Optimal habitat configurations in the diffusive logistic equation *

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1 Introduction

This article is concerned with the following reaction-diffusion equation arising in a population model:

$$\begin{cases} \partial_t u = d\Delta u + u(m(x) - u) & \text{in } \Omega \times (0, \infty), \\ \mathfrak{B}_\theta u := \theta \partial_\nu u + (1-\theta)u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\,\cdot\,, 0) = u_0(\geq 0\,) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$; $\Delta := \sum_{j=1}^{n} \partial^2/\partial x_j^2$ is the usual Laplace operator; ∂_ν denotes the directional derivative in the direction of the outward unit normal vector ν on $\partial\Omega$; $d > 0$ and $0 \leq \theta \leq 1$ are given constants; $m(x)$ is a nonnegative measurable function. In the research field of reaction-diffusion equations in biological models, (1.1) is referred as the diffusive logistic equation in which the unknown function $u(x,t)$ represents the population density of a species at location x in the bounded habitat Ω and time $t > 0$. The diffusion coefficient $d > 0$ represents the degree of random movement of each individual of the species. The nonnegative function $m(x)$ can be interpreted as the amount of resources (feed) for the species at location $x \in \Omega$. In the boundary condition, the homogeneous Dirichlet type is corresponding to $\theta = 0$, where the habitat Ω is assumed to be surrounded by a hostile environment for the species; the homogeneous Neumann type is corresponding to $\theta = 1$, where the flux of u across the boundary is assumed to be zero; the Robin type is corresponding to $0 < \theta < 1$, where the flux of u across the boundary is assumed to be proportional

*This article is a summary of works [3, 4] by Jumpei Inoue (Waseda University) and a joint work with him [5].

to u. Throughout this article, we assume that the resource function $m(x)$ belongs to the following functional class:

$$L^\infty_+(\Omega) := \{\, m \in L^\infty(\Omega) \,:\, m \geq 0 \ \text{ a.e. in } \Omega, \ \ \|m\|_\infty > 0 \,\},$$

where $\|\cdot\|_p$ denotes the usual $L^p(\Omega)$ norm for $p \in [1,\infty]$. Concerning (1.1), the global well-posedness and the long-time behavior of solutions are known as follows (e.g., [2, Sections 1.6.5-1.6.7], [7, Chapter 5] and [10, Theorem 3.6]):

Lemma 1.1. *If $u_0 \in L^\infty_+(\Omega) \cap C(\overline{\Omega})$ satisfies $\mathfrak{B}_\theta u = 0$ on $\partial\Omega$, then (1.1) admits a unique solution $u(x,t)$ in the class $u \in C^{1+\gamma, \frac{1+\gamma}{2}}(\overline{\Omega} \times (0,\infty)) \cap C(\overline{\Omega} \times [0,\infty))$ with any $\gamma \in (0,1)$. Furthermore, as $t \to \infty$, $u(x,t)$ converges to the maximal nonnegative steady-state $u^*(x)$ uniformly in $\overline{\Omega}$.*

Therefore, the profile of the maximal solution to the following nonlinear elliptic equation can approximate the spatial configuration of the species after a long time.

$$\begin{cases} d\Delta u + u(m(x) - u) = 0, \quad u \geq 0 \ \text{ in } \Omega, \\ \mathfrak{B}_\theta u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

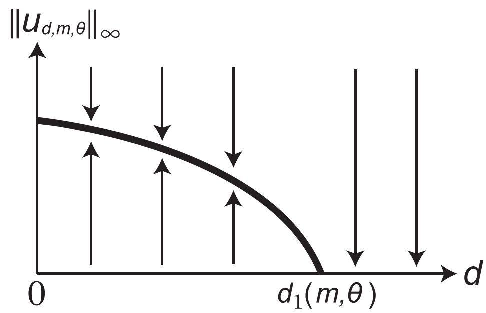
It is easily checked that all nontrivial nonnegative solutions are positive solutions by the strong maximum principle and all weak solutions can be in class $W^{2,p}(\Omega)$ for any $p \geq 1$. By the Sobolev embedding theorem, such solutions are in class $C^{1+\gamma}(\overline{\Omega})$ for any $\gamma \in (0,1)$. Furthermore, the global bifurcation structure of positive solutions of (1.2) is known. For the sake of the expression of the structure, we introduce the following eigenvalue problem with weight:

$$-\Delta\phi = \lambda m(x)\phi \ \text{ in } \Omega, \qquad \mathfrak{B}_\theta \phi = 0 \ \text{ on } \partial\Omega. \tag{1.3}$$

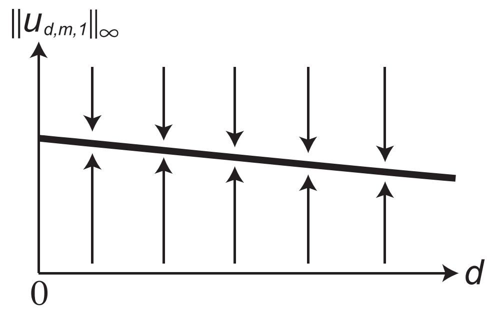
It is well-known that all eigenvalues of (1.3) consist of a monotone increasing sequence $\{\lambda_j(m,\theta)\}_{j=1}^\infty \subset [0,\infty)$ with $\lambda_j(m,\theta) \to \infty$ as $j \to \infty$, and moreover, the least eigenvalue $\lambda_1(m,\theta)$ can be characterized by the following variational formula (e.g., [2, Theorem 2.4]):

$$\lambda_1(m,\theta) = \begin{cases} \displaystyle\min_{\phi \in H^1(\Omega),\, \phi \neq 0} \frac{\|\nabla\phi\|_2^2 + \frac{1-\theta}{\theta}\int_{\partial\Omega}\phi^2}{\int_\Omega m(x)\phi^2} & \text{if } \theta \in (0,1], \\ \displaystyle\min_{\phi \in H^1_0(\Omega),\, \phi \neq 0} \frac{\|\nabla\phi\|_2^2}{\int_\Omega m(x)\phi^2} & \text{if } \theta = 0. \end{cases}$$

It is possible to verify that, for each fixed $m \in L^\infty_+(\Omega)$, $\lambda_1(m,\theta)$ is a monotone decreasing continuous function with respect to $\theta \in (0,1)$ with $\lambda_1(m,1) = 0$. Here we define $d_1(m,\theta) := 1/\lambda_1(m,\theta)$, and hence, $d_1(m,\theta)$ is monotone increasing with respect to $\theta \in (0,1)$ with $d_1(m,1) = \infty$. It is known that $d_1(m,\theta)$ gives the threshold for the existence/nonexistence of positive solutions of (1.2) (e.g., [2, Corollary 3.14]):



(a) $0 \le \theta < 1$



(b) $\theta = 1$

Figure 1: Bifurcation diagram

Lemma 1.2. *For each $m \in L^\infty_+(\Omega)$, the set of all positive solutions of* (1.2) *forms a simple curve parameterized by $d \in (0, d_1(m,\theta))$ as follows*

$$\Gamma(m,\theta) := \{\, (d, u_{d,m,\theta}) \in (0, d_1(m,\theta)) \times C^{1+\gamma}(\overline{\Omega})\,\},$$

where the map $(0, d_1(m,\theta)) \ni d \mapsto u_{d,m,\theta} \in C^{1+\gamma}(\overline{\Omega})$ is continuous and satisfies $\lim_{d\searrow 0} u_{d,m,\theta}(x) = m(x)$ *for each* $(x,\theta) \in \Omega \times [0,1]$ *and* $\lim_{d\nearrow d_1(m,\theta)} u_{d,m,\theta} = 0$ *uniformly in $\overline{\Omega}$ for each $\theta \in [0,1)$.*

Hence, Lemma 1.2 asserts that, if $\theta \in [0,1)$ (except for the Neumann type), the set $\Gamma(m,\theta)$ of all positive solutions to (1.2) forms a bifurcation curve, which bifurcates from the trivial solution at $d = d_1(m,\theta)$ and extends in the direction $d < d_1(m,\theta)$. Then, the necessary and sufficient range of d for the existence of positive solutions is $0 < d < d_1(m,\theta)$, and moreover, for each $d \in (0, d_1(m,\theta))$, the positive solution is uniquely determined by $u_{d,m,\theta}(x)$. Combining Lemmas 1.1 and 1.2, one can see that, for any $u_0 \in L^\infty_+(\Omega) \cap C(\overline{\Omega})$ with $\mathfrak{B}_\theta u = 0$ on $\partial\Omega$, the solution $u(x,t)$ of (1.1) satisfies

$$\lim_{t\to\infty} u(x,t) = \begin{cases} u_{d,m,\theta}(x) & \text{uniformly for } x \in \overline{\Omega} \text{ if } d \in (0, d_1(m,\theta)), \\ 0 & \text{uniformly for } x \in \overline{\Omega} \text{ if } d \in [d_1(m,\theta), \infty), \end{cases}$$

where it is noted that the latter case is empty if $\theta = 1$ (see Figure 1). Therefore, it can be said that $u_{d,m,\theta}$ gives important information on the population density of the species after a long time. In this sense, many mathematicians have studied qualitative properties of $u_{d,m,\theta}$ from various viewpoints.

Among other things, this article focuses on the following optimal problem.

Problem. Evaluate $S(\Omega, \theta) := \sup_{d \in (0, d_1(m,\theta)), m \in L^\infty_+(\Omega)} \dfrac{\|u_{d,m,\theta}\|_1}{\|m\|_1}$.

This problem was proposed by Ni (e.g., [1, Abstract], [6, (8.36)]) especially for the Neumann problem ($\theta = 1$). From a viewpoint of the biological model, this problem asks for the maximum (or supremum) of the total population of the species relative to the total amount of feed. From another viewpoint of the nonlinear PDEs, this problem asks whether there exists a constant C such that

$$\|u_{d,m,\theta}\|_{L^1(\Omega)} \leq C \|m\|_{L^1(\Omega)} \quad \text{for any } d \in (0, d_1(m, \theta)) \text{ and } m \in L^\infty_+(\Omega), \tag{1.4}$$

and moreover, if such a C exists, the best possible constant, denoted by $S(\Omega, \theta)$, is naturally required. For the one-dimensional Neumann case where $\Omega = (a, b)$ with $\theta = 1$, Bai, He and Li [1] gave the following answer:

Theorem 1.3 ([1])**.** *It holds that $S((a, b), 1) = 3$ for any $-\infty < a < b < \infty$. Furthermore, this supremum is not achieved by any solution of* (1.2) *with $\Omega = (a, b)$ and $\theta = 1$.*

Subsequently, in papers by Inoue [3, 4], Inoue and the author [5], it was shown that, in the one-dimensional case, the supremum is still equal to 3 for any other boundary condition, whereas, in the multi-dimensional case, there is no constant C that satisfies (1.4). Precisely stated, the following result was obtained:

Theorem 1.4 ([3, 4, 5])**.** *The following properties hold true:*

(i) $S((a, b), \theta) = 3$ *for any* $-\infty < a < b < \infty$ *and* $\theta \in [0, 1)$. *Furthermore, this supremum is not achieved by any solution of* (1.2) *with* $\Omega = (a, b)$*;*

(ii) $S(\Omega, \theta) = \infty$ *for any bounded domain* $\Omega \subset \mathbb{R}^n$ *with* $n \geq 2$ *and* $\theta \in [0, 1]$.

The purpose of this article is to express a mathematical motivation of the studies on Problem and introduce some ideas in the proof of Theorem 1.4.

2 Neumann boundary condition

2.1 Effect of heterogeneity of resources on the biomass

This section concerns with (1.2) in the Neumann boundary condition case where $\theta = 1$. We should begin with a mathematical observation by Lou [8], which led to the global consideration of (1.2) with $\theta = 1$. His observation gave the following fine proof the fact that heterogeneity of resources can increase the total population of the species when the homogeneous Neumann boundary condition is imposed:

Let u be any positive solution of (1.2) with $\theta = 1$. Dividing the elliptic equation of (1.2) with $\theta = 1$ by u and integrating the resulting expression over Ω, one can get

$$d \int_\Omega \frac{\Delta u}{u}\,\mathrm{d}x + \|m\|_1 - \|u\|_1 = 0. \tag{2.1}$$

Applying the integration by parts to the first term, one can see

$$\int_\Omega \frac{\Delta u}{u}\,\mathrm{d}x = \int_{\partial\Omega} \frac{1}{u}\partial_\nu u\,\mathrm{d}S + \left\|\frac{\nabla u}{u}\right\|_2^2 = \left\|\frac{\nabla u}{u}\right\|_2^2 \geq 0$$

due to the Neumann boundary condition. Therefore, we know that

$$\|m\|_1 \leq \|u\|_1, \quad \text{or equivalently,} \quad \frac{\|u\|_1}{\|m\|_1} \geq 1$$

and the equality holds if and only if $m(x)$ is a positive constant over Ω. That is to say, the heterogeneity of $m(x)$ can increase the total population of the species.

2.2 The one-dimensional case

In this subsection, we introduce the outline of the proof of Theorem 1.3 by Bai, He and Li [1].

Proof of Theorem 1.3. Owing to a suitable scaling, we may choose $\Omega = (a, b)$ arbitrarily for the study of $S((a, b), 1)$. Then we consider the following Neumann problem of a nonlinear ODE:

$$\begin{cases} du'' + u(m(x) - u) = 0, \quad u > 0 \qquad (0 < x < 1), \\ u'(0) = u'(1) = 0, \end{cases} \tag{2.2}$$

where the prime symbol represents the derivative by x. As mentioned in Lemma 1.2, there exists a unique solution $u_{d,m,1}(x)$ to (2.2) for any $(d, m) \in (0, \infty) \times L^\infty_+(0, 1)$. For simplicity, we restrict ourselves to the case where $m(x)$ is a non-increasing and non-constant function over $(0, 1)$. It is easily checked that the corresponding solution $u(x) := u_{d,m,1}(x)$ is monotone decreasing for $x \in (0, 1)$. Following the usual energy procedure, one multiplies (2.2) by u' and integrates the resulting expression over $(x, 1)$ to get

$$du'(x)^2 = \frac{2\,u(x)^3}{3} - \frac{2\,u(1)^3}{3} + 2\int_x^1 m(y)\,u(y)\,u'(y)\,\mathrm{d}y. \tag{2.3}$$

At the same time, we recall (2.1) to note

$$\int_0^1 \frac{du'(x)^2}{u(x)^2}\,\mathrm{d}x + \|m\|_1 - \|u\|_1 = 0. \tag{2.4}$$

Then, by substituting (2.3) into the first term of (2.4), one can see

$$\|m\|_1 - \frac{\|u\|_1}{3} \underbrace{-\frac{2}{3}u(1)\int_0^1 \left(\frac{u(1)}{u(x)}\right)^2 \mathrm{d}x + 2\int_0^1 \frac{\mathrm{d}x}{u(x)^2}\int_x^1 m(y)\,u(y)\,u'(y)\,\mathrm{d}y}_{<0} = 0. \tag{2.5}$$

Then it follows that $\|m\|_1 - \|u\|_1/3 > 0$, thereby, $\|u\|_1/\|m\|_1 < 3$. It was shown in [1] that this estimate holds true even unless $m(x)$ is non-increasing, that is,

$$\frac{\|u_{d,m,1}\|_1}{\|m\|_1} < 3 \quad \text{for any } (d,m) \in (0,\infty) \times L^\infty_+(0,1). \tag{2.6}$$

The core idea of the proof of Theorem 1.3 by [1] is to find that the bracket term in (2.5) tends to zero as $\varepsilon \to +0$ in the setting

$$d = \sqrt{\varepsilon}, \quad m(x) = \frac{1}{\varepsilon}\mathbb{1}_{[0,\varepsilon)} := \begin{cases} 1/\varepsilon & (0 \le x < \varepsilon), \\ 0 & (\varepsilon \le x \le 1). \end{cases} \tag{2.7}$$

Actually, the first term of the bracket vanishes as $\varepsilon \to +0$ since the solution $u_\varepsilon(x)$ to (2.2) with (2.7) satisfies $\lim_{\varepsilon\to+0} u_\varepsilon \to 0$ uniformly in any compact subset of $(0,1]$. Furthermore, the second term of the bracket in (2.5)

$$J := \int_0^1 \frac{\mathrm{d}x}{u(x)^2}\int_x^1 m(y)\,u(y)\,u'(y)\,\mathrm{d}y$$

also vanishes as follows: Since $u_\varepsilon(y) < u_\varepsilon(x)$ for any $y \in (x,\varepsilon)$, hence one use the Schwarz inequality to get

$$|J| = \int_0^1 \frac{\mathrm{d}x}{u_\varepsilon(x)^2}\int_x^\varepsilon \frac{u_\varepsilon(y)\,|u'_\varepsilon(y)|}{\varepsilon}\mathrm{d}y < \varepsilon\int_0^\varepsilon \frac{1}{\varepsilon}\frac{|u'_\varepsilon(y)|}{u_\varepsilon(y)}\,\mathrm{d}y \le \sqrt{\varepsilon}\left(\int_0^\varepsilon \left(\frac{u'_\varepsilon(x)}{u_\varepsilon(x)}\right)^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Substituting (2.4) into the integrand, one can find

$$\begin{aligned} |J| &< \sqrt{\varepsilon}\left(\int_0^1 \left(\frac{u'_\varepsilon(x)}{u_\varepsilon(x)}\right)^2 \mathrm{d}x\right)^{\frac{1}{2}} < \sqrt{\varepsilon}\left(\frac{1}{\sqrt{\varepsilon}}\int_0^1 (u_\varepsilon(x) - \varepsilon^{-1}\mathbb{1}_{[0,\varepsilon)})\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &< \sqrt{\varepsilon}\left(\frac{3-1}{\sqrt{\varepsilon}}\right)^{\frac{1}{2}} \to 0 \quad \text{as } \varepsilon \to +0, \end{aligned}$$

where the last inequality is due to (2.6). The proof of Theorem 1.3 is complete. □

For the solutions $\{u_\varepsilon\}_{\varepsilon>0}$ found by [1] we see that $\|u_\varepsilon\|_1 \nearrow 3$ as $\varepsilon \to +0$ because $\|m_\varepsilon\|_1 = 1$ for any $\varepsilon \in (0,1)$.

Concerning more detailed information on $\{u_\varepsilon\}$, Inoue [3] obtained the following asymptotic profiles as $\varepsilon \to +0$:

Theorem 2.1 ([3]). *Let $u_\varepsilon(x)$ be the solution of* (2.2) *with* (2.7). *Then, for any fixed $k > 0$, $u_\varepsilon(x)$ satisfies the following properties:*

(i) $\displaystyle\lim_{\varepsilon\to+0} \sqrt{\varepsilon}\, u_\varepsilon(0) = \lim_{\varepsilon\to+0} \sqrt{\varepsilon}\, u_\varepsilon(\varepsilon) = \frac{3}{2}$;

(ii) $\displaystyle\lim_{\varepsilon\to+0} \frac{u_\varepsilon(1)}{\sqrt{\varepsilon}} = 2\sqrt{3}\left(\int_0^1 \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-\frac{2-\sqrt{3}}{4}z^2)}}\right)^2$;

(iii) $\displaystyle\lim_{\varepsilon\to+0} \sqrt{\varepsilon}\, u_\varepsilon(\varepsilon + k\sqrt{\varepsilon}y) = \frac{6}{(ky+2)^2}$ *for each $y \geq 0$;*

(iv) $\displaystyle\lim_{\varepsilon\to+0} \int_\varepsilon^{\varepsilon+k\sqrt{\varepsilon}} u_\varepsilon(x)\,\mathrm{d}x = \frac{3}{1+2k^{-1}}$.

From the maximum principle, we know the fundamental property of any non-constant solution u of (1.2) that $0 < u(x) < \|m\|_\infty$ for all $x \in \Omega$. In Theorem 2.1, the assertion (i) says that the maximum value of u_ε, $u_\varepsilon(0)$, is shown to remain about $3/(2\sqrt{\varepsilon})$ being much less than the maximum value $1/\varepsilon$ of m. Furthermore, it can be seen that the height of u remains approximately the same in the presence range $[0, \varepsilon]$ of resources. The assertion (ii) says that the minimum value $u_\varepsilon(1)$ of $u_\varepsilon(x)$ is shown to be about $C\sqrt{\varepsilon}$, where C is the constant expressed in the right-hand side. As mentioned above, the fundamental property implies that u_ε decays to zero in any compact subset of $(0, 1]$ as $\varepsilon \to +0$. The assertion (iii) shows that, if the solution u_ε is scaled down in height by a factor of $\sqrt{\varepsilon}$ and simultaneously extended by a factor of $k/\sqrt{\varepsilon}$ starting from $x = \varepsilon$, then the scaling function $\varphi_\varepsilon(y) := \sqrt{\varepsilon}u_\varepsilon(\varepsilon + k\sqrt{\varepsilon}y)$ converges to the function $6/(ky+2)^2$ as $\varepsilon \to +0$. The assertion (iv) says that most of the total population 3 of u_ε is occupied in the interval $(\varepsilon, \varepsilon + k\sqrt{\varepsilon})$ if $\varepsilon > 0$ is small and $k > 0$ is large.

2.3 The multi-dimensional case

Next we consider (1.2) for the case where $\theta = 1$ and $\Omega = B_1^n := \{\, x \in \mathbb{R}^n : |x| < 1 \,\}$ with $n \geq 2$. This subsection introduces our idea in [5] concerning the proof of

$$S(B_1^n, 1) = \infty \quad \text{if } n \geq 2. \tag{2.8}$$

For the verification of (2.8). it suffices to show the following proposition with

$$m_\varepsilon(x) := \frac{1}{\varepsilon^n}\mathbb{1}_{B_\varepsilon^n} = \begin{cases} 1/\varepsilon^n & (\,|x| < \varepsilon\,), \\ 0 & (\,\varepsilon \leq |x| \leq 1\,). \end{cases} \tag{2.9}$$

Proposition 2.2. *If $n \geq 2$, there exist positive constants c_1 and c_2 depending only on n such that the solution $u_\varepsilon(x)$ of*

$$\begin{cases} \dfrac{c_1}{\varepsilon^{n-2}} \Delta u + u\,(m_\varepsilon(x) - u) = 0, \quad u > 0 & \text{in } B_1^n, \\ \partial_\nu u = 0 & \text{on } \partial B_1^n \end{cases} \tag{2.10}$$

satisfies

$$\frac{\|u_\varepsilon\|_{L^1(B_1^n)}}{\|m_\varepsilon\|_{L^1(B_1^n)}} \geq c_2 \left(1 - \frac{1}{e} + \frac{n}{e} |\log \varepsilon|\right). \tag{2.11}$$

By the fact that right-hand side of (2.11) tends to infinity as $\varepsilon \to +0$, (2.8) immediately follows. The idea of the proof of Proposition 2.2 is to construct an L^1 unbounded sequence of sub-solutions.

Before stating the proof, we review the sub-super solution method. By the uniqueness of solutions of (1.2), it suffices to consider the radial solution represented as $v(r) := u(x)$ with $r = |x|$. Hence $v(r)$ satisfies the following boundary value problem of the following nonlinear ODE:

$$\begin{cases} d\left(v''(r) + \dfrac{n-1}{r} v'(r)\right) + v(r)(\widetilde{m}(r) - v(r)) = 0 \quad (0 < r < 1), \\ v'(0) = 0 = \theta v'(1) + (1-\theta) v(1), \end{cases} \tag{2.12}$$

where the prime symbol denotes the derivative by r and $\widetilde{m}(r) := m(x)$ with $r = |x|$. Although this section is devoted to the case where $\theta = 1$, we summarize the sub-super solution method in the general θ case for the sake of the discussion after this subsection, see also e.g., [9, Section 2].

Definition 2.1. Let $\underline{v} : [0,1] \to \mathbb{R}$ be continuous in $[0,1]$ and of class C^2 in $(a_0, a_1) \cup (a_1, a_2) \cup \cdots \cup (a_N, a_{N+1})$ with $a_0 = 0$, $a_{N+1} = 1$. Then $\underline{v}$ is called a **sub-solution** of (2.12) if $\underline{v}$ satisfies the following conditions (i)-(iii):

(i) $d\left(\underline{v}''(r) + \dfrac{n-1}{r} \underline{v}'(r)\right) + \underline{v}(r)(\widetilde{m}(r) - \underline{v}(r)) \geq 0$ in (a_i, a_{i+1}) for $0 \leq i \leq N$;

(ii) $\lim\limits_{r \nearrow a_i} \underline{v}'(r) \leq \lim\limits_{r \searrow a_i} \underline{v}'(r)$ for each $1 \leq i \leq N$;

(iii) $\underline{v}'(0) \geq 0 \geq \theta \underline{v}'(1) + (1-\theta)\underline{v}(1)$.

If $\overline{v}$ belongs to to the same class as $\underline{v}$ and satisfies (i)-(iii) with reverse inequalities, then $\overline{v}$ is called a **super-solution** of (2.12).

Lemma 2.3 (the sub-super solution method)**.** *If there exist a super-solution $\overline{v}(r)$ and a sub-solution $\underline{v}(r)$ of (2.12) such that $\underline{v}(r) \leq \overline{v}(r)$ for all $0 \leq r \leq 1$, then there exists a solution $v(r)$ of (2.12) such that $\underline{v}(r) \leq v(r) \leq \overline{v}(r)$ for all $0 \leq r \leq 1$.*

Proof of Proposition 2.2. It is easy to check that $\overline{v}(r) = 1/\varepsilon^n$ is a super-solution of (2.12) with $\theta = 1$. Our crucial task is to construct an L^1 unbounded sequence of sub-solutions of (2.2) with $\theta = 1$. For any $\varepsilon \in (0,1)$, we define $\underline{v}(r) := \underline{v}_\varepsilon(r)$ by

$$\underline{v}(r) := \begin{cases} \dfrac{c_2}{\varepsilon^n} e^{-(\frac{r}{\varepsilon})^n} & (0 \le r < \varepsilon), \\ \dfrac{c_2}{er^n} & (\varepsilon \le r \le 1). \end{cases} \tag{2.13}$$

Then by a straightforward calculation, one can verify that

$$\underline{v}'(r) := \begin{cases} -\dfrac{c_2 n r^{n-1}}{\varepsilon^{2n}} e^{-(\frac{r}{\varepsilon})^n} & (0 \le r < \varepsilon), \\ -\dfrac{c_2 n}{er^{n+1}} & (\varepsilon \le r \le 1) \end{cases}$$

and

$$\underline{v}''(r) := \begin{cases} -\dfrac{c_2 n r^{n-2}}{\varepsilon^{2n}} \left(n - 1 - \dfrac{nr^n}{\varepsilon^n} \right) e^{-(\frac{r}{\varepsilon})^n} & (0 \le r < \varepsilon), \\ \dfrac{c_2 n(n+1)}{er^{n+2}} & (\varepsilon \le r \le 1), \end{cases}$$

and moreover, $\underline{v} \in C^2([0,\varepsilon) \cup (\varepsilon, 1]) \cap C^1([0,1])$. To find a sufficient region of (c_1, c_2) such that $\underline{v}(r)$ becomes a sub-solution, we derive the following lower estimate of the left-hand side of the ODE corresponding to (2.9):

$$\begin{aligned} & \frac{c_1}{\varepsilon^{n-2}} \left(\underline{v}''(r) + \frac{n-1}{r} \underline{v}'(r) \right) + \underline{v}(r) \left(\frac{1}{\varepsilon^n} - \underline{v}(r) \right) \\ &= \left\{ -\frac{c_1 c_2 n r^{n-2}}{\varepsilon^{3n-2}} \left(n - 1 - \frac{nr^n}{\varepsilon^n} \right) - \frac{c_1 c_2 n(n-1) r^{n-2}}{\varepsilon^{3n-2}} + \frac{c_2}{\varepsilon^{2n}} - \frac{c_2^2}{\varepsilon^{2n}} e^{-(\frac{r}{\varepsilon})^n} \right\} e^{-(\frac{r}{\varepsilon})^n} \\ &\ge \left\{ -\frac{2c_1 c_2 n(n-1) r^{n-2}}{\varepsilon^{3n-2}} + \frac{c_2}{\varepsilon^{2n}} (1 - c_2 e^{-(\frac{r}{\varepsilon})^n}) \right\} e^{-(\frac{r}{\varepsilon})^n} \\ &> \frac{c_2}{\varepsilon^{2n}} \{ -2n(n-1)c_1 - c_2 + 1 \} e^{-(\frac{r}{\varepsilon})^n} \qquad \text{for any } r \in [0, \varepsilon), \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} & \frac{c_1}{\varepsilon^{n-2}} \left(\underline{v}''(r) + \frac{n-1}{r} \underline{v}'(r) \right) - \underline{v}(r)^2 \\ &= \frac{c_2}{er^{n+2}} \left\{ \frac{c_1 n(n+1)}{\varepsilon^{n-2}} - \frac{c_1 n(n-1)}{\varepsilon^{n-2}} - \frac{c_2}{er^{n-2}} \right\} \\ &\ge \frac{c_2}{e\varepsilon^{n-2} r^{n+2}} \left(2nc_1 - \frac{c_2}{e} \right) \qquad \text{for any } r \in [\varepsilon, 1]. \end{aligned} \tag{2.15}$$

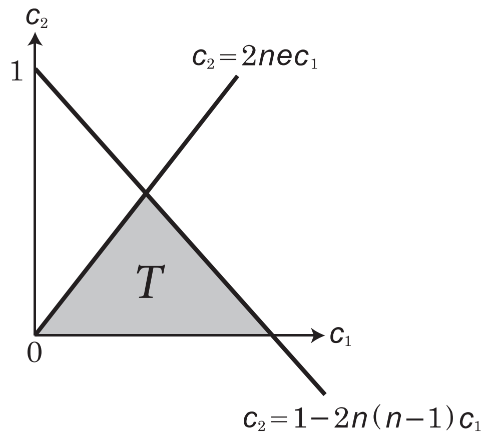


Figure 2: Triangle T

Together with $\underline{v}'(0) = 0$ and $\underline{v}'(1) = -c_2 n/e < 0$, we deduce that, if

$$0 < c_2 \leq \min\{\, 1, 1 - 2n(n-1)c_1, 2enc_1 \,\}, \tag{2.16}$$

then $\underline{v}(r)$ is a sub-solution of (2.12) with $\theta = 1$ and satisfies $\underline{v}(r) < \overline{v}(r)$ for any $r \in (0,1)$. Here it is noted that the region of (c_1, c_2) satisfying (2.16) is corresponding to the interior of the triangle T whose vertexes are

$$(0,0), \quad \left(\frac{1}{2n(n-1)}, 0\right), \quad \left(\frac{1}{2n(n+e-1)}, \frac{e}{n+e-1}\right)$$

(see Figure 2). Therefore, if $(c_1, c_2) \in \operatorname{int}(T)$, then Lemma 2.3 gives a solution $v(r)$ of (2.12) with $\theta = 1$ satisfying

$$\underline{v}(r) \leq v(r) \leq \overline{v}(r) \quad \text{for any } r \in [0,1]. \tag{2.17}$$

Hence $u(x) = v(r)$ with $r = |x|$ is a positive solution of (3.9). By a straightforward computation, one can verify that $\underline{u}_\varepsilon(x) := \underline{v}_\varepsilon(r)$ with $r = |x|$ satisfies

$$\frac{\|\underline{u}_\varepsilon\|_1}{\|m_\varepsilon\|_1} = c_2\left(1 - \frac{1}{e} + \frac{n}{e}|\log \varepsilon|\right). \tag{2.18}$$

Together with (2.17), we obtain (2.11). The proof of Proposition 2.2 is complete. $\square$

3 Dirichlet boundary condition

3.1 The one-dimensional case

Concerning the one-dimensional Dirichlet problem, this subsection introduces the proof by Inoue [4] for the assertion (i) of Theorem 1.4 in the case $\theta = 0$:

Proposition 3.1 ([4]). *It holds that $S((a,b),0) = 3$ for any $-\infty < a < b < \infty$. Furthermore, this supremum is not achieved by any solution of* (1.2) *with $\Omega = (a,b)$ and $\theta = 0$.*

Proof. By the usual scaling argument, it suffices to show $S((-1,1),0) = 3$. Let $u := u_{d,m,\theta}$ be the positive solution of (1.2) with $(\Omega,\theta) = ((-1,1),0)$ in case $d \in (0, d_1(m,0))$. Furthermore, let $U := u_{d,m,1}$ be the positive solution of (1.2) with $(\Omega,\theta) = ((-1,1),1)$. Then, the usual comparison argument using Lemma 2.3. enables us to verify that $u(x) < U(x)$ for any $x \in [-1,1]$ and $d \in (0, d_1(m,0))$. Together with (2.6), one can see that

$$\frac{\|u\|_1}{\|m\|_1} < \frac{\|U\|_1}{\|m\|_1} < 3 \quad \text{for any } (d,m) \in (0, d_1(m,0)) \times L^\infty_+(-1,1).$$

We consider (1.2) with

$$\Omega = (-1,1), \quad \theta = 0, \quad d = \sqrt{\varepsilon}, \quad m(x) = \frac{1}{\varepsilon}\mathbb{1}_{(-\varepsilon,\varepsilon)}.$$

It is easy to check that any solution $u(x)$ to this Dirichlet problem satisfies $u(x) = u(-x)$ for any $x \in [-1,1]$. Then it is convenient to consider the following Neumann-Dirichlet problem over $(0,1)$:

$$\begin{cases} \sqrt{\varepsilon}u'' + u(\widetilde{m}_\varepsilon(x) - u) = 0, \quad u > 0 \qquad (0 < x < 1), \\ u'(0) = u(1) = 0, \end{cases} \tag{3.1}$$

where $\widetilde{m}_\varepsilon(x) = \varepsilon^{-1}\mathbb{1}_{[0,\varepsilon)}$. Obviously, for the completion of the proof, it is sufficient to show that (3.1) has positive solutions u_ε for small $\varepsilon > 0$ and they satisfy

$$\|u_\varepsilon\|_1 \nearrow 3 \quad \text{as } \varepsilon \to +0. \tag{3.2}$$

Let $v_\varepsilon(x)$ be the positive solution of (3.1) with the boundary conditions replaced by $v'_\varepsilon(0) = v'_\varepsilon(1) = 0$. It is clear that $v_\varepsilon(x)$ is a super-solution of (3.1). In view of the proof of Theorem 1.3, we recall that

$$\|v_\varepsilon\|_1 \nearrow 3 \quad \text{as } \varepsilon \to +0. \tag{3.3}$$

We construct a sub-solution $\underline{u}_\varepsilon(x)$ by shifting $v_\varepsilon(x)$ downward to satisfy the Dirichlet boundary condition and multiplying the shifted function by an undetermined factor as follows:

$$\underline{u}_\varepsilon(x) := k_\varepsilon(v_\varepsilon(x) - v_\varepsilon(1)),$$

where k_ε will be determined later such that $k_\varepsilon \nearrow 1$ as $\varepsilon \to +0$, and $u_\varepsilon(x)$ becomes a sub-solution. Then it follows that

$$\begin{aligned} F_\varepsilon(x) &:= \sqrt{\varepsilon}\underline{u}_\varepsilon''(x) + \underline{u}_\varepsilon(x)(\widetilde{m}_\varepsilon(x) - \underline{u}_\varepsilon(x)) \\ &= k_\varepsilon \left[\sqrt{\varepsilon}v_\varepsilon'' + (v_\varepsilon(x) - v_\varepsilon(1))\left\{\, \widetilde{m}_\varepsilon(x) - k_\varepsilon(v_\varepsilon(x) - v_\varepsilon(1))\,\right\}\right]. \end{aligned}$$

Substituting $\sqrt{\varepsilon}v_\varepsilon''(x) = -v_\varepsilon(x)(m_\varepsilon(x) - v_\varepsilon(x))$ into the right-hand side, we have

$$F_\varepsilon(x) = k_\varepsilon\left[v_\varepsilon(1)\left\{\, k_\varepsilon(2v_\varepsilon(x) - v_\varepsilon(1)) - \widetilde{m}_\varepsilon(x)\,\right\} + (1-k_\varepsilon)v_\varepsilon(x)^2\right]. \tag{3.4}$$

Since $\widetilde{m}_\varepsilon(x) = 1/\varepsilon$ and $v_\varepsilon(x) > v_\varepsilon(\varepsilon)$ for any $x \in [0,\varepsilon)$, then (3.4) implies

$$F_\varepsilon(x) > k_\varepsilon\left[v_\varepsilon(1)\left\{\, k_\varepsilon(2v_\varepsilon(\varepsilon) - v_\varepsilon(1)) - \frac{1}{\varepsilon}\,\right\} + (1-k_\varepsilon)v_\varepsilon(\varepsilon)^2\right] \tag{3.5}$$

for any $x \in [0,\varepsilon)$. By (i) of Theorem 2.1, we see that

$$v_\varepsilon(\varepsilon) = \left(\frac{3}{2} + o(1)\right)\frac{1}{\sqrt{\varepsilon}} \quad \text{and} \quad v_\varepsilon(1) = (C + o(1))\sqrt{\varepsilon} \quad \text{as } \varepsilon \to +0,$$

where C is the positive constant in the right-hand side of (ii) of Theorem 2.1. Then we know from (3.5) that

$$F_\varepsilon(x) > k_\varepsilon\left\{-\frac{C + o(1)}{\sqrt{\varepsilon}} + \left(\frac{9}{4} + o(1)\right)\frac{1-k_\varepsilon}{\varepsilon}\right\} \quad \text{as } \varepsilon \to +0$$

for any $x \in [0,\varepsilon)$. Then, if we determine k_ε as

$$1 - k_\varepsilon = \varepsilon^{1/4}, \quad \text{namely,} \quad k_\varepsilon = 1 - \varepsilon^{1/4}$$

for sufficiently small $\varepsilon > 0$, then $k_\varepsilon \nearrow 1$ as $\varepsilon \to +0$, and

$$F_\varepsilon(x) > 0 \quad \text{for any } x \in [0,\varepsilon). \tag{3.6}$$

Since $\widetilde{m}_\varepsilon(x) = 0$ and $v_\varepsilon(x) > v_\varepsilon(1)$ for any $x \in [\varepsilon, 1)$, then (3.4) implies

$$F_\varepsilon(x) > k_\varepsilon(1-k_\varepsilon)v_\varepsilon(1)^2 > 0 \quad \text{for any } x \in [\varepsilon, 1). \tag{3.7}$$

It follows from (3.6) and (3.7) that $\underline{u}_\varepsilon(x) = (1-\varepsilon^{1/4})(v_\varepsilon(x) - v_\varepsilon(1))$ is a sub-solution of (3.1) if $\varepsilon > 0$ is sufficiently small. Therefore, Lemma 2.3 ensures a solution u_ε of (3.1) such that $\underline{u}_\varepsilon(x) \le u_\varepsilon(x) \le v_\varepsilon(x)$ for any $x \in [0,1]$ if $\varepsilon > 0$ is sufficiently small. With (3.3) and the definition of $\underline{u}_\varepsilon$, we obtain (3.2). The proof of Proposition 3.1 is complete. $\square$

3.2 The multi-dimensional case

For the function $\underline{v}(r)$ by (2.13), we define $w(r)$ by shifting $\underline{v}(r)$ downward to satisfy the Dirichlet boundary condition at $r = 1$:

$$w(r) := \underline{v}(r) - \frac{c_2}{e} \quad \text{for } r \in [0, 1]. \tag{3.8}$$

Proposition 3.2 ([4]). *There exist positive constants c_1 and c_2 depending only on n such that, if $\varepsilon > 0$ is sufficiently small, then $\underset{\sim}{u}(x) = w(r)$ with $r = |x|$ is a sub-solution to*

$$\begin{cases} \dfrac{c_1}{\varepsilon^{n-2}} \Delta u + u\,(m_\varepsilon(x) - u) = 0, \quad u > 0 & \textit{in} \ \ B_1^n, \\ u = 0 & \textit{on} \ \ \partial B_1^n, \end{cases} \tag{3.9}$$

where $m_\varepsilon(x)$ is the function defined by (2.9).

Proof. We set $\widetilde{m}_\varepsilon(r) = \varepsilon^{-n} \mathbb{1}_{[0,\varepsilon)}$. For any $r \in [0, \varepsilon)$, we know from (2.14) that

$$\begin{aligned} G_\varepsilon(r) :=& \frac{c_1}{\varepsilon^{n-2}} \left(w''(r) + \frac{n-1}{r} w'(r) \right) + w(r)(\widetilde{m}_\varepsilon(r) - w(r)) \\ =& \frac{c_1}{\varepsilon^{n-2}} \left(\underline{v}''(r) + \frac{n-1}{r} \underline{v}'(r) \right) + \underline{v}(r) \left(\frac{1}{\varepsilon^n} - \underline{v}(r) \right) + \frac{c_2}{e} \left(2\underline{v}(r) - \frac{1}{\varepsilon^n} - \frac{c_2}{e} \right) \\ >& \frac{c_2}{\varepsilon^{2n}} \{ -2n(n-1)c_1 - c_2 + 1 \} e^{-(\frac{r}{\varepsilon})^n} - \frac{c_2}{e} \left(\frac{1}{\varepsilon^n} + \frac{c_2}{e} \right). \end{aligned}$$

We note $e^{-(\frac{r}{\varepsilon})^n} > e^{-1}$ for any $r \in [0, \varepsilon)$. Hence, if $0 < c_2 < 1 - 2n(n-1)c_1$ and $\varepsilon > 0$ is sufficiently small, then

$$G_\varepsilon(r) > \frac{c_2}{e\varepsilon^{2n}} \left\{ -2n(n-1)c_1 - \left(1 + \frac{\varepsilon^{2n}}{e} \right) c_2 + 1 - \varepsilon^n \right\} \quad \text{for } r \in [0, \varepsilon). \tag{3.10}$$

On the other hand, for any $r \in [\varepsilon, 1]$, we know form (2.15) that

$$\begin{aligned} G_\varepsilon(r) =& \frac{c_1}{\varepsilon^{n-2}} \left(w''(r) + \frac{n-1}{r} w'(r) \right) - w(r)^2 \\ =& \frac{c_1}{\varepsilon^{n-2}} \left(\underline{v}''(r) + \frac{n-1}{r} \underline{v}'(r) \right) - \underline{v}(r)^2 + \frac{2c_2}{e} \underline{v}(r) - \left(\frac{c_2}{e} \right)^2 \\ \geq& \frac{c_2}{e\varepsilon^{n-2} r^{n+2}} \left(2nc_1 - \frac{c_2}{e} \right) + \left(\frac{c_2}{e} \right)^2 \left(\frac{2}{r^n} - 1 \right). \end{aligned} \tag{3.11}$$

Consequently, (3.10) and (3.11) enable us to deduce that, for any small $\delta > 0$, if ε is sufficiently small and

$$0 < c_2 < \min\{\, 1 - \delta - 2n(n-1)c_1, 2nec_1 \,\},$$

then $G_\varepsilon(r) \geq 0$ for any $r \in [0,1]$. Hence, in such a situation, $\underset{\sim}{u}(x) = w(r)$ with $r = |x|$ becomes a sub-solution of (3.9). The proof of Proposition 3.2 is accomplished. □

Corollary 3.3 ([4]). *If $n \geq 2$, then $S(B_1^n, 0) = \infty$.*

Proof. Obviously, $\overline{u}_\varepsilon(x) = 1/\varepsilon^n$ is a super-solution of (3.9), and moreover, it satisfies $\underset{\sim}{u}_\varepsilon(x) < \overline{u}_\varepsilon(x)$ for any $x \in B_1^n$, where $\underset{\sim}{u}_\varepsilon$ is the sub-solution obtained in Proposition 3.2. Thanks to Lemma 2.3, there exists a solution u_ε to (3.9) such that $\underset{\sim}{u}_\varepsilon(x) \leq u_\varepsilon(x) \leq \overline{u}_\varepsilon(x)$ for any $x \in B_1^n$. Since $\underset{\sim}{u}_\varepsilon = \underline{u}_\varepsilon - c_2/e$, then (2.18) gives $\|\underset{\sim}{u}_\varepsilon\|_1/\|m_\varepsilon\|_1 \to \infty$. Therefore we obtain $S(B_1^n, 0) = \infty$. □

4 General domains and boundary conditions

In this section, we prove (ii) of Theorem 1.4. In the study of $S(\Omega, \theta)$ with general domains and $\theta \in [0,1]$, the following well-known parabolic version of the sub-super solution method will play an important role:

Definition 4.1. If $\underline{u} \in C^{1+\gamma, \frac{1+\gamma}{2}}(\overline{\Omega} \times (0,\infty)) \cap C(\overline{\Omega} \times [0,\infty))\ (=: X)$ satisfies

$$\begin{cases} \partial_t \underline{u} \leq d\Delta \underline{u} + \underline{u}(m(x) - \underline{u}) & \text{in } \Omega \times (0,\infty), \\ \mathfrak{B}_\theta \underline{u} := \theta \partial_\nu \underline{u} + (1-\theta)\underline{u} \leq 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{4.1}$$

then $\underline{u}$ is called a **sub-solution** of (1.1). If $\overline{u} \in X$ satisfies (4.1) with reverse inequalities, then $\overline{u}$ is called a **super-solution** of (1.1).

Lemma 4.1 (the sub-super solution method). *Suppose that $\underline{u} \in X$ and $\overline{u} \in X$ are a sub-solution and a super-solution of* (1.1)*, respectively. Then, if $\underline{u}(x,0) \leq (\not\equiv)\, \overline{u}(x,0)$ for all $x \in \overline{\Omega}$, then $\underline{u}(x,t) < \overline{u}(x,t)$ for all $(x,t) \in \overline{\Omega} \times (0,\infty)$.*

Proof of (ii) of Theorem 1.4. We may assume $0 \in \Omega$ without loss of generality. For any small $\varepsilon > 0$, we consider the time-depending solution to a diffusive logistic equation in a general bounded domain Ω:

$$\begin{cases} \partial_t u = \dfrac{c_1}{\varepsilon^{n-2}} \Delta u + u(m_\varepsilon(x) - u) & \text{in } \Omega \times (0,\infty), \\ \mathfrak{B}_\theta u = 0 & \text{on } \Omega \times (0,\infty), \\ u(\,\cdot\,,0) = u_0 & \text{in } \Omega, \end{cases} \tag{4.2}$$

where $m_\varepsilon(x) = \varepsilon^{-n}\mathbb{1}_{B^1_\varepsilon}$. We take any $\rho > 0$ such that $B^n_\rho := \{\, x \in \mathbb{R}^n \,:\, |x| < \rho \,\}$ satisfies $\overline{B}^n_\rho := \{\, x \in \mathbb{R}^n \,:\, |x| \leq \rho \,\} \subset \Omega$. Next we set a nonnegative initial data $u_0 \in C(\overline{\Omega})$ such that $\operatorname{supp} u_0 = \overline{B}^n_\rho$. Then, Lemma 1.1 implies that the solution $u_\varepsilon(x,t) = u_\varepsilon(x,t;\Omega,\theta)$ satisfies

$$\lim_{t\to\infty} u_\varepsilon(x,t) = u^*_\varepsilon(x) \quad \text{uniformly for } x \in \overline{\Omega}, \tag{4.3}$$

where $u^*_\varepsilon(x) = u^*_\varepsilon(x;\Omega,\theta)$ is the maximal nonnegative stationary solution of (4.2).

On the other hand, we consider solutions to the same parabolic equation with the homogeneous Dirichlet boundary condition on $\partial B^n_\rho \times (0,\infty)$:

$$\begin{cases} \partial_t v = \dfrac{c_1}{\varepsilon^{n-2}}\Delta v + v(m_\varepsilon(x) - v) & \text{in } B^n_\rho \times (0,\infty), \\ v = 0 & \text{on } \partial B^n_\rho \times (0,\infty), \\ v(\,\cdot\,,0) = u_0 & \text{in } B^n_\rho, \end{cases} \tag{4.4}$$

where the initial data u_0 is taken same as that in (4.2). Similarly, Lemma 1.1 ensures the solution $v_\varepsilon(x,t) = v_\varepsilon(x,t;B^n_\rho,0)$ satisfies

$$\lim_{t\to\infty} v_\varepsilon(x,t) = v^*_\varepsilon(x) \quad \text{uniformly for } x \in \overline{B}^n_\rho, \tag{4.5}$$

where $v^*_\varepsilon(x) = v^*_\varepsilon(x;B^n_\rho,0)$ is the maximal stationary solution of (4.4). In spite of $\rho \neq 1$, one can verify

$$\|v^*_\varepsilon\|_{1;B^n_\rho} := \int_{B^n_\rho} v^*_\varepsilon(x;B^n_\rho,0)\,\mathrm{d}x \to \infty \quad \text{as } \varepsilon \to +0 \tag{4.6}$$

in the same manner as in the argument in the previous subsection.

Here we note that the solution $u_\varepsilon(x,t;\Omega,\theta)$ of (4.2) satisfies $u_\varepsilon(x,t) > 0$ for any $(x,t) \in \Omega \times (0,\infty)$ by the strong maximum principle. Hence $u_\varepsilon(x,t) > 0$ for any $(x,t) \in \partial B^n_\rho \times (0,\infty)$, and thereby, $u_\varepsilon(x,t)$ (restricted in $\overline{B}^n_\rho \times (0,\infty)$) is a super-solution of (4.4). Then Lemma 4.1 implies that

$$v_\varepsilon(x,t) < u_\varepsilon(x,t) \quad \text{for any } (x,t) \in \overline{B}^n_\rho \times (0,\infty).$$

Setting $t \to \infty$ in the above inequality, we know from (4.3) and (4.5) that

$$v^*_\varepsilon(x) \leq u^*_\varepsilon(x) \quad \text{for any } x \in \overline{B}^n_\rho.$$

Together with (4.6), we set $\varepsilon \to +0$ in the above inequality to get

$$\|u_\varepsilon\|_{1;\Omega} := \int_\Omega u^*_\varepsilon(x;\Omega,\theta)\,\mathrm{d}x \to \infty \quad \text{as } \varepsilon \to +0.$$

This fact obviously completes the proof for (ii) of Theorem 1.4. $\square$

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