

Scattering by the local perturbation of an open periodic waveguide in the half plane

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1 Introduction

Let $k > 0$ be the wave number, and let $\mathbb{R}_+^2 := \mathbb{R} \times (0, \infty)$ be the upper half plane, and let $W := \mathbb{R} \times (0, h)$ be the waveguide in \mathbb{R}_+^2 . We denote by $\Gamma_a := \mathbb{R} \times \{a\}$ for $a > 0$. Let $n \in L^\infty(\mathbb{R}_+^2)$ be real valued, 2π -periodic with respect to x_1 (that is, $n(x_1 + 2\pi, x_2) = n(x_1, x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}_+^2$), and equal to one for $x_2 > h$. We assume that there exists a constant $n_0 > 0$ such that $n \geq n_0$ in \mathbb{R}_+^2 . Let $q \in L^\infty(\mathbb{R}_+^2)$ be real valued with the compact support $\text{supp } q$ in W . We denote by $Q := \text{supp } q$. In this paper, we consider the following scattering problem: For fixed $y \in \mathbb{R}_+^2 \setminus \overline{W}$, determine the scattered field $u^s \in H_{loc}^1(\mathbb{R}_+^2)$ such that

$$\Delta u^s + k^2(1 + q)nu^s = -k^2 qnu^i(\cdot, y) \text{ in } \mathbb{R}_+^2, \quad (1.1)$$

$$u^s = 0 \text{ on } \Gamma_0, \quad (1.2)$$

Here, the incident field u^i is given by $u^i(x, y) = G_n(x, y)$, where G_n is the Dirichlet Green's function in the upper half plane \mathbb{R}_+^2 for $\Delta + k^2n$, that is,

$$G_n(x, y) := G(x, y) + \tilde{u}^s(x, y), \quad (1.3)$$

where $G(x, y) := \Phi_k(x, y) - \Phi_k(x, y^*)$ is the Dirichlet Green's function in \mathbb{R}_+^2 for $\Delta + k^2$, and $y^* = (y_1, -y_2)$ is the reflected point of y at $\mathbb{R} \times \{0\}$. Here, $\Phi_k(x, y)$ is the fundamental solution to Helmholtz equation in \mathbb{R}^2 , that is,

$$\Phi_k(x, y) := \frac{i}{4} H_0^{(1)}(k|x - y|), \quad x \neq y, \quad (1.4)$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order one. \tilde{u}^s is the scattered field of the unperturbed problem by the incident field $G(x, y)$, that is, \tilde{u}^s vanishes for $x_2 = 0$ and solves

$$\Delta \tilde{u}^s + k^2n\tilde{u}^s = k^2(1 - n)G(\cdot, y) \text{ in } \mathbb{R}_+^2. \quad (1.5)$$

If we impose a suitable radiation condition introduced in [8], the unperturbed solution \tilde{u}^s is uniquely determined. Later, we will explain the exact definition of this radiation condition (see Definition 2.4).

In order to show the well-posedness of the perturbed scattering problem (1.1)–(1.2), we make the following assumption.

Assumption 1.1. We assume that k^2 is not the point spectrum of $\frac{1}{(1+q)n}\Delta$ in $H_0^1(\mathbb{R}_+^2)$, that is, every $v \in H^1(\mathbb{R}_+^2)$ which satisfies

$$\Delta v + k^2(1+q)nv = 0 \text{ in } \mathbb{R}_+^2, \quad (1.6)$$

$$v = 0 \text{ on } \Gamma_0, \quad (1.7)$$

has to vanish for $x_2 > 0$.

If we assume that q and n satisfy in addition that $\partial_2((1+q)n) \geq 0$ in W , then v which satisfies (1.6)–(1.7) vanishes, that is, under this assumption all of k^2 is not the point spectrum of $\frac{1}{(1+q)n}\Delta$ (see Section 6). Our aim in this paper is to show the following theorem.

Theorem 1.2. Let Assumptions 1.1 and 2.1 hold and let $k > 0$ be regular in the sense of Definition 2.3 and let $f \in L^2(\mathbb{R}_+^2)$ such that $\text{supp} f = Q$. Then, there exists a unique solution $u \in H_{loc}^1(\mathbb{R}_+^2)$ such that

$$\Delta u + k^2(1+q)nu = f \text{ in } \mathbb{R}_+^2, \quad (1.8)$$

$$u = 0 \text{ on } \Gamma_0, \quad (1.9)$$

and u satisfies the radiation condition in the sense of Definition 2.4.

Roughly speaking, the radiation condition of Definition 2.4 requires that we have a decomposition of the solution u into $u^{(1)}$ which decays in the direction of x_1 , and a finite combination $u^{(2)}$ of propagative modes which does not decay, but it exponentially decays in the direction of x_2 .

This paper is organized as follows. In Section 2, we briefly recall a radiation condition introduced in [8]. Under the radiation condition in the sense of Definition 2.4, we show the uniqueness of $u^{(2)}$ and $u^{(1)}$ in Section 3 and 4, respectively. In Section 5, we show the existence of u . In Section 6, we give an example of n and q with respect to Assumption 1.1.

2 A radiation condition

In Section 2, we briefly recall a radiation condition introduced in [8]. Let $f \in L^2(\mathbb{R}_+^2)$ have the compact support in W . First, we consider the following problem: Find $u \in H_{loc}^1(\mathbb{R}_+^2)$ such that

$$\Delta u + k^2nu = f \text{ in } \mathbb{R}_+^2, \quad (2.1)$$

$$u = 0 \text{ on } \Gamma_0. \quad (2.2)$$

(2.1) is understood in the variational sense, that is,

$$\int_{\mathbb{R}_+^2} [\nabla u \cdot \nabla \bar{\varphi} - k^2nu\bar{\varphi}] dx = - \int_W f\bar{\varphi} dx, \quad (2.3)$$

for all $\varphi \in H^1(\mathbb{R}_+^2)$, with compact support. In such a problem, it is natural to impose the *upward propagating radiation condition*, that is, $u(\cdot, h) \in L^\infty(\mathbb{R})$ and

$$u(x) = 2 \int_{\Gamma_h} u(y) \frac{\partial \Phi_k(x, y)}{\partial y_2} ds(y) = 0, \quad x_2 > h. \quad (2.4)$$

However, even with this condition we can not expect the uniqueness of this problem. (see Example 2.3 of [8].) In order to introduce a *suitable radiation condition*, [8] discussed limiting absorption solution of this problem, that is, the limit of the solution u_ϵ of $\Delta u_\epsilon + (k + i\epsilon)^2 n u_\epsilon = f$ as $\epsilon \rightarrow 0$. For the details of an introduction of this radiation condition, we refer to [5, 6, 7, 8].

Let us prepare for the exact definition of the radiation condition. First we recall that the *Floquet Bloch transform* $T_{per} : L^2(\mathbb{R}) \rightarrow L^2((0, 2\pi) \times (-1/2, 1/2))$ is defined by

$$T_{per} f(t, \alpha) = \tilde{f}_\alpha(t) := \sum_{m \in \mathbb{Z}} f(t + 2\pi m) e^{-i\alpha(t + 2\pi m)}, \quad (2.5)$$

for $(t, \alpha) \in (0, 2\pi) \times (-1/2, 1/2)$. The inverse transform is given by

$$T_{per}^{-1} g(t) = \int_{-1/2}^{1/2} g(t, \alpha) e^{i\alpha t} d\alpha, \quad t \in \mathbb{R}. \quad (2.6)$$

By taking the Floquet Bloch transform with respect to x_1 in (2.1)–(2.2), we have for $\alpha \in (-1/2, 1/2]$

$$\Delta \tilde{u}_\alpha + 2i\alpha \frac{\partial \tilde{u}_\alpha}{\partial x_1} + (k^2 n - \alpha^2) \tilde{u}_\alpha = \tilde{f}_\alpha \text{ in } (0, 2\pi) \times (0, \infty). \quad (2.7)$$

$$\tilde{u}_\alpha = 0 \text{ on } (0, 2\pi) \times \{0\}. \quad (2.8)$$

By taking the Floquet Bloch transform with respect to x_1 in (2.4), \tilde{u}_α satisfies the *Rayleigh expansion* of the form

$$\tilde{u}_\alpha(x) = \sum_{n \in \mathbb{Z}} u_n(\alpha) e^{inx_1 + i\sqrt{k^2 - (n+\alpha)^2}(x_2 - h)}, \quad x_2 > h, \quad (2.9)$$

where $u_n(\alpha) := (2\pi)^{-1} \int_0^{2\pi} u_\alpha(x_1, h) e^{-inx_1} dx_1$ are the Fourier coefficients of $u_\alpha(\cdot, h)$, and $\sqrt{k^2 - (n+\alpha)^2} = i\sqrt{(n+\alpha)^2 - k^2}$ if $n+\alpha > k$.

We denote by $C_R := (0, 2\pi) \times (0, R)$ for $R \in (0, \infty]$, and $H_{per}^1(C_R)$ the subspace of the 2π -periodic function in $H^1(C_R)$. We also denote by $H_{0,per}^1(C_R) := \{u \in H_{per}^1(C_R) : u = 0 \text{ on } (0, 2\pi) \times \{0\}\}$ that is equipped with $H^1(C_R)$ norm. The space $H_{0,per}^1(C_R)$ has the inner product of the form

$$\langle u, v \rangle_* = \int_{C_h} \nabla u \cdot \nabla \bar{v} dx + 2\pi \sum_{n \in \mathbb{Z}} \sqrt{n^2 + 1} u_n \bar{v}_n, \quad (2.10)$$

where $u_n = (2\pi)^{-1} \int_0^{2\pi} u(x_1, R) e^{-inx_1} dx_1$. The problem (2.7)–(2.9) is equivalent to the following operator equation (see section 3 in [8]),

$$\tilde{u}_\alpha - K_\alpha \tilde{u}_\alpha = \tilde{f}_\alpha \text{ in } H_{0,per}^1(C_h), \quad (2.11)$$

where the operator $K_\alpha : H_{0,per}^1(C_h) \rightarrow H_{0,per}^1(C_h)$ is defined by

$$\begin{aligned} \langle K_\alpha u, v \rangle_* &= - \int_{C_h} \left[i\alpha \left(u \frac{\partial \bar{v}}{\partial x_1} - \bar{v} \frac{\partial u}{\partial x_1} \right) + (\alpha^2 - k^2 n) u \bar{v} \right] dx \\ &+ 2\pi i \sum_{|n+\alpha| \leq k} u_n \bar{v}_n (\sqrt{k^2 - (n+\alpha)^2} - i\sqrt{n^2 + 1}) \\ &+ 2\pi \sum_{|n+\alpha| > k} u_n \bar{v}_n (\sqrt{n^2 + 1} - \sqrt{(n+\alpha)^2 - k^2}). \end{aligned} \quad (2.12)$$

For several $\alpha \in (-1/2, 1/2]$, the uniqueness of this problem fails. We call these α *exceptional values* if the operator $I - K_\alpha$ fails to be injective. For the difficulty of treatment of α such that $|\alpha + l| = k$ for some $l \in \mathbb{Z}$ in an periodic scattering problem, we set $A_k := \{\alpha \in (-1/2, 1/2] : \exists l \in \mathbb{Z} \text{ s.t. } |\alpha + l| = k\}$, and make the following assumption:

Assumption 2.1. *For every $\alpha \in A_k$, $I - K_\alpha$ has to be injective.*

The following properties of exceptional values was shown in Lemmas 4.2 and 5.6 of [8].

Lemma 2.2. *Let Assumption 2.1 hold. Then, there exists only finitely many exceptional values $\alpha \in (-1/2, 1/2]$. Furthermore, if α is an exceptional value, then so is $-\alpha$. Therefore, the set of exceptional values can be described by $\{\alpha_j : j \in J\}$ where some $J \subset \mathbb{Z}$ is finite and $\alpha_{-j} = -\alpha_j$ for $j \in J$. For each exceptional value α_j we define*

$$X_j := \left\{ \begin{array}{l} \Delta\phi + 2i\alpha_j \frac{\partial\phi}{\partial x_1} + (k^2 n - \alpha^2)\phi = 0 \text{ in } \mathbb{R}_+^2, \\ \phi \in H_{loc}^1(\mathbb{R}_+^2) : \phi = 0 \text{ for } x_2 = 0, \quad \phi \text{ is } 2\pi\text{-periodic for } x_1, \\ \phi \text{ satisfies the Rayleigh expansion (2.9)} \end{array} \right\}$$

Then, X_j are finite dimensional. We set $m_j = \dim X_j$. Furthermore, $\phi \in X_j$ is evanescent, that is, there exists $c > 0$ and $\delta > 0$ such that $|\phi(x)|, |\nabla\phi(x)| \leq ce^{-\delta|x_2|}$ for all $x \in \mathbb{R}_+^2$.

Next, we consider the following eigenvalue problem in X_j : Determine $d \in \mathbb{R}$ and $\phi \in X_j$ such that

$$\int_{C_\infty} \left[-i \frac{\partial\phi}{\partial x_1} + \alpha_j \phi \right] \bar{\psi} dx = dk \int_{C_\infty} n \phi \bar{\psi} dx, \quad (2.13)$$

for all $\psi \in X_j$. We denote by the eigenvalues $d_{l,j}$ and the eigenfunction $\phi_{l,j}$ of this problem, that is,

$$\int_{C_\infty} \left[-i \frac{\partial\phi_{l,j}}{\partial x_1} + \alpha_j \phi_{l,j} \right] \bar{\psi} dx = d_{l,j} k \int_{C_\infty} n \phi_{l,j} \bar{\psi} dx, \quad (2.14)$$

for every $l = 1, \dots, m_j$ and $j \in J$. We normalize the eigenfunction $\{\phi_{l,j} : l = 1, \dots, m_j\}$ such that

$$k \int_{C_\infty} n \phi_{l,j} \overline{\phi_{l',j}} dx = \delta_{l,l'}, \quad (2.15)$$

for all l, l' . We will assume that the wave number $k > 0$ is *regular* in the following sense.

Definition 2.3. $k > 0$ is *regular* if $d_{l,j} \neq 0$ for all $l = 1, \dots, m_j$ and $j \in J$.

Now we are ready to define the radiation condition.

Definition 2.4. Let Assumptions 2.1 hold, and let $k > 0$ be regular in the sense of Definition 2.3. We set

$$\psi^\pm(x_1) := \frac{1}{2} \left[1 \pm \frac{2}{\pi} \int_0^{x_1/2} \frac{\sin t}{t} dt \right], \quad x_1 \in \mathbb{R}. \quad (2.16)$$

Then, $u \in H_{loc}^1(\mathbb{R}_+^2)$ satisfies the *radiation condition* if u satisfies the upward propagating radiation condition (2.4), and has a decomposition in the form $u = u^{(1)} + u^{(2)}$ where $u^{(1)}|_{\mathbb{R} \times (0, R)} \in H^1(\mathbb{R} \times (0, R))$ for all $R > 0$, and $u^{(2)} \in L^\infty(\mathbb{R}_+^2)$ has the following form

$$u^{(2)}(x) = \psi^+(x_1) \sum_{j \in J} \sum_{d_{l,j} > 0} a_{l,j} \phi_{l,j}(x) + \psi^-(x_1) \sum_{j \in J} \sum_{d_{l,j} < 0} a_{l,j} \phi_{l,j}(x) \quad (2.17)$$

where some $a_{l,j} \in \mathbb{C}$, and $\{d_{l,j}, \phi_{l,j} : l = 1, \dots, m_j\}$ are normalized eigenvalues and eigenfunctions of the problem (2.8).

Remark 2.5. We can replace ψ^\pm by any smooth functions $\tilde{\psi}^\pm$ such that $|\psi^\pm(x_1) - \tilde{\psi}^\pm(x_1)| \rightarrow 0$, and $|\frac{d}{dx_1}\psi^\pm(x_1) - \frac{d}{dx_1}\tilde{\psi}^\pm(x_1)| \rightarrow 0$ as $|x_1| \rightarrow \infty$ because (2.12) is of the form

$$\begin{aligned} u^{(2)}(x) &= \tilde{\psi}^+(x_1) \sum_{j \in J} \sum_{d_{l,j} > 0} a_{l,j} \phi_{l,j}(x) + \tilde{\psi}^-(x_1) \sum_{j \in J} \sum_{d_{l,j} < 0} a_{l,j} \phi_{l,j}(x) \\ &+ \left(\psi^+(x_1) - \tilde{\psi}^+(x_1) \right) \sum_{j \in J} \sum_{d_{l,j} > 0} a_{l,j} \phi_{l,j}(x) + \left(\psi^-(x_1) - \tilde{\psi}^-(x_1) \right) \sum_{j \in J} \sum_{d_{l,j} < 0} a_{l,j} \phi_{l,j}(x), \end{aligned} \quad (2.18)$$

where the second term in the right-hand side of (2.13) is a H^1 -function, which is the role of $u^{(1)}$.

The following was shown in Theorems 2.2, 6.6, and 6.8 of [8].

Theorem 2.6. *Let Assumptions 2.1 hold and let $k > 0$ be regular in the sense of Definition 2.3. For every $f \in L^2(\mathbb{R}_+^2)$ with the compact support in W , there exists a unique solution $u_{k+i\epsilon} \in H^1(\mathbb{R}_+^2)$ of the problem (2.1)–(2.2) replacing k by $k+i\epsilon$. Furthermore, $u_{k+i\epsilon}$ converge as $\epsilon \rightarrow +0$ in $H_{loc}^1(\mathbb{R}_+^2)$ to some $u \in H_{loc}^1(\mathbb{R}_+^2)$ which satisfy (2.1)–(2.2) and the radiation condition in the sense of Definition 2.4. Furthermore, the solution u of this problem is uniquely determined.*

Finally in this section, we will show the following integral representation.

Lemma 2.7. *Let $f \in L^2(\mathbb{R}_+^2)$ have a compact support in W , and let u be a solution of (2.1)–(2.2) which satisfying the radiation condition in the sense of Definition 2.4. Then, u has an integral representation of the form*

$$u(x) = k^2 \int_W (n(y) - 1)u(y)G(x, y)dy - \int_W f(y)G(x, y)dy, \quad x \in \mathbb{R}_+^2 \quad (2.19)$$

Proof of Lemma 2.7. Let $\epsilon > 0$ be small enough and let $u_\epsilon \in H^1(\mathbb{R}_+^2)$ be a solution of the problem (2.1)–(2.2) replacing k by $k + i\epsilon$, that is, u_ϵ satisfies

$$\Delta u_\epsilon + (k + i\epsilon)^2 n u_\epsilon = f \text{ in } \mathbb{R}_+^2, \quad (2.20)$$

$$u_\epsilon = 0 \text{ on } \Gamma_0. \quad (2.21)$$

Let $G_\epsilon(x, y)$ be the Dirichlet Green's function in the upper half plane \mathbb{R}_+^2 for $\Delta + (k + i\epsilon)^2$. Let $x \in \mathbb{R}_+^2$ be always fixed such that $x_2 > R$. Let $r > 0$ be large enough such that $x \in B_r(0)$ where $B_r(0) \subset \mathbb{R}^2$ be a open ball with center 0 and radius $r > 0$. By Green's representation theorem in $B_r(0) \cap \mathbb{R}_+^2$ we have

$$\begin{aligned} u_\epsilon(x) &= \int_{\partial B_r(0) \cap \mathbb{R}_+^2} \left[\frac{\partial u_\epsilon}{\partial \nu}(y) G_\epsilon(x, y) - u_\epsilon(y) \frac{\partial G_\epsilon}{\partial \nu}(x, y) \right] ds(y) \\ &- \int_{B_r(0) \cap \mathbb{R}_+^2} \left[\Delta u_\epsilon(y) + (k + i\epsilon)^2 u_\epsilon(y) \right] G_\epsilon(x, y) dy \\ &= \int_{\partial B_r(0) \cap \mathbb{R}_+^2} \left[\frac{\partial u_\epsilon}{\partial \nu}(y) G_\epsilon(x, y) - u_\epsilon(y) \frac{\partial G_\epsilon}{\partial \nu}(x, y) \right] ds(y) \\ &+ (k + i\epsilon)^2 \int_{B_r(0) \cap \mathbb{R}_+^2} (n(y) - 1) u_\epsilon(y) G_\epsilon(x, y) dy \\ &- \int_{B_r(0) \cap \mathbb{R}_+^2} f(y) G_\epsilon(x, y) dy. \end{aligned} \quad (2.22)$$

Since $u_\epsilon \in H^1(\mathbb{R}_+^2)$, the first term of the right hand side converges to zero as $r \rightarrow \infty$. Therefore, as $r \rightarrow \infty$ we have for $x \in \mathbb{R}_+^2$

$$u_\epsilon(x) = (k + i\epsilon)^2 \int_W (n(y) - 1)u_\epsilon(y)G_\epsilon(x, y)dy - \int_W f(y)G_\epsilon(x, y)dy. \quad (2.23)$$

We will show that (2.23) converges as $\epsilon \rightarrow 0$ to

$$u(x) = k^2 \int_W (n(y) - 1)u(y)G(x, y)dy - \int_W f(y)G(x, y)dy. \quad (2.24)$$

Indeed, by the argument in (3.8) and (3.9) of [2], $G_\epsilon(x, y)$ is of the estimation

$$|G_\epsilon(x, y)| \leq C \frac{x_2 y_2}{1 + |x - y|^{3/2}}, \quad |x - y| > 1, \quad (2.25)$$

where above C is independent of $\epsilon > 0$. Then, by Lebesgue dominated convergence theorem we have the second integral in (2.23) converges as $\epsilon \rightarrow 0$ to one in (2.24). So, we will consider the convergence of the first integral in (2.23).

By the beginning of the proof of Theorem 6.6 in [8], u_ϵ can be of the form $u_\epsilon = u_\epsilon^{(1)} + u_\epsilon^{(2)}$ where $u_\epsilon^{(1)}$ converges to $u^{(1)}$ in $H^1(W)$, and $u_\epsilon^{(2)}$ is of the form for $x \in W$

$$u_\epsilon^{(2)}(x) = \sum_{j \in J} \sum_{l=1}^{m_j} y_{l,j} \int_{-1/2}^{1/2} \frac{e^{i\alpha x_1}}{i\epsilon - d_{l,j}\alpha} d\alpha \phi_{l,j}(x), \quad (2.26)$$

which converges pointwise to $u^{(2)}(x)$. Here, $y_{l,j} \in \mathbb{C}$ is some constant. From the convergence of $u_\epsilon^{(1)}$ in $H^1(W)$ we obtain that $\int_W (n(y) - 1)u_\epsilon^{(1)}(y)G_\epsilon(x, y)dy$ converges $\int_W (n(y) - 1)u^{(1)}(y)G(x, y)dy$ as $\epsilon \rightarrow 0$.

By the argument of (b) in Lemma 6.1 of [8] we have

$$\begin{aligned} \psi_{l,j,\epsilon}(x_1) &:= \int_{-1/2}^{1/2} \frac{e^{i\alpha x_1}}{i\epsilon - d_{l,j}\alpha} d\alpha \\ &= -\frac{i}{|d_{l,j}|} \int_{-|d_{l,j}|/(2\epsilon)}^{|d_{l,j}|/(2\epsilon)} \frac{\cos(t\epsilon x_1/|d_{l,j}|)}{1 + t^2} dt - 2id_{l,j} \int_0^{x_1/2} \frac{t \operatorname{sint}}{x_1^2 \epsilon^2 + d_{l,j}^2 t^2} dt, \end{aligned} \quad (2.27)$$

which implies that for all $x_1 \in \mathbb{R}$

$$\begin{aligned} |\psi_{l,j,\epsilon}(x_1)| &\leq C \left(\int_{-\infty}^{\infty} \frac{dt}{1 + t^2} + \int_0^{|x_1|/2} \left| \frac{\operatorname{sint}}{t} \right| dt \right) \\ &\leq C \left(\int_{-\infty}^{\infty} \frac{dt}{1 + t^2} dt + \int_0^1 \left| \frac{\operatorname{sint}}{t} \right| dt + \int_1^{|x_1|+1} \frac{1}{t} dt \right) \\ &\leq C(1 + \log(|x_1| + 1)), \end{aligned} \quad (2.28)$$

where above C is independent of $\epsilon > 0$. Then, we have that for $y \in W$

$$|(n(y) - 1)u_\epsilon^{(2)}(y)G_\epsilon(x, y)| \leq \frac{C(1 + \log(|y_1| + 1))}{1 + |x - y|^{3/2}}, \quad (2.29)$$

where above C is independent of y and ϵ . Then, right hand side of (2.29) is an integrable function in W with respect to y . Then, by Lebesgue dominated convergence theorem $\int_W (n(y) - 1)u_\epsilon^{(2)}(y)G_\epsilon(x, y)dy$ converges to $\int_W (n(y) - 1)u^{(2)}(y)G(x, y)dy$ as $\epsilon \rightarrow 0$. Therefore, (2.24) has been shown. \square

3 Uniqueness of $u^{(2)}$

In Section 3, we will show the uniqueness of $u^{(2)}$ in Theorem 1.2.

Lemma 3.1. *Let Assumptions 2.1 hold and let $k > 0$ be regular in the sense of Definition 2.3. If $u \in H_{loc}^1(\mathbb{R}_+^2)$ such that*

$$\Delta u + k^2(1+q)nu = 0, \text{ in } \mathbb{R}_+^2, \quad (3.1)$$

$$u = 0 \text{ on } \Gamma_0, \quad (3.2)$$

and u satisfies the radiation condition in the sense of Definition 2.4, then $u^{(2)} = 0$ in \mathbb{R}_+^2 .

Proof of Lemma 3.1. By the definition of the radiation condition, u is of the form $u = u^{(1)} + u^{(2)}$ where $u^{(1)}|_{\mathbb{R} \times (0, R)} \in H^1(\mathbb{R} \times (0, R))$ for all $R > 0$, and $u^{(2)} \in L^\infty(\mathbb{R}_+^2)$ has the form

$$u^{(2)}(x) = \psi^+(x_1) \sum_{j \in J} \sum_{d_{l,j} > 0} a_{l,j} \phi_{l,j}(x) + \psi^-(x_1) \sum_{j \in J} \sum_{d_{l,j} < 0} a_{l,j} \phi_{l,j}(x), \quad (3.3)$$

where some $a_{l,j} \in \mathbb{C}$, and $\{d_{l,j}, \phi_{l,j} : l = 1, \dots, m_j\}$ are normalized eigenvalues and eigenfunctions of the problem (2.13). Here, by Remark 2.5 the function ψ^+ is chosen as a smooth function such that $\psi^+(x_1) = 1$ for $x_1 \geq \eta$ and $\psi^+(x_1) = 0$ for $x_1 \leq -\eta$, and $\psi^- := 1 - \psi^+$ where $\eta > 0$ is some positive number.

Step1 (Green's theorem in Ω_N): We set $\Omega_N := (-N, N) \times (0, \phi(N))$ where $\psi(N) := N^s$. Later we will choose a appropriate $s \in (0, 1)$. Let $R > h$ be large and always fixed, and let N be large enough such that $\phi(N) > R$. We denote by $I_{\pm N}^R := \{\pm N\} \times (0, R)$, $I_{\pm N}^{\phi(N)} := \{\pm N\} \times (R, \phi(N))$, and $\Gamma_{\phi(N), N} := (-N, N) \times \{\phi(N)\}$. (see the figure below.) We set $I_{\pm N} := I_{\pm N}^R \cup I_{\pm N}^{\phi(N)}$.

By Green's first theorem in Ω_N and $u = 0$ on $(-N, N) \times \{0\}$, we have

$$\begin{aligned} & \int_{\Omega_N} \{-k^2(1+q)n|u|^2 + |\nabla u|^2\} dx = \int_{\Omega_N} \{\bar{u}\Delta u + |\nabla u|^2\} dx \\ & = \int_{I_N} \bar{u} \frac{\partial u}{\partial x_1} ds - \int_{I_{-N}} \bar{u} \frac{\partial u}{\partial x_1} ds + \int_{\Gamma_{\phi(N), N}} \bar{u} \frac{\partial u}{\partial x_2} ds \\ & = \int_{I_N} \overline{u^{(2)}} \frac{\partial u^{(2)}}{\partial x_1} ds - \int_{I_{-N}} \overline{u^{(2)}} \frac{\partial u^{(2)}}{\partial x_1} ds \\ & + \int_{I_N} \overline{u^{(1)}} \frac{\partial u^{(1)}}{\partial x_1} ds + \int_{I_N} \overline{u^{(1)}} \frac{\partial u^{(2)}}{\partial x_1} ds + \int_{I_N} \overline{u^{(2)}} \frac{\partial u^{(1)}}{\partial x_1} ds \\ & - \int_{I_{-N}} \overline{u^{(1)}} \frac{\partial u^{(1)}}{\partial x_1} ds - \int_{I_{-N}} \overline{u^{(1)}} \frac{\partial u^{(2)}}{\partial x_1} ds - \int_{I_{-N}} \overline{u^{(2)}} \frac{\partial u^{(1)}}{\partial x_1} ds \\ & + \int_{\Gamma_{\phi(N), N}} \bar{u} \frac{\partial u}{\partial x_2} ds. \end{aligned} \quad (3.4)$$

By the same argument in Theorem 4.6 of [7] and Lemma 6.3 of [8], we can show that

$$\begin{aligned}
& \int_{I_N} \overline{u^{(2)}} \frac{\partial u^{(2)}}{\partial x_1} ds - \int_{I_{-N}} \overline{u^{(2)}} \frac{\partial u^{(2)}}{\partial x_1} ds \\
& + \int_{I_N^R} \overline{u^{(1)}} \frac{\partial u^{(1)}}{\partial x_1} ds + \int_{I_N^R} \overline{u^{(1)}} \frac{\partial u^{(2)}}{\partial x_1} ds + \int_{I_N^R} \overline{u^{(2)}} \frac{\partial u^{(1)}}{\partial x_1} ds \\
& - \int_{I_{-N}^R} \overline{u^{(1)}} \frac{\partial u^{(1)}}{\partial x_1} ds - \int_{I_{-N}^R} \overline{u^{(1)}} \frac{\partial u^{(2)}}{\partial x_1} ds - \int_{I_{-N}^R} \overline{u^{(2)}} \frac{\partial u^{(1)}}{\partial x_1} ds \\
& = \frac{1}{2\pi} \sum_{j \in J} \sum_{d_{l,j}, d_{l',j} > 0} \overline{a_{l,j}} a_{l',j} \int_{C_{\phi(N)}} \overline{\phi_{l,j}} \frac{\partial \phi_{l',j}}{\partial x_1} dx \\
& - \frac{1}{2\pi} \sum_{j \in J} \sum_{d_{l,j}, d_{l',j} < 0} \overline{a_{l,j}} a_{l',j} \int_{C_{\phi(N)}} \overline{\phi_{l,j}} \frac{\partial \phi_{l',j}}{\partial x_1} dx + o(1), \tag{3.5}
\end{aligned}$$

and the first and second term in the right hand side converge as $N \rightarrow \infty$ to $\frac{ik}{2\pi} \sum_{j \in J} \sum_{d_{l,j} > 0} |a_{l,j}|^2 d_{l,j}$ and $-\frac{ik}{2\pi} \sum_{j \in J} \sum_{d_{l,j} < 0} |a_{l,j}|^2 d_{l,j}$ respectively. Therefore, taking an imaginary part in (3.4) yields that

$$\begin{aligned}
0 & = \text{Im} \left[\frac{1}{2\pi} \sum_{j \in J} \sum_{d_{l,j}, d_{l',j} > 0} \overline{a_{l,j}} a_{l',j} \int_{C_{\phi(N)}} \overline{\phi_{l,j}} \frac{\partial \phi_{l',j}}{\partial x_1} dx \right] \\
& - \text{Im} \left[\frac{1}{2\pi} \sum_{j \in J} \sum_{d_{l,j}, d_{l',j} < 0} \overline{a_{l,j}} a_{l',j} \int_{C_{\phi(N)}} \overline{\phi_{l,j}} \frac{\partial \phi_{l',j}}{\partial x_1} dx \right] \\
& + \text{Im} \int_{I_N^{\phi(N)}} \overline{u^{(1)}} \frac{\partial u^{(1)}}{\partial x_1} ds + \text{Im} \int_{I_N^{\phi(N)}} \overline{u^{(1)}} \frac{\partial u^{(2)}}{\partial x_1} ds + \text{Im} \int_{I_N^{\phi(N)}} \overline{u^{(2)}} \frac{\partial u^{(1)}}{\partial x_1} ds \\
& - \text{Im} \int_{I_{-N}^{\phi(N)}} \overline{u^{(1)}} \frac{\partial u^{(1)}}{\partial x_1} ds - \text{Im} \int_{I_{-N}^{\phi(N)}} \overline{u^{(1)}} \frac{\partial u^{(2)}}{\partial x_1} ds - \text{Im} \int_{I_{-N}^{\phi(N)}} \overline{u^{(2)}} \frac{\partial u^{(1)}}{\partial x_1} ds \\
& + \text{Im} \int_{\Gamma_{\phi(N), N}} \overline{u} \frac{\partial u}{\partial x_2} ds + o(1). \tag{3.6}
\end{aligned}$$

We set

$$J_{\pm}(N) := \pm \text{Im} \int_{I_{\pm N}^{\phi(N)}} \overline{u^{(1)}} \frac{\partial u^{(1)}}{\partial x_1} ds \pm \text{Im} \int_{I_{\pm N}^{\phi(N)}} \overline{u^{(1)}} \frac{\partial u^{(2)}}{\partial x_1} ds \pm \text{Im} \int_{I_{\pm N}^{\phi(N)}} \overline{u^{(2)}} \frac{\partial u^{(1)}}{\partial x_1} ds, \tag{3.7}$$

and we will show that $\limsup_{N \rightarrow \infty} J_{\pm}(N) \geq 0$.

Step2 ($\limsup_{N \rightarrow \infty} J_{\pm}(N) \geq 0$): By the Cauchy Schwarz inequality we have

$$\begin{aligned}
|J_+(N)| &\leq \left(\int_R^{\phi(N)} |u^{(1)}(N, x_2)|^2 dx_2 \right)^{1/2} \left(\int_R^{\phi(N)} \left| \frac{\partial u^{(1)}}{\partial x_1}(N, x_2) \right|^2 dx_2 \right)^{1/2} \\
&+ \left(\int_R^{\phi(N)} |u^{(1)}(N, x_2)|^2 dx_2 \right)^{1/2} \left(\int_R^{\phi(N)} \left| \frac{\partial u^{(2)}}{\partial x_1}(N, x_2) \right|^2 dx_2 \right)^{1/2} \\
&+ \left(\int_R^{\phi(N)} |u^{(2)}(N, x_2)|^2 dx_2 \right)^{1/2} \left(\int_R^{\phi(N)} \left| \frac{\partial u^{(1)}}{\partial x_1}(N, x_2) \right|^2 dx_2 \right)^{1/2} \\
&\leq \left(\int_R^{\phi(N)} |u^{(1)}(N, x_2)|^2 dx_2 \right)^{1/2} \left(\int_R^{\phi(N)} \left| \frac{\partial u^{(1)}}{\partial x_1}(N, x_2) \right|^2 dx_2 \right)^{1/2} \\
&+ C(\phi(N) - R)^{1/2} \left(\int_R^{\phi(N)} |u^{(1)}(N, x_2)|^2 dx_2 \right)^{1/2} \\
&+ C(\phi(N) - R)^{1/2} \left(\int_R^{\phi(N)} \left| \frac{\partial u^{(1)}}{\partial x_1}(N, x_2) \right|^2 dx_2 \right)^{1/2}. \tag{3.8}
\end{aligned}$$

In order to estimate $u^{(1)}$, we will show the following lemma.

Lemma 3.2. $u^{(1)}$ has an integral representation of the form

$$\begin{aligned}
u^{(1)}(x) &= \int_{y_2 > 0} \sigma(y) G(x, y) dy + k^2 \int_W (n(y)(1 + q(y)) - 1) u^{(1)}(y) G(x, y) dy \\
&+ k^2 \int_Q n(y) q(y) u^{(2)}(y) G(x, y) dy, \quad x_2 > 0, \tag{3.9}
\end{aligned}$$

where $\sigma := \Delta u^{(2)} + k^2 n u^{(2)}$.

Proof of Lemma 3.2. First, we will consider an integral representation of $u^{(2)}$. Let $N > 0$ be large enough. By Green's representation theorem in $(-N, N) \times (0, N^{1/4})$, we have

$$\begin{aligned}
u^{(2)}(x) &= \int_{(-N, N) \times \{N^{1/4}\}} [u^{(2)}(y) \frac{\partial G}{\partial y_2}(x, y) - G(x, y) \frac{\partial u^{(2)}}{\partial y_2}(y)] ds(y) \\
&+ \left(\int_{\{N\} \times (0, N^{1/4})} - \int_{\{-N\} \times (0, N^{1/4})} \right) [u^{(2)}(y) \frac{\partial G}{\partial y_1}(x, y) - G(x, y) \frac{\partial u^{(2)}}{\partial y_1}(y)] ds(y) \\
&- \int_{(-N, N) \times (0, N^{1/4})} [\sigma(y) + k^2(1 - n(y))u^{(2)}(y)] G(x, y) dy. \tag{3.10}
\end{aligned}$$

By Lemma 3.1 of [2], the Dirichlet Green's function $G(x, y)$ is of the estimation

$$|G(x, y)|, |\nabla_y G(x, y)| \leq C \frac{x_2 y_2}{1 + |x - y|^{3/2}}, \quad |x - y| > 1. \tag{3.11}$$

By Lemma 2.2 we have that $|u^{(2)}(x)|, \left| \frac{\partial u^{(2)}(x)}{\partial x_2} \right| \leq c e^{-\delta|x_2|}$ for all $x \in \mathbb{R}_+^2$, and some $c, \delta > 0$. Then, we obtain

$$\begin{aligned}
&\left| \int_{(-N, N) \times \{N^{1/4}\}} [u^{(2)}(y) \frac{\partial G}{\partial y_2}(x, y) - G(x, y) \frac{\partial u^{(2)}}{\partial y_2}(y)] ds(y) \right| \\
&\leq C \int_{-N}^N \frac{x_2 e^{-\delta N^{1/4}}}{|N^{1/4} - x_2|^{3/2}} dy_2 \leq C \frac{x_2 N e^{-\delta N^{1/4}}}{|N^{1/4} - x_2|^{3/2}}. \tag{3.12}
\end{aligned}$$

Furthermore,

$$\begin{aligned} & \left| \int_{\{\pm N\} \times (0, N^{1/4})} [u^{(2)}(y) \frac{\partial G}{\partial y_1}(x, y) - G(x, y) \frac{\partial u^{(2)}}{\partial y_1}(y)] ds(y) \right| \\ & \leq C \int_0^{N^{1/4}} \frac{x_2 y_2}{|\pm N - x_1|^{3/2}} dy_2 \leq C \frac{x_2 N^{1/2}}{|\pm N - x_1|^{3/2}}. \end{aligned} \quad (3.13)$$

Therefore, as $N \rightarrow \infty$ in (3.10) we get

$$u^{(2)}(x) = - \int_{y_2 > 0} \sigma(y) G(x, y) dy + k^2 \int_W (n(y) - 1) u^{(2)}(y) G(x, y) dy. \quad (3.14)$$

By Lemma 2.7, we have (substitute $-k^2 qnu$ for f in (2.19))

$$u(x) = k^2 \int_W (n(y) - 1) u(y) G(x, y) dy + k^2 \int_Q q(y) n(y) u(y) G(x, y) dy. \quad (3.15)$$

Combining (3.14) with (3.15) we have

$$\begin{aligned} u^{(1)}(x) &= -u^{(2)}(x) + k^2 \int_W (n(y) - 1) u(y) G(x, y) dy + k^2 \int_Q q(y) n(y) u(y) G(x, y) dy \\ &= \int_{y_2 > 0} \sigma(y) G(x, y) dy - k^2 \int_W (n(y) - 1) u^{(2)}(y) G(x, y) dy \\ &+ k^2 \int_W (n(y) - 1) u(y) G(x, y) dy + k^2 \int_Q q(y) n(y) u(y) G(x, y) dy \\ &= \int_{\mathbb{R}_+^2} \sigma(y) G(x, y) dy + k^2 \int_W (n(y)(1 + q(y)) - 1) u^{(1)}(y) G(x, y) dy \\ &+ k^2 \int_Q n(y) q(y) u^{(2)}(y) G(x, y) dy. \end{aligned} \quad (3.16)$$

Therefore, Lemma 3.2 has been shown. \square

We set $u^\pm(x) := \sum_{j \in J} \sum_{d_{l,j} \leq 0} a_{l,j} \phi_{l,j}(x)$. Then, by a simple calculation we can show

$$\sigma(y) = \frac{d^2 \psi^+(y_1)}{dy_1^2} u^+(y) + 2 \frac{d\psi^+(y_1)}{dy_1} \frac{\partial u^+(y)}{\partial y_1} + \frac{d^2 \psi^-(y_1)}{dy_1^2} u^-(y) + 2 \frac{d\psi^-(y_1)}{dy_1} \frac{\partial u^-(y)}{\partial y_1}, \quad (3.17)$$

which implies that $\text{supp } \sigma \subset (-\eta, \eta) \times (0, \infty)$. By Lemma 3.2 we have for $R < x_2 < \phi(N)$

$$\begin{aligned} & |u^{(1)}(N, x_2)|, \left| \frac{\partial u^{(1)}}{\partial x_1}(N, x_2) \right| \leq C \int_{(-\eta, \eta) \times (0, \infty)} |\sigma(y)| \frac{\phi(N) y_2}{|N - \eta|^{3/2}} dy \\ & + C \int_W |u^{(1)}(y)| \frac{\phi(N) h}{(1 + |N - y_1|)^{3/2}} dy + C \int_Q \frac{\phi(N) |u^{(2)}(y)|}{|N - y_1|^{3/2}} dy \\ & \leq C \frac{\phi(N)}{N^{3/2}} + C \phi(N) \int_W \frac{|u^{(1)}(y)|}{(1 + |N - y_1|)^{3/2}} dy. \end{aligned} \quad (3.18)$$

We have to estimate the second term in right hand side. The following lemma was shown in Lemma 4.12 of [1].

Lemma 3.3. *Assume that $\varphi \in L^2_{loc}(\mathbb{R})$ such that*

$$\sup_{A>0} \left\{ (1+A^2)^{-\epsilon} \int_{-A}^A |\varphi(t)|^2 dt \right\} < \infty, \quad (3.19)$$

for some $\epsilon > 0$. Then, for every $\alpha \in [0, \frac{1}{2} - \epsilon)$ there exists a constant $C > 0$ and a sequence $\{A_m\}_{m \in \mathbb{N}}$ such that $A_m \rightarrow \infty$ as $m \rightarrow \infty$ and

$$\int_{K_{A_m}} |\varphi(t)|^2 dt \leq C A_m^{-\alpha}, \quad m \in \mathbb{N}, \quad (3.20)$$

where $K_A := K_A^+ \cup K_A^-$, $K_A^+ := (-A^+, A^+) \setminus (-A, A)$, $K_A^- := (-A, A) \setminus (-A^-, A^-)$, and $A^\pm := A \pm A^{1/2}$ for $A \in [1, \infty)$.

Applying Lemma 3.3 to $\varphi = \left(\int_0^h |u^{(1)}(\cdot, y_2)|^2 dy_2 \right)^{1/2} \in L^2(\mathbb{R})$, there exists a sequence $\{N_m\}_{m \in \mathbb{N}}$ such that $N_m \rightarrow \infty$ as $m \rightarrow \infty$ and

$$\int_{K_{N_m}} \int_0^h |u^{(1)}(y_1, y_2)|^2 dy_1 dy_2 \leq C N_m^{-1/4}, \quad m \in \mathbb{N}. \quad (3.21)$$

Then, by the Cauchy Schwarz inequality we have

$$\begin{aligned} \int_W \frac{|u^{(1)}(y)|}{(1+|N-y_1|)^{3/2}} dy &= \left(\int_{-N_m^-}^{N_m^-} + \int_{K_{N_m}} + \int_{\mathbb{R} \setminus [-N_m^+, N_m^+]} \right) \int_0^h \frac{|u^{(1)}(y)|}{(1+|N_m-y_1|)^{3/2}} dy \\ &\leq C \left(\int_{-N_m^-}^{N_m^-} \frac{dy_1}{(1+N_m-|y_1|)^3} \right)^{1/2} + C \left(\int_{K_{N_m}} \int_0^h |u^{(1)}(y_1, y_2)|^2 dy_1 dy_2 \right)^{1/2} \\ &\quad + C \left(\int_{\mathbb{R} \setminus [-N_m^+, N_m^+]} \frac{dy_1}{(1+|y_1|-N_m)^3} \right)^{1/2} \\ &\leq C \left(\int_0^{N_m^-} \frac{dy_1}{(1+N_m-y_1)^3} \right)^{1/2} + C N_m^{-1/8} + C \left(\int_{N_m^+}^{\infty} \frac{dy_1}{(1+y_1-N_m)^3} \right)^{1/2} \\ &\leq C N_m^{-1/8}. \end{aligned} \quad (3.22)$$

With (3.18) we have for $m \in \mathbb{N}$,

$$|u^{(1)}(N_m, x_2)|, \left| \frac{\partial u^{(1)}}{\partial x_1}(N_m, x_2) \right| \leq C \frac{\phi(N_m)}{N_m^{1/8}}. \quad (3.23)$$

Therefore, by (3.8) we have

$$\begin{aligned} |J_+(N_m)| &\leq C(\phi(N_m) - R) \frac{\phi(N_m)^2}{N_m^{1/4}} + C(\phi(N_m) - R) \frac{\phi(N_m)}{N_m^{1/8}} \\ &\leq C(\phi(N_m) - R) \frac{\phi(N_m)^2}{N_m^{1/8}} \leq C \frac{\phi(N_m)^3}{N_m^{1/8}}. \end{aligned} \quad (3.24)$$

Since $\phi(N) = N^s$, if we choose $s \in (0, 1)$ such that $3s < \frac{1}{8}$, that is, $0 < s < \frac{1}{24}$ the right hand side in (3.24) converges to zero as $m \rightarrow \infty$. Therefore, $\limsup_{N \rightarrow \infty} J_+(N) \geq 0$. By the same argument

of J_+ , we can show that $\limsup_{N \rightarrow \infty} J_-(N) \geq 0$, which yields Step 2.

Next, we discuss the last term in (3.6). By the same argument in Lemma 3.2 that we apply Green's representation theorem in $x_2 > h$ and use the Dirichlet Green's function G_h of $\mathbb{R}_{x_2 > h}^2$ ($:= \mathbb{R} \times (h, \infty)$) instead of G , $u^{(1)}$ can also be of another integral representation for $x_2 > h$

$$\begin{aligned} u^{(1)}(x) &= \int_{y_2 > h} \sigma(y) G_h(x, y) dy + 2 \int_{\Gamma_h} u^{(1)}(y) \frac{\partial \Phi_k(x, y)}{\partial y_2} ds(y) \\ &=: v^1(x) + v^2(x), \end{aligned} \quad (3.25)$$

where G_h is defined by $G_h(x, y) := \Phi_k(x, y) - \Phi_k(x, y_h^*)$ where $y_h^* = (y_1, 2h - y_2)$. We define approximation $u_N^{(1)}$ of $u^{(1)}$ by

$$\begin{aligned} u_N^{(1)}(x) &:= \int_{y_2 > 0} \chi_{\phi(N)-1}(y_2) \sigma(y) G(x, y) dy + 2 \int_{\Gamma_h} \chi_N(y_1) u^{(1)}(y) \frac{\partial \Phi_k(x, y)}{\partial y_2} ds(y) \\ &=: v_N^1(x) + v_N^2(x), \quad x_2 > h, \end{aligned} \quad (3.26)$$

where χ_a is defined by for $a > 0$,

$$\chi_a(t) := \begin{cases} 1 & \text{for } |t| \leq a \\ 0 & \text{for } |t| > a. \end{cases} \quad (3.27)$$

By Lemma 3.4 of [4] and Lemma 2.1 of [3] we can show that v_N^1 and v_N^2 satisfy the upward propagating radiation condition, which implies that so does $u_N^{(1)}$. Furthermore, by the definition of $u_N^{(1)}$ we can show that $u_N^{(1)}(\cdot, \phi(N) - 1) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, by Lemma 6.1 of [4] we have that

$$\operatorname{Im} \int_{\Gamma_{\phi(N)}} \overline{u_N^{(1)}} \frac{\partial u_N^{(1)}}{\partial x_2} ds \geq 0. \quad (3.28)$$

Combining (3.6) with (3.28) we have

$$\begin{aligned} 0 &\geq -\operatorname{Im} \int_{\Gamma_{\phi(N)}} \overline{u_N^{(1)}} \frac{\partial u_N^{(1)}}{\partial x_2} ds \\ &= \operatorname{Im} \left[\frac{1}{2\pi} \sum_{j \in J} \sum_{d_{l,j}, d_{l',j} > 0} \overline{a_{l,j}} a_{l',j} \int_{C_{\phi(N)}} \overline{\phi_{l,j}} \frac{\partial \phi_{l',j}}{\partial x_1} dx \right] \\ &\quad - \operatorname{Im} \left[\frac{1}{2\pi} \sum_{j \in J} \sum_{d_{l,j}, d_{l',j} < 0} \overline{a_{l,j}} a_{l',j} \int_{C_{\phi(N)}} \overline{\phi_{l,j}} \frac{\partial \phi_{l',j}}{\partial x_1} dx \right] + J_+(N) + J_-(N) \\ &\quad + \operatorname{Im} \int_{\Gamma_{\phi(N), N}} \overline{u} \frac{\partial u}{\partial x_2} - \operatorname{Im} \int_{\Gamma_{\phi(N)}} \overline{u_N^{(1)}} \frac{\partial u_N^{(1)}}{\partial x_2} ds + o(1). \end{aligned} \quad (3.29)$$

We observe the last term

$$\operatorname{Im} \int_{\Gamma_{\phi(N), N}} \overline{u} \frac{\partial u}{\partial x_2} - \operatorname{Im} \int_{\Gamma_{\phi(N)}} \overline{u_N^{(1)}} \frac{\partial u_N^{(1)}}{\partial x_2} ds =: L(N) + M(N), \quad (3.30)$$

where

$$L(N) := \operatorname{Im} \int_{\Gamma_{\phi(N),N}} \overline{u^{(1)}} \frac{\partial u^{(1)}}{\partial x_2} ds - \operatorname{Im} \int_{\Gamma_{\phi(N)}} \overline{u_N^{(1)}} \frac{\partial u_N^{(1)}}{\partial x_2} ds, \quad (3.31)$$

$$M(N) := \operatorname{Im} \int_{\Gamma_{\phi(N),N}} \overline{u^{(1)}} \frac{\partial u^{(2)}}{\partial x_2} ds + \operatorname{Im} \int_{\Gamma_{\phi(N),N}} \overline{u^{(2)}} \frac{\partial u^{(1)}}{\partial x_2} ds + \operatorname{Im} \int_{\Gamma_{\phi(N),N}} \overline{u^{(2)}} \frac{\partial u^{(2)}}{\partial x_2} ds. \quad (3.32)$$

By Lemma 3.2 we can show $|u^{(1)}(x_1, \phi(N))|, \left| \frac{\partial u^{(1)}}{\partial x_2}(x_1, \phi(N)) \right| \leq C\phi(N)$ for $x_1 \in \mathbb{R}$, and by Lemma 2.2 we have $|u^{(2)}(x_1, \phi(N))|, \left| \frac{\partial u^{(2)}}{\partial x_2}(x_1, \phi(N)) \right| \leq Ce^{-\delta\phi(N)}$ for $x_1 \in \mathbb{R}$. Then, we have

$$\begin{aligned} |M(N)| &\leq \int_{-N}^N |u^{(1)}(x_1, \phi(N))| \left| \frac{\partial u^{(2)}}{\partial x_2}(x_1, \phi(N)) \right| dx_1 \\ &\quad + \int_{-N}^N |u^{(2)}(x_1, \phi(N))| \left| \frac{\partial u^{(1)}}{\partial x_2}(x_1, \phi(N)) \right| dx_1 \\ &\quad + \int_{-N}^N |u^{(2)}(x_1, \phi(N))| \left| \frac{\partial u^{(2)}}{\partial x_2}(x_1, \phi(N)) \right| dx_1 \\ &\leq C(N\phi(N)e^{-\delta\phi(N)} + Ne^{-2\delta\phi(N)}) \\ &\leq CN\phi(N)e^{-\delta\phi(N)}, \end{aligned} \quad (3.33)$$

which implies that $M(N) = o(1)$ as $N \rightarrow \infty$. Hence, we will show that $\limsup_{N \rightarrow \infty} L(N) \geq 0$.

Step3 ($\limsup_{N \rightarrow \infty} L(N) \geq 0$): First, we observe that

$$\begin{aligned} |L(N)| &\leq \left| \operatorname{Im} \int_{\Gamma_{\phi(N),N}} \overline{u^{(1)}} \frac{\partial u^{(1)}}{\partial x_2} ds - \operatorname{Im} \int_{\Gamma_{\phi(N),N}} \overline{u_N^{(1)}} \frac{\partial u_N^{(1)}}{\partial x_2} ds \right| \\ &\quad + \left| \operatorname{Im} \int_{\Gamma_{\phi(N),N}} \overline{u^{(1)}} \frac{\partial u_N^{(1)}}{\partial x_2} ds - \operatorname{Im} \int_{\Gamma_{\phi(N),N}} \overline{u_N^{(1)}} \frac{\partial u^{(1)}}{\partial x_2} ds \right| \\ &\quad + \left| \operatorname{Im} \int_{\Gamma_{\phi(N)} \setminus \Gamma_{\phi(N),N}} \overline{u_N^{(1)}} \frac{\partial u_N^{(1)}}{\partial x_2} ds \right| \\ &\leq \int_{-N}^N |u^{(1)}(x_1, \phi(N))| \left| \frac{\partial u^{(1)}}{\partial x_2}(x_1, \phi(N)) - \frac{\partial u_N^{(1)}}{\partial x_2}(x_1, \phi(N)) \right| ds \\ &\quad + \int_{-N}^N |u^{(1)}(x_1, \phi(N)) - u_N^{(1)}(x_1, \phi(N))| \left| \frac{\partial u_N^{(1)}}{\partial x_2}(x_1, \phi(N)) \right| ds \\ &\quad + \int_{\mathbb{R} \setminus (-N, N)} |u_N^{(1)}(x_1, \phi(N))| \left| \frac{\partial u_N^{(1)}}{\partial x_2}(x_1, \phi(N)) \right| ds. \end{aligned} \quad (3.34)$$

By Lemma 2.2 σ has a exponential decay in y_2 . Then, we have for $x_1 \in \mathbb{R}$,

$$\begin{aligned} |v^1(x_1, \phi(N))|, \left| \frac{\partial v^1}{\partial x_2}(x_1, \phi(N)) \right|, |v_N^1(x_1, \phi(N))|, \left| \frac{\partial v_N^1}{\partial x_2}(x_1, \phi(N)) \right| \\ \leq C \int_{(-\eta, \eta) \times (0, \infty)} \frac{e^{-\delta y_2} \phi(N) y_2}{(1 + |x_1 - y_1|)^{3/2}} dy \leq C \frac{\phi(N)}{(1 + |x_1|)^{3/2}}, \end{aligned} \quad (3.35)$$

and

$$\begin{aligned}
& |v^1(x_1, \phi(N)) - v_N^1(x_1, \phi(N))|, \quad \left| \frac{\partial v^1}{\partial x_2}(x_1, \phi(N)) - \frac{\partial v_N^1}{\partial x_2}(x_1, \phi(N)) \right| \\
& \leq C \int_{(-\eta, \eta) \times (\phi(N)-1, \infty)} \frac{e^{-\delta y_2} \phi(N) y_2}{(1 + |x_1 - y_1|)^{3/2}} dy \\
& \leq C \left(\int_{\phi(N)}^{\infty} e^{-\delta y_2} y_2 dy_2 \right) \frac{\phi(N)}{(1 + |x_1|)^{3/2}} dy \leq \frac{e^{-\delta \phi(N)} \phi(N)}{(1 + |x_1|)^{3/2}}. \tag{3.36}
\end{aligned}$$

Since the fundamental solution to Helmholtz equation $\Phi(x, y)$ is of the following estimation (see e.g., [2]) for $|x - y| \geq 1$

$$\left| \frac{\partial \Phi}{\partial y_2}(x, y) \right| \leq C \frac{|x_2 - y_2|}{1 + |x - y|^{3/2}}, \quad \left| \frac{\partial^2 \Phi}{\partial x_2 \partial y_2}(x, y) \right| \leq C \frac{|x_2 - y_2|^2}{1 + |x - y|^{3/2}}, \tag{3.37}$$

we can show that for $x_1 \in \mathbb{R}$

$$|v^2(x_1, \phi(N))| \leq C \phi(N) W_\infty(x_1), \quad |v_N^2(x_1, \phi(N))| \leq C \phi(N) W_N(x_1), \tag{3.38}$$

and

$$\left| \frac{\partial v^2}{\partial x_2}(x_1, \phi(N)) \right| \leq C \phi(N)^2 W_\infty(x_1), \quad \left| \frac{\partial v_N^2}{\partial x_2}(x_1, \phi(N)) \right| \leq C \phi(N)^2 W_N(x_1), \tag{3.39}$$

and

$$|v^2(x_1, \phi(N)) - v_N^2(x_1, \phi(N))| \leq C \phi(N) (W_\infty(x_1) - W_N(x_1)), \tag{3.40}$$

and

$$\left| \frac{\partial v^2}{\partial x_2}(x_1, \phi(N)) - \frac{\partial v_N^2}{\partial x_2}(x_1, \phi(N)) \right| \leq C \phi(N)^2 (W_\infty(x_1) - W_N(x_1)), \tag{3.41}$$

where W_N is defined by for $N \in (0, \infty]$

$$W_N(x_1) := \int_{-N}^N \frac{|u^{(1)}(y_1, h)|}{(1 + |x_1 - y_1|)^{3/2}} dy_1, \quad x_1 \in \mathbb{R}. \tag{3.42}$$

Using (3.35)–(3.41), we continue to estimate (3.34). By the Cauchy Schwarz inequality we have

$$\begin{aligned}
|L(N)| & \leq C \int_{-N}^N \left\{ \frac{\phi(N)}{(1 + |x_1|)^{3/2}} + \phi(N) W_\infty(x_1) \right\} \\
& \quad \times \left\{ \frac{\phi(N) e^{-\sigma \phi(N)}}{(1 + |x_1|)^{3/2}} + \phi(N)^2 (W_\infty(x_1) - W_N(x_1)) \right\} dx_1 \\
& + \int_{-N}^N \left\{ \frac{\phi(N) e^{-\sigma \phi(N)}}{(1 + |x_1|)^{3/2}} + \phi(N) (W_\infty(x_1) - W_N(x_1)) \right\} \\
& \quad \times \left\{ \frac{\phi(N)}{(1 + |x_1|)^{3/2}} + \phi(N)^2 W_N(x_1) \right\} dx_1 \\
& + \int_{\mathbb{R} \setminus (-N, N)} \left\{ \frac{\phi(N)}{(1 + |x_1|)^{3/2}} + \phi(N) W_N(x_1) \right\} \left\{ \frac{\phi(N)}{(1 + |x_1|)^{3/2}} + \phi(N)^2 W_N(x_1) \right\} dx_1
\end{aligned}$$

$$\begin{aligned}
&\leq C\phi(N)^3 \int_{-N}^N W_\infty(x_1)(W_\infty(x_1) - W_N(x_1))dx_1 \\
&+ C\phi(N)^3 \int_{-N}^N \frac{1}{(1+|x_1|)^{3/2}}(W_\infty(x_1) - W_N(x_1))dx_1 \\
&+ C\phi(N)^2 \int_{\mathbb{R} \setminus (-N, N)} \frac{1}{(1+|x_1|)^3} dx_1 + C\phi(N)^2 \int_{\mathbb{R} \setminus (-N, N)} \frac{1}{(1+|x_1|)^{3/2}} W_N(x_1) dx_1 \\
&+ C\phi(N)^3 \int_{\mathbb{R} \setminus (-N, N)} |W_N(x_1)|^2 dx_1 + o(1) \\
&\leq C\phi(N)^3 \left\{ \left(\int_{-N}^N (W_\infty(x_1) - W_N(x_1))^2 dx_1 \right)^{1/2} + \left(\int_{\mathbb{R} \setminus (-N, N)} W_N(x_1)^2 dx_1 \right)^{1/2} \right\} \\
&\quad + o(1). \tag{3.43}
\end{aligned}$$

Finally, we will estimate $(W_\infty(x_1) - W_N(x_1))$ and $W_N(x_1)$. Since $u^{(1)}(\cdot, h) \in L^2(\mathbb{R})$, by Lemma 3.3 there exists a sequence $\{N_m\}_{m \in \mathbb{N}}$ such that $N_m \rightarrow \infty$ as $m \rightarrow \infty$ and

$$\int_{K_{N_m}} |u^{(1)}(y_1, h)|^2 dy_1 \leq CN_m^{-\frac{1}{4}}, \quad m \in \mathbb{N}, \tag{3.44}$$

where $K_A := K_A^+ \cup K_A^-$, $K_A^+ := (-A^+, A^+) \setminus (-A, A)$, $K_A^- := (-A, A) \setminus (-A^-, A^-)$, and $A^\pm := A \pm A^{1/2}$ for $A \in [1, \infty)$.

By the Cauchy Schwarz inequality we have for $|x_1| > N_m$,

$$\begin{aligned}
\int_{-N_m^-}^{N_m^-} \frac{|u^{(1)}(y_1, h)|}{(1+|x_1-y_1|)^{3/2}} dy_1 &\leq \left(\int_{-N_m^-}^{N_m^-} |u^{(1)}(y_1, h)|^2 dy_1 \right)^{1/2} \left(\int_{-N_m^-}^{N_m^-} \frac{dy_1}{(1+|x_1-y_1|)^3} \right)^{1/2} \\
&\leq \frac{C}{1-|x_1|-N_m^-}, \tag{3.45}
\end{aligned}$$

and

$$\begin{aligned}
\int_{K_{N_m^-}} \frac{|u^{(1)}(y_1, h)|}{(1+|x_1-y_1|)^{3/2}} dy_1 &\leq \left(\int_{K_{N_m^-}} |u^{(1)}(y_1, h)|^2 dy_1 \right)^{1/2} \left(\int_{K_{N_m^-}} \frac{dy_1}{(1+|x_1-y_1|)^3} \right)^{1/2} \\
&\leq \frac{C}{N_m^{1/8}(1+|x_1|-N_m)}. \tag{3.46}
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\int_{\mathbb{R} \setminus (-N_m, N_m)} W_N(x_1)^2 dx_1 \\
&\leq C \int_{N_m}^{\infty} \frac{dx_1}{(1-|x_1|-N_m^-)^2} + \frac{C}{N_m^{1/4}} \int_{N_m}^{\infty} \frac{dx_1}{(1-|x_1|-N_m)^2} \\
&\leq \frac{C}{1+N_m^{1/2}} + \frac{C}{N_m^{1/4}} \leq \frac{C}{N_m^{1/4}}. \tag{3.47}
\end{aligned}$$

By the Cauchy Schwarz inequality we have for $|x_1| < N_m$,

$$\begin{aligned} & \int_{\mathbb{R} \setminus (-N_m^+, N_m^+)} \frac{|u^{(1)}(y_1, h)|}{(1 + |x_1 - y_1|)^{3/2}} dy_1 \\ & \leq \left(\int_{\mathbb{R} \setminus (-N_m^+, N_m^+)} |u^{(1)}(y_1, h)|^2 dy_1 \right)^{1/2} \left(\int_{\mathbb{R} \setminus (-N_m^+, N_m^+)} \frac{dy_1}{(1 + y_1 - |x_1|)^3} \right)^{1/2} \\ & \leq \frac{C}{1 + N_m^+ - |x_1|}, \end{aligned} \quad (3.48)$$

and

$$\begin{aligned} \int_{K_{N_m^+}} \frac{|u^{(1)}(y_1, h)|}{(1 + |x_1 - y_1|)^{3/2}} dy_1 & \leq \left(\int_{K_{N_m^+}} |u^{(1)}(y_1, h)|^2 dy_1 \right)^{1/2} \left(\int_{K_{N_m^+}} \frac{dy_1}{(1 + y_1 - |x_1|)^3} \right)^{1/2} \\ & \leq \frac{C}{N_m^{1/8} (1 + N_m - |x_1|)}. \end{aligned} \quad (3.49)$$

Therefore, we obtain

$$\begin{aligned} & \int_{-N_m}^{N_m} (W_\infty(x_1) - W_N(x_1))^2 dx_1 \\ & \leq C \int_{-N_m}^{N_m} \frac{dx_1}{(1 + N_m^+ - |x_1|)^2} + \frac{C}{N_m^{1/4}} \int_{-N_m}^{N_m} \frac{dx_1}{(1 + N_m - |x_1|)^2} \\ & \leq \frac{C}{1 + N_m^{1/2}} + \frac{C}{N_m^{1/4}} \leq \frac{C}{N_m^{1/4}}. \end{aligned} \quad (3.50)$$

Therefore, Collecting (3.43), (3.47), and (3.50) we conclude that $|L(N_m)| \leq C \frac{\phi(N_m)^3}{N_m^{1/8}}$. Since $\phi(N) = N^s$, if we choose $s \in (0, 1)$ such that $3s < \frac{1}{8}$, that is, $0 < s < \frac{1}{24}$, the term $\frac{\phi(N_m)^3}{N_m^{1/8}}$ converges to zero as $m \rightarrow \infty$. Therefore, $\limsup_{N \rightarrow \infty} L(N) \geq 0$, which yields Step 3.

By taking $\limsup_{N \rightarrow \infty}$ in (3.29) we have that

$$\begin{aligned} 0 & \geq \frac{k}{2\pi} \sum_{j \in J} \left[\sum_{d_{l,j} > 0} |a_{l,j}|^2 d_{l,j} - \sum_{d_{l,j} < 0} |a_{l,j}|^2 d_{l,j} \right] \\ & \quad + \limsup_{N \rightarrow \infty} (J_+(N) + J_-(N) + L(N)). \end{aligned} \quad (3.51)$$

By Steps 2 and 3 and choosing $0 < s < \frac{1}{24}$ the right hand side is non-negative. Therefore, $a_{l,j} = 0$ for all l, j , which yields $u^{(2)} = 0$. Lemma 3.1 has been shown, and in next section we will show the uniqueness of $u^{(1)}$. \square

4 Uniqueness of $u^{(1)}$

In Section 4, we will show the following lemma.

Lemma 4.1. *If $u \in H_{loc}^1(\mathbb{R}_+^2)$ satisfies*

- (i) $u \in H^1(\mathbb{R} \times (0, R))$ for all $R > 0$,
 - (ii) $\Delta u + k^2(1+q)nu = 0$ in \mathbb{R}_+^2 ,
 - (iii) u vanishes for $x_2 = 0$,
 - (iv) There exists $\phi \in L^\infty(\Gamma_h) \cap H^{1/2}(\Gamma_h)$ with $u(x) = 2 \int_{\Gamma_h} \phi(y) \frac{\partial \Phi_k(x,y)}{\partial y_2} ds(y)$ for $x_2 > h$,
- then, $u \in H_0^1(\mathbb{R}_+^2)$.

If we can use Lemma 4.1, we have the uniqueness of the solution in Theorem 1.2.

Theorem 4.2. *Let Assumptions 1.1 and 2.1 hold and let $k > 0$ be regular in the sense of Definition 2.3. If $u \in H_{loc}^1(\mathbb{R}_+^2)$ satisfies (3.1), (3.2), and the radiation condition in the sense of Definition 2.4, then u vanishes for $x_2 > 0$.*

Proof of Theorem 4.2. Let $u \in H_{loc}^1(\mathbb{R}_+^2)$ satisfy (3.1), (3.2), and the radiation condition in the sense of Definition 2.4. By Lemma 3.1, $u^{(2)} = 0$ for $x_2 > 0$. Then, $u^{(1)}$ satisfies the assumptions (i)–(iv) of Lemma 4.1, which implies that $u^{(1)} \in H_0^1(\mathbb{R}_+^2)$. By Assumption 1.1, $u^{(1)}$ vanishes for $x_2 > 0$, which yields the uniqueness. \square

Finally in this section we will show Theorem 4.2.

Proof of Lemma 4.1. Let $R > h$ be fixed. We set $\Omega_{N,R} := (-N, N) \times (0, R)$ where $N > 0$ is large enough. We denote by $I_{\pm N}^R := \{\pm N\} \times (0, R)$, $\Gamma_{R,N} := (-N, N) \times \{R\}$, and $\Gamma_R := (-\infty, \infty) \times \{R\}$. By Green's first theorem in $\Omega_{N,R}$ and assumptions (ii), (iii) we have

$$\begin{aligned} \int_{\Omega_{N,R}} \{-k^2(1+q)n|u|^2 + |\nabla u|^2\} dx &= \int_{\Omega_{N,R}} \{\bar{u}\Delta u + |\nabla u|^2\} dx \\ &= \int_{I_N^R} \bar{u} \frac{\partial u}{\partial x_1} ds - \int_{I_{-N}^R} \bar{u} \frac{\partial u}{\partial x_1} ds + \int_{\Gamma_{R,N}} \bar{u} \frac{\partial u}{\partial x_2} ds. \end{aligned} \quad (4.1)$$

By the assumption (i), the first and second term in the right hands side of (4.1) go to zero as $N \rightarrow \infty$. Then, by taking an imaginary part and as $N \rightarrow \infty$ in (4.1) we have

$$\text{Im} \int_{\Gamma_R} \bar{u} \frac{\partial u}{\partial x_2} ds = 0. \quad (4.2)$$

By considering the Floquet Bloch transform with respect to x_1 (see the notation of (2.5)), we can show that

$$\int_{\Gamma_R} \bar{u} \frac{\partial u}{\partial x_2} ds = \int_{-1/2}^{1/2} \int_0^{2\pi} \bar{\tilde{u}}_\alpha(x_1, R) \frac{\partial \tilde{u}_\alpha(x_1, R)}{\partial x_2} dx_1 d\alpha. \quad (4.3)$$

Since the upward propagating radiation condition is equivalent to the Rayleigh expansion by the Floquet Bloch transform (see the proof of Theorem 6.8 in [8]), we can show that

$$\tilde{u}_\alpha(x) = \sum_{n \in \mathbb{Z}} u_n(\alpha) e^{inx_1 + i\sqrt{k^2 - (n+\alpha)^2}(x_2-h)}, \quad x_2 > h, \quad (4.4)$$

where $u_n(\alpha) := (2\pi)^{-1} \int_0^{2\pi} u_\alpha(x_1, h) e^{-inx_1} dx_1$. From (4.2)–(4.4) we obtain that

$$\begin{aligned} 0 &= \operatorname{Im} \int_{-1/2}^{1/2} \int_0^{2\pi} \overline{u_\alpha}(x_1, R) \frac{\partial \tilde{u}_\alpha(x_1, R)}{\partial x_2} dx_1 d\alpha \\ &= \operatorname{Im} \sum_{n \in \mathbb{Z}} \int_{-1/2}^{1/2} 2\pi |u_n(\alpha)|^2 i \sqrt{k^2 - (n + \alpha)^2}, \end{aligned} \quad (4.5)$$

Here, we denote by $k = n_0 + r$ where $n_0 \in \mathbb{N}_0$ and $r \in [-1/2, 1/2)$. Then by (4.5) we have

$$\begin{aligned} u_n(\alpha) &= 0 \text{ for } |n| < n_0, \text{ a.e. } \alpha \in (-1/2, 1/2), \\ u_{n_0}(\alpha) &= 0 \text{ for } \alpha \in (-1/2, r), \\ u_{-n_0}(\alpha) &= 0 \text{ for } \alpha \in (-r, 1/2). \end{aligned} \quad (4.6)$$

By (4.6) we have

$$\begin{aligned} & \int_{-1/2}^{1/2} \int_0^{2\pi} \int_R^\infty |\tilde{u}_\alpha(x)|^2 dx_2 dx_1 d\alpha \\ &= 2\pi \int_{-1/2}^{1/2} \sum_{|n| > n_0} |u_n(\alpha)|^2 \int_R^\infty e^{-\sqrt{(n+\alpha)^2 - k^2}(x_2 - h)} dx_2 d\alpha \\ &+ 2\pi \int_r^{1/2} |u_{n_0}(\alpha)|^2 \int_R^\infty e^{-\sqrt{(n_0+\alpha)^2 - k^2}(x_2 - h)} dx_2 d\alpha \\ &+ 2\pi \int_{-1/2}^{-r} |u_{-n_0}(\alpha)|^2 \int_R^\infty e^{-\sqrt{(-n_0+\alpha)^2 - k^2}(x_2 - h)} dx_2 d\alpha \\ &\leq 2\pi \sum_{|n| > n_0} \int_{-1/2}^{1/2} \frac{|u_n(\alpha)|^2 e^{-\sqrt{(n+\alpha)^2 - k^2}(R-h)}}{\sqrt{(n+\alpha)^2 - k^2}} d\alpha \\ &+ 2\pi \int_r^{1/2} \frac{|u_{n_0}(\alpha)|^2 e^{-\sqrt{(n_0+\alpha)^2 - k^2}(R-h)}}{\sqrt{(n_0+\alpha)^2 - k^2}} d\alpha \\ &+ 2\pi \int_{-1/2}^{-r} \frac{|u_{-n_0}(\alpha)|^2 e^{-\sqrt{(-n_0+\alpha)^2 - k^2}(R-h)}}{\sqrt{(-n_0+\alpha)^2 - k^2}} d\alpha \\ &\leq C \sum_{|n| > n_0} \int_{-1/2}^{1/2} |u_n(\alpha)|^2 d\alpha \\ &+ C \int_r^{1/2} \frac{|u_{n_0}(\alpha)|^2}{\sqrt{\alpha - r}} d\alpha + C \int_{-1/2}^{-r} \frac{|u_{-n_0}(\alpha)|^2}{\sqrt{-\alpha - r}} d\alpha, \end{aligned} \quad (4.7)$$

and

$$\begin{aligned}
& \int_{-1/2}^{1/2} \int_0^{2\pi} \int_R^\infty |\partial_{x_1} \tilde{u}_\alpha(x)|^2 dx_2 dx_1 d\alpha \\
= & 2\pi \sum_{|n|>n_0} \int_{-1/2}^{1/2} \frac{|u_n(\alpha)|^2 n^2 e^{-\sqrt{(n+\alpha)^2 - k^2}(R-h)}}{\sqrt{(n+\alpha)^2 - k^2}} d\alpha \\
& + 2\pi \int_r^{1/2} \frac{|u_{n_0}(\alpha)|^2 n_0^2 e^{-\sqrt{(n_0+\alpha)^2 - k^2}(R-h)}}{\sqrt{(n_0+\alpha)^2 - k^2}} d\alpha \\
& + 2\pi \int_{-1/2}^{-r} \frac{|u_{-n_0}(\alpha)|^2 n_0^2 e^{-\sqrt{(-n_0+\alpha)^2 - k^2}(R-h)}}{\sqrt{(-n_0+\alpha)^2 - k^2}} d\alpha \\
\leq & C \sum_{|n|>n_0} \int_{-1/2}^{1/2} |u_n(\alpha)|^2 d\alpha \\
& + C \int_r^{1/2} \frac{|u_{n_0}(\alpha)|^2}{\sqrt{\alpha-r}} d\alpha + C \int_{-1/2}^{-r} \frac{|u_{-n_0}(\alpha)|^2}{\sqrt{-\alpha-r}} d\alpha. \tag{4.8}
\end{aligned}$$

By the same argument in (4.8) we have

$$\begin{aligned}
& \int_{-1/2}^{1/2} \int_0^{2\pi} \int_R^\infty |\partial_{x_2} \tilde{u}_\alpha(x)|^2 dx_2 dx_1 d\alpha \leq C \sum_{|n|>n_0} \int_{-1/2}^{1/2} |u_n(\alpha)|^2 d\alpha \\
& + C \int_r^{1/2} \frac{|u_{n_0}(\alpha)|^2}{\sqrt{\alpha-r}} d\alpha + C \int_{-1/2}^{-r} \frac{|u_{-n_0}(\alpha)|^2}{\sqrt{-\alpha-r}} d\alpha. \tag{4.9}
\end{aligned}$$

It is well known that the Floquet Bloch Transform is an isomorphism between $H^1(\mathbb{R}_+^2)$ and $L^2((-1/2, 1/2)_\alpha; H^1((0, 2\pi) \times \mathbb{R})_x)$ (e.g., see Theorem 4 in [9]). Therefore, we obtain from (4.7)–(4.9)

$$\begin{aligned}
\|u\|_{H^1(\mathbb{R} \times (R, \infty))}^2 & \leq C \int_{-1/2}^{1/2} \int_0^{2\pi} \int_R^\infty (|\tilde{u}_\alpha(x)|^2 + |\partial_{x_1} \tilde{u}_\alpha(x)|^2 + |\partial_{x_2} \tilde{u}_\alpha(x)|^2) dx_2 dx_1 d\alpha \\
& \leq C \sum_{|n|>n_0} \int_{-1/2}^{1/2} |u_n(\alpha)|^2 d\alpha \\
& + C \int_r^{1/2} \frac{|u_{n_0}(\alpha)|^2}{\sqrt{\alpha-r}} d\alpha + C \int_{-1/2}^{-r} \frac{|u_{-n_0}(\alpha)|^2}{\sqrt{-\alpha-r}} d\alpha \\
& \leq C \int_{-1/2}^{1/2} \int_0^{2\pi} |\tilde{u}_\alpha(x_1, h)|^2 dx_1 d\alpha \\
& + C \int_r^{1/2} \frac{|u_{n_0}(\alpha)|^2}{\sqrt{\alpha-r}} d\alpha + C \int_{-1/2}^{-r} \frac{|u_{-n_0}(\alpha)|^2}{\sqrt{-\alpha-r}} d\alpha. \tag{4.10}
\end{aligned}$$

If we can show that

$$\exists \delta > 0 \text{ and } \exists C > 0 \text{ s.t. } |u_{\pm n_0}(\alpha)| \leq C \text{ for all } \alpha \in (-\delta \pm r, \delta \pm r), \tag{4.11}$$

then the right hands side of (4.10) is finite, which yields Lemma 4.1.

Finally, we will show (4.11). By the same argument in section 3 of [8] we have

$$(I - K_\alpha)\tilde{u}_\alpha = f_\alpha \text{ in } H_{0,per}^1(C_h), \quad (4.12)$$

where the operator K_α is defined by (2.12) and $f_\alpha := -(T_{per}k^2nqu)(\cdot, \alpha)$. Since the function k^2nqu has a compact support, $\|f_\alpha\|_{H^1(C_h)}^2$ is bounded with respect to α . By Assumption 2.1 and the operator K_α is compact, $(I - K_\alpha)$ is invertible if $\alpha \in A_k$. Since $\pm r \in A_k$, $(I - K_\pm)$ is invertible. Since the exceptional values are finitely many (see Lemma 2.2), $(I - K_\alpha)$ is also invertible if α is close to $\pm r$. Therefore, there exists $\delta > 0$ such that $(I - K_\alpha)$ is invertible for all $\alpha \in (-\delta + r, \delta + r) \cup (-\delta - r, \delta - r)$.

The operator $(I - K_\alpha)$ is of the form

$$(I - K_\alpha) = (I - K_{\pm r}) \left(I - (I - K_{\pm r})^{-1} [I - K_{\pm r} - (I - K_\alpha)] \right) = (I - K_{\pm r})(I - M_\alpha), \quad (4.13)$$

where $M_\alpha := (I - K_{\pm r})^{-1}(K_\alpha - K_{\pm r})$. Next, we will estimate $(K_\alpha - K_{\pm r})$. By the definition of K_α we have for all $v, w \in H_{0,per}^1(C_h)$,

$$\begin{aligned} \langle (K_\alpha - K_{\pm r})v, w \rangle_* &= - \int_{C_h} \left[i(\alpha \mp r) \left(v \frac{\partial \bar{w}}{\partial x_1} - \bar{v} \frac{\partial w}{\partial x_1} \right) + (\alpha^2 - r^2)v\bar{w} \right] dx \\ &+ 2\pi i \sum_{|n| \neq n_0} v_n \bar{w}_n (\sqrt{k^2 - (n + \alpha)^2} - \sqrt{k^2 - (n \pm r)^2}) \\ &+ 2\pi i \sum_{|n|=n_0} v_n \bar{w}_n (\sqrt{k^2 - (n + \alpha)^2} - \sqrt{k^2 - (n \pm r)^2}). \end{aligned} \quad (4.14)$$

Since

$$\begin{aligned} |\sqrt{k^2 - (n + \alpha)^2} - \sqrt{k^2 - (n \pm r)^2}| &= \left| \frac{\pm 2nr + r^2 - 2n\alpha - \alpha^2}{\sqrt{k^2 - (n + \alpha)^2} + \sqrt{k^2 - (n \pm r)^2}} \right| \\ &\leq \begin{cases} \frac{|n||\alpha \pm r| + |r^2 - \alpha^2|}{\sqrt{|k^2 - (n \pm r)^2|}} & \text{for } |n| \neq n_0 \\ \frac{|n||\alpha \pm r| + |r^2 - \alpha^2|}{\sqrt{|r + \alpha||r - \alpha|}} & \text{for } |n| = n_0, \end{cases} \end{aligned} \quad (4.15)$$

we have for all $\alpha \in (-\delta + r, \delta + r) \cup (-\delta - r, \delta - r)$

$$\begin{aligned} |\langle (K_\alpha - K_{\pm r})v, w \rangle_*| &\leq C|\alpha \mp r| \|v\|_{H^1(C_h)} \|w\|_{H^1(C_h)} \\ &+ C \sum_{|n| \neq n_0} |v_n| |w_n| \frac{|n||\alpha \mp r|}{\sqrt{|k^2 - (n \pm r)^2|}} \\ &+ C \sum_{|n|=n_0} |v_n| |w_n| n_0 \sqrt{|\alpha \mp r|} \\ &\leq C\sqrt{|\alpha \mp r|} \|v\|_{H^1(C_h)} \|w\|_{H^1(C_h)}. \end{aligned} \quad (4.16)$$

(we retake very small $\delta > 0$ if needed.) This implies that there is a constant number $C > 0$ which is independent of α such that $\|K_\alpha - K_{\pm r}\| \leq C\sqrt{|\alpha \mp r|}$. Therefore, by the property of Neumann series, there is a small $\delta > 0$ such that for all $\alpha \in (-\delta + r, \delta + r) \cup (-\delta - r, \delta - r)$

$$(I - M_\alpha)^{-1} = \sum_{n=0}^{\infty} M_\alpha^n \text{ and } \|M_\alpha\| \leq 1/2. \quad (4.17)$$

By the Cauchy Schwarz inequality, the boundedness of trace operator, and (4.17) we have

$$\begin{aligned}
|u_{\pm n_0}(\alpha)| &\leq \int_0^{2\pi} |\tilde{u}_\alpha(x_1, h)| dx_1 \leq C \|\tilde{u}_\alpha\|_{H^1(C_h)} \\
&= C \|(I - M_\alpha)^{-1}(I - K_{\pm r})^{-1} f_\alpha\|_{H^1(C_h)} \\
&\leq C \|(I - M_\alpha)^{-1}\| \|(I - K_{\pm r})^{-1} f_\alpha\| \\
&\leq C \sum_{n=0}^{\infty} \|M_\alpha\|^n < C \sum_{n=0}^{\infty} (1/2)^j < \infty,
\end{aligned} \tag{4.18}$$

where constant number $C > 0$ is independent of α . Therefore, we have shown (4.11). \square

5 Existence

In previous sections we discussed the uniqueness of Theorem 1.2. In Section 5, we will show the existence. Let Assumptions 1.1 and 2.1 hold and let $k > 0$ be regular in the sense of Definition 2.3. Let $f \in L^2(\mathbb{R}_+^2)$ such that $\text{supp} f = Q$. We define the solution operator $S : L^2(Q) \rightarrow L^2(Q)$ by $Sg := v|_Q$ where v satisfies the radiation condition and

$$\Delta v + k^2 n v = g, \text{ in } \mathbb{R}_+^2, \tag{5.1}$$

$$v = 0 \text{ on } \Gamma_0. \tag{5.2}$$

Remark that by Theorem 2.6 we can define such a operator S , and S is a compact operator since the restriction to Q of the solution v is in $H^1(Q)$. We define the multiplication operator $M : L^2(Q) \rightarrow L^2(Q)$ by $Mh := k^2 n q h$. We will show the following lemma.

Lemma 5.1. $I_{L^2(Q)} + SM$ is invertible.

Proof of Lemma 5.1. By the definition of operators S and M we have $SMg = v|_Q$ where v is a radiating solution of (5.1)–(5.2) replacing g by $k^2 n q g$. If we assume that $(I_{L^2(Q)} + SM)g = 0$, then $g = -v|_Q$, which implies that v satisfies $\Delta v + k^2 n(1 + q)v = 0$ in \mathbb{R}_+^2 . By the uniqueness we have $v = 0$ in \mathbb{R}_+^2 , which implies that $I_{L^2(Q)} + SM$ is injective. Since the operator SM is compact, by Fredholm theory we conclude that $I_{L^2(Q)} + SM$ is invertible. \square

We define u as the solution of

$$\Delta u + k^2 n u = f - M(I_{L^2(Q)} + SM)^{-1} S f, \text{ in } \mathbb{R}_+^2. \tag{5.3}$$

satisfying the radiation condition and $u = 0$ on Γ_0 . Since

$$\begin{aligned}
u|_Q &= S(f - M(I_{L^2(Q)} + SM)^{-1} S f) \\
&= (I_{L^2(Q)} + SM)(I_{L^2(Q)} + SM)^{-1} S f - SM(I_{L^2(Q)} + SM)^{-1} S f \\
&= (I_{L^2(Q)} + SM)^{-1} S f,
\end{aligned} \tag{5.4}$$

we have that

$$\Delta u + k^2 n u = f - k^2 n q u, \text{ in } \mathbb{R}_+^2, \tag{5.5}$$

and u is a radiating solution of (1.8)–(1.9). Therefore, Theorem 1.2 has been shown.

References

- [1] S. N. Chandler-Wilde, *The impedance boundary value problem for the Helmholtz equation in a half-plane*, Math. Methods Appl. Sci. **20**, (1997), no. 10, 813–840.
- [2] S. N. Chandler-Wilde, R. R. Christopher, *Scattering by rough surfaces: the Dirichlet problem for the Helmholtz equation in a non-locally perturbed half-plane*, Math. Methods Appl. Sci. **19**, (1996), no. 12, 959–976.
- [3] S. N. Chandler-Wilde, B. Zhang, *A uniqueness result for scattering by infinite rough surfaces*, SIAM J. Appl. Math. **58**, (1998), no. 6, 1774–1790.
- [4] S. N. Chandler-Wilde, B. Zhang, *Electromagnetic scattering by an inhomogeneous conducting or dielectric layer on a perfectly conducting plate*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **454**, (1998), no. 1970, 519–542.
- [5] A. Kirsch, *Scattering by a periodic tube in \mathbb{R}^3 : part i. The limiting absorption principle*, Inverse Prob., **35**, (2019), 104004.
- [6] A. Kirsch, *Scattering by a periodic tube in \mathbb{R}^3 : part ii. A radiation condition*, Inverse Prob., **35**, (2019), 104005.
- [7] A. Kirsch, A. Lechleiter, *A radiation condition arising from the limiting absorption principle for a closed full- or half-waveguide problem*, Math. Methods Appl. Sci. **41**, (2018), no. 10, 3955–3975.
- [8] A. Kirsch, A. Lechleiter, *The limiting absorption principle and a radiation condition for the scattering by a periodic layer*, SIAM J. Math. Anal. **50**, (2018), no. 3, 2536–2565.
- [9] A. Lechleiter, *The Floquet-Bloch transform and scattering from locally perturbed periodic surfaces*, J. Math. Anal. Appl. **446**, (2017), no. 1, 605–627.

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