We often write this point simply as $B(\{x_i\}, \{\alpha_i\})$. We call the point $B(\{x_i\}, \{\alpha_i\})$ a *balanced* 1*-convex combination* of $x_1, x_2, \ldots, x_m$ on S. The 1-convex combination is the case where $m = 2$ for the balanced 1-convex combination.

Theorem 5.1. *Let S be a nonempty convex subspace of S_H such that $d(u,v) < \pi$ for any $u, v \in S$, and take $x_1, x_2, \ldots, x_m \in S$ arbitrarily. Then a balanced 1-convex combination $B(\{x_i\}, \{\alpha_i\}) \in S$ is well-defined, and it is expressed by*

$$B(\{x_i\}, \{\alpha_i\}) = \sum_{i=1}^m \alpha_i x_i \Bigg/ \left\| \sum_{i=1}^m \alpha_i x_i \right\|$$

for any $\alpha_1, \alpha_2, \ldots, \alpha_m \in [0,1]$ such that $\sum_{i=1}^m \alpha_i = 1$.

Proof. By the definition of $B(\{x_i\}, \{\alpha_i\})$, we have

$$B(\{x_i\}, \{\alpha_i\}) = \operatorname*{argmax}_{z \in S} \sum_{i=1}^m \alpha_i \cos d(x_i, z) = \operatorname*{argmax}_{z \in S} \left\langle \sum_{i=1}^m \alpha_i x_i, z \right\rangle.$$

Put $p = \sum_{i=1}^m \alpha_i x_i$ and $w = p/\|p\| \in S$. Then for any $z \in S \setminus \{p\}$, we obtain

$$\left\langle \sum_{i=1}^m \alpha_i x_i, w \right\rangle - \left\langle \sum_{i=1}^m \alpha_i x_i, z \right\rangle = \|p\| - \langle p, z \rangle = \|p\|\|z\| - \langle p, z \rangle > 0$$

and hence we get the conclusion. □

Theorem 5.1 is a generalization of Theorem 4.13.

Theorem 5.2. *Let S be a nonempty convex subspace of S_H such that $d(u,v) < \pi$ for any $u, v \in S$, and let $\triangle(x,y,z)$ be a geodesic triangle on S. Take $\alpha_1, \alpha_2, \alpha_3 \in]0,1[$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and let $u = B(\{x,y,z\},\{\alpha_1,\alpha_2,\alpha_3\})$. Put $\beta = \alpha_2/(\alpha_2+\alpha_3)$ and let $w = \beta y \overset{1}{\oplus} (1-\beta)z$. Then $u \in [x,w]$.*

Proof. Put $p = \beta y + (1-\beta)z$ and $q = \alpha_1 x + \alpha_2 y + \alpha_3 z$. Then, from Theorem 4.13 and Theorem 5.1, we obtain $w = p/\|p\|$ and $u = q/\|q\|$. Since $1 - \alpha_1 = \alpha_2 + \alpha_3$, we also have $q = \alpha_1 x + (1-\alpha_1)p$. Thus, putting $\gamma = \alpha_1/(\alpha_1 + (1-\alpha_1)\|p\|)$, we get $q = (\alpha_1 + (1-\alpha_1)\|p\|)(\gamma x + (1-\gamma)w)$. It implies

$$u = \frac{q}{\|q\|} = \frac{\gamma x + (1-\gamma)w}{\|\gamma x + (1-\gamma)w\|} = \gamma x \overset{1}{\oplus} (1-\gamma)w \in [x,w]$$

from Corollary 4.7. □

We consider that Theorem 5.2 is a crucial result that shows the suitability of the 1-convex combination on the unit sphere. Indeed, if we only use the traditional convex combination $\oplus$ on a unit sphere, then we do not obtain simple results like Theorem 5.2.

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