Approximation of a solution to equilibrium problems in geodesic spaces using projection methods
測地空間における 均衡問題の射影法を用いた解近似定理

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1 Introduction

We use an operator called resolvent defined for an equilibrium problem, which is one of the most common nonlinear problems. The resolvent of an equilibrium problem is a fundamental concept since the solution to the problem coincides with the set of fixed points of the resolvent. In this study, we prove an approximation theorem for the solution to the equilibrium problem in CAT(1) space using the resolvent with the CQ projection method.

Kimura and Kishi proposed the notion of resolvent for equilibrium problems in Hadamard spaces as follows:

Theorem 1 (Kimura and Kishi [3]). *Let X be a Hadamard space with the convex hull finite property, and let K be a nonempty closed convex subset of X. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies the following.*

(E1) $f(x,x) = 0$ for every $x \in K$;
(E2) $f(x,y) + f(y,x) \leq 0$ for every $x, y \in K$;
(E3) for every $x \in K$, $f(x,\cdot)\colon K \to \mathbb{R}$ is lower semicontinuous and convex;
(E4) for every $y \in K$, $f(\cdot,y)\colon K \to \mathbb{R}$ is upper hemicontinuous.

Define $J_f\colon X \to X$ *by*

$$J_f x = \left\{ z \in K \;\middle|\; \inf_{y\in K} \left(f(z,y) + \frac{1}{2}d(x,y)^2 - \frac{1}{2}d(x,z)^2 \right) \geq 0 \right\}$$

for $x \in X$. *Then,*

(i) $D(J_f) = X$;
(ii) J_f *is single-valued and firmly nonexpansive;*
(iii) $F(J_f) = S(f)$;
(iv) $S(f)$ *is closed and convex.*

Motivated by this result, the second author introduced a resolvent on CAT(1) spaces.

Theorem 2 (Kimura [1]). *Let X be an admissible complete CAT(1) space with the convex hull finite property, and let K be a nonempty closed convex subset of X. Suppose that $f\colon K\times K \to \mathbb{R}$ satisfies (E1)–(E4) in the theorem above. Define $T_f x \subset K$ by*

$$T_f x \;= \left\{ z \in K \;\middle|\; \inf_{y\in K} (f(z,y) - \log\cos d(x,y) + \log\cos d(x,z)) \geq 0 \right\}$$

for $x \in X$. *Then,*

(i) T_f *is single-valued;*
(ii) $T_f\colon X \to K$ *satisfies*

$$\frac{\cos d(x,T_f y)}{\cos d(x,T_f x)} + \frac{\cos d(y,T_f x)}{\cos d(y,T_f y)} \leq 2\cos d(T_f x, T_f y)$$

for $x, y \in X$;
(iii) $F(T_f) = S(f)$;
(iv) $S(f)$ *is closed and convex.*

The CQ projection method for a nonexpansive mapping was firstly proposed by Nakajo and Takahashi.

Theorem 3 (Nakajo and Takahashi [4]). *Let H be a Hilbert space. Let $T\colon H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For given $x = x_1 \in H$, $C_1 = Q_1 = H$, define $\{x_n\}$ by*

$$\begin{aligned} C_{n+1} &= \{z \in H \mid \|Tx_n - z\| \leq \|x_n - z\|\}, \\ Q_{n+1} &= \{z \in H \mid \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{C_{n+1}\cap Q_{n+1}} x. \end{aligned}$$

Then $x_n \to P_{F(T)}x$, where $P_K\colon H \to K$ is the metric projection of H onto a nonempty closed convex subset K of H.

In this article, we apply the resolvent of the equilibrium problem in CAT(1) space to the CQ projection method, which is a scheme for generating a sequence that converges to a fixed point. We prove an approximation theorem of the solution to the equilibrium problem.

2 Preliminaries

Let X be a metric space and $T\colon X \to X$. Then, the set of all fixed points of T is denoted by $F(T)$, that is,

$$F(T) = \{z \in X \mid z = Tz\}.$$

T is said to be quasinonexpansive, if $F(T) \neq \emptyset$ and $d(Tx, z) \leq d(x, z)$ for $x \in X$ and $z \in F(T)$.

Let X be a metric space. For $x, y \in X$, a mapping $c\colon [0, d(x, y)] \to X$ is called a geodesic if c satisfies $c(0) = x, c(d(x, y)) = y$, and $d(c(s), c(t)) = |s - t|$ for every $s, t \in [0, d(x, y)]$. If for any $x, y \in X$, there exists a unique geodesic with endpoints x and y, then X is called a uniquely geodesic space. For a uniquely geodesic space X, the image of the geodesic with endpoints $x, y \in X$ is denoted by $[x, y]$. In this case, there exists a unique $z \in [x, y]$ such that

$$d(x, z) = (1 - t)d(x, y) \text{ and } d(z, y) = td(x, y).$$

We denote it by $z = tx \oplus (1 - t)y$ and we call it a convex combination of x and y.

Let (X, d) be a uniquely geodesic space. The triangle $\triangle(x, y, z)$ formed by $x, y, z \in X$ satisfying $d(x, y)+d(y, z)+d(z, x) < 2\pi$ is called a geodesic triangle. Consider the two-dimensional unit sphere $\mathbb{S}^2$ as a model space of X. Then for a point $x, y, z \in X$ satisfying $d(x, y)+d(y, z)+d(z, x) < 2\pi$, a comparison triangle $\overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$ of $\triangle(x, y, z)$ is defined as a triangle on $\mathbb{S}^2$ such that $d(x, y) = d_{\mathbb{S}^2}(\overline{x}, \overline{y}), d(y, z) = d_{\mathbb{S}^2}(\overline{y}, \overline{z}), d(z, x) = d_{\mathbb{S}^2}(\overline{z}, \overline{x})$. A comparison point of $p = tx \oplus (1-t)y \in [x, y]$ is defined by $\overline{p} = t\overline{x} \oplus (1-t)\overline{y} \in [\overline{x}, \overline{y}]$. If X satisfies that

$$d(p, q) \leq d_{\mathbb{S}^2}(\overline{p}, \overline{q})$$

for any $\triangle(x, y, z)$, $p, q \in \triangle(x, y, z)$ and $\overline{p}, \overline{q} \in \triangle(\overline{x}, \overline{y}, \overline{z})$, then it is called a CAT(1) space and this inequality is called the CAT(1) inequality.

Theorem 2.1. *Let X be a CAT(1) space. Then*

$$\cos d(tx \oplus (1 - t)y, z) \sin d(x, y) \geq \cos d(x, z) \sin td(x, y) + \cos d(y, z) \sin(1 - t)d(x, y)$$

for $x, y, z \in X$ such that $d(x, y) + d(y, z) + d(z, x) < 2\pi$, and $t \in [0, 1]$.

Corollary 2.1. *Let X be a CAT(1) space. Suppose $d(x, y) + d(y, z) + d(z, x) < 2\pi$ for $x, y, z \in X$. Then*

$$\cos d(tx \oplus (1 - t)y, z) \geq t \cos d(x, z) + (1 - t) \cos d(y, z)$$

for $t \in [0, l]$.

Let X be a CAT(1) space. X is said to be admissible if $d(u,v) < \pi/2$ for any $u, v \in X$.

Let X be an admissible complete CAT(1) space. Let $C \subset X$ be a nonempty closed convex set. Then, there exists a unique $y_x \in C$ satisfying

$$d(x, y_x) = \inf_{y \in C} d(x, y)$$

for $x \in X$. We define $P_C \colon X \to C$ by $P_C x = y_x$ for $x \in X$. We call it the metric projection onto C.

Let X be a CAT(1) space. The set $\mathrm{AC}(\{x_n\})$ of all asymptotic conters of a bounded sequence $\{x_n\}$ is defined by

$$\mathrm{AC}(\{x_n\}) = \left\{ z \in X \;\middle|\; \inf_{x \in X} \limsup_{n\to\infty} d(x, x_n) = \limsup_{n\to\infty} d(z, x_n) \right\}.$$

Let X be a CAT(1) space and $\{x_n\} \subset X$. If $\mathrm{AC}(\{x_{n_k}\}) = \{x_0\}$ for all subsequence $\{x_{n_k}\}$ of $\{x_n\}$, then we say $\{x_n\}$ is Δ-convergent to x_0, and we denote it by $x_n \overset{\Delta}{\rightharpoonup} x_0$. The point x_0 is called a Δ-limit of $\{x_n\}$.

Let X be a CAT(1) space. A sequence $\{x_n\} \subset X$ is said to be spherically bounded if

$$\inf_{x \in X} \limsup_{n\to\infty} d(x, x_n) < \frac{\pi}{2}$$

holds.

3 Approximation of a solution to a equilibrium problem

Let X be an admissible complete CAT(1) space. Let $K \subset X$ be a nonempty set. An equilibrium problem for $f \colon K \times K \to \mathbb{R}$ is the problem of finding $z_0 \in K$ such that $f(z_0, y) \geq 0$ for all $y \in K$. The solution set $S(f)$ is defined by

$$S(f) = \left\{ z \in K \;\middle|\; \inf_{y \in K} f(z, y) \geq 0 \right\}.$$

We suppose the four conditions for f as follows:

(E1) $f(x, x) = 0$ for all $x \in K$;
(E2) $f(x, y) + f(y, x) \leq 0$ for all $x, y \in K$;
(E3) $f(x, \cdot) \colon K \to \mathbb{R}$ is lower semicontinuous and convex for every $x \in K$;
(E4) $f(\cdot, y) \colon K \to \mathbb{R}$ is upper hemicontinuous for every $y \in K$.

Theorem 3.1 (Kimura [2]). *Let X be an admissible complete CAT(1) space with the convex hull finite property and let $K \subset X$ be a nonempty closed convex set. Suppose that $f \colon K \times K \to \mathbb{R}$ satisfies (E1)–(E4). Define $T_{\lambda f} \colon X \to K$ by*

$$T_{\lambda f} x = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left(\lambda f(z, y) - \log\cos d(y, x) + \log\cos d(z, x) \right) \geq 0 \right\}$$

for $x \in X$. *Then*

$$(\lambda + \mu)\cos d(T_{\lambda f}u, T_{\mu f}v) \geq \lambda \frac{\cos d(T_{\mu f}v, u)}{\cos d(T_{\lambda f}u, u)} + \mu \frac{\cos d(T_{\mu f}u, v)}{\cos d(T_{\mu f}v, v)}$$

for $\lambda, \mu > 0$ *and* $u, v \in X$.

Theorem 3.2. *Let* X *be an admissible complete CAT(1) space with the convex hull finite property. Suppose that* X *satisfies the following:*

- $\{z \in X \mid d(u,z) \leq d(v,z)\}$ *is convex for* $u, v \in X$;
- $\{z \in X \mid \cos d(u,v) \cos d(v,z) \geq \cos d(u,z)\}$ *is convex for* $u, v \in X$.

Let $K \subset X$ *be a nonempty cloded convex set. Suppose that* $f \colon K \times K \to \mathbb{R}$ *satisfies (E1)–(E4) and* $S(f) \neq \emptyset$. *Define* $T_f \colon X \to K$ *by*

$$T_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} (f(z,y) - \log \cos d(x,y) + \log \cos d(x,z)) \geq 0 \right\}$$

for every $x \in X$. *Let* $\{\lambda_n\} \subset [a, \infty[$ *and* $0 < a < \infty$. *Generate* $\{x_n\}$ *by* $x_1 \in X, C_1 = Q_1 = X$, *and*

$$\begin{aligned} C_{n+1} &= \{z \in X \mid d(T_{\lambda_n f}x_n, z) \leq d(x_n, z)\}, \\ Q_{n+1} &= \{z \in X \mid \cos d(x, x_n) \cos d(x_n, z) \geq \cos d(x,z)\}, \\ x_{n+1} &= P_{C_{n+1} \cap Q_{n+1}} x \end{aligned}$$

for $n \in \mathbb{N}$. *Then* $x_n \to P_{S(f)}x \in K$.

Proof. First, we prove $\{x_n\}$ is well-defined by induction. $C_1 = Q_1 = X$ is a closed convex set and $S(f) \subset C_1 \cap Q_1$. For $k \in \mathbb{N}$, assume that C_k, Q_k are closed convex sets and they satisfy $S(f) \subset C_k \cap Q_k$. Since $\{z \in X \mid d(T_f x_k, z) \leq d(x_k, z)\}$ is convex by assumption, we know that C_{k+1} is closed and convex. Similarly, since $\{z \in X \mid \cos d(x, x_k) \cos d(x_k, z) \geq \cos d(x,z)\}$ is convex by assumption, we also know that Q_{k+1} is closed and convex. Next, we prove $S(f) \subset C_{k+1} \cap Q_{k+1}$. Let $z \in S(f) = F(T_f)$. Since T_f is quasinonexpansive, $d(T_f x_k, z) \leq d(x_k, z)$ holds, and we obtain $z \in C_{k+1}$. This implies $S(f) \subset C_{k+1}$. Moreover, we can show $S(f) \subset Q_{k+1}$. Since $S(f) \subset C_k \cap Q_k$ from the assumption of induction, it is sufficient to show $C_k \cap Q_k \subset Q_{k+1}$. Fix $z \in C_k \cap Q_k$ arbitrarily. Then,

$$tz \oplus (1-t)x_k = tz \oplus (1-t)P_{C_k \cap Q_k} x \in C_k \cap Q_k$$

for $t \in]0,1[$. Therefore,

$$\begin{aligned} &2\cos d(x, x_k) \cos\left(\left(1 - \frac{t}{2}\right) d(x_k, z)\right) \sin\left(\frac{t}{2} d(x_k, z)\right) \\ &\quad = \cos d(x, x_k)(\sin d(x_k, z) - \sin((1-t)d(x_k, z))) \\ &\quad = \cos d(x, P_{C_k \cap Q_k} x) \sin d(x_k, z) - \cos d(x, x_k) \sin((1-t)d(x_k, z)) \end{aligned}$$

$$
\begin{aligned}
&\geq \cos d(x, tz \oplus (1-t)x_k) \sin d(x_k, z) - \cos d(x, x_k) \sin((1-t)d(x_k, z)) \\
&= \cos d(x, z) \sin(td(x_k, z)) \\
&= 2 \cos d(x, z) \cos\left(\frac{t}{2} d(x_k, z)\right) \sin\left(\frac{t}{2} d(x_k, z)\right).
\end{aligned}
$$

When $z \neq x_k$, dividing by $2\sin(td(x_k, z)/2)$ and letting $t \to 0$, we have

$$\cos d(x, x_k) \cos d(x_k, z) \geq \cos d(x, z).$$

From the definition of Q_{k+1}, we have $z \in Q_{k+1}$. If $z = x_k$, then obviously $z \in Q_{k+1}$. Therefore, we get $C_k \cap Q_k \subset Q_{k+1}$. Hence we have C_{k+1} and Q_{k+1} are closed convex sets and $S(f) \subset C_{k+1} \cap Q_{k+1}$. Since the intersection of closed convex sets is a closed convex set, there exists the metric projection to $C_{k+1} \cap Q_{k+1}$ and $x_{k+1} = P_{C_{k+1} \cap Q_{k+1}} x$ can be defined. Therefore $\{x_n\}$ is well-defined. It is also shown that $P_{S(f)} x \in S(f) \subset C_n \cap Q_n$ and $C_n \cap Q_n \subset Q_{n+1}$, for arbitrary $n \in \mathbb{N}$.

Next, we prove $d(T_{\lambda_n f} x_n, x_n) \to 0$. For arbitrary $n \in \mathbb{N}$, since $P_{S(f)} x \in S(f) \subset C_n \cap Q_n$, from the definition of the metric projection, we get

$$d(x, x_n) = d(x, P_{C_n \cap Q_n} x) \leq d(x, P_{S(f)} x) < \frac{\pi}{2}.$$

Therefore, $\sup_{n \in \mathbb{N}} d(x, x_n) \leq d(x, P_{S(f)} x) < \pi/2$. Fix $z \in Q_n$ arbitrarily. From the definition of Q_n, we have

$$\cos d(x, x_n) \cos d(x_n, z) \geq \cos d(x, z)$$

and then,

$$\cos d(x, x_n) \geq \cos d(x, z).$$

It follows that

$$\inf_{y \in Q_n} d(x, y) \leq d(x, x_n) \leq d(x, z).$$

It implies that $d(x, x_n) = \inf_{y \in Q_n} d(x, y)$. Therefore, we have $P_{Q_n} x = x_n = P_{C_n \cap Q_n} x \in C_n \cap Q_n \subset Q_{n+1}$. Thus, we obtain

$$
\begin{aligned}
d(x, x_n) &= d(x, P_{C_n \cap Q_n} x) = d(x, P_{Q_n} x) \\
&\geq d(x, P_{Q_{n+1}} x) = d(x, P_{C_{n+1} \cap Q_{n+1}} x) = d(x, x_{n+1}),
\end{aligned}
$$

for $n \in \mathbb{N}$. This implies that $\{d(x, x_n)\}$ is a decreasing sequence. Thus, $\{\cos d(x, x_n)\}$ is increasing and bounded above, so we get

$$c = \lim_{n \to \infty} \cos d(x, x_n) > \cos \frac{\pi}{2} = 0.$$

Also, since $x_{n+1} \in C_{n+1} \cap Q_{n+1} \subset Q_{n+1}$, we have

$$\cos d(x, x_n) \cos d(x_n, x_{n+1}) \geq \cos d(x, x_{n+1})$$

for $n \in \mathbb{N}$. Letting $n \to \infty$, we have

$$c \liminf_{n\to\infty} \cos d(x_n, x_{n+1}) \geq c.$$

Thus, dividing by $c > 0$, we get

$$\liminf_{n\to\infty} \cos d(x_n, x_{n+1}) \geq 1,$$

and since

$$1 \leq \liminf_{n\to\infty} \cos d(x_n, x_{n+1}) \leq \limsup_{n\to\infty} \cos d(x_n, x_{n+1}) \leq 1,$$

we get $\lim_{n\to\infty} \cos d(x_n, x_{n+1}) = 1$. This implies $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Furthermore, since $x_{n+1} \in C_{n+1} \cap Q_{n+1} \subset C_{n+1}$, we have $d(T_{\lambda_n f} x_n, x_{n+1}) \leq d(x_n, x_{n+1})$ for $n \in \mathbb{N}$. Thus we get,

$$0 \leq d(T_{\lambda_n f} x_n, x_n) \leq d(T_{\lambda_n f} x_n, x_{n+1}) + d(x_{n+1}, x_n) \leq 2d(x_n, x_{n+1}) \to 0.$$

Finally, we show $x_n \to P_{S(f)} x$. Since $\sup_{n\in\mathbb{N}} d(x, x_n) < \pi/2$, $\{x_n\}$ is a spherically bounded sequence. Fix $\{x_{n_i}\} \subset \{x_n\}$ arbitrarily. There are $\{\lambda_{n_{i_j}}\} \subset \{\lambda_{n_i}\}$ and $\{x_{n_{i_j}}\} \subset \{x_{n_i}\}$ such that $\lambda_{n_{i_j}} \to \lambda_0 \in [a, \infty]$ and $x_{n_{i_j}} \xrightarrow{\Delta} w_0$. Suppose $\lambda_{n_{i_j}} \to \infty$. For any $y \in X$, we have

$$\begin{aligned} d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, y) &\leq d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, x_{n_{i_j}}) + d(x_{n_{i_j}}, y) \\ &\leq 2d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, x_{n_{i_j}}) + d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, y). \end{aligned}$$

Then,

$$\limsup_{j\to\infty} d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, y) = \limsup_{j\to\infty} d(x_{n_{i_j}}, y).$$

We also have

$$\begin{aligned} &(\lambda_{n_{i_j}} + 1) \cos d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, T_f w_0) \\ &\qquad \geq \frac{\cos d(T_f w_0, x_{n_{i_j}})}{\cos d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, x_{n_{i_j}})} + \lambda_{n_{i_j}} \frac{\cos d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, w_0)}{\cos d(T_f w_0, w_0)}. \end{aligned}$$

Thus,

$$\begin{aligned} &\cos d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, T_f w_0) \\ &\qquad \geq \frac{1}{\lambda_{n_{i_j}} + 1} \frac{\cos d(T_f w_0, x_{n_{i_j}})}{\cos d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, x_{n_{i_j}})} + \frac{\lambda_{n_{i_j}}}{\lambda_{n_{i_j}} + 1} \frac{\cos d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, w_0)}{\cos d(T_{f w_0}, w_0)} \\ &\qquad \geq \frac{\lambda_{n_{i_j}}}{\lambda_{n_{i_j}} + 1} \frac{\cos d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, w_0)}{\cos d(T_f w_0, w_0)}. \end{aligned}$$

It follows that

$$\liminf_{j\to\infty} \cos d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, T_f w_0) \geq \liminf_{j\to\infty} \frac{\cos d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, w_0)}{\cos d(T_f w_0, w_0)},$$

and

$$\cos \limsup_{j\to\infty} d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, T_f w_0) \geq \frac{\cos \limsup_{j\to\infty} d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, w_0)}{\cos d(T_f w_0, f w_0)}.$$

Therefore

$$\cos \limsup_{j\to\infty} d(x_{n_{i_j}}, T_f w_0) \geq \frac{\cos \limsup_{j\to\infty} d(x_{n_{i_j}}, w_0)}{\cos d(T_f w_0, w_0)}.$$

On the other hand since $w_0 \in \mathrm{AC}(\{x_{n_{i_j}}\})$, we have

$$\cos \limsup_{j\to\infty} d(x_{n_{i_j}}, w_0) \geq \cos \limsup_{j\to\infty} d(x_{n_{i_j}}, T_f w_0).$$

Thus,

$$\cos \limsup_{j\to\infty} d(x_{n_{i_j}}, w_0) \geq \frac{\cos \limsup_{j\to\infty} d(x_{n_{i_j}}, w_0)}{\cos d(T_f w_0, w_0)},$$

which implies

$$\cos d(T_f w_0, w_0) \geq 1.$$

Therefore, we get $w_0 \in F(T_f) = S(f)$.

Next, suppose $\lambda_{n_{i_j}} \to \lambda_0$. We also have

$$\begin{aligned}
(\lambda_{n_{i_j}} + \lambda_0) &\cos d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, T_{\lambda_0 f} x_{n_{i_j}}) \\
&\geq \lambda_{n_{i_j}} \frac{\cos d(T_{\lambda_0 f} x_{n_{i_j}}, x_{n_{i_j}})}{\cos d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, x_{n_{i_j}})} + \lambda_0 \frac{\cos d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, x_{n_{i_j}})}{\cos(T_{\lambda_0 f} x_{n_{i_j}}, x_{n_{i_j}})} \\
&\geq 2\sqrt{\lambda_{n_{i_j}} \lambda_0}
\end{aligned}$$

Thus,

$$1 \geq \cos d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, T_{\lambda_0} x_{n_{i_j}}) \geq \frac{2\sqrt{\lambda_{n_{i_j}} \lambda_0}}{\lambda_{n_{i_j}} + \lambda_0} \to \frac{2\sqrt{\lambda_0^2}}{2\lambda_0} = 1.$$

Then we have,

$$d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, T_{\lambda_0 f} x_{n_{i_j}}) \to 0.$$

Since $d(T_{\lambda_0 f} x_{n_{i_j}}, T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}) \to 0$ and $d(x_{n_{i_j}}, T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}) \to 0$, we have $d(T_{\lambda_0 f} x_{n_{i_j}}, x_{n_{i_j}}) \to 0$. Also, since $x_{n_{i_j}} \xrightarrow{\Delta} w_0$, $d(x_{n_{i_j}}, T_{\lambda_0 f} x_{n_{i_j}}) \to 0$, and T_{λ_0} is Δ-demiclosed, we get $w_0 \in F(T_{\lambda f}) = S(f)$.

Then we have

$$\begin{aligned} d(x, P_{S(f)}x) \le d(x, w_0) &\le \liminf_{j\to\infty} d(x, x_{n_{i_j}}) \\ &\le \limsup_{j\to\infty} d(x, x_{n_{i_j}}) \\ &\le \sup_{n\in\mathbb{N}} d(x, x_n) \\ &\le d(x, P_{S(f)}x). \end{aligned}$$

Thus, $d(x, P_{S(f)}x) = d(x, w_0)$, and hence $w_0 = P_{S(f)}x$. We also have $\lim_{j\to\infty} d(x, x_{n_{i_j}}) = d(x, P_{S(f)}x)$, and then $x_{n_{i_j}} \to P_{S(f)}x$. Consequently, we have

$$x_n \to P_{S(f)}x,$$

which is the desired result. □

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