Robustness of multi-valued optimization problems via set relations

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Abstract

This paper focuses on the feasibility of multi-valued optimization problems under perturbations. By using set relations and their scalarization, a modified version of theorems of the alternative can characterize the robustness of the feasibility. Especially under some assumptions, we show algorithms for evaluating the robustness which computers could deal with.

1 Introduction

We usually solve optimization problems to make decisions for planning, scheduling, or matching. However, mathematical optimization models can't reflect every part of issues which has many errors and perturbations. Perturbation theory deals with problems called "robust optimization problems" that contain parameters giving them small deviation. If a solution remains itself under such deviation, it is said to be "robust." In general, the difficulty of solving a robust optimization problem is strongly dependent on the way giving perturbations. This paper investigates criteria for the robustness of multi-valued optimization problems via set-valued analysis.

Set relations, originally given in [8], are kinds of binary relations between two sets and used to determine which one is prefer to the other. To ease set-to-set comparisons, We usually quantify set relations with scalarization commonly done in two ways: scalarizaion functions or oriented distance functions. The relationship between the relations and the functions have been studied as dual expressions or theorems of the alternative (e.g., see [3, 9–11, 14] for scalarization functions, [5–7] for oriented distance functions). Recently, Hui et. al. studied calculability of scalarization functions and they proved the value of the functions can be computed by solving linear programming problems when given sets are polyhedra ([4]). Their results enable computers to find out which a preferred set is.

We sould like to show more relaxed scalarization theorems of the alternative in a topological vector space based on ones in [11]. For proving this kind of theorems, some topological assumptions are required such as compactness, boundedness, closedness. As opposed to former researches, we use convex cone properties inspired by [1]. And as an application, we introduce criteria for robustness of feasibility of a multi-valued optimization problems and their calculation algorithms by using set comparisons.

2 Basic notations

Unless otherwise specified, we let X be a topological vector space, $C \subset X$ a convex cone satisfying int$C \neq \emptyset$ throughout the thesis. For two vectors $x, y \in X$, $x \leq_C y$ is defined to be $x \in y - C$. For two sets $A, B \subset X \setminus \{\emptyset\}$ and $\alpha \in \mathbb{R}$,

- $A + B := \{a + b \mid a \in A, b \in B\}$;
- $\alpha A := \{\alpha a \mid a \in A\}$.

We use convex cone properties with respect to C: A is C-closed if $A + C$ is closed, A is C-bounded if it holds that $A \subset U + C$ for any open neighborhood U of the zero, A is C-compact if any cover of S being like $\{U_\lambda + C \mid U_\lambda \text{ is open}\}$ admits a finite subcover. We clearly see C-compactness leads to $C-$closedness and C-boundedness.

At first, we introduce the six types of set relations originally proposed in [8]: for nonempty sets $A, B \subset X \setminus \{\emptyset\}$ and $i = 1, \ldots, 6$, the relations $\preceq_C^{(\mathrm{i})}$ are defined by

- $A \preceq_C^{(1)} B \iff A \subset \bigcap_{b \in B}(b - C)$;
- $A \preceq_C^{(2)} B \iff A \cap \bigcap_{b \in B}(b - C) \neq \emptyset$;
- $A \preceq_C^{(3)} B \iff B \subset A + C$;
- $A \preceq_C^{(4)} B \iff A \cap \bigcap_{a \in A}(a + C) \neq \emptyset$;
- $A \preceq_C^{(5)} B \iff A \subset B - C$;
- $A \preceq_C^{(6)} B \iff B \cap (A + C) \neq \emptyset$.

Note that $\preceq_C^{(1)}$ implies $\preceq_C^{(2)}$ and $\preceq_C^{(4)}$, which lead to $\preceq_C^{(3)}$ and $\preceq_C^{(5)}$ respectively. The last relation $\preceq_C^{(6)}$ is implied by the others. Moreover, these relations $\preceq_C^{(\mathrm{i})}$ for $i = 1, \ldots, 6$ coincide with $\leq_C$ when two compared set A, B are both singleton.

3 Scalarization functions

In the thesis, we use the following Minkovski-type Garstewitz functional $\varphi_{C,d} : X \to \mathbb{R} \cup \{\infty\}$ given in [2] defined by

$$\varphi_{C,d}(x) := \inf\{\gamma \in \mathbb{R} \mid x \leq_C \gamma d\}.$$

for a given vector $x \in X$ and a fixed direction $d \in X$. This function coincides with the linear functional $f \in X^*$ where $C := \{x \in X \mid f(x) \geq 0\}$ is a half space. This functional is utilized in set scalarization.some set inequality. Nishizawa et.al. proposed theorems of the alternative for set-valued maps with the function in [10]. Moreover, [3,9] generalized the functional and introduced a set scalarization functional $\Phi^{(\mathrm{i})}_{C,B,d} : 2^X \to \mathbb{R} \cup \{\infty\}$ defined by

$$\Phi^{(\mathrm{i})}_{C,B,d}(A) := \inf\{\gamma \in \mathbb{R} \mid A \preceq^{(\mathrm{i})}_C B + \gamma d\}.$$

for a given set A, a fixed reference set B, and a fixed direction d. All the functions coincide with $\varphi_{C,d}$ when $A = \{x\}$, $B = \{\mathbf{0}_X\}$. Also, it holds that $A \preceq^{(\mathrm{i})}_C B$ implies $\Phi^{(\mathrm{i})}_{C,B,d}(A) \leq 0$.

Speaking of set scalarization, set relation-based characterization theorems have been proposed ([11,14]) under the compactness of given sets.

Proposition 3.1 ([11])**.** Let $A, B \in 2^X \backslash \{\emptyset\}$. Then the following assertions hold.

- If A is compact, then $A \preceq^{(2)}_{\mathrm{cl}C} B$ and $A \preceq^{(3)}_{\mathrm{cl}C} B$ follow from $\Phi^{(2)}_{\mathrm{cl}C,B,d}(A) \leq 0$ and $\Phi^{(3)}_{\mathrm{cl}C,B,d}(A) \leq 0$ for some $d \in X$, respectively.
- If B is compact, then $A \preceq^{(4)}_{\mathrm{cl}C} B$ and $A \preceq^{(4)}_{\mathrm{cl}C} B$ follow from $\Phi^{(5)}_{\mathrm{cl}C,B,d}(A) \leq 0$ and $\Phi^{(5)}_{\mathrm{cl}C,B,d}(A) \leq 0$ for some $d \in X$, respectively.
- If both A, B are compact, then $A \preceq^{(6)}_{\mathrm{cl}C} B$ follows from $\Phi^{(6)}_{\mathrm{cl}C,B,d}(A) \leq 0$ for some $d \in X$.

Proposition 3.2 ([14])**.** Let $A, B \in 2^X \backslash \{\emptyset\}$.

- If both A, B are compact, then $A \preceq^{(1)}_{\mathrm{int}C} B$ follows from $\Phi^{(1)}_{\mathrm{int}C,B,d}(A) \leq 0$ for some $d \in X$.
- If B is compact, then $A \preceq^{(2)}_{\mathrm{int}C} B$ and $A \preceq^{(3)}_{\mathrm{int}C} B$ follow from $\Phi^{(2)}_{\mathrm{int}C,B,d}(A) \leq 0$ and $\Phi^{(3)}_{\mathrm{int}C,B,d}(A) \leq 0$ for some $d \in X$, respectively.
- If A is compact, then $A \preceq^{(4)}_{\mathrm{int}C} B$ and $A \preceq^{(4)}_{\mathrm{int}C} B$ follow from $\Phi^{(5)}_{\mathrm{int}C,B,d}(A) \leq 0$ and $\Phi^{(5)}_{\mathrm{int}C,B,d}(A) \leq 0$ for some $d \in X$, respectively.

One can see the case $i = 1$ for Proposition 3.1, and the case $i = 6$ for Proposition 3.2 hold without any compactness.

Theorem 3.1 ([12,13])**.** Let $A, B \in 2^X \backslash \{\emptyset\}$. Then,

$$A \preceq^{(\mathrm{i})}_{\mathrm{cl}C} B \iff \exists k \in \mathrm{int}C \text{ s.t. } \Phi^{(\mathrm{i})}_{C,k}(A, B) \leq 0$$

where

- A is C-compact for case $i = 2$;
- A is C-closed for case $i = 3$;
- B is $(-C)$-compact for case $i = 4$;
- B is $(-C)$-closed for case $i = 5$;
- A is C-closed and B is $(-C)$-compact,
  or A is C-compact and B is $(-C)$-closed for case $i = 6$.

4 Application

Let S be a nonempty set and consider the following optimization problem:

$$\text{(P) Minimize } f(x) \text{ subject to } g(x) \leq_C r$$

where $f : S \to \mathbb{R}^n$, $g : S \to \mathbb{R}^m$, $r \in \mathbb{R}^m$.

We assume that g and r are perturbed in the sets G and R, respectively. Moreover, we let $G(x) := \{g(x) \mid g \in G\}$.

Proposition 4.1 ([13]). We assume (P) is feasible. Then, the following statements hold on G and R:

- (P) is still feasible for all $g \in G$ and $r \in R$ if and only if $\Phi^{(1)}_{C,k}(G(x), R) \leq 0$ for some $k \in \text{int}C$;
- there exists $g \in G$ such that (P) is feasible for all $r \in R$ if and only if $\Phi^{(2)}_{C,k}(G(x), R) \leq 0$ for some $k \in \text{int}C$;
- for all $r \in R$, we can find $g \in G$ to make (P) remain feasible if and only if $\Phi^{(3)}_{C,k}(G(x), R) \leq 0$ for some $k \in \text{int}C$;
- there exists $r \in R$ such that (P) is feasible for all $g \in G$ if and only if $\Phi^{(4)}_{C,k}(G(x), R) \leq 0$ for some $k \in \text{int}C$;
- for all $g \in G$, we can find $r \in R$ to make (P) remain feasible if and only if $\Phi^{(5)}_{C,k}(G(x), R) \leq 0$ for some $k \in \text{int}C$;
- (P) is feasible for some $g \in G$ and some $r \in R$ if and only if $\Phi^{(6)}_{C,k}(G(x), R) \leq 0$ for some $k \in \text{int}C$.

The above proposition implies the values of scalarization functional indicate the robustness of feasibility for a multi-valued optimization problem. Moreover, each value is calculated by solving linear programming problems.

Proposition 4.2 ([4]). Let $G(x), R, C$ be polyhedral for all $x \in S$, that is,

$$\begin{aligned} G(x) &= \{z \in \mathbb{R}^m \mid P_G(x)z \leq q_G\}, \\ R &= \{z \in \mathbb{R}^m \mid P_R z \leq q_R\}, \\ C &= \{z \in \mathbb{R}^m \mid \langle p_j, z\rangle \geq 0 \text{ for all } j = 1, \dots, J\} \end{aligned}$$

where $P_G(x)$ is an $\alpha \times n$ matrix for all $x \in S$, P_R is a $\beta \times n$ matrix, $q_G \in \mathbb{R}^\alpha$, $q_R \in \mathbb{R}^\beta$, and $p_j \in \mathbb{R}^m$ for all $j = 1, \dots, J$. Then, the following statements hold for $x \in S$:

- $\Phi^{(1)}_{C,k}(G(x), R) = \max_{j=1,\dots,J} \{\text{Val}(P1(x)_j)\}$ for $(\text{P1}(x)_j)$ defined by

$$(\text{P1}(x)_i) \text{ Maximize } \frac{\langle p_j, z_G - z_R\rangle}{\langle p_j, k\rangle} \text{ subject to } P_G(x)z_G \leq q_G \text{ and } P_R z_R \leq q_R;$$

- $\Phi^{(2)}_{C,k}(G(x), R) = \mathrm{Val}(P2(x))$ for (P2(x)) and (P2$_j$) defined by

(P2(x)) Minimize $t \in \mathbb{R}$ subject to $\dfrac{\langle p_j, z\rangle}{\langle p_j, k\rangle} + \mathrm{Val}(P2_j) \leq t$ for all $j = 1, \ldots, J$ and $P_G(x)z \leq q_G$,

(P2$_j$) Maximize $\dfrac{\langle p_j, -z\rangle}{\langle p_j, k\rangle}$ subject to $P_R z \leq q_R$;

- $\Phi^{(3)}_{C,k}(G(x), R) = \max\limits_{z \in R} \mathrm{Val}(P3(x, z))$ for (P3(x, z)) defined by

(P3(x, z)) Minimize $t \in \mathbb{R}$ subject to $\dfrac{\langle p_j, z_G - z\rangle}{\langle p_j, k\rangle} \leq t$ for all $j = 1, \ldots, J$ and $P_G(x)z_G \leq q_G$

- $\Phi^{(4)}_{C,k}(G(x), R) = \mathrm{Val}(P4(x))$ for (P4(x)) and (P4(x)$_j$) defined by

(P4(x)) Minimize $t \in \mathbb{R}$ subject to $\mathrm{Val}(P4(x)_j) + \dfrac{\langle p_j, -z\rangle}{\langle p_j, k\rangle} \leq t$ for all $j = 1, \ldots, J$ and $P_R z \leq q_R$,

(P4(x)$_j$) Maximize $\dfrac{\langle p_j, z\rangle}{\langle p_j, k\rangle}$ subject to $P_G(x)z \leq q_G$;

- $\Phi^{(5)}_{C,k}(G(x), R) = \max\limits_{z \in G(x)} \mathrm{Val}(P5(z))$ for (P5(z)) defined by

(P5(z)) Minimize $t \in \mathbb{R}$ subject to $\dfrac{\langle p_j, z - z_R\rangle}{\langle p_j, k\rangle} \leq t$ for all $j = 1, \ldots, J$ and $P_R z_R \leq q_R$;

- $\Phi^{(6)}_{C,k}(G(x), R) = \mathrm{Val}(P6(x))$ for (P6(x)) defined by

(P6(x)) Minimize $t \in \mathbb{R}$ subject to $\dfrac{\langle p_j, z_1 - z_2\rangle}{\langle p_j, k\rangle} \leq t$ for all $j = 1, \ldots, J$, $P_G(x)z_1 \leq q_G$, and $P_R z_2 \leq q_R$.

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