An approximation theorem to a solution to an equilibrium problem in complete CAT(1) spaces
完備 CAT(1) 空間における均衡問題の解の近似定理

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1 Introduction

Let K be a nonempty set and $f\colon K \times K \to \mathbb{R}$ a function. Equilibrium problems are defined as follows: Find $z_0 \in K$ such that

$$f(z_0, y) \geq 0$$

for all $y \in K$.

In 1994, Blum and Oettli defined a mapping called resolvent for equilibrium problems on Banach spaces. In 2005, Combettes and Hirstoaga showed many important properties about resolvents of equilibrium problems in Hilbert spaces and made contribution to the development of the approximation methods for equilibrium problem. The following is one of the most important theorems.

Theorem 1 (Combettes–Hirstoaga [**?**]). *Let H be a Hilbert space, and K a nonempty, closed convex subset of H. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies the conditions (E1)–(E4).*

(E1) $f(x, x) = 0$ *for all* $x \in K$;
(E2) $f(x, y) + f(y, x) \leq 0$ *for all* $x, y \in K$;
(E3) $f(x, \cdot)\colon K \to \mathbb{R}$ *is lower semicontinuous and convex for all* $x \in K$;
(E4) $f(\cdot, y)\colon K \to \mathbb{R}$ *is upper hemicontinuous for all* $y \in K$.

The set of solutions to the equilibrium problem is denoted by $\operatorname{Equil} f$*, that is,*

$$\operatorname{Equil} f = \left\{ x \in K \;\middle|\; \inf_{y \in K} f(x,y) \geq 0 \right\}.$$

Define the resolvent J_f *by*

$$J_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} (f(z,y) + \langle z - x, y - z \rangle) \geq 0 \right\}$$

for $x \in H$*. Then* J_f *has the following properties:*

1. *The domain of* J_f *is* H*;*
2. J_f *is single-valued and firmly nonexpansive;*
3. *the set of all fixed points of* J_f *coincides with* $\operatorname{Equil} f$ *and it is closed and convex.*

A basic approach to solving fixed point problems in general normed spaces is Mann's type iteration.

Further in 2018, Kimura and Kishi proposed the resolvent of CAT(0) space. One the other hand, in 2012, He, Fang, López and Li showed a convergence theorem to a fixed point of a nonexpansive mapping in a CAT(κ) spaces with Mann's type iteration.

In this paper, we obtain a convergence theorem of an iterative sequence to a solution of an equilibrium problem on a CAT(1) space. We use Mann's type iterative method to generate the approximate sequence.

2 Preliminaries

Let $x, y \in X$ and γ a mapping of $[0, d(x,y)]$ into X. A mapping γ is called a geodesic with endpoints x and y if $\gamma(0) = x$, $\gamma(d(x,y)) = y$, $d(\gamma(u), \gamma(v)) = |u - v|$ for all $u, v \in [0, d(x,y)]$. X is called a uniquely π-geodesic space if for any $x, y \in X$ with $d(x,y) < \pi$, a geodesic with endpoints x and y exists uniquely. For $x, y \in X$ and $t \in [0,1]$, there exists $z \in [x,y]$ such that $d(x,z) = (1-t)d(x,y)$ and $d(z,y) = td(x,y)$ which is denoted by $tx \oplus (1-t)y$. X is called a CAT(1) space if $\cos d(tx \oplus (1-t)y, z) \sin d(x,y) \geq \cos d(x,z) \sin td(x,y) + \cos d(y,z) \sin(1-t)d(x,y)$ for all $x, y, z \in X$ and $t \in [0,1]$. A CAT(1) space X is said to be admissible if $d(u,v) < \pi/2$ for any $u, v \in X$.

Let X be a metric space and $T\colon X \to X$. We call $x \in X$ a fixed point of T if $x = Tx$, and denote the set of all fixed points of T by $F(T)$. An admissible complete CAT(1) space X has the convex hull finite property if every continuous selfmapping on $\operatorname{cl}\operatorname{co} E$ has a fixed point for every finite subset E of X, where $\operatorname{cl}\operatorname{co} E$ is the closure of the convex hull of X.

Theorem 2.1 (Kimura [**?**])**.** *Let* X *be an admissible complete CAT(1) space having the convex hull finite property and* $K \subset X$ *a nonempty closed convex set. Suppose that* $f\colon K \times K \to \mathbb{R}$ *satisfies the conditions (E1)–(E4). Define the resolvent* R_f *of* f

by

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} (f(z,y) - \log\cos d(x,y) + \log\cos d(x,z)) \geq 0 \right\}$$

for all $x \in X$. Then the following hold:

1. *$R_f \colon X \to K$ is defined as a single-valued mapping;*
2. *R_f satisfies the following inequality for any $x, y \in X$:*

$$\frac{\cos d(x, R_f y)}{\cos d(x, R_f x)} + \frac{\cos d(y, R_f x)}{\cos d(y, R_f y)} \leq 2\cos d(R_f x, R_f y);$$

3. *$F(R_f) = \mathrm{Equil}\, f$ and it is closed and convex.*

Let X be CAT(1) space and $\{x_n\} \subset X$ a sequence. An asymptotic center $\mathrm{AC}(\{x_n\})$ of $\{x_n\}$ is defined by

$$\mathrm{AC}(\{x_n\}) = \left\{ z \in X \;\middle|\; \inf_{x \in X} \limsup_{n \to \infty} d(x, x_n) = \limsup_{n \to \infty} d(z, x_n) \right\}.$$

A sequence $\{x_n\} \subset X$ is said to be Δ-convergent to $x_0 \in X$ if $\mathrm{AC}(\{x_{n_i}\}) = \{x_0\}$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. It is denoted by $x_n \overset{\Delta}{\rightharpoonup} x_0$.

3 Main result

We begin with the basic result of the resolvent on CAT(1) spaces.

Theorem 3.1 (Kimura [?]). *Let X be an admissible complete CAT(1) space having the convex hull finite property. Let K is nonempty closed convex subset of X, suppose that $f \colon K \times K \to \mathbb{R}$ satisfies (E1)–(E4). Let $R_f \colon X \to K$ be the resolvent of f. Then the following the inequality*

$$\lambda \frac{\cos d(v, R_{\lambda f} u)}{\cos d(v, R_{\mu f} v)} + \mu \frac{\cos d(u, R_{\mu f} v)}{\cos d(u, R_{\lambda f} u)} \leq (\lambda + \mu) \cos d(R_{\lambda f} u, R_{\mu f} v)$$

holds for all $\lambda, \mu \in \,]0, \infty[$ and $u, v \in X$.

Lemma 3.1. *Let X be an admissible complete CAT(1) space having the convex hull finite property. Let K is nonempty closed convex subset of X, suppose that $f \colon K \times K \to \mathbb{R}$ satisfies (E1)–(E4). Let $\{\lambda_n\} \subset \,]0, \infty[$ and $\{x_n\} \subset X$ be sequences satisfying $\inf_n \lambda_n > 0$, $x_n \overset{\Delta}{\rightharpoonup} x_0$ and $d(x_n, R_{\lambda_n f} x_n) \to 0$. Then, $x_0 \in \mathrm{Equil}\, f$.*

Proof. Let $\{n_i\} \subset \mathbb{N}$ be an increasing sequence and $z \in X$. Since

$$\begin{aligned} d(z, x_{n_i}) &\leq d(z, R_{\lambda_{n_i} f} x_{n_i}) + d(R_{\lambda_{n_i} f} x_{n_i}) \\ &\leq d(z, x_{n_i}) + 2d(R_{\lambda_{n_i} f} x_{n_i}), \end{aligned}$$

we get

$$\limsup_{i\to\infty} d(z, x_{n_i}) = \limsup_{i\to\infty} d(z, R_{\lambda_{n_i} f} x_{n_i}).$$

We first suppose that $\sup_{n\in\mathbb{N}} \lambda_n < \infty$. There exists $\{\lambda_{n_j}\} \subset \{\lambda_n\}$ such that $\lambda_{n_j} \to \lambda_0 \in [\inf_{n\in\mathbb{N}} \lambda_n, \sup_{n\in\mathbb{N}} \lambda_n]$. Fix $j \in \mathbb{N}$. Then,

$$\lambda_{n_j} \cos d(x_0, R_{\lambda_{n_j} f} x_{n_j}) + \cos d(x_{n_j}, R_f x_0) \leq (\lambda_{n_j} + 1) \cos d(R_{\lambda_{n_j} f} x_{n_j}, R_f x_0).$$

Letting $j \to \infty$, we have

$$\begin{aligned}\lambda_0 \liminf_{j\to\infty} \cos d(x_0, R_{\lambda_{n_j} f} x_{n_j}) + \liminf_{j\to\infty} \cos d(x_{n_j}, R_f x_0)& \\ \leq \liminf_{j\to\infty} (\lambda_{n_j} + 1) \cos d(R_{\lambda_{n_j} f} x_{n_j}, R_f x_0)&\end{aligned}$$

and hence

$$\begin{aligned}\lambda_0 \cos \limsup_{j\to\infty} d(x_0, R_{\lambda_{n_j} f} x_{n_j}) + \cos \limsup_{j\to\infty} d(x_{n_j}, R_f x_0)& \\ \leq (\lambda_0 + 1) \cos \limsup_{j\to\infty} d(R_{\lambda_{n_j} f} x_{n_j}, R_f x_0).&\end{aligned}$$

Thus,

$$\begin{aligned}\lambda_0 \cos \limsup_{j\to\infty} d(x_0, x_{n_j}) + \cos \limsup_{j\to\infty} d(x_{n_j}, R_f x_0)& \\ \leq (\lambda_0 + 1) \cos \limsup_{j\to\infty} d(x_{n_j}, R_f x_0).&\end{aligned}$$

and therefore

$$\cos \limsup_{j\to\infty} d(x_0, x_{n_j}) \leq \cos \limsup_{j\to\infty} d(x_{n_j}, R_f x_0).$$

Consequently, we have

$$\limsup_{j\to\infty} d(x_0, x_{n_j}) \geq \limsup_{j\to\infty} d(x_{n_j}, R_f x_0).$$

Since $\{x_0\} = \mathrm{AC}(\{x_{n_j}\})$, we have $x_0 = R_f x_0$. It means that $x_0 \in \operatorname{Equil} f$.

We next suppose that $\sup_{n\in\mathbb{N}} \lambda_n = \infty$. Then, there exists $\{\lambda_{n_j}\} \subset \{\lambda_n\}$ such that $\lambda_{n_j} \to \infty$. Fix $j \in \mathbb{N}$. Then,

$$\lambda_{n_j} \cos d(x_0, R_{\lambda_{n_j} f} x_{n_j}) + \cos d(x_{n_j}, R_f x_0) \leq (\lambda_{n_j} + 1) \cos d(R_{\lambda_{n_j} f} x_{n_j}, R_f x_0).$$

and thus

$$\cos d(x_0, R_{\lambda_{n_j} f} x_{n_j}) + \frac{\cos d(x_{n_j}, R_f x_0)}{\lambda_{n_j}} \leq \left(1 + \frac{1}{\lambda_{n_j}}\right) \cos d(R_{\lambda_{n_j} f} x_{n_j}, R_f x_0).$$

Letting $j \to \infty$, we get

$$\liminf_{j\to\infty} d\cos d(x_0, R_{\lambda_{n_j} f} x_{n_j}) \le \liminf_{j\to\infty} d\cos d(R_f x_0, R_{\lambda_{n_j} f} x_{n_j})$$

and hence

$$\limsup_{j\to\infty} d(x_0, R_{\lambda_{n_j} f} x_{n_j}) \ge \limsup_{j\to\infty} d(R_f x_0, R_{\lambda_{n_j} f} x_{n_j}).$$

It implies that

$$\limsup_{j\to\infty} d(x_0, x_{n_j}) \ge \limsup_{j\to\infty} d(R_f x_0, x_{n_j}).$$

Since $\{x_0\} = \mathrm{AC}(\{X_{n_j}\})$, we have $x_0 = R_f x_0$. Therefore $x_0 \in \operatorname{Equil} f$. □

Theorem 3.2. *Let X be an admissible complete* CAT(1) *space having the convex hull finite property and $K \subset X$ a nonempty closed convex set. Suppose that $f\colon K\times K \to \mathbb{R}$ satisfies the conditions (E1)–(E4), and* $\operatorname{Equil} f \neq \varnothing$.

Let $\{\lambda_n\} \subset]0,\infty[$ and $\{\alpha_n\} \subset [0,1[$ be sequences satisfying $\inf_n \lambda_n > 0$ and $\sup_n \alpha_n < 1$. Let $R_{\lambda_n f}\colon X \to K$ be the resolvent of $\lambda_n f$ for each $n \in \mathbb{N}$. Let $\{x_n\}$ be a sequence defined by $x_1 \in X$, and

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) R_{\lambda_n f} x_n$$

for each $n \in \mathbb{N}$. Then, $x_n \stackrel{\Delta}{\rightharpoonup} x_0 \in \operatorname{Equil} f$.

Proof. Since $\operatorname{Equil} f \neq \varnothing$, let $u \in \operatorname{Equil} f$. From Theorem **??**,

$$\frac{\cos d(u, R_{\lambda_n f} x_n)}{\cos d(u, R_{\lambda_n f} u)} + \frac{\cos d(x_n, R_{\lambda_n f} u)}{\cos d(x_n, R_{\lambda_n f} x_n)} \le 2\cos d(R_{\lambda_n f} u, R_{\lambda_n f} x_n).$$

Since $u \in \operatorname{Equil} f$, we have $d(u, R_{\lambda_n f} u) = 0$ and thus

$$\cos d(u, R_{\lambda_n f} x_n) + \frac{\cos d(x_n, u)}{\cos d(x_n, R_{\lambda_n f} x_n)} \le 2\cos d(u, R_{\lambda_n f} x_n).$$

It implies that $\cos d(u, R_{\lambda_n f} x_n)\cos d(x_n, R_{\lambda_n f} x_n) \ge \cos d(x_n, u)$ and hence $d(u, R_{\lambda_n f} x_n) \le d(u, x_n)$. From the parallelogram law, we get

$$\cos d(u, x_{n+1}) \ge \alpha_n \cos d(u, x_n) + (1-\alpha_n)\cos d(u, R_{\lambda_n f} x_n) \ge \cos d(u, x_n)$$

and hence

$$d(u, x_{n+1}) \le d(u, x_n) \le d(u, x_1) < \frac{\pi}{2}.$$

Therefore $\{x_n\}$ is spherically bounded and $d(u, x_n) \to c \in [0, \frac{\pi}{2}[$.

Further, we have

$$\begin{aligned}\cos d(u, x_{n+1}) &\geq \alpha_n \cos d(u, x_n) + (1-\alpha_n)\cos d(u, R_{\lambda_n f} x_n)\\ &\geq \alpha_n \cos d(u, x_n) + (1-\alpha_n)\frac{\cos d(u, x_n)}{\cos d(R_{\lambda_n f} x_n, x_n)}\\ &= \cos d(u, x_n) + (1-\alpha_n)\cos d(u, x_n)\left(\frac{1}{\cos d(R_{\lambda_n f} x_n, x_n)} - 1\right),\end{aligned}$$

which implies that

$$0 \leq (1-\alpha_n)\left(\frac{1}{\cos d(R_{\lambda_n f} x_n, x_n)} - 1\right) \leq \frac{\cos d(u, x_{n+1})}{\cos d(u, x_n)} - 1.$$

Since $d(u, x_n) \to c \in [0, \frac{\pi}{2}[$ and $\sup_{n\in\mathbb{N}} \alpha_n < 1$, we know $d(R_{\lambda_n f} x_n, x_n) \to 0$. Since $\{x_n\}$ is spherically bounded, any subsequence $\{x_{n_i}\}$ is spherically bounded. Let $\{x_0\} = \mathrm{AC}(\{x_n\})$ and $\{w_0\} = \mathrm{AC}(\{x_{n_i}\})$. We can take a subsequence $\{x_{n_{i_j}}\}$ such that $x_{n_{i_j}} \overset{\Delta}{\rightharpoonup} z_0 \in X$. From Lemma **??** and since $d(R_{\lambda_n f} x_{n_{i_j}}, x_{n_{i_j}}) \to 0$, we obtain $z_0 \in \operatorname{Equil} f$. Further, we have

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, z_0) &= \limsup_{j\to\infty} d(x_{n_{i_j}}, z_0)\\ &\leq \limsup_{j\to\infty} d(x_{n_{i_j}}, w_0)\\ &\leq \limsup_{i\to\infty} d(x_{n_i}, w_0)\\ &\leq \limsup_{i\to\infty} d(x_{n_i}, x_0)\\ &\leq \limsup_{n\to\infty} d(x_n, x_0) \leq \limsup_{n\to\infty} d(x_n, z_0).\end{aligned}$$

Therefore $x_0 = w_0 = z_0$ and thus we get $\{x_0\} = \mathrm{AC}(\{x_{n_i}\})$ for all $\{x_{n_i}\} \subset \{x_n\}$. Consequently, $x_n \overset{\Delta}{\rightharpoonup} x_0 \in \operatorname{Equil} f$. □

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