On Affine Structures Which Come from Berkovich Geometry for K-trivial Finite Quotients of Abelian Varieties

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ABSTRACT. In the context of Strominger-Yau-Zaslow mirror symmetry (after [SYZ96]), SYZ fibrations are often studied. It is known that an SYZ fibration from a polarized Calabi-Yau manifold gives two affine structures and one metric to its base space. The base space endowed with the triple is expected to play an important role to obtain a mirror pair of the Calabi-Yau manifold. This approach makes extensive use of complex differential geometrical tools such as SYZ fibrations.

On the other hand, Kontsevich and Soibelman proposed a construction of affine manifolds for maximal degenerations of Calabi-Yau manifolds using a non-Archimedean analog of the SYZ fibration. Nowadays, such an analog of the SYZ fibration is called a non-Archimedean SYZ fibration. Furthermore, they predicted that two affine manifolds, one coming from SYZ fibrations and the other from a non-Archimedean SYZ fibration, are equivalent. Actually, to be precise, they adopted a somewhat different construction (called Collapse picture) for the SYZ-side than the one described as above.

Main theme of this thesis is to prove the above Kontsevich-Soibelman conjecture in an enhanced form for maximal degenerations of K-trivial finite quotients of polarized abelian varieties. This thesis is based on three papers ([Got20], [Got22] and [GO22]) written by the author, Keita Goto. The last paper is a joint work with the author's advisor, Yuji Odaka.

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1. Introduction

1.1. At the end of the 20th century, in order to formulate what is called mirror symmetry, several approaches have been proposed. One of them is due to Strominger, Yau and Zaslow [SYZ96]. In *op.cit.*, they gave a geometric interpretation for mirror symmetry and proposed a conjecture called the SYZ conjecture. Gross and Siebert provided an algebro-geometric interpretation of the SYZ conjecture [GS06]. It is known as the Gross-Siebert program. In this program, it is important to associate an integral affine manifold with singularities (IAMS, for short) with a degeneration of polarized Calabi-Yau manifolds, and vice versa. For a (toric) degeneration of polarized Calabi-Yau manifolds, they extracted the polyhedral decomposition and the fan structure for each vertex and gave an IAMS structure to the dual intersection complex based on them, and vice versa.

Kontsevich and Soibelman associated an IAMS structure of the dual intersection complex in a non-Archimedean way [KS06]. The precise definition will be given later (§5), but for now, we call it *NA SYZ picture*. In [*op.cit.*, §4.2], they mentioned the specific IAMS structure for the degeneration of K3 surfaces defined by

$$\{x_0x_1x_2x_3 + tP_4(x) = 0\} \subset \mathbb{P}^3 \times \Delta,$$

where $x = [x_0 : x_1 : x_2 : x_3]$ are homogeneous coordinates on $\mathbb{P}^3$, Δ is a (formal) disk with a (formal) parameter t and P_4 is a generic homogeneous polynomial of degree 4. For general degenerations of K3 surfaces, however, the specific affine structures associated in this way are not well known.

1.2. In the context of Strominger-Yau-Zaslow mirror symmetry, it is well known that SYZ fibrations associate IAMS's with maximal degenerations of polarized Calabi-Yau varieties. We discuss the following three kinds of 'SYZ fibrations' from §4.

(i) Special Lagrangian fibrations from complex Calabi-Yau manifolds (often called 'SYZ fibrations') as defined in Definition 4.2.
(ii) Non-Archimedean SYZ fibrations as defined in Definition 3.86.
(iii) Hybrid SYZ fibrations as defined in Theorem 6.7.

Their historic origins are quite different. The fibration (i) dates back at least to Harvey-Lawson [HL82, III] in the context of calibrated geometry, and after Strominger-Yau-Zaslow [SYZ96], often studied in the context of the mirror symmetry. On the other hand, the fibration (ii) first appeared in the basic general theory of non-archimedean geometry [Ber99]. Note that in both situations, affine manifolds with singularities underly the fibrations. Such a heuristic similarity of these two fibrations is first pointed out in Kontsevich-Soibelman [KS06], which in particular lead to [KS06, Conjecture 3] roughly predicting the interesting coincidence of two affine structures. Both fibrations are also regarded roughly as “tropicalization".

[Got22, §5] proves the conjecture [KS06, Conjecture 3] for abelian varieties and Kummer surfaces in an explicit manner, with a particular emphasis on the non-archimedean SYZ fibration side.

The fibration (i) i.e., special Lagrangian fibrations, on which we place more emphasis in this thesis, are expected to exist for any maximal degenerations in the context of the mirror symmetry. This fairly nontrivial prediction is occasionally referred to as (a part of) “SYZ conjecture", after [SYZ96].

1.3. In this thesis, we calculate two IAMS structures, one from a family of SYZ fibrations (i) and the other from a non-Archimedean SYZ fibration (ii), for degenerations of K-trivial finite quotients of polarized abelian varieties. In addition, we unify these two types of SYZ fibrations by introducing what we call a hybrid SYZ fibration (iii). In other words, we partially prove the following Conjecture 1.4 and Conjecture 1.6 motivated by [KS06, Conjecture 1 and Conjecture 3] for maximal degenerations of K-trivial finite quotients of polarized abelian varieties.

Conjecture 1.4 (Existence of special Lagrangian fibrations)**.** *Take any flat proper family* $f\colon \mathcal{X}^* \to \Delta^* = \{t \in \mathbb{C} \mid 0 < |t| < 1\}$ *which is a maximal degeneration, that is, with the maximal unipotency index of the monodromy. Also, take any relatively ample line bundle* $\mathcal{L}^*$ *and consider the famliy of*

Ricci-flat Kähler metrics $g_{\mathrm{KE}}(\mathcal{X}_t)$ on $f^{-1}(t) = \mathcal{X}_t$ for $t \neq 0$, whose Kähler classes are $c_1(\mathcal{L}_t := \mathcal{L}|_{\mathcal{X}_t})$. Then for any $|t| \ll 1$, there is a special Lagrangian fibration on $\mathcal{X}_t$ (with respect to a certain meromorphic relative section of $K_{\mathcal{X}^*/\Delta^*}$).

1.5. In §4.2, we prove the above Conjecture 1.4 for K-trivial surfaces, abelian varieties of any dimension and their finite quotients. Recall that already in [OO21], Conjecture 1.4 was proven for the case of K3 surfaces (*op.cit.* Chapter 4) and higher dimensional irreducible holomorphic symplectic varieties of K3$^{[n]}$-type and generalized Kummer varieties (*op.cit.* Chapter 8).

Conjecture 1.6 (Equivalence between SYZ picture and NA SYZ picture (cf.[KS06, Conjecture 3]))**.** *Consider the maximal degenerating polarized algebraic Calabi-Yau varieties $\mathcal{X}^*$ over a sufficiently small punctured disc $\Delta^* := \{t \in \mathbb{C} \mid 0 < |t| < \epsilon\}$ such that there exists a special Lagrangian fibration for each fiber $\mathcal{X}_t$ as Conjecture 1.4. Then these fibrations for any fiber $\mathcal{X}_t$ extend by continuity to a non-Archimedean SYZ fibration as we see later (Definition 3.86). Further, the IAMS structure induced by the family of SYZ fibrations coincides with the IAMS structure induced by the non-Archimedean SYZ fibration.*

That is, we prove the following theorems in this thesis.

Theorem 1.7 (= Theorem 4.20 and Corollary 4.24, An affirmative answer to Conjecture 1.4 for K-trivial finite quotients of polarized abelian varieties, cf. [GO22, Theorem 2.8])**.** *For maximal degenerations of K-trivial finite quotients of polarized abelian varieties, there is a special Lagrangian fibration on each fiber. Furthermore, IAMS induced by the family of these special Lagrangian fibrations can be described explicitly.*

Remark 1.8. The last assertion of Theorem 1.7 gives an affirmative answer to [KS06, Conjecture 1] for K-trivial finite quotients of polarized abelian varieties although it is already proved in [OO21] partially.

Theorem 1.9 (= Theorem 5.32, cf. [Got22, Theorem 5.31])**.** *For maximal degenerations of K-trivial finite quotients of polarized abelian varieties, an IAMS structure induced by a family of SYZ fibrations coincides with an IAMS structure induced by a non-Archimedean SYZ fibration up to scaling.*

1.10. In the process of proving this, we prove that the IAMS induced by the non-Archimedean SYZ fibration for K-trivial finite quotients of polarized abelian varieties is explicitly described by the degeneration data as in [FC90] (= Theorem 5.27).

Theorem 1.11 (=Theorem 6.7 and Theorem 6.23, An affirmative answer to Conjecture 1.6 for K-trivial finite quotients of abelian varieties, cf. [GO22, Theorem 3.2 and Theorem 4.9]). *Consider the maximally degenerating K-trivial finite quotients of polarized abelian varieties* $\mathcal{X}^*$ *over a sufficiently small punctured disc* $\Delta^* := \{t \in \mathbb{C} \mid 0 < |t| < \epsilon\}$*. Then there exists a special Lagrangian fibration for each fiber* $\mathcal{X}_t$ *as Conjecture 1.4, and these fibrations for any fiber* $\mathcal{X}_t$ *extends to a hybrid SYZ fibration* $f^{\mathrm{hyb}} \colon \mathcal{X}^{\mathrm{hyb}} \to \mathcal{B}$ *as we will see in Theorem 6.7. Here, the fibration* f^{hyb} *satisfies the following:*

(i) *for* $t \in \Delta^*$*,* $f^{\mathrm{hyb}}|_{\mathcal{X}_t} = f_t$ *that is, coincides with the special Lagrangian fibrations.*
(ii) $f^{\mathrm{hyb}}|_{t=0} = f^{\mathrm{hyb}}|_{X^{\mathrm{an}}}$ *is a non-Archimedean SYZ fibration.*

In particular, the fibration f^{hyb} *implies that the IAMS structure induced by the family of SYZ fibrations coincides with the IAMS structure induced by the non-Archimedean SYZ fibration. Further, the finite group action on the maximal degeneration of polarized abelian varieties, that we consider when taking its quotient, can be described explicitly.*

1.12. Here is a brief description of the structure of this thesis:

§2 In this section, we recall classical theory of abelian varieties. In particular, the period theory, Faltings-Chai theory (cf. [FC90]) and Künnemann theory (cf. [Kün98]) are quite improtant for this thesis. Künnemann theory states the way to construct an SNC model of a polarized abelian variety over a complete discrete valuation field. It is a modification of Mumford's construction by which we can construct a semiabelian degeneration from degeneration data. In Künnemann's construction, it is important to construct a cone decomposition associated with the degeneration. Further, the cone decomposition is also important for NA SYZ picture as we will see in §2.3. In applying Künnemann's construction to the proofs of our main theorems, we will modify his method due to technical problems ([Got22, Lemma 3.17]= Lemma 2.55).

§3 This section is intended to recall the classical theory of Berkovich analytic spaces. In this context, we also introduce the results of [Got20, Theorem 3.5, Theorem 6.11]. In latter half of this section, we recall non-Archimedean SYZ fibrations originally introduced in [KS06], following [NXY19].

So far, we have basically shared the assumed knowledge. From §4, we get to the main problem.

§4 The main theme of §4 is to prove Theorem 4.20, following [GO22, §2]. In this section, we consider SYZ picture for K-trivial finite quotients of polarized abelian varieties, and compare it with Gromov-Hausdorff limit picture.

§5 In [Got22], we treated non-Archimedean SYZ fibrations only for Kummer surfaces or abelian varieties. However, in this section, we study non-Archimedean SYZ fibrations for K-trivial finite quotients of polarized abelian varieties, beyond *op.cit.*.

§6 The main theme of this section is to show how to “merge" two fibrations (i) and (ii) as in (1.2) with different origins, to construct what we call a *hybrid SYZ fibration* (the fibration (iii) as in (1.2)), following [GO22]. This provides some enhanced answer to [KS06, Conjecture 3], not only at the level of integral affine structures. Our construction is inspired by the recent technology of hybrid norms that originated in [Ber10] and re-explored in [BJ17]. We prove [KS06, Conjecture 3] for K-trivial finite quotients of abelian varieties in arbitrary dimension, generalizing [Got22, §5]. In §6.3, we *explicitly* explore the possible symmetry of maximally degenerating abelian varieties and its inducing symmetry on the limit object given by SYZ fibrations, which fits well to the previous sections.

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2. Classical Theory of Abelian Varieties

In this chapter, we give a brief introduction to the classical theory with respect to abelian varieties that play important roles in this thesis.

2.1. Definitions and basic properties.

This section is mainly based on [Mum70].

Definition 2.1. An abelian variety X is a complete algebraic variety over a field k with a group law $m : X \times X \to X$ such that m and the inverse map are both morphisms of varieties.

2.2. In particular, an abelian variety is connected algebraic group. Further, it is well known that any algebraic group scheme over k is quasi-projective over k. It implies that any abelian variety over k is projective over k. (cf. Corollary 2.19) If $k = \mathbb{C}$, then the underlying complex analytic space of an abelian variety is actually a complex torus. Now review this.

Let X be a compact connected complex Lie group of dimension g and V the tangent space to X at the identity point $0 \in X$. Then $V(\cong \mathbb{C}^g)$ is a complex vector space and the *exponential map* $\exp : V \to X$ is uniquely defined. Moreover, the map $\exp : V \to X$ is a surjective homomorphism of complex Lie groups with kernel a lattice $\Lambda(\cong \mathbb{Z}^{2g})$, where a *lattice* in V means a subgroup generated by a real basis of V, and induces an isomorphism $V/\Lambda \cong X$. Therefore, X is a complex torus.

However, complex tori are not necessarily projective, that is, abelian varieties. To summarize, an abelian variety over $\mathbb{C}$ is equivalent to a complex torus with some polarization.

Unless otherwise noted, an abelian variety shall mean a complex abelian variety as above until the end of this section.

2.3. Let X be a complex torus of the form V/Λ as (2.2). The lattice Λ is an important invariant. In fact, there are canonical isomorphisms

$$\mathrm{H}^r(X, \mathbb{Z}) \cong \bigwedge^r \mathrm{Hom}(\Lambda, \mathbb{Z}).$$

Furthermore, it is also important when considering the moduli of principally polarized abelian varieties, as we see later (2.28).

The cohomology groups $\mathrm{H}^q(X, \Omega^p)$ are also significant invariants. By transition with respect to the group law on X, we can see that

$$\mathcal{O}_X \otimes_{\mathbb{C}} \bigwedge^p V^* \cong \Omega^p,$$

where $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is the cotangent vector space to X at 0. Further, it is well-known that

$$\mathrm{H}^q(X, \mathcal{O}_X) \cong \bigwedge^q \overline{V^*},$$

where $\overline{V^*}$ is the complex conjugate of V^*. It implies that

$$\mathrm{H}^q(X, \Omega^p) \cong \mathrm{H}^q(X, \mathcal{O}_X \otimes_{\mathbb{C}} \bigwedge^p V^*) \cong \mathrm{H}^q(X, \mathcal{O}_X) \otimes \Lambda^p V^* \cong \bigwedge^q \overline{V^*} \otimes \bigwedge^p V^*.$$

Note that $\mathrm{H}^r(X, \mathbb{C}) \cong \mathrm{H}^r(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \bigwedge^r \mathrm{Hom}(\Lambda, \mathbb{C})$ holds. Since $\mathrm{Hom}(\Lambda, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong V^* \oplus \overline{V^*}$, it holds that

$$\mathrm{H}^r(X, \mathbb{C}) \cong \bigwedge^r (V^* \oplus \overline{V^*}) \cong \bigoplus_{p+q=r} (\bigwedge^q \overline{V^*} \otimes \bigwedge^p V^*) \cong \bigoplus_{p+q=r} \mathrm{H}^q(X, \Omega^p).$$

It means Hodge decomposition.

2.4. Let X be a complex torus of the form V/Λ as (2.2), L a line bundle on X and $\pi(= \exp) : V \to X$ an universal covering of X. Since $H^1(V, \mathcal{O}_V^*) \cong \mathrm{Pic}(V) = 0$, the pull back $\pi^*(L)$ is trivial on V. It gives a trivialization $\chi : \pi^*(L) \cong \mathbb{C} \times V$. Furthermore, the canonical action of Λ on $\pi^*(L)$

induces an action of $\mathbb{C} \times V$ through χ. The induced action of $\lambda \in \Lambda$ is given by

$$\mathbb{C} \times V \ni (u, z) \mapsto \phi_\lambda(u, z) := (e_\lambda(z) \cdot u, z + \lambda) \in \mathbb{C} \times V,$$

where $e_\lambda \in H^* := \mathrm{H}^0(V, \mathcal{O}_V^*)$. Then it holds that

$$e_{\lambda+\lambda'}(z) = e_\lambda(z + \lambda') \cdot e_{\lambda'}(z).$$

It means that the map $\lambda \mapsto e_\lambda$ is a 1-cocycle for Λ with coefficients in H^*. Such a function $e_- : \Lambda \to H^*$ is called a *factor of automorphy*. Further, if the trivialization χ is altered, $\{e_\lambda\}$ is replaced by a cohomologous cocycle. Therefore, we obtain the map $\mathrm{Pic}(X) \to \mathrm{H}^1(\Lambda, H^*)$. In fact, this map is an isomorphism. It means that any line bundle on X is isomorphic to the quotient of $\mathbb{C} \times V$ for the action of Λ corresponding to $\{e_\lambda\} \in \mathrm{H}^1(\Lambda, H^*)$.

Let H be a hermitian form on V such that $\mathrm{Im}H(\Lambda \times \Lambda) \subset \mathbb{Z}$. Then there is a map $\alpha : \Lambda \to U(1) \in \mathbb{C}^\times$ such that

$$\alpha(\lambda_1 + \lambda_2) = e^{\sqrt{-1}\pi \mathrm{Im}H(\lambda_1, \lambda_2)} \cdot \alpha(\lambda_1)\alpha(\lambda_2).$$

Such a map α is called a *semi-character* with respect to H. If we put

$$e_\lambda(z) = \alpha(\lambda) \cdot e^{\pi H(z,\lambda) + \frac{1}{2}\pi H(\lambda,\lambda)},$$

then $\{e_\lambda\} \in \mathrm{H}^1(\Lambda, H^*)$. Denote the associated line bundle by $L(H, \alpha)$. Furthermore, the first Chern class of the line bundle $L(H, \alpha)$ is

$$\mathrm{Im}H \in \bigwedge^2 \mathrm{Hom}(\Lambda, \mathbb{Z}) \cong \mathrm{H}^2(X, \mathbb{Z}).$$

Note that if we take $L(H_1, \alpha_1)$ and $L(H_2, \alpha_2)$ as above, then it holds that

$$L(H_1, \alpha_1) \otimes L(H_2, \alpha_2) \cong L(H_1 + H_2, \alpha_1\alpha_2).$$

More precisely, the following theorem is well known.

Theorem 2.5 (Appel-Humbert's Theorem)**.** *Let X be a complex torus of the form V/Λ as* (2.2)*, $\{H\}$ the group of hermitian $H : V \times V \to \mathbb{C}$ with $\mathrm{Im}H(\Lambda \times \Lambda) \subset \mathbb{Z}$ and $\{(H, \alpha)\}$ the group of data (H, α) consisting of the above H and semi-character α with respect to H. Then the following diagram commutes.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(\Lambda, U(1)) & \longrightarrow & \{(H,\alpha)\} & \longrightarrow & \{H\} & \longrightarrow & 0 \\
 & & \wr\!\!\downarrow & & \wr\!\!\downarrow f & & \wr\!\!\downarrow g & & \\
0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}X & \longrightarrow & \mathrm{H}^{1,1}(X, \mathbb{Z}) & \longrightarrow & 0
\end{array}$$

where $f(H, \alpha) := L(H, \alpha)$ and $g(H) = \mathrm{Im}H$.

2.6. Let $L = L(H, \alpha)$ be the associated line bundle on X as Theorem 2.5. Then it is well-known that the following are equivalent.

- L is ample.
- H is positive definite.

2.7. Now consider morphisms between abelian varieties. Before that, we recall the following important lemma.

Lemma 2.8 (Rigidity lemma)**.** *Let X be a complete variety, Y and Z any varieties, and $f : X \times Y \to Z$ a morphism such that for some $y_0 \in Y$, the image $f(X \times \{y_0\})$ is a single point $z_0 \in Z$. Then there is a morphism $g : Y \to Z$ such that $f = g \circ p_2$, where $p_2 : X \times Y \to Y$ is the second projection.*

Corollary 2.9. *If X and Y are abelian varieties and $f : X \to Y$ is any morphism, then $f(x) = g(x) + a$, where g is a homomorphism of X into Y and $a \in Y$.*

Proof of Corollary 2.9. Replacing f by $f - f(0)$, we may assume $f(0) = 0$. It suffices to show that f is a homomorphism. Then the assertion holds by applying Lemma 2.8 to the morphism $\phi : X \times X \to Y$ defined by $\phi(x_1, x_2) := f(x_1 + x_2) - f(x_1) - f(x_2)$. □

2.10. Let X be an abelian variety of the form V/Λ as (2.2), $\pi : V \to X$ the universal covering and $\mathrm{Aut}(X)$ the automorphism group of X. We denote by $t_x : X \to X$ the *translation morphism* defined by $t_x(y) = x + y$. Any homomorphism of X into itself comes from some linear map of V into itself. By Corollary 2.9, if $f \in \mathrm{Aut}(X)$, then $f(\pi(x)) = t_v(\pi(Mx))$, where $M \in \mathrm{GL}(V, \mathbb{C})$ and $v \in V$. In particular, $\mathrm{Aut}(X) \hookrightarrow \mathrm{GL}(V, \mathbb{C}) \ltimes V$ holds.

The following theorems are important.

Theorem 2.11 (Seesaw Theorem)**.** *Let X be a complete variety, B any variety and L a line bundle on $X \times B$. Then the set*

$$B_1 = \{b \in B \mid L|_{X \times \{b\}} \text{ is trivial on } X \times \{b\}\}$$

is closed in B, and if on $X \times B_1$, $p_2 : X \times B_1 \to B_1$ is the second projection, then $L|_{X \times B_1} \cong p_2^* M$ for some line bundle M on B_1.

Theorem 2.12 (The theorem of the cube)**.** *Let X and Y be complete varieties, Z a connected scheme, and L a line bundle on $X \times Y \times Z$ whose restrictions to $\{x_0\} \times Y \times Z$, $X \times \{y_0\} \times Z$ and $X \times Y \times \{z_0\}$ are trivial for some $x_0 \in X$, $y_0 \in Y$ and $z_0 \in Z$. Then L is trivial.*

Corollary 2.13. *Let X_1, X_2 and X_3 be complete varieties and L a line bundle on $X_1 \times X_2 \times X_3$. Then*

$$L \cong p_{12}^*(L_{12}) \otimes p_{23}^*(L_{23}) \otimes p_{13}^*(L_{13}),$$

where $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ is the projection and L_{ij} is some line bundle on $X_i \times X_j$.

Proof of Corollary 2.13. Take a point $x_i \in X_i$ for each i. Let $s_{12} : X_1 \times X_2 \to X_1 \times X_2 \times X_3$ be the inclusion such that $\mathrm{Im}(s_{12}) = X_1 \times X_2 \times \{x_3\}$. Likewise, we can define s_{ij} if $i \neq j$. Further, let $s_1 : X_1 \to X_1 \times X_2 \times X_3$ be the inclusion such that $\mathrm{Im}(s_1) = X_1 \times \{x_2\} \times \{x_3\}$. Likewise, we can define s_i for each i. Then s_i is a section of the projection $p_i : X_1 \times X_2 \times X_3 \to X_i$. Now consider

$$L \otimes p_{12}^* s_{12}^*(L^{-1}) \otimes p_{23}^* s_{23}^*(L^{-1}) \otimes p_{13}^* s_{13}^*(L^{-1}) \otimes p_1^* s_1^*(L) \otimes p_2^* s_2^*(L) \otimes p_3^* s_3^*(L).$$

The restriction of this line bundle to each $X_i \times X_j$ is trivial. By Theorem 2.12, this line bundle is trivial on $X_1 \times X_2 \times X_3$. Note that p_i factors through p_{ij} (or p_{ji}, which is the same thing). In other words, it holds that $p_1 = q_1 \circ p_{12}$, $p_2 = q_2 \circ p_{23}$ and $p_3 = q_3 \circ p_{13}$, where q_i is the appropriate projection. Now we set $L_{12} = s_{12}^*(L) \otimes q_1^* s_1^*(L)$, $L_{23} = s_{23}^*(L) \otimes q_2^* s_2^*(L)$ and $L_{13} = s_{13}^*(L) \otimes q_3^* s_3^*(L)$. Then we obtain desired isomorphism. □

Corollary 2.14. *Let X be any variety, Y an abelian variety, and $f, g, h : X \to Y$ morphisms. Then for all $L \in \mathrm{Pic}(Y)$, we have*

$$(f+g+h)^*L \cong (f+g)^*L \otimes (g+h)^*L \otimes (h+f)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}.$$

Proof of Corollary 2.14. Let $p_i : Y \times Y \times Y \to Y$ be the projection onto the i^{th} factor. Set $m := p_1 + p_2 + p_3 : Y \times Y \times Y \to Y$. By considering i^{th} factor of $Y \times Y \times Y$ as X_i and $0 \in Y$ as $x_i \in X_i$, we can apply Corollary 2.13 to this case. Then it holds that

$$m^*L \cong (p_1+p_2)^*L \otimes (p_2+p_3)^*L \otimes (p_3+p_1)^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1} \otimes p_3^*L^{-1}$$

by the argument in the proof of Corollary 2.13. Therefore, the assertion holds by considering the pull back of m^*L by the map $(f, g, h) : X \to Y \times Y \times Y$. □

Corollary 2.15 (The theorem of the square)**.** *Let X be an abelian variety. For all line bundles L on X and $x, y \in X$,*

$$t_{x+y}^*L \otimes L \cong t_x^*L \otimes t_y^*L.$$

In particular, the map $\phi_L : X \to \mathrm{Pic}X$ defined by

$$x \mapsto \phi_L(x) := [t_x^*L \otimes L^{-1}] \in \mathrm{Pic}X$$

is a homomorphism.

Proof of Corollary 2.15. By considering Y as X, f, g as constant maps with images $x, y \in X$ respectively, and h as the identity map, the former assertion directly follows from Corollary 2.14.

The desired isomorphism implies $\phi_L(x+y) = \phi_L(x) + \phi_L(y)$ in $\mathrm{Pic}X$. Hence, the latter assertion also holds. □

2.16. In the rest of this thesis, we will always keep the notation ϕ_L as appeared in Corollary 2.15. We can easily see that $\phi_{L_1 \otimes L_2} = \phi_{L_1} + \phi_{L_2}$ and $\phi_{t_x^* L} = \phi_L$. In addition, the followings hold.

Proposition 2.17. $\ker(\phi_L)$ *is a Zariski-closed subgroup of* X.

Proof. Let $m : X \times X \to X$ be the addition morphism, $p_2 : X \times X \to X$ the second porjection. Apply Theorem 2.11 to the line bundle $m^*L \otimes p_2^* L^{-1}$. Then the set consisting of points $x \in X$ such that $m^*L \otimes p_2^* L^{-1}|_{\{x\} \times X}$ is Zariski closed. Since $m^*L \otimes p_2^* L^{-1}|_{\{x\} \times X} \cong t_x^* L \otimes L^{-1}$, the assertion holds. □

Proposition 2.18. *Let* X *be an abelian variety,* L *a line bundle on* X *with* $h^0(X, L) > 0$ *and* D *an associated effective divisor. The following are equivalent.*

(i) L *is ample on* X.
(ii) $\ker \phi_L$ *is finite.*
(iii) $H = \{x \in X \mid t_x^*(D) = D\}$ *is finite.*

Proof. First, we show $(1) \Rightarrow (2)$. If $\ker \phi_L$ is not finite, there is a positive dimensional abelian variety Y such that $Y \hookrightarrow \ker \phi_L \hookrightarrow X$. Consider the line bundle $M := m^*(L|_Y)^{-1} \otimes p_1^*(L|_Y) \otimes p_2^*(L|_Y)$ on $Y \times Y$, where $m : Y \times Y \to Y$ is the addition and $p_i : Y \times Y \to Y$ is the proojection onto i^{th} factor. Since $Y \subset \ker \phi_L$, $t_y^*(L|_Y) \cong L|_Y$ holds for each $y \in Y$. It implies that M is trivial by Theorem 2.11. Now consider the morphism $f : Y \to Y \times Y$ defined by $y \mapsto (y, -y)$. Then $f^*M \cong L|_Y \otimes (-1)^*(L|_Y)$ is trivial on Y. However, if L is ample on X, the restriction $L|_Y$ is also ample. It means that $L|_Y \otimes (-1)^*(L|_Y)$ is ample on Y. This is a contradiction since $\dim Y > 0$. In other words, L is not ample.

It follows from $H \subset \ker \phi_L$ that $(2) \Rightarrow (3)$.

We now show $(3) \Rightarrow (1)$. By Corollary 2.15, the complete linear system $|2L|$ contains the divisors $t_x^*(D) + t_{-x}^*(D)$. For any $u \in X$, we can find an $x \in X$ such that $u \pm x \notin \mathrm{Supp} D$. It also means that $u \notin \mathrm{Supp}(t_x^*(D) + t_{-x}^*(D))$. Hence, $|2L|$ has no base points and we obtain the associated morphism $\varphi_{|2L|} : X \to |2L| = \mathbb{P}(\mathrm{H}^0(X, 2L))$. If L is not ample, then $\varphi_{|2L|}$ is not finite [Gro61, Proposition (2.6.1)]. Then we can find an irreducible curve C such that $\varphi_{|2L|}(C) = \{1\mathrm{pt}\}$. It means that C is disjoint from $E \in |2L|$ if C is not contained in E. In particular, $t_x^*(D) + t_{-x}^*(D)$ is disjoint from C for almost all $x \in X$. Now we take such a point $x \in X$. Since $t_x^*(D)$ and $t_{-x}^*(D)$ are effective divisors, it holds that

$$\mathrm{Supp}(t_x^*(D) + t_{-x}^*(D)) = \mathrm{Supp}(t_x^*(D)) \cup \mathrm{Supp}(t_{-x}^*(D)).$$

It implies that $t_x^*(D)$ are disjoint from C. Now we use the following claim.

Claim. *If C is a curve on an abelian variety X and E is a prime divisor on X such that $C \cap E = \emptyset$, then E is invariant under translation by $x_1 - x_2$ for any $x_1, x_2 \in C$.*

If $D = \sum n_i D_i$, where D_i is a prime divisor, then $t_x^*(D) = \sum n_i t_x^*(D_i)$. By the above claim, each $t_x^*(D_i)$ is invariant under translation by all points $x_1 - x_2$ for any $x_1, x_2 \in C$. It means $t_{x_1-x_2+x}^*(D) = t_x^*(D)$. It implies that $t_{x_1-x_2}^*(D) = D$. Hence, H is at least 1-dimensional. In other words, H is not finite.

To finish the proof, we show the above claim. If $L = L(E)$, then L is trivial on C since E is disjoint from C. Consider m^*L, where $m : X \times X \to X$ is the addition. Here we can find an irreducible nonsingular curve C_x in X connecting any x and $0 \in X$. Then $m^*L|_{C_x \times X}$ is flat on C_x. Since $m^*L|_{\{0\}\times X} \cong L$ and $m^*L|_{\{x\}\times X} \cong t_x^*L$, t_x^*L is algebraically equivalent to L. It implies that $(t_x^*L)|_C$ has degree 0 for all $x \in X$. If E and t_x^*C intersected non-trivially, $(t_x^*L)|_C$ would have positive degree. That is, $t_x^*C \subset E$ if $t_x^*C \cap E \neq \emptyset$ for all $x \in X$. Take $x_1, x_2 \in C$ and $y \in E$. Then $y \in t_{y-x_2}^*(C) \cap E$. It implies $t_{y-x_2}^*(C) \subset E$. Hence, $x_1 - x_2 + y \in E$. It concludes the claim. □

Corollary 2.19. *An abelian variety X is projective.*

Proof. Let U be any affine open subscheme of X such that $0 \in U$, and $D_1, \ldots, D_r$ the irreducible components of $X \setminus U$. Set $D := \sum D_i$. Then U is stable under t_x for any $x \in H$. It follows from $0 \in U$ and $x = t_x^*(0) \in U$ for any $x \in H$ that $H \subset U$. That is, H is affine. In addition, H is proper since H is closed in X. Therefore, H is finite. By Proposition 2.18, it means that $L = L(D)$ is ample. Hence X is projective. □

2.20. By definition, $\mathrm{Pic}^0 X$ means the subgroup of $\mathrm{Pic} X$ consisting of all line bundles algebraically equivalent to the trivial line bundle. Let X be an abelian variety of the form V/Λ, and $p : V \to X$ the universal cover. If $L = L(H, \alpha) \in \mathrm{Pic}(X)$ as in Theorem 2.5, then we obtain

$$t_{p(v)}^*(L) \cong L(H, \alpha \cdot \gamma_v),$$

where $\gamma_v(\lambda) := e^{2\pi\sqrt{-1}\mathrm{Im}H(v,\lambda)}$ by definition of $L(H, \alpha)$. In particular, it implies $\phi_{L(H,\alpha)}(p(v)) = L(0, \gamma_v) \in \mathrm{Pic}^0 X$. Further, if $L = L(H, \alpha) \in \mathrm{Pic}^0 X$ (that is, $H = 0$), then $\gamma_v \equiv 1$. Therefore, $\mathrm{Pic}^0 X$ can be rephrased as follows:

$$\mathrm{Pic}^0 X = \{L \in \mathrm{Pic} X \mid \phi_L = 0 \in \mathrm{Hom}(X, \mathrm{Pic} X)\}.$$

In other words, we obtain an exact seqence

$$0 \longrightarrow \mathrm{Pic}^0 X \longrightarrow \mathrm{Pic} X \xrightarrow{\phi_-} \mathrm{Hom}(X, \mathrm{Pic}^0 X).$$

Set $\Lambda^{\perp} := \{v \in V \mid \mathrm{Im}H(v, \lambda) \in \mathbb{Z},\ \text{for all}\ \lambda \in \lambda\}$. Then we can easily see that $\ker \phi_{L(H,\alpha)} = \Lambda^{\perp}/\Lambda$. Hence, the following are equivalent.

- $\ker \phi_{L(H,\alpha)}$ is finite.
- $\Lambda^{\perp}/\Lambda$ is finite.
- $\Lambda^{\perp}$ is a lattice.
- $\mathrm{Im}H$ is non-degenerated.
- H is non-degenerated.

Note that it is slightly different from Proposition 2.18. The above does not need any global section of $L = L(H, \alpha)$.

The following is important to define the dual abelian variety.

Theorem 2.21. *Let L be ample on an abelian variety X. Then $\phi_L : X \to \mathrm{Pic}^0 X$ is surjective.*

2.22. By Theorem 2.21, the group $\mathrm{Pic}^0 X$ can be viewed as a group coming from an abelian variety $X/\ker \phi_L$.

Conversely, we can consider an abelian variety $\hat{X}$ isomorphic to $\mathrm{Pic}^0 X$ as an abstract group. Such an abelian variety $\hat{X}$ is not unique, however, can be uniquely characterized by considering some line bundle $\mathscr{P}$ on $X \times \hat{X}$ that we now describe, up to canonical isomorphisms.

In other words, there exists an unique pair $(\hat{X}, \mathscr{P})$ consisting of an abelian variety $\hat{X}$ with a group isomorphism $\hat{X} \cong \mathrm{Pic}^0 X$, and a line bundle $\mathscr{P}$ on $X \times \hat{X}$ up to canonical isomorphisms satisfying the following properties.

- $\mathscr{P}|_{\{0\}\times\hat{X}}$ is trivial.
- For any $\alpha \in \hat{X}$, the restriction $\mathscr{P}_\alpha := \mathscr{P}|_{X\times\{\alpha\}}$ of $\mathscr{P}$ represents the element of $\mathrm{Pic}^0 X$ given by $\alpha \in \hat{X}$ under the group isomorphism $\hat{X} \cong \mathrm{Pic}^0 X$. In particular, the map $\alpha \mapsto \mathscr{P}_\alpha$ is to be the group isomorphism $\hat{X} \to \mathrm{Pic}^0 X$.
- (Universality.) For every normal variety S, and every line bundle K on $X \times S$ such that
  - (i) $K_s := K|_{X\times\{s\}}$ is in $\mathrm{Pic}^0 X$ for one (and hence all) $s \in S$,
  - (ii) $K|_{\{0\}\times S}$ is trivial,

  the unique set-theoretic map $f : S \to \hat{X}$ such that $K_s \cong \mathscr{P}_{f(s)}$, is to be a morphism, and K is to be isomorphic to $(1_X \times f)^*\mathscr{P}$.

Such an $\hat{X}$ is called a *dual abelian variety* of X, and such a $\mathscr{P}$ is called a *Poincaré bundle*. The above properties mean that $\hat{X}$ is the moduli space parametrizing $\mathrm{Pic}^0 X$. In particular, an ample line bundle L on X gives the morphism $\phi_L : X \twoheadrightarrow \hat{X}$ as abelian varieties.

2.2. **Periods of Abelian Varieties.**

This section is mainly based on [Kob05]. The aim of this section is to describe the moduli of principally polarized abelian varieties explicitly.

2.23. According to (2.2) and (2.18), for any complex torus X of the form V/Λ as in (2.2), the following are equivalent:

- X is an abelian variety.
- There exists a positive definite hermitian form H on V such that $\mathrm{Im}H(\Lambda,\Lambda) \subset \mathbb{Z}$.

2.24. We now re-describe the above criterion in terms of some matrix. Take an isomorphism $V \cong \mathbb{C}^n$. Let $\{\lambda_1,\dots,\lambda_{2n}\}$ be a basis of Λ. Then

$$\lambda_j = (c_j^1,\dots,c_j^n) \in \mathbb{C}^n$$

via $\Lambda \hookrightarrow V \cong \mathbb{C}^n$. Now consider the $n \times 2n$-matrix

$$C = \begin{pmatrix} c_1^1 & \cdots & c_{2n}^1 \\ \vdots & c_j^i & \vdots \\ c_1^n & \cdots & c_{2n}^n \end{pmatrix}$$

and the $2n \times 2n$-matrix

$$\tilde{C} = \begin{pmatrix} C \\ \overline{C} \end{pmatrix}.$$

Such a C is called a *period matrix*. Now we can easily see that $\det \tilde{C} \neq 0$. Set $\tilde{B} := \tilde{C}^{-1} = (B, B')$, where B and B' are $2n \times n$-matrices. It is easy to see that $B' = \overline{B}$. Note that $\tilde{B}$ is described as follows:

For given $X = V/\Lambda$ and the basis $\{\lambda_i\}$ of Λ, we take a basis $\{\omega_i\}$ of $\Lambda^* \cong \mathrm{H}^1(X,\mathbb{Z})$ so that real 1-from ω_i is invariant under translations and

$$\int_{\lambda_i} \omega_j = \delta_{ij}$$

for each $i,j = 1,\dots,2n$. Then, in $\mathrm{H}^1(X,\mathbb{C})$, the following holds.

$$\begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{2n} \end{pmatrix} = \tilde{B} \cdot {}^t(dz_1,\dots,dz_n,d\overline{z}_1,\dots,d\overline{z}_n).$$

Let $Q = (q_{jk})$ be the $2n \times 2n$ skew symmetric matrix with

$$q_{jk} := \mathrm{Im}H(\lambda_j,\lambda_k) = \frac{\sqrt{-1}}{2}(H(\lambda_k,\lambda_j) - H(\lambda_j,\lambda_k)) \in \mathbb{Z},$$

where H is a hermitian form on V. Now we obtain

$$Q = \frac{\sqrt{-1}}{2}({}^tCH\overline{C} - {}^t\overline{C}HC) = \frac{\sqrt{-1}}{2}{}^t\tilde{C}\begin{pmatrix} H & 0 \\ 0 & -{}^tH \end{pmatrix}\overline{\tilde{C}}.$$

Hence, ${}^t\tilde{B}Q\overline{\tilde{B}} = \frac{\sqrt{-1}}{2}\begin{pmatrix} H & 0 \\ 0 & -{}^tH \end{pmatrix}$. It implies that ${}^tBQB = 0$ and

$$-\sqrt{-1}{}^tBQ\overline{B} = \frac{1}{2}H.$$

If X is an abelian variety, we can find a Q such that $-\sqrt{-1}{}^tBQ\overline{B} > 0$ by (2.23). Conversely, if there exists a Q with $-\sqrt{-1}{}^tBQ\overline{B} > 0$, we can see that X is an abelian variety by considering $H := -2\sqrt{-1}{}^tBQ\overline{B}$. Then, note that

$$H^{-1} = \frac{-\sqrt{-1}}{2}\overline{C} \cdot {}^tQ^{-1} \cdot {}^tC$$

also holds. Hence, the following holds.

Theorem 2.25. *For a complex torus $X = V/\Lambda$, we set B and C as in* (2.24)*. Then the following are equivalent.*

- *X is an abelian variety.*
- *(Riemann conditions) There exists a skew symmetric matrix $Q \in \mathrm{Mat}(2n, \mathbb{Z})$ such that ${}^tBQB = 0$ and $-\sqrt{-1}{}^tBQ\overline{B} > 0$.*
- *(Riemann conditions) There exists a skew symmetric matrix $Q \in \mathrm{Mat}(2n, \mathbb{Z})$ such that Q is non-degenerated, $C \cdot Q^{-1} \cdot {}^tC = 0$ and $-\sqrt{-1} \cdot \overline{C} \cdot {}^tQ^{-1} \cdot {}^tC > 0$.*

2.26. In the argument in (2.24), the form of Q depends on the choice of the basis $\{\lambda_i\}$ of Λ and the basis $\{z_j\}$ of V. Actually, if Q is non-degenerated, we can obtain Q of the form

$$Q = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}, \text{ where } \Delta = \begin{pmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{pmatrix}$$

with $\delta_i \in \mathbb{Z}_{>0}$ such that $\delta_i | \delta_{i+1}$ for each i, by taking appropriate basis $\{\lambda_i\}$ of Λ. Such a choice of the basis $\{\lambda_i\}$ of Λ is not unique. Such bases are shifted by matrices $M \in \mathrm{Sp}(Q, \mathbb{Z})$, where

$$\mathrm{Sp}(Q, \mathbb{Z}) := \{M \in \mathrm{SL}(2n, \mathbb{Z}) \mid {}^tMQM = Q\}.$$

Then the basis $\{\lambda_i\}$ of Λ such that Q is of the above form is unique up to actions by $\mathrm{Sp}(Q, \mathbb{Z})$.

In addition, we obtain the form $(\Delta\ \Omega)$ of the period matrix C, where $\Omega \in \mathrm{Mat}(n, \mathbb{C})$, by taking the basis $\{z_j\}$ of V defined by $z_j := \lambda_j/\delta_j$ for each $j = 1, \ldots, n$. Note that the choice of such a basis $\{z_j\}$ of V is unique up to $\mathrm{Sp}(Q, \mathbb{Z})$. In other words, the form $(\Delta\ \Omega)$ of a period matrix C is unique once we take a basis $\{\lambda_i\}$ of Λ such that Q is of the above form. Such a form $(\Delta\ \Omega)$ of a period matrix C is called the *normalized* period matrix of C.

Then the Riemann conditions as in Theorem 2.25 are rephrased as

$$\Omega \in \mathbb{H}_n := \{\Omega \in \mathrm{Mat}(n, \mathbb{C}) \mid \Omega = {}^t\Omega,\ \mathrm{Im}\Omega > 0\}.$$

Here, $\mathbb{H}_n$ is called the *Siegel upper half-space*. Actually, replacing the basis $\{\lambda_i\}$ of Λ by the actions of $\mathrm{Sp}(Q, \mathbb{Z})$ induces the actions on $\mathbb{H}_n$. It implies that $Q = \mathrm{Im}H$ and Λ determine a point of $\mathrm{Sp}(Q, \mathbb{Z})\backslash\mathbb{H}_n$. Conversely, a point of $\mathrm{Sp}(Q, \mathbb{Z})\backslash\mathbb{H}_n$ uniquely determines Q and Λ, or more simply, the abelian variety of the form $\mathbb{C}^n/\Lambda$ with $Q \sim \mathrm{Im}H \in \mathrm{H}^{1,1}(X, \mathbb{Z})$ such that $H > 0$, where $Q \sim \mathrm{Im}H$ means that Q and $\mathrm{Im}H$ are shifted together by the appropriate coordinate transformation as (2.24). In particular, such a pair $(\mathbb{C}^n/\Lambda, Q)$ determines a flat metric on $\mathbb{C}^n/\Lambda$ defined by the Hermitian matrix $H = (\mathrm{Im}\Omega)^{-1}$ with respect to the basis $\{z_j(= \lambda_j/\delta_j)\}$ of $\mathbb{C}^n$. Indeed, it is enough to substitute $C = (\Delta\ \Omega)$ and $Q = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$ for the equation

$$H^{-1} = \frac{-\sqrt{-1}}{2}\overline{C} \cdot {}^tQ^{-1} \cdot {}^tC.$$

As a result, we obtain the following:

$$H^{-1} = \frac{-\sqrt{-1}}{2}(\Delta\ \overline{\Omega}) \begin{pmatrix} 0 & \Delta^{-1} \\ -\Delta^{-1} & 0 \end{pmatrix} \begin{pmatrix} \Delta \\ {}^t\Omega \end{pmatrix} = \mathrm{Im}\Omega.$$

In particular, for the basis $\{\lambda_j\}$ of $\mathbb{C}^n$ which we often use later (cf. §4.3, §6.3), the above period matrix C (resp., the above Hermitian matrix H) is represented by $C = (I\ \Delta^{-1}\Omega)$ (resp., $H = \Delta(\mathrm{Im}\Omega)^{-1}\Delta$).

Definition 2.27. Let X be an abelian variety. A *polarization* of X is a homomorphism $\phi_L : X \to \hat{X}$ for some ample line bundle L on X.

2.28. Note that a polarization of X (or $\phi_{L(H,\alpha)}$) is determined by the first Chern class $c_1(L(H, \alpha)) = \mathrm{Im}H \in \mathrm{H}^{1,1}(X, \mathbb{Z})$ rather than $L(H, \alpha)$ as we saw in (2.20). More precisely, there exists a one-to-one correspondence between polarizations of X and first Chern classes $\mathrm{H}^{1,1}(X, \mathbb{Z})$ coming from ample line bundles on X. The *type* of the polarization $\phi = \phi_{L(H,\alpha)}$ (or $L(H, \alpha)$) is the vector $(\delta_1, \ldots, \delta_n) \in \mathbb{Z}^n$ defined by $Q = \mathrm{Im}H$ of the form

$$Q = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$$

as in (2.26), where $\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_n)$. If a polarization $\phi : X \to \hat{X}$ is an isomorphism, the polarization ϕ is called a *principal polarization*. In particular, $\phi_{L(H,\alpha)}$ is principally polarized if and only if

$$Q = \mathrm{Im}H = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Indeed, it follows from $|\ker \phi_{L(H,\alpha)}| = |\Lambda^{\perp}/\Lambda| = \det \Delta$. A pair (X, L) is called a (resp., *principally*) *polarized abelian variety* if X is an abelian variety and ϕ_L is (resp., principal) polarization.

Set

$$Q_0 := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},\ \mathrm{Sp}(2g, \mathbb{Z}) := \mathrm{Sp}(Q_0, \mathbb{Z}).$$

Note that $\mathrm{Sp}(2g, \mathbb{Z})$ is the usual symplectic group with $\mathbb{Z}$-coefficients. Then it follows from the argument in (2.26) that

$$\mathcal{A}_g := \mathrm{Sp}(2g, \mathbb{Z})\backslash\mathbb{H}_g$$

is the *moduli space of principally polarized abelian varieties of dimension* g. $\mathcal{A}_g$ is not just a complex analytic space, but also has functorial properties, which will not be introduced here.

2.3. **Degenerations of Abelian Varieties.**

This section is mainly based on [Got22, §2 and §3].

2.29. The aim of this section is to introduce some important results from [Kün98] with respect to degenerations of abelian varieties.

2.30. Unless otherwise noted, we fix the notation as follows: Let R be a complete discrete valuation ring (cDVR, for short) with uniformizing parameter t and algebraically closed residue field k (cf. (3.7)). We note that we start with the residue field k of an arbitrary characteristic, but later we make the condition stronger. Let $S = \mathrm{Spec} R$, and let η be the generic point of S. We denote by $K = \mathcal{O}_{S,\eta}$ the fraction field of R. Let $|\cdot|$ be the valuation on K uniquely determined by $|t| = e^{-1}$.

Definition 2.31. Let X be a locally Noetherian scheme and let D be an effective Cartier divisor on X. Let $D_1, ..., D_r$ be the irreducible components of D endowed with the reduced induced closed subscheme structure. For each subset $J \subseteq \{1, ..., r\}$, we denote by D_J the scheme-theoretic intersection $\cap_{j\in J} D_j$. If $J = \emptyset$, we note $D_{\emptyset} := X$.
An effective divisor D on X is said to be with *strict normal crossings* if it satisfies the following.

(i) D is reduced.
(ii) For each point x of D, the stalk $\mathcal{O}_{X,x}$ is regular.
(iii) For each nonempty set $J \subseteq \{1, ..., r\}$, the scheme D_J is regular and of codimension $|J|$ in X.

2.32. Let X be a smooth K-variety. A *model* of X is a flat R-algebraic space $\mathscr{X}$ endowed with an isomorphism $\mathscr{X}_K (= \mathscr{X} \times_S \mathrm{Spec} K) \to X$. (We do not assume properness and quasi-compactness.) An *snc-model of* X (or, a *semistable model of* X) is a regular model $\mathscr{X}$ of X such that $\mathscr{X}$ is a scheme and the central fiber $\mathscr{X}_k (= \mathscr{X} \times_S \mathrm{Spec} k)$ is a divisor with strict normal crossings. By the semistable reduction theorem [KKMS73, Chapter 4 §3], there exists a finite extension K' of K such that $X \times_K K'$ has an snc-model over the integral closure of R in K'. Further, if X is projective, then we can obtain a projective snc-model.

Definition 2.33 (Kulikov Model)**.** Let X be a geometrically integral smooth projective variety over K with $\omega_X \cong \mathcal{O}_X$. A *Kulikov model* of X is a regular algebraic space $\mathscr{X}$ that is proper and flat over S with the following properties:

- The algebraic space $\mathscr{X}$ is a model of X.
- The special fiber $\mathscr{X}_k$ of $\mathscr{X}$ is a reduced scheme.
- The special fiber $\mathscr{X}_k$ has strict normal crossings on $\mathscr{X}$.
- $\omega_{\mathscr{X}/S}$ is trivial.

2.34. If a Kulikov model $\mathscr{X}$ of X is a scheme, then $\mathscr{X}$ is an snc-model.

Definition 2.35. A *stratification* of a scheme X is a *not necessarily* finite set $\{X_\alpha\}_{\alpha \in I}$ of locally closed subsets, called the *strata*, such that every point of X is in exactly one stratum, and such that the closure of a stratum is a finite union of strata. We note that a stratification in the sense of [Kün98, (1.3)] (or [KKMS73, p.56]) had to be a finite set.

2.36. For an snc-model $\mathscr{X}$, the special fiber $\mathscr{X}_k$ induces a stratification of $\mathscr{X}_k$ naturally. We denote by $\Delta(\mathscr{X})$ the *dual intersection complex* of the special fiber $\mathscr{X}_k$ with respect to this stratification.

2.37. For an R-scheme $\mathscr{X}$, we denote by $\mathscr{X}_{\mathbf{for}}$ *the formal completion of* $\mathscr{X}$ *along the special fiber* $\mathscr{X}_k$. If $\mathscr{X}$ is covered by open affine subschemes of the form $\mathrm{Spec} A_\alpha$, the formal completion $\mathscr{X}_{\mathbf{for}}$ is obtained by gluing open formal subschemes of the form $\mathrm{Spf} \hat{A}_\alpha$ together, where $\hat{A}_\alpha$ is the t-adic completion of A_α. In particular, for flat R-scheme $\mathscr{X}$ locally of finite type, the formal completion $\mathscr{X}_{\mathbf{for}}$ is a flat formal R-scheme locally of finite type. Here, a *flat formal R-scheme locally of finite type* (resp., admissible formal R-scheme) means that it is covered by *not necessarily* finitely many (resp., finitely many) open formal subschemes of the form $\mathrm{Spf} \mathscr{A}_\alpha$, where $\mathscr{A}_\alpha$ is an admissible R-algebra (cf. (3.52)).

It is time to introduce some important results from [Kün98] (cf. [FC90]).

2.38. Let G be a *semiabelian scheme* over R. That is, *G is a smooth separated group scheme of finite type over S whose geometric fibers are extensions of Abelian varieties by algebraic tori*. We assume that G_η is Abelian variety. Let $\mathscr{L}$ be a line bundle on G such that $\mathscr{L}_\eta$ is ample on G_η. Then we obtain *the Raynaud extension*

$$0 \to T \to \tilde{G} \xrightarrow{\pi} A \to 0$$

associated with G and $\mathscr{L}$, where T is an algebraic torus, A an Abelian scheme, and $\tilde{G}$ a semiabelian scheme over S. If the abelian part A is trivial, G is called *maximally degenerated*. We note that we need to choose such a line bundle $\mathscr{L}$ to obtain this Raynaud extension. However, this extension is independent of the choice of $\mathscr{L}$. The line bundle $\mathscr{L}$ induces a line bundle $\tilde{\mathscr{L}}$ on $\tilde{G}$. We assume that all line bundles have cubical structures as well as [Kün98, (1.7)]. In this thesis, we shall use the categories $\mathbf{DEG}^{\mathrm{split}}_{\mathrm{ample}}$ and $\mathbf{DD}^{\mathrm{split}}_{\mathrm{ample}}$ introduced by [Kün98]. Each category is a subcategory of $\mathbf{DEG}_{\mathrm{ample}}$ and $\mathbf{DD}_{\mathrm{ample}}$ as constructed in [FC90], respectively. In particular, there is an equivalence of categories $\mathbf{M}_{\mathrm{ample}} : \mathbf{DD}_{\mathrm{ample}} \to \mathbf{DEG}_{\mathrm{ample}}$ (See [FC90, Chapter III, Corollary 7.2]). We denote F_{ample} by the inverse of this functor. Originally, F_{ample} is a more naturally determined functor, and its inverse, $\mathbf{M}_{\mathrm{ample}}$, is the non-trivial functor.

Objects of the category $\mathbf{DEG}^{\mathrm{split}}_{\mathrm{ample}}$ of split ample degenerations are triples $(G, \mathscr{L}, \mathscr{M})$, where G is a semiabelian scheme over S such that T is a *split torus* over S, $\mathscr{L}$ a cubical invertible sheaf on G such that $\mathscr{L}_\eta$ is ample on G_η, and $\mathscr{M}$ a cubical ample invertible sheaf on A such that $\tilde{\mathscr{L}} = \pi^*\mathscr{M}$. In particular, $\mathscr{M}$ is trivial when G is maximally degenerated. By definition of the algebraic torus, every ample degeneration $(G, \mathscr{L})$ becomes split after a finite extension of the base scheme S.

On the other hand, objects of the category $\mathbf{DD}^{\mathrm{split}}_{\mathrm{ample}}$ of split ample degeneration data are tuples

$$(A, M, L, \phi, c, c^t, \tilde{G}, \iota, \tau, \tilde{\mathscr{L}}, \mathscr{M}, \lambda_A, \psi, a, b).$$

Here, M and L are free Abelian groups of the same finite rank r, and $\phi : L \to M$ is an injective homomorphism. Functions $a : L \to \mathbb{Z}$ and $b : L \times M \to \mathbb{Z}$ are determined by ψ and τ, respectively. We note that M reflects the information of the Raynaud extension (or more precisely, its split torus part), ϕ reflects the information of polarization, a and b reflect the information of G_η-action. In particular, $(G, \mathscr{L})$ is called *principally polarized* if the morphism ϕ induced by $\mathbf{M}_{\mathrm{ample}}$ is an isomorphism. This convention is the same as in (2.28). Since we will not use the rest in this thesis, the rest is omitted. Please refer to [Kün98] for more details.

We note that there is an equivalence of categories $F : \mathbf{DEG}^{\mathrm{split}}_{\mathrm{ample}} \to \mathbf{DD}^{\mathrm{split}}_{\mathrm{ample}}$ (cf. [Kün98, (2.8)]). This functor is defined by the restriction of $F_{\mathrm{ample}} = \mathbf{M}^{-1}_{\mathrm{ample}} : \mathbf{DEG}_{\mathrm{ample}} \to \mathbf{DD}_{\mathrm{ample}}$ to $\mathbf{DEG}^{\mathrm{split}}_{\mathrm{ample}}$.

2.39. The key idea of [Kün98] is to construct rational polyhedral cone decompositions that give us the relatively complete model as in [Mum72]. To construct them, we shall use the category $\mathcal{C}$ introduced by [Kün98, §3] (cf. [Ove21]).

Objects of the category $\mathcal{C}$ are tuples (M, L, ϕ, a, b), where M and L are free Abelian groups of the same finite rank, $\phi : L \to M$ is an injective homomorphism, $a : L \to \mathbb{Z}$ is a function with $a(0) = 0$, and $b : L \times M \to \mathbb{Z}$ is a bilinear pairing such that $b(-, \phi(-))$ is symmetric, positive definite, and satisfies

$$a(l + l') - a(l) - a(l') = b(l, \phi(l')).$$

Here, we set $B(l_i, l_j) := b(l_i, \phi(l_j))$, where $\{l_i\}$ is a basis of L. By definition, $B : L \times L \to \mathbb{Z}$ is a symmetric positive definite quadric form.

There is a natural forgetful functor $\mathbf{For} : \mathbf{DD}^{\mathrm{split}}_{\mathrm{ample}} \to \mathcal{C}$. This function extracts the information necessary to construct rational polyhedral cone decompositions from the degeneration data $\mathbf{DD}^{\mathrm{split}}_{\mathrm{ample}}$.

2.40. We set $S' = \mathrm{Spec} R'$, where R' is another cDVR and η' is its generic point. Let $f : S' \to S$ be a finite flat morphism, let ν be the ramification index of $f^* : K = \mathcal{O}_{S,\eta} \hookrightarrow K' = \mathcal{O}_{S',\eta'}$.

In fact, the two categories $\mathbf{DEG}^{\mathrm{split}}_{\mathrm{ample}}$ and $\mathbf{DD}^{\mathrm{split}}_{\mathrm{ample}}$ depend on the base field K. That is, $\mathbf{DEG}^{\mathrm{split}}_{\mathrm{ample}}$ (resp., $\mathbf{DD}^{\mathrm{split}}_{\mathrm{ample}}$) should have been written as $\mathbf{DEG}^{\mathrm{split}}_{\mathrm{ample},K}$ (resp., $\mathbf{DD}^{\mathrm{split}}_{\mathrm{ample},K}$). In particular, these categories are not closed under base change along $f : S' \to S$. However, since we are dealing with degenerations after sufficient finite extension, these abbreviations do not cause any problems.

On the other hand, $\mathcal{C}$ does not depend on the base field K. Let us see what happens when we take the base change along $f : S' \to S$.

Given $(G, \mathscr{L}, \mathscr{M}) \in \mathbf{DEG}^{\mathrm{split}}_{\mathrm{ample},K}$, let $(G', \mathscr{L}', \mathscr{M}') \in \mathbf{DEG}^{\mathrm{split}}_{\mathrm{ample},K'}$ be the base change of $(G, \mathscr{L}, \mathscr{M})$ along $f : S' \to S$. If $\mathbf{For}(F(G, \mathscr{L}, \mathscr{M})) = (M, L, \phi, a, b) \in \mathcal{C}$, then $\mathbf{For}(F(G', \mathscr{L}', \mathscr{M}')) \cong (M, L, \phi, \nu \cdot a, \nu \cdot b)$ (cf. [Kün98, (2.9)]).

2.41. Let H be a finite group acting on $(G, \mathscr{L}, \mathscr{M}) \in \mathbf{DEG}^{\mathrm{split}}_{\mathrm{ample}}$. It means that we can regard each $h \in H$ as the S-automorphism

$$h : (G, \mathscr{L}, \mathscr{M}) \to (G, \mathscr{L}, \mathscr{M})$$

and these morphisms are compatible in a natural way. We note that we can define *the action of H on $(G, \mathscr{L}, \mathscr{M})$ over its action on S*, as in [Kün98, (2.10)], more generally, although we will not use it this time. In that definition, the condition that H acts trivially on S is not imposed. Conversely, we assume that H acts trivially on S in this thesis. Further, we also define the action of H on $F((G, \mathscr{L}, \mathscr{M})) \in \mathbf{DD}^{\mathrm{split}}_{\mathrm{ample}}$ (resp., $\mathbf{For}(F((G, \mathscr{L}, \mathscr{M}))) \in \mathcal{C}$).

2.42. Given an object $\mathbf{For}(F(G, \mathscr{L}, \mathscr{M})) = (M, L, \phi, a, b) \in \mathcal{C}$ on which the finite group H acts as 2.41, we obtain an action (from the left) of H on L, and an action (from the right) of H on M. We set $\Gamma := L \rtimes H$ and

$\tilde{M} := M \oplus \mathbb{Z}$. Then we denote by N (resp., $\tilde{N}$) the dual of M (resp., $\tilde{M}$). Let $\langle -, - \rangle : \tilde{M} \times \tilde{N} \to \mathbb{Z}$ be *the canonical pairing*.

Now we define the action of Γ on $\tilde{N} = N \oplus \mathbb{Z}$ via

$$S_{(l,h)}((n,s)) := (n \circ h + sb(l,-), s),$$

as in [Kün98, p.181]. As we will now explain, this action reflects the natural action of Γ on $T_\eta = \mathrm{Spec}K[M]$, where T is a split torus part of $\tilde{G}$. At first, we identify $\tilde{m} = (m,k) \in \tilde{M}$ with $t^k X^m \in K[M]$. In the proof of [Kün98, Lemma 3.7], the action of L on $T_\eta = \mathrm{Spec}K[M]$ induced by the natural action of T_η is defined as follows:

$$l : \tilde{M} \to \tilde{M}, \quad (m,s) \mapsto (m, b(l,m) + s).$$

We can easily verify that this action is dual to the action $S_{(l,\mathrm{Id})}$ in the sense of $\langle l \cdot \tilde{m}, \tilde{n} \rangle = \langle \tilde{m}, S_{(l,\mathrm{Id})}(\tilde{n}) \rangle$. In the same way, we can easily check that the action of $h \in H$ on T is dual to $S_{(0,h)}$. Hence, the action of $\gamma \in \Gamma$ on T corresponds to S_γ on $\tilde{N}$.

In addition, we consider the function $\chi : \Gamma \times \tilde{N}_{\mathbb{R}} \to \mathbb{R}$ defined by

$$\chi((l,h),(n,s)) = sa(l) + n \circ \phi \circ h^{-1}(l)$$

as in [Kün98, p.181].

In $\tilde{N}_{\mathbb{R}} = N_{\mathbb{R}} \oplus \mathbb{R}$, we have the cone $\mathscr{C} := (N_{\mathbb{R}} \oplus \mathbb{R}_{>0}) \cup \{0\}$. The cone $\mathscr{C}$ is stable under the action of Γ. We shall consider a smooth Γ-admissible rational polyhedral cone decomposition $\Sigma := \{\sigma_\alpha\}_{\alpha \in I}$ which admits a Γ-admissible κ-twisted polarization function $\varphi \colon \mathscr{C} = \bigcup_{\alpha \in I} \sigma_\alpha \to \mathbb{R}$ for some $\kappa \in \mathbb{N}$. Let us take a moment to recall these definitions.

Definition 2.43. A rational polyhedral cone decomposition $\Sigma := \{\sigma_\alpha\}_{\alpha \in I}$ of $\mathscr{C}$ is called Γ-*admissible* if the action of Γ causes the bijections from I to itself (that is, the decomposition Σ invariant under the action of Γ) and we can take a system of finitely many representatives $\{\sigma_\alpha\}$ for the action of Γ (that is, there are at most finitely many orbits).

A function $\varphi \colon \mathscr{C} = \bigcup_{\alpha \in I} \sigma_\alpha \to \mathbb{R}$ is called *polarization function* associated with Σ if it satisfies the following properties:

- φ is continuous function that satisfies $\varphi(\tilde{N} \cap \mathscr{C}) \subset \mathbb{Z}$
- $\varphi(rx) = r\varphi(x)$, for any $r \in \mathbb{R}_{\geq 0}$
- The restriction $\varphi|_{\sigma_\alpha}$ to each cone σ_α is a linear function
- φ is strictly convex function for Σ. That is, for any $\sigma \in \Sigma$, there exists $r \in \mathbb{N}$ and $\tilde{m} \in \tilde{M}$ such that $\langle \tilde{m}, \tilde{n} \rangle \geq r\varphi(\tilde{n})$ for all $\tilde{n} \in \mathscr{C}$ and $\sigma = \{\tilde{n} \in \mathscr{C} \mid \langle \tilde{m}, \tilde{n} \rangle = r\varphi(\tilde{n})\}$

A polarization function $\varphi \colon \mathscr{C} \to \mathbb{R}$ is called κ-*twisted* Γ-*admissible* for some $\kappa \in \mathbb{N}$ if it satisfies $\varphi(x) - \varphi \circ S_\gamma(x) = \kappa\chi(\gamma, x)$ for all $\gamma \in \Gamma$, $x \in \mathscr{C}$.

When κ is not important, it is often referred to as Γ-*admissible polarization* for short.

We denote by $I^d \subset I$ the set of the indices corresponding to the d-dimensional cones of Σ. We set $I^+ := \bigcup_{d>0} I^d$. Since Σ is Γ-admissible, the group Γ acts on each I^d. Overkamp combines various Theorems and Propositions in [Kün98] into the following result [Ove21, Theorem 2.2]:

Theorem 2.44 ([Kün98], [Ove21, Theorem 2.2])**.** *We set a semiabelian degeneration* $(G, \mathscr{L}, \mathscr{M}) \in \mathbf{DEG}^{\mathrm{split}}_{\mathrm{ample}}$ *and assume that* H *acts on this object as* (2.41)*. We denote by* $\mathscr{A}$ *the Néron model of the Abelian variety* $A := G_\eta$*. Let* $(M, L, \phi, a, b) := \mathbf{For}(F((G, \mathscr{L}, \mathscr{M})))$ *and suppose we have a smooth* Γ*-admissible rational polyhedral cone decomposition* $\Sigma := \{\sigma_\alpha\}_{\alpha \in I}$ *of* $\mathscr{C} \subset \tilde{N}_{\mathbb{R}}$*. Furthermore, we assume that this decomposition* Σ *has the following properties:*

(a) There exists a κ*-twisted* Γ*-admissible polarization function* φ *for the decomposition* Σ*.*
(b) The decomposition Σ *is* semistable*. That is, the primitive element of any one-dimensional cone of the decomposition* Σ *is of the form* $(n, 1)$ *for some* $n \in N$*.*
(c) The cone $\sigma_T = \{0\} \times \mathbf{R}_{\geq 0}$ *is contained in the decomposition* Σ*.*
(d) For all $l \in L \backslash \{0\}$ *and* $\alpha \in I$*, it holds that*

$$\sigma_\alpha \cap S_{(l,\mathrm{Id})}(\sigma_\alpha) = \{0\}.$$

Then there exists a projective snc model $\mathscr{P}$ *of* A *over* S *associated to* Σ *and a line bundle* $\mathscr{L}_{\mathscr{P}}$ *such that the following holds:*

(i) The canonical morphism $\mathscr{P}^{\mathrm{sm}} \to \mathscr{A}$ *is an isomorphism.*
(ii) The action of H *on* $G = \mathscr{A}^0$ *extends uniquely to* $\mathscr{P}$*, and the restriction of* $\mathscr{L}_{\mathscr{P}}$ *to* G *is isomorphic to* $\mathscr{L}^{\otimes \kappa}$*, where* $\mathscr{A}^0$ *means the identity component of* $\mathscr{A}$*.*
(iii) Let I^+_L *be the set of orbits* $I^+_L := I^+/L$*. Then the reduced special fiber of* $\mathscr{P}$ *has a stratification indexed by* I^+_L*. This stratification is preserved by the action of* H*, and the induced action of* H *on the set of strata is determined by the action of* H *on* I^+_L*.*
(iv) The strata corresponding to one-dimensional cones are smooth over k*.*

2.45. Let us discuss $\mathscr{P}$, which appears in Theorem 2.44. For each cone $\sigma \in \Sigma$, we define the affine scheme $U_\sigma := \mathrm{Spec} R[\sigma^\vee \cap \tilde{M}]$, where $\sigma^\vee := \mathrm{Hom}_{\mathrm{monoid}}(\sigma, \mathbb{R}_{\geq 0})$ and we identify $\tilde{m} = (m, k) \in \tilde{M}$ with $t^k X^m \in K[M]$. Then we can define $\tilde{\mathscr{P}}$ by gluing these U_σ together as in [Kün98, 1.13]. In particular, we obtain the toroidal embedding $T_\eta = \mathrm{Spec} k[M] \hookrightarrow \tilde{\mathscr{P}}$ as in *loc.cit*. This $\tilde{\mathscr{P}}$ is called the *toroidal compactification* of $T_\eta = \mathrm{Spec} K[M]$

over R associated with Σ. Further, the cone σ_T induces the embedding $T_\eta \hookrightarrow T = U_{\sigma_T} = \mathrm{Spec} R[M]$. It implies that the toroidal embedding $T_\eta \hookrightarrow \tilde{\mathscr{P}}$ extends to a T-equivariant embedding $T \hookrightarrow \tilde{\mathscr{P}}$. The special fiber of $\tilde{\mathscr{P}}$ is a reduced divisor with strict normal crossings on $\tilde{\mathscr{P}}$ and has a stratification indexed by I^+.

If $(G, \mathscr{L}, \mathscr{M}) \in \mathbf{DEG}^{\mathrm{split}}_{\mathrm{ample}}$ *is maximally dagenerated, then the above* $\mathscr{P}$ *of Theorem 2.44 satisfies* $\mathscr{P}_{\mathbf{for}} \cong \tilde{\mathscr{P}}_{\mathbf{for}}/L$. Then, this $\tilde{\mathscr{P}}$ is also called a relatively complete model as in [Mum72]. In general, the above $\mathscr{P}$ is constructed by taking a contraction product $\tilde{G} \times^T \tilde{\mathscr{P}}$, which we do not use in this thesis. See [Kün98, §3.6] for the details.

2.46. In [HN17, Theorem 5.1.6], they proved this $\mathscr{P}$ is a Kulikov model of A (cf. [Ove21, Corollary 2.8]).

2.47. For the tuple $(M, L, \phi, a, b) := \mathbf{For}(F((G, \mathscr{L}, \mathscr{M})))$, b gives the injective homomorphism $\tilde{b} : L \to N = M^\vee$ defined by $\tilde{b}(l) = b(l, -)$. We identify L with $\tilde{b}(L)$. That is, we regard L as the sublattice of N. As we see before, Γ act on $\tilde{N}$ as follows:

$$S_{(l,h)}((n, s)) = (n \circ h + s\tilde{b}(l), s).$$

In particular,

$$S_{(l,h)}((n, 1)) = (n \circ h + \tilde{b}(l), 1).$$

2.48. Künnemann proved the existence of the cone decomposition Σ which satisfies the assumption of Theorem 2.44 as follows:

Proposition 2.49 ([Kün98, Proposition 3.3 and Theorem 4.7])**.** *We set the tuple* $(G, \mathscr{L}, \mathscr{M}) \in \mathbf{DEG}^{\mathrm{split}}_{\mathrm{ample}}$ *, and assume that the finite group* H *acts on this object. Let* $(M, L, \phi, a, b) := \mathbf{For}(F((G, \mathscr{L}, \mathscr{M})))$. *After taking a base change along* $f : S' \to S$ *as in* (2.40) *if necessary, there exists a smooth rational polyhedral cone decomposition* $\Sigma := \{\sigma_\alpha\}_{\alpha \in I}$ *which has the properties (a)-(d) listed in Theorem 2.44.*

2.50. Now we recall Künnemann's proof of the above Proposition 2.49. Please refer to *loc.cit.* for more details. We consider the function $\varphi : \mathscr{C} \to \mathbb{R}$ defined by

$$\tilde{n} \mapsto \min_{l \in L} \chi(l, \tilde{n}),$$

where $\chi(l, \tilde{n})$ means $\chi((l, \mathrm{Id}), \tilde{n})$. This φ gives the decomposition $\Sigma = \{\sigma_\alpha\}$ defined by

$$\sigma_\alpha = \{\tilde{n} \in \mathscr{C} \mid \varphi(\tilde{n}) = \chi(\alpha_i, \tilde{n}) \ {}^\forall \alpha_i \in \alpha\},$$

where $\alpha = \{\alpha_i\}$ is a finite set of L. Then φ is a 1-twisted polarization function associated with this Σ as in [Kün98, Proposition 3.2]. In particular, it holds that $S_{(l,h)}(\sigma_\alpha) = \sigma_{h(\alpha)-l}$. Now we consider the cone

$$\sigma_{\{0\}} = \{\tilde{n} \in \mathscr{C} \mid \varphi(\tilde{n}) = \chi(0, \tilde{n}) = 0\}.$$

It is clear that $\mathscr{C} = \bigcup S_l(\sigma_{\{0\}})$. We will now show that the desired subdivision can be obtained in three steps as follows:

First step : For this cone $\sigma_{\{0\}}$, we can subdivide it and obtain an H-invariant finite cone decomposition $\{\tau_\beta\}$ of $\sigma_{\{0\}}$ such that each cone τ_β is a simplex and the stabilizer of τ_β in H acts trivially on τ_β. Further we can subdivide the whole Σ by transporting the above subdivision on $\sigma_{\{0\}}$ via L-action on $\mathscr{C}$ and obtain an H-invariant cone decomposition $\{\tau_\alpha\}$ of $\mathscr{C}$. In addition, we can modify the polarization function φ and obtain a 1-twisted polarization function for this subdivision $\{\tau_\alpha\}$ after replacing K by a finite extension.

Second step : We choose a system $\{\tau_1, ..., \tau_n\}$ of representatives for the action of Γ on the decomposition $\{\tau_\alpha\}$. According to [KKMS73, I.2, proof of Theorem 11], *for any subdivision* Σ_i *of each* τ_i, there is a subdivision of the subdivision Σ_i such that it has a κ-twisted polarization function on τ_i for sufficiently large $\kappa \in \mathbb{N}$. In the same way as above, we can extend these subdivisions to the whole via L-action. Further, we can modify the polarization function on $\mathscr{C}$ and obtain a κ-twisted polarization function for this subdivision Σ' after replacing K by a finite extension. Hence, we consider a subdivision that satisfies (c), (d) to obtain a subdivision that satisfies (a), (c), (d).

Third step : We choose a system $\{\tau_1, ..., \tau_n\}$ of representatives for the action of Γ on the decomposition Σ'. By using the semistable reduction theorem [KKMS73, II.2, proof of Theorem11], we can subdivide each τ_i so that the resulting decomposition Σ'' is smooth. In addition, we can obtain a κ'-twisted polarization function for this subdivision Σ'' after replacing K by a finite extension. Hence, the desired decomposition is constructed. □

2.51. Let B be a topological space endowed with a simplicial complex structure. We denote by $\Sigma := \{\sigma_\alpha\}_{\alpha \in I}$ the set of all faces of B. Let σ° be the relative open set of $\sigma \in \Sigma$. We define the open star $\mathrm{Star}(\sigma_\alpha)$ of $\sigma_\alpha \in \Sigma$ as follows:

$$\mathrm{Star}(\sigma_\alpha) := \bigcup_{\beta \succ \alpha} \sigma_\beta^\circ,$$

where $\beta \succ \alpha$ means that σ_α is a face of σ_β. Then $\mathrm{Star}(\sigma)$ is an open set of B. In particular, $\{\mathrm{Star}(\sigma_\alpha)\}_{\alpha \in I}$ is a open cover of B.

2.52. The decomposition $\Sigma := \{\sigma_\alpha\}_{\alpha \in I}$ of $\mathscr{C}$ as Theorem 2.44 gives the smooth rational polyhedral decomposition $\overline{\Sigma}$ in $N_\mathbb{R}$ obtained by intersectiong the cones in Σ with $N_\mathbb{R} \times \{1\}$. Let $\overline{\sigma}_\alpha \in \overline{\Sigma}$ be the intersection of σ_α with $N_\mathbb{R} \times \{1\}$. Then this decomposition $\overline{\Sigma} = \{\overline{\sigma}_\alpha\}_{\alpha \in I}$ gives a simplicial complex structure to $N_{\mathbb{R}}$. Moreover the dual intersection complex $\Delta(\tilde{\mathscr{P}})$ of $\tilde{\mathscr{P}}_k$ coincides with $\overline{\Sigma}$ as we see in (2.45). Theorem 2.44 implies that the dual intersection complex $\Delta(\mathscr{P})$ of $\mathscr{P}_k$ has the simplicial complex structure of $\overline{\Sigma}/L := \{\overline{\sigma}_\alpha\}_{\alpha \in I_L^+}$.

2.53. To make it easier to see the covering map, which is the key to Theorem 5.22 as we will look at later, we refine Proposition 2.49.

Lemma 2.54. *Let F be the fixed locus of H-action on $N_{\mathbb{R}}/L$. Then $N_{\mathbb{R}}/L$ has a simplicial complex structure such that any 0-vertex of $N_{\mathbb{R}}/L$ is included in $N_{\mathbb{Q}}/L$ and F is a compact sub simplicial complex.*

Proof. It follows from Proposition 2.49 that $N_{\mathbb{R}}/L$ has a simplicial complex structure such that the canonical projection $N_{\mathbb{R}} \twoheadrightarrow N_{\mathbb{R}}/L$ is a simplicial map and any 0-simplex in $N_{\mathbb{R}}/L$ is included in $N_{\mathbb{Q}}/L$. We show that F is compact and has a simplicial complex structure such that any 0-simplex of F is included in $N_{\mathbb{Q}}/L$. Note that F might not be connected. Then the assertion follows from [RS82, 2.12 Addendum].

From now on, we prove the claim. By definition, F is denoted as follows:

$$F = \bigcup_{h \in H \setminus \{0\}} F_h,$$

where $F_h := \{x \in N_{\mathbb{R}}/L \mid x = h(x)\}$. Under the setting as we considered in (2.42), H acts on $N_{\mathbb{R}}/L$ via $H \to (\mathrm{GL}(L) \ltimes L) \cap (\mathrm{GL}(N) \ltimes N)$. In other words, h is determined by $\tilde{h} \in (\mathrm{GL}(L) \ltimes L) \cap (\mathrm{GL}(N) \ltimes N)$ via the canonical projection $N_{\mathbb{R}} \to N_{\mathbb{R}}/L$. Here, we identify h with $\tilde{h}$. In particular, denote $h : N_{\mathbb{R}} \to N_{\mathbb{R}}$ by $h(y) = Ay + b$, where $A \in \mathrm{GL}(L) \cap \mathrm{GL}(N)$ and $b \in L (\subset N)$. Set $g : N_{\mathbb{R}} \to N_{\mathbb{R}}$ as $g(y) := h(y) - y$. For any $y_1, y_2 \in N_{\mathbb{R}}$, $g(y_1 + y_2) = g(y_1) + Ay_2 - y_2$ holds. It implies that $g(y) \in L$ is equivalent to $g(y + a) \in L$ for some $a \in L$. Taka a basis $\{l_i\}$ of L in $N_{\mathbb{R}}$ and set a fundamental domain D of $N_{\mathbb{R}}/L$ as follows:

$$D := \sum_{i=1}^{\dim N} [0,1] \cdot l_i \subset N_{\mathbb{R}}.$$

Then it holds that

$$F_h' := \{y \in D \mid g(y) \in L\} \twoheadrightarrow F_h$$

by the canonical projection $N_{\mathbb{R}} \to N_{\mathbb{R}}/L$. Since L is discrete in $N_{\mathbb{R}}$ and $g(D)$ is compact, $V := g(D) \cap L$ is a finite set. Here, we can denote F_h' by

$$F_h' = \bigcup_{v \in V} F_{h,v}',$$

where $F'_{h,v} := \{y \in D \mid g(y) = v\}$. Since g is an integral affine map (or $g \in \mathrm{Hom}(N, N) \ltimes N$), $F'_{h,v}$ is a closed set of a subaffine space in $N_{\mathbb{R}}$ with rational slopes for the coordinates of $N_{\mathbb{R}}$ containing some point in $N_{\mathbb{Q}}$. It implies that $F'_{h,v}$ has a simplicial structure such that each vertex of $F'_{h,v}$ is in $N_{\mathbb{Q}}$. In particular, the inclusion $F'_{h,v} \to N_{\mathbb{R}}$ is a piecewise linear map. Since the canonical projection $N_{\mathbb{R}} \twoheadrightarrow N_{\mathbb{R}}/L$ is a simplicial map, the composition $F'_{h,v} \to N_{\mathbb{R}} \twoheadrightarrow N_{\mathbb{R}}/L$ is also a piecewise linear map. By [RS82, 2.14 Theorem], there are subdivisions of $N_{\mathbb{R}}/L$ and $F'_{h,v}$ such that $F'_{h,v} \to N_{\mathbb{R}}/L$ is a simplicial map and any 0-simplex of the subdivision of $N_{\mathbb{R}}/L$ is included in $N_{\mathbb{Q}}/L$. Hence, the image $F_{h,v}$ of $F'_{h,v}$ has a simplicial structure induced by the simplicial map $F'_{h,v} \to N_{\mathbb{R}}/L$. Here, any intersection (as a cell complex) between two cell complexes is also a cell complex. Since $F'_{h,v}$ is an intersection of D and some subaffine space in $N_{\mathbb{R}}$ with rational slopes intersecting $N_{\mathbb{Q}}$, for any $F_{h,v}$ and $F_{h',v'}$, it holds that any 0-cell of the cell complex of the intersection is included in $N_{\mathbb{Q}}/L$. Hence, the union of $F_{h,v}$ and $F_{h',v'}$ is a cell complex such that all 0-cells are in $N_{\mathbb{Q}}/L$ by gluing together along the intersection cell complex. It is well-known that any cell complex can be subdivided to a simplicial complex without introducing any new vertices. That is, the union of $F_{h,v}$ and $F_{h',v'}$ has a simplicial complex structure such that all 0-simplexes are in $N_{\mathbb{Q}}/L$. Since F is a finite union of simplicial complexes, more precisely

$$F = \bigcup_{h \in H \setminus \{e\}} \bigcup_{v \in V} F_{h,v},$$

then it follows inductively that F has a simplicial structure such that any vertex of F is included in $N_{\mathbb{Q}}/L$. Besides, since $F_{h,v}$ is compact, so is F. Hence, the claim follows. □

Lemma 2.55. *Let F be the fixed locus of H-action on $N_{\mathbb{R}}/L$. Let $\tilde{F}$ be the inverse image of the fixed locus F by the quotient map $N_{\mathbb{R}} \to N_{\mathbb{R}}/L$. After taking a base change along $f : S' \to S$ as in* (2.40) *if necessary, there exists a smooth rational polyhedral cone decomposition $\Sigma = \{\sigma_\alpha\}_{\alpha \in I}$ which has not only the properties (a)-(d) listed in Theorem 2.44 but also the following (e)-(g).*

- *(e) For all $l \in L \setminus \{0\}$ and $\alpha \in I_+$, we have*
$$\mathrm{Star}(\overline{\sigma}_\alpha) \cap S_{(l,\mathrm{Id})}(\mathrm{Star}(\overline{\sigma}_\alpha)) = \emptyset.$$
- *(f) F has a simplicial structure $\mathscr{F}$ such that F is a subcomplex of the complex $\overline{\Sigma}/L$ as appeared in* (2.52)*. In particular, for any simplex τ of the induced simplicial structure $\tilde{\mathscr{F}}$ on $\tilde{F}$, a cone generated by $(\tau, 1) \subset \tilde{N}_{\mathbb{R}} := N_{\mathbb{R}} \times \mathbb{R}$ corresponds to some index in I. We denote by $I_{\mathrm{sing}} \subset I_+$ the set of indices corresponding to $\tilde{\mathscr{F}}$.*

(g) For all $\gamma \in \Gamma\backslash\{0\}$ *and* $\alpha \in I_+ \setminus I_{\mathrm{sing}}$*, we have*

$$\mathrm{Star}(\overline{\sigma}_\alpha) \cap S_\gamma(\mathrm{Star}(\overline{\sigma}_\alpha)) = \emptyset.$$

Proof. It follows from Proposition 2.49 that there is a smooth rational polyhedral cone decomposition Σ which satisfies the conditions (a)-(d) after replacing K by a finite extension. Then we refine Σ to obtain a desired decomposition as follows: In the second step of (2.50), we consider a subdivision that satisfies (e), (f), (g). Note that each stabilizer of H on each $\tau \in \Sigma$ acts trivially on the cone τ and L acts on τ by transporting via $\tilde{b}(L)$. Then it is easily verified that such subdivisions exist by Lemma 2.54. Afterward, we apply the third step of (2.50) to this decomposition. Then the resulting decomposition is a desired one. □

Example 2.56. If $H = \{\pm 1\}$, then $\tilde{F} = \frac{1}{2}L$ and $F = \frac{1}{2}L/L$. In particular, it holds that $|F| = 2^{\dim N}$. Further, $|F/H| = 2^{\dim N}$ follows.

2.57. For the rest of this section, we assume that *the residue field* k *of* R *is of characteristic* $p \neq 2$, We set that $H = \{\pm 1\}$ and the action of H on M is determined by $-1 : m \mapsto -m$. In particular, $H = \{\pm 1\}$ also acts on $N = M^\vee$ by $-1 : n \mapsto -n$.

2.58. Let $\mathscr{P}$ be the projective model of A and $\mathscr{A}$ be the Néron model of A as Theorem 2.44. For an abelian variety Z, we denote by $Z[2]$ the 2-torsion of Z, that is the kernel of the morphism $[2] : Z \to Z$ defined by $x \mapsto 2x$. After replacing K by finite extension, we may assume that $A[2]$ is constant over K without loss of generality. Overkamp proved this $\mathscr{A}[2]$ coincides with the fixed locus of the action of H on $\mathscr{P}$ when A is of 2-dimensional [Ove21, Theorem 3.7]. Then the action of $H = \{\pm 1\}$ on $\mathscr{P}$ extends to the blow-up $\tilde{\mathscr{X}} := \mathrm{Bl}_{\mathscr{A}[2]}\mathscr{P}$ along the closed subscheme $\mathscr{A}[2]$. Hence we obtain $\mathscr{X} := \tilde{\mathscr{X}}/H$. Let X be the Kummer surface associated with A. Overkamp proves this $\mathscr{X}$ is a Kulikov model of X [Ove21, Theorem 3.12].

2.59. We fix the same notation as (2.52) and (2.58). The dual intersection complex $\Delta(\tilde{\mathscr{X}})$ of $\tilde{\mathscr{X}}_k$ has the same stratification as the dual intersection complex $\Delta(\mathscr{P})$ of $\mathscr{P}_k$. Indeed, Overkamp proved that the special fiber $\tilde{\mathscr{X}}$ is $\mathrm{Bl}_{\mathscr{A}_k[2]}\mathscr{P}_k$ [Ove21, Lemma 3.10] and $\mathscr{A}_k[2]$ is a finite set lying on top dimensional strata of $\mathscr{P}_k$ [Ove21, Lemma 3.6]. We can also check the latter by using Lemma 2.55. Hence, the blow-up along $\mathscr{A}_k[2]$ does not change the dual intersection complex. It implies that $\Delta(\mathscr{P}) \cong \Delta(\tilde{\mathscr{X}})$ as simplicial complexes.

We denote by I_Γ^+ the set of orbits $I_\Gamma^+ := I^+/\Gamma$. Theorem 2.44 says that H acts on $\Delta(\mathscr{P}) \cong \Delta(\tilde{\mathscr{X}})$ preserving the simplicial complex structure. It implies that the map $\Delta(\tilde{\mathscr{X}}) \twoheadrightarrow \Delta(\mathscr{X})$ is double branched cover as simplicial complexes. The dual intersection complex $\Delta(\mathscr{X})$ of $\mathscr{X}_k$ has a stratification indexed by I_Γ^+. In particular, $\Delta(\mathscr{X})$ has the simplicial complex structure of $\overline{\Sigma}/\Gamma := \{\overline{\sigma}_\alpha\}_{\alpha \in I_\Gamma^+}$.

3. Basic Theory of Berkovich Analytic Spaces

3.1. **Definitions and basic properties.**

This section is essentially based on [Ber90, §1, §2, §3], and its structure is based on [Got20, §2 and §3]. In particular, a part of our original results proved in [Got20] is discussed in the latter half of this section.

Definition 3.1. Let A be a commutative ring with identity 1. A *seminorm* on A is a function $|\cdot| : A \to \mathbb{R}_{\geq 0}$ possessing the following properties:

(i) $|0| = 0$,
(ii) $|1| \leq 1$,
(iii) $|f - g| \leq |f| + |g|$,
(iv) $|fg| \leq |f||g|$,

for all $f, g \in A$. Furthermorea seminorm $|\cdot|$ on A is called

- a *norm* if the equality $|f| = 0$ implies $f = 0$.
- *non-Archimedean* if $|f - g| \leq \max\{|f|, |g|\}$ for all $f, g \in A$.
- *multiplicative* if $|1| = 1$ and $|fg| = |f||g|$ for all $f, g \in A$.

3.2. For each seminorm $|\cdot|$ on A, the following are equivalent:

- $|\cdot|$ is a norm on A.
- The induced topology on A is Hausdorff.

Definition 3.3. A *Banach ring* $\mathscr{A} = (\mathscr{A}, ||\cdot||)$ is a normed ring $\mathscr{A}$ that is complete with respect to its norm $||\cdot||$.

3.4. We can consider any ring A as a Banach ring by the *trivial norm* $|\cdot|_0$ defined as below.

For each $f \in A$,

$$|f|_0 := \begin{cases} 1 & (\text{if } f \neq 0), \\ 0 & (\text{if } f = 0). \end{cases}$$

The trivial norm is a non-Archimedean norm. Moreover, it is clear that the normed ring $(A, |\cdot|_0)$ is complete. In other words, $(A, |\cdot|_0)$ is a Banach ring. Further, when A is a domain, the norm is multiplicative.

3.5. A norm $|\cdot|$ is called a *valuation* if it is multiplicative. By the argument in (3.4), for any field K, the trivial norm $|\cdot|_0$ is a valuation. Then $(K, |\cdot|_0)$ is called a *trivially valued field*.

3.6. Let $|\cdot|_\infty$ be the usual absolute value on $\mathbb{C}$. Then the *hybrid norm* $|\cdot|_{\mathrm{hyb}}$ on $\mathbb{C}$ is defined by

$$z \mapsto |z|_{\mathrm{hyb}} := \max\{|z|_\infty, |z|_0\}.$$

As the name implies, the map $|\cdot|_{\mathrm{hyb}}$ is a norm on $\mathbb{C}$. However, $|\cdot|_{\mathrm{hyb}}$ is neither multiplicative nor non-Archimedean.

3.7. We recall the definition of DVR. R is called a *DVR* if R is a noetherian local domain of dimension 1 with the principally maximal ideal $\mathfrak{m} = (t)$. Then we obtain the map $|\cdot| : R \to \mathbb{R}$ defined by

$$|f| := \inf\{e^{-n} \mid f \in \mathfrak{m}^n\}.$$

It is obvious that the map $|\cdot|$ is a non-Archimedean valuation on R. The pair $(R, |\cdot|)$ is called a *complete DVR* if $(R, |\cdot|)$ is a Banach ring, that is, the induced topology on R is complete.

Definition 3.8. Let K be a field.

- $(K, ||\cdot||)$ is called a *Banach field* if $(K, ||\cdot||)$ is a Banach ring.
- $(K, ||\cdot||)$ is called a *complete valuation field* if $(K, ||\cdot||)$ is a Banach field whose norm is a valuation.
- $(K, ||\cdot||)$ is called a *non-Archimedean field* if $(K, ||\cdot||)$ is a complete valuation field whose norm is non-Archimedean.

Example 3.9. Any trivially valued field is a non-Archimedean field. If R is a complete DVR, then the fractional field $K = \mathrm{Frac}(R)$ is a non-Archimedean field. Such a K is called a *complete DVF*.

3.10. For any complete valuation field $K = (K, |\cdot|)$, the *value group* of K is defined by

$$|K^\times| := \{|f| \in \mathbb{R} \mid f \in K^\times (= K \setminus \{0\})\}.$$

Further, we set

$$\sqrt{|K^\times|} := \{a \in \mathbb{R}_{\geq 0} \mid a^n \in |K^\times| \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

Here, $|K^\times|$ can be considered as a $\mathbb{Z}$-module by taking the logarithm. In the same way, $\sqrt{|K^\times|}$ can be considered as a $\mathbb{Q}$-vector space. In addition, we can easily see that $\sqrt{|K^\times|} \cong |K^\times| \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 3.11. Let $(\mathscr{A}, ||\cdot||)$ be a Banach ring. A seminorm $|\cdot|$ on $\mathscr{A}$ is *bounded* if there exists $C > 0$ such that $|f| \leq C||f||$ for all $f \in \mathscr{A}$.

3.12. Let $(\mathscr{A}, ||\cdot||)$ be a Banach ring and I be an ideal of $\mathscr{A}$. Then the *residue seminorm* $|\cdot| : \mathscr{A}/I \to \mathbb{R}$ on $\mathscr{A}/I$ is defined as follows: For any $f \in \mathscr{A}/I$,

$$|f| := \inf\{||g|| \in \mathbb{R}_{\geq 0} \mid g \in \mathscr{A}, f = g + I \in \mathscr{A}/I\}.$$

We can easily see that the map is a seminorm. However, in general, it is not a norm. I is called *closed* if the residue seminorm is norm. If I is closed, the quotient $\mathscr{A}/I$ again becomes a Banach ring by the residue seminorm.

Definition 3.13. Let $(\mathscr{A}, ||\cdot||_{\mathscr{A}})$ and $(\mathscr{B}, ||\cdot||_{\mathscr{B}})$ be Banach rings. A ring homomorphism $\varphi : \mathscr{A} \to \mathscr{B}$ is *bounded* if there exists $C > 0$ such that

$$||\varphi(f)||_{\mathscr{B}} \leq C||f||_{\mathscr{A}}$$

for each $f \in \mathscr{A}$. Further, a ring homomorphism $\varphi : \mathscr{A} \to \mathscr{B}$ is *admissible* if the residue seminorm of $\mathscr{A}/\ker\varphi$ is equivalent to the restriction of the norm $||\cdot||_{\mathscr{B}}$ to $\mathrm{Im}\varphi$ through $\mathscr{A}/\ker\varphi \cong \mathrm{Im}\varphi \subset \mathscr{B}$.

3.14. A bounded homomorphism is the most fundamental morphism between two Banach rings. An admissible homomorphism is a bounded homomorphism that holds the fundamental theorem on homomorphisms as Banach rings.

Definition 3.15. Let $\mathscr{A}$ be a commutative Banach ring with identity. The *spectrum* $\mathscr{M}(\mathscr{A})$ is the set of all bounded multiplicative seminorms on $\mathscr{A}$ provided with the weakest topology with respect to which all real valued functions on $\mathscr{M}(\mathscr{A})$ of the form $|\cdot| \mapsto |f|,\ f \in \mathscr{A}$, are continuous.

Theorem 3.16 ([Ber90, Theorem 1.2.1])**.** *Let $\mathscr{A}$ be a non-zero commutative Banach ring with identity. The spectrum $\mathscr{M}(\mathscr{A})$ is a nonempty, compact Hausdorff space.*

3.17. It is well-known that, if k is a complete valuation field, then $\mathscr{M}(k) = \{1\mathrm{pt}\}$. Let $|\cdot|_x$ be the multiplicative seminorm on $\mathscr{A}$ corresponding to $x \in \mathscr{M}(\mathscr{A})$. In the rest of this thesis, we will keep the notation $|\cdot|_x$. For any bounded homomorphism $\varphi : \mathscr{A} \to \mathscr{B}$, it induces the continuous map $\varphi^\sharp : \mathscr{M}(\mathscr{B}) \to \mathscr{M}(\mathscr{A})$ defined by $|f|_{\varphi^\sharp(x)} := |\varphi(f)|_x$ for all $f \in \mathscr{A}$ for each $x \in \mathscr{M}(\mathscr{B})$.

3.18. For each $x \in \mathscr{M}(\mathscr{A})$, we denote by $\mathfrak{p}_x$ the kernel of $|\cdot|_x$. This is a prime ideal of $\mathscr{A}$. Then the multiplicative seminorm $|\cdot|_x$ on $\mathscr{A}$ induces a valuation $|\cdot|_{\overline{x}}$ on $\mathscr{A}/\mathfrak{p}_x$ defined by

$$|\overline{f}|_{\overline{x}} := |f|_x$$

for each $f \in \mathscr{A}$. By abuse of language, we denote by x the induced valuation $\overline{x}$. The completion of the fraction field of $\mathscr{A}/\mathfrak{p}_x$ with respect to this valuation x is denoted by $\mathscr{H}(x)$. In this thesis, we call $\mathscr{H}(x)$ the *Berkovich residue field* of x although this is not a common way to call it. Further, we call $\dim_{\mathbb{Q}} \sqrt{|\mathscr{H}(x)^\times|}$ the *rational rank* of x.

3.19. For each $x \in \mathscr{M}(\mathscr{A})$, we obtain the Berkovich residue field $\mathscr{H}(x)$ as in (3.18). Now consider the valuation ring with respect to the valuation $|\cdot|_x$ on $\mathscr{H}(x)$ as follows:

$$\mathscr{H}(x)^\circ := \{f \in \mathscr{H}(x) \mid |f|_x \leq 1\}.$$

Then we obtain the residue field of $\mathscr{H}(x)^\circ$ as follows:

$$\widetilde{\mathscr{H}(x)} := \mathscr{H}(x)^\circ / \mathscr{H}(x)^{\circ\circ},$$

where $\mathscr{H}(x)^{\circ\circ}$ is the maximal ideal of $\mathscr{H}(x)^\circ$. Such a $\widetilde{\mathscr{H}(x)}$ is called the *double residue field* of $x \in \mathscr{M}(\mathscr{A})$.

From now on, we review the construction of Berkovich analytification X^{an} for any scheme X of locally finite type over a non-Archimedean field K. At first, we recall the Banach ring corresponding to a closed disc.

Definition 3.20. Let $(K, |\cdot|)$ be a non-Archimedean field.

For any $r_1, \ldots, r_n \in \mathbb{R}_{>0}$, we set:

$$K\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} := \{f = \sum_{I \in \mathbb{Z}_{\geq 0}^n} a_I T^I \mid a_I \in K, \limsup_{|I| \to \infty} |a_I| r^I = 0\},$$

where $|I| = i_1 + \cdots + i_n$, $T^I = T_1^{i_1} \cdots T_n^{i_n}$ and $r^I = r_1^{i_1} \cdots r_n^{i_n}$ for each $I = (i_1, \ldots, i_n) \in \mathbb{Z}_{\geq 0}^n$. For brevity, this algebra will also be denoted by $K\{r^{-1}T\}$.

3.21. $K\{r^{-1}T\}$ is a commutative Banach ring with respect to the valuation

$$||f|| = \max_I |a_I| r^I.$$

If $r_i = 1$ for all i, then the valuation is called the *Gauss norm*. Further, $K\{T\}$ is called the *Tate algebra* over K. Now consider $E(0, r) := \mathscr{M}(K\{r^{-1}T\})$. The spectrum $E(0, r)$ can be considered as an analog of the complex closed disc at the origin with radii $r = (r_1, \ldots, r_n)$.

Example 3.22. Let us assume that the valuation on K is trivial. If $r_i \geq 1$ for all $1 \leq i \leq n$, then $K\{r^{-1}T\}$ coincides with the polynomial ring $K[T_1, \ldots, T_n]$. On the other hands, if $r_i < 1$ for all $1 \leq i \leq n$, then $K\{r^{-1}T\}$ coincides with the ring of formal power series $K[[T_1, \ldots, T_n]]$.

Definition 3.23. Let $\mathscr{A}$ be a Banach ring. $\mathscr{A}$ is a *Banach K-algebra* if there is a $K \hookrightarrow \mathscr{A}$ is admissible injective.

Example 3.24. Now consider $K\{r^{-1}T\}$. By the definition of $K\{r^{-1}T\}$, it is obvious that there is the natural admissible injection $K \hookrightarrow K\{r^{-1}T\}$. That is, $K\{r^{-1}T\}$ is a Banach K-algebra.

Definition 3.25. Let $\mathscr{A}$ be a Banach K-algebra. $\mathscr{A}$ is called a *K-affinoid* algebra (resp., *strictly K-affinoid algebra*) if there exists an admissible surjective K-homomorphism $K\{r^{-1}T\} \twoheadrightarrow \mathscr{A}$ (resp., $K\{T\} \twoheadrightarrow \mathscr{A}$). Further, X is called a (resp., *strictly*) *K-affinoid space* if $X = \mathscr{M}(\mathscr{A})$ for some (resp., *strictly*) K-affinoid algebra $\mathscr{A}$.

Example 3.26. By definition, $K\{r^{-1}T\}$ is a K-affinoid algebra. It implies that $E(0, r) = \mathscr{M}(K\{r^{-1}T\})$ is a K-affinoid space.

The following is important.

Fact 3.27 ([Ber90, Proposition 2.1.3])**.** *Any K-affinoid algebra is noetherian and all of its ideals are closed.*

Definition 3.28. Let $(\mathscr{A}, |\cdot|)$ be a K-affinoid algebra. For any $r_1, \ldots, r_n \in \mathbb{R}_{>0}$, we set:

$$\mathscr{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} := \{f = \sum_{I \in \mathbb{Z}^n_{\geq 0}} a_I T^I \mid a_I \in \mathscr{A}, \limsup_{|I| \to \infty} |a_I| r^I = 0\},$$

where$|I| = i_1 + \cdots + i_n$, $T^I = T_1^{i_1} \cdots T_n^{i_n}$ and $r^I = r_1^{i_1} \cdots r_n^{i_n}$ for each $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$. This is a commutative Banach ring with respect to the valuation $||f|| = \max_I |a_I| r^I$. For brevity, this algebra will also be denoted by $\mathscr{A}\{r^{-1}T\}$.

3.29. By the definition, there is the natural inclusion $\mathscr{A} \to \mathscr{A}\{r^{-1}T\}$. It is obvious that $\mathscr{A}\{r^{-1}T\}$ is K-affinoid algebra.

Definition 3.30. Let $\mathscr{A}$ be a K-affinoid algebra. Take $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_m)$ for some $f_i, g_j \in \mathscr{A}$. Further, take $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_m)$ for some $p_i, q_j \in \mathbb{R}_{>0}$. Then $\mathscr{A}\{p^{-1}f, qg^{-1}\}$ is defined to be a K-affinoid algebra of the form

$$\mathscr{A}\{p^{-1}T, qS\}/(T_1 - f_1, \ldots, T_n - f_n, g_1 S_1 - 1, \ldots, g_m S_m - 1).$$

3.31. It follows from Fact 3.27 that any quotient of a K-affinoid algebra is also a K-affinoid algebra with the residue seminorm. In particular, it implies that $\mathscr{A}\{p^{-1}f, qg^{-1}\}$ is K-affinoid.

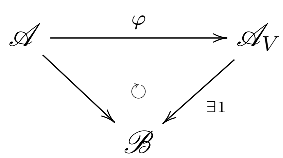
The natural morphism $\mathscr{A} \to \mathscr{A}\{p^{-1}f, qg^{-1}\}$ induces the closed immersion $\mathscr{M}(\mathscr{A}\{p^{-1}f, qg^{-1}\}) \hookrightarrow \mathscr{M}(\mathscr{A})$. If we set $X = \mathscr{M}(\mathscr{A})$, then the image of the closed immersion $\mathscr{M}(\mathscr{A}\{p^{-1}f, qg^{-1}\}) \hookrightarrow \mathscr{M}(\mathscr{A})$ coincides with the closed set

$$X\{p^{-1}f, qg^{-1}\} = \{x \in X \mid |f_i|_x \leq p_i, |g_j|_x \geq q_j, 1 \leq i \leq n, 1 \leq j \leq m\}.$$

In particular, it holds that $X\{p^{-1}f, qg^{-1}\} \cong \mathscr{M}(\mathscr{A}\{p^{-1}f, qg^{-1}\})$. Such an affinoid space of the form $X\{p^{-1}f, qg^{-1}\}$ is called a *Laurent domain* in X.

Fact 3.32 ([Ber90, § 2.2]). *Let X be a K-affinoid space. The Laurent neighborhoods of a point $x \in X$ form a basis of closed neighborhoods of x.*

Definition 3.33. A closed set $V \subset \mathscr{M}(\mathscr{A})$ is said to be a *K-affinoid domain* in X if there exists a bounded homomorphism of K-affinoid algebras $\varphi : \mathscr{A} \to \mathscr{A}_V$ satisfying the following universal mapping property: Given a bounded homomorphism of affinoid K-algebras $\mathscr{A} \to \mathscr{B}$ such that the image of $\mathscr{M}(\mathscr{B})$ in X lies in V, there exists a unique bounded homomorphism $\mathscr{A}_V \to \mathscr{B}$ making the following diagram commutative.



In particular, $\mathscr{A}_V$ is a K-affinoid flat $\mathscr{A}$-algebra. A K-affinoid domain V is called a *strictly K-affinoid domain* if the corresponding Banach K-algebra $\mathscr{A}_V$ is a strictly K-affinoid algebra.

Example 3.34. Any Laurent domain $X\{p^{-1}f, qg^{-1}\}$ in $X = \mathscr{M}(\mathscr{A})$ is an affinoid domain in X. In particular, it holds that

$$\mathscr{A}_{X\{p^{-1}f, qg^{-1}\}} \cong \mathscr{A}\{p^{-1}f, qg^{-1}\}.$$

3.35. A finite union of affinoid domains is called a *special subset* in X. For a special subset $V = \bigcup V_i$, where V_i is an affinoid domain in X, we set

$$\mathscr{A}_V := \ker\left(\prod \mathscr{A}_{V_i} \to \prod \mathscr{A}_{V_i \cap V_j}\right).$$

Then the Banach K-algebra $\mathscr{A}_V$ does not depend on the covering of V. By [Ber90, Corollary 2.2.6], V is an affinoid domain in X if and only if $\mathscr{A}_V$ is a K-affinoid algebra.

3.36. We considered K-affinoid space $X = \mathscr{M}(\mathscr{A})$ just as a topological space so far. However, X is also a locally ringed space. Actually, X has the structure sheaf $\mathcal{O}_X$ defined as follows: For an open set $U \subset X$, we set

$$\Gamma(U, \mathcal{O}_X) := \varprojlim \mathscr{A}_V,$$

where the limit is taken over all special subsets $V \subset U$. In particular, $\mathcal{O}_{X,x}$ is a local ring with the maximal ideal $\mathfrak{m}_x = \{f \in \mathcal{O}_{X,x} \mid |f(x)| = 0\}$.

Any open set $U \subset X = \mathscr{M}(\mathscr{A})$ is also a locally ringed space by the restriction of the structure sheaf $\mathcal{O}_X$. Such a $(U, \mathcal{O}_X|_U)$ is called a *K-quasiaffinoid space*.

A locally ringed space $(X, \mathcal{O}_X)$ is called a *K-analytic space* if there exists a covering $\{U_i\}$ of X such that each U_i is a K-quasiaffinoid space. More precisely, we have to care about morphisms between K-quasiaffinoid spaces. Please refer to [Ber90, §3.1] for more details.

3.37. A non-archimedean field K' is called a *non-Archimedean field over* K if K' is also a Banach K-algebra. For any K-affinoid algebra $\mathscr{A}$, the given norm on $\mathscr{A}$ can be extended to a norm on the tensor product $\mathscr{A} \otimes_K K'$. Then the *complete tensor product* $\mathscr{A}_{K'} := \mathscr{A} \hat{\otimes}_K K'$ is defined by the completion of $\mathscr{A} \otimes_K K'$ with respect to the induced norm. In particular, there is a natural morphism $\mathscr{A} \to \mathscr{A} \hat{\otimes}_K K'$. In other words, we obtain a natural morphism $\mathscr{M}(\mathscr{A}_{K'}) \to \mathscr{M}(\mathscr{A})$. Actually, it follows from [Ber90, Corollary 2.1.8] that $\mathscr{A}_{K'}$ is a K'-affinoid algebra. For any K-quasiaffinoid space $U \hookrightarrow \mathscr{M}(\mathscr{A})$, we consider the fiber product

$$U \hat{\otimes}_K K' := U \times_{\mathscr{M}(\mathscr{A})} \mathscr{M}(\mathscr{A}_{K'}).$$

Since $U \hat{\otimes}_K K' \hookrightarrow \mathscr{M}(\mathscr{A}_{K'})$ is an open immersion, the fiber product $U \hat{\otimes}_K K'$ is a K'-quasiaffinoid space. Hence, for any K-analytic space X, we obtain a K'-analytic space $X \hat{\otimes}_K K'$ as follows: For each K-quasiaffinoid space $U \subset X$, we obtain the K'-quasiaffinoid space $U \hat{\otimes}_K K'$. Then the desired K'-analytic space $X \hat{\otimes}_K K'$ is obtained by gluing together these K'-quasiaffinoid spaces $U \hat{\otimes}_K K'$.

Definition 3.38. Let X be a K-analytic space.

- X is called *separated* if X is Hausdorff.
- X is called *projective* if there is a closed immersion $X \hookrightarrow (\mathbb{P}^n_K)^{\mathrm{an}}$ for some $n \in \mathbb{N}$.
- X is called *smooth* if, for any non-Archimedean field K' over K and any point $x' \in X' := X \hat{\otimes}_K K'$, the local ring $\mathcal{O}_{X',x'}$ is regular.
- X is called a *strictly K-analytic space* if any compact subset of X lies in a finite union of strictly K-affinoid domains.

3.39. It is time to construct the Berkovich analytification concretely. We now construct it in three steps.

At first, when $X = \mathbb{A}^n_K$, the *Berkovich analytification* of X is defined by

$$X^{\mathrm{an}} := \bigcup_{r \in \mathbb{R}^n_{>0}} E(0,r) = \bigcup_{r \in \mathbb{R}^n_{>0}} D(0,r),$$

where $D(0,r) = \{x \in E(0,r) \mid |T_i|_x < r_i, 1 \leq i \leq n\}$. $D(0,r)$ is a K-quasiaffinoid space as a open set in $E(0,r)$. The structure of X^{an} as K-analytic spaces is defined as follows: There is a natural open immersion $D(0,r) \hookrightarrow X^{\mathrm{an}}$ for each $r \in \mathbb{R}^n_{>0}$. It means that

$$\mathcal{O}_X|_{D(0,r)} := \mathcal{O}_{E(0,r)}|_{D(0,r)}.$$

Further, if $r_1 \geq r_2$ (that is, $r_1 - r_2 \in \mathbb{R}^n_{\geq 0}$), then these two open immersions $D(0,r_i) \hookrightarrow X^{\mathrm{an}}$ $(i = 1,2)$ are compatible with the natural open immersion

$D(0,r_2) \hookrightarrow D(0,r_1)$. That is, the following diagram is commutative.

$$\begin{array}{ccc} D(0,r_2) & \longrightarrow & D(0,r_1) \\ & \searrow \quad \circlearrowleft \quad \swarrow & \\ & X^{\mathrm{an}} & \end{array}$$

It means that $\mathcal{O}_X$ is well-defined.

Next, let A be a finitely generated K-algebra. That is, A can be rephrased as $K[T_1, \ldots, T_n]/I$ for some ideal $I \subset A$. When $X = \mathrm{Spec}A$, the *Berkovich analytification* of X is defined by

$$X^{\mathrm{an}} := \bigcup_{r \in \mathbb{R}^n_{>0}} \mathscr{M}(K\{r^{-1}T\}/I \cdot K\{r^{-1}T\}).$$

The structure sheaf of X^{an} is defined similarly.

Finally, let X be a locally algebraic scheme over K. Then the Berkovich analytification X^{an} is obtained by gluing together the K-analytic spaces U^{an} for each affine open set $U \subset X$.

Remark 3.40. Let A be a finitely generated K-algebra. From a set-theoretic point of view, the Berkovich analytification $(\mathrm{Spec}A)^{\mathrm{an}}$ of $\mathrm{Spec}A$ can be identified with the set of all multiplicative seminorms on A whose restrictions to K coincide with the given valuation on K.

Proposition 3.41. *Let X be a locally algebraic scheme over K as in* (3.39). *Then, for any $x \in X^{\mathrm{an}}$, there is a K-affinoid domain $V \subset X^{\mathrm{an}}$ such that $x \in V$.*

Proof. It follows from the construction of X^{an}. □

Remark 3.42. Of course, for any point x of a K-analytic space X, we can take some K-quasiaffinoid neighborhood of $x \in X$. However, it does not mean that we can take some K-affinoid neighborhood of x. Indeed, we sometimes cannot take any K-affinoid neighborhood at a point of a K-analytic space.

3.43. We can obtain the canonical continuous map $\pi_X : X^{\mathrm{an}} \to X$ defined as follows: For each affine open set $U \subset X$, where $U = \mathrm{Spec}A$, the restriction $\pi_X|_{U^{\mathrm{an}}}$ is defined by sending a multiplicative seminorm on A to its kernel that becomes a prime ideal of A.

The Berkovich analytification $X \mapsto X^{\mathrm{an}}$ satisfies many properties including GAGA type theorems. We list some properties below.

Fact 3.44 ([Ber90, § 3]). *Let X be a locally algebraic scheme over K. The following hold.*

- *X is connected if and only if X^{an} is arcwise connected.*
- *X is separated if and only if so is X^{an}.*
- *X is proper if and only if X^{an} is Hausdorff and compact.*
- *X is projective if and only if so is X^{an}.*
- *X is smooth if and only if so is X^{an}.*

Fact 3.45 ([Ber90, Proposition 3.3.23])**.** *If K is not a trivially valued field, any projective K-analytic space is given by the analytification of some K-variety.*

Fact 3.46 ([Ber90, § 3])**.** *Let K be a non-Archimedean field. For any morphism $\varphi : X \to Y$ between two locally algebraic schemes over K, there exists the morphism $\varphi^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ as K-analytic spaces such that the following diagram is commutative.*

$$\begin{array}{ccc} X^{\mathrm{an}} & \overset{\varphi^{\mathrm{an}}}{\dashrightarrow} & Y^{\mathrm{an}} \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{\varphi} & Y \end{array}$$

Fact 3.47 ([Ber90, § 3])**.** *Let φ and φ^{an} be morphisms as in Fact 3.46. The following hold.*

- *If φ is open immersion, then so is φ^{an}.*
- *If φ is closed immersion, then so is φ^{an}.*
- *If φ is surjective, then so is φ^{an}.*

From now on, we see general properties of the Berkovich residue field $\mathscr{H}(x)$ and the double residue field $\widetilde{\mathscr{H}(x)}$ for each point x of a K-analytic space.

Proposition 3.48. *Let $\mathscr{A}$ be a K-affinoid algebra. Then $\mathscr{H}(x)$ for $x \in \mathscr{M}(\mathscr{A})$ does not depend on the choice of Laurent neighborhood of x.*

Proof. Let $V = \mathscr{M}(\mathscr{B}) \subset \mathscr{M}(\mathscr{A}) =: X$ be a Laurent neiborhood of x. Here, we may assume $\mathscr{B} = \mathscr{A}\{p^{-1}f, qg^{-1}\}$. That is, $x \in X\{p^{-1}f, qg^{-1}\}$. Define $\varphi : \mathscr{A} \to \mathscr{B}$ as the corresponding morphism to $V \hookrightarrow X$. Then we define $y \in \mathscr{M}(\mathscr{B})$ by $\varphi^\sharp(y) = x$. Note that $\left|g^J\right|_y \neq 0$ for all $J \in \mathbb{Z}^m_{\geq 0}$.

It is enough to show that the completion of $\mathscr{A}/\mathfrak{p}_x$ coincides with the completion of $\mathscr{B}/\mathfrak{p}_y$, where $\mathfrak{p}_x$ (resp., $\mathfrak{p}_y$) is the kernel of $|\cdot|_x : \mathscr{A} \to \mathbb{R}$ (resp., $|\cdot|_y : \mathscr{B} \to \mathbb{R}$). Since $\varphi^\sharp(y) = x$, the bounded homomorphism $\varphi : \mathscr{A} \to \mathscr{B}$ induces the injection $\overline{\varphi} : \mathscr{A}/\mathfrak{p}_x \hookrightarrow \mathscr{B}/\mathfrak{p}_y$. Hence, it holds that

$$\iota : \mathrm{Frac}(\mathscr{A}/\mathfrak{p}_x) \subset \mathrm{Frac}(\mathscr{B}/\mathfrak{p}_y).$$

By Fact 3.27, the residue seminorm on a quotient ring of K-affinoid algebra is a norm. Therefore, the residue norm on a quotient ring of K-affinoid algebra extends to the unique norm on the fraction field of this quotient ring.

Since $\varphi : \mathscr{A} \to \mathscr{B}$ is admissible, so is $\overline{\varphi} : \mathscr{A}/\mathfrak{p}_x \hookrightarrow \mathscr{B}/\mathfrak{p}_y$ with respect to their residue norms. For each $\overline{h} \in \mathscr{B}/\mathfrak{p}_y$, we set $h \in \mathscr{B} = \mathscr{A}\{p^{-1}f, qg^{-1}\}$ as

$$h = \sum_{I,J} a_{IJ} f^I g^{-J}.$$

Then we set $\overline{h_n} \in \mathrm{Frac}(\mathscr{A}/\mathfrak{p}_x)$ as

$$\overline{h_n} = \sum_{|I|+|J|\leq n} \overline{a_{IJ}}\overline{f}^I (\overline{g})^{-J}.$$

Here, $|g^J|_y \neq 0$ implies $(\overline{g})^J \neq 0$. Hence, $\overline{h_n}$ is well-defined. Now let us assume $\overline{\mathscr{B}} := \mathscr{B}/\mathfrak{p}_y$. Then it is clear that y induces a bounded multiplicative norm on $\overline{\mathscr{B}}$ naturally. It is also denoted by y. Therefore,

$$\left|\overline{h} - \iota(\overline{h_n})\right|_y = \left| \sum_{|I|+|J|>n} \overline{a_{IJ}}\overline{f}^I (\overline{g})^{-J} \right|_y$$

$$\leq \left\| \sum_{|I|+|J|>n} \overline{a_{IJ}}\overline{f}^I (\overline{g})^{-J} \right\|_{\overline{\mathscr{B}}} \leq \left\| \sum_{|I|+|J|>n} a_{IJ} f^I g^{-J} \right\|_{\mathscr{B}} \to 0$$

as $n \to \infty$. This convergence follows from the definition of $\mathscr{A}\{p^{-1}f, qg^{-1}\}$. Since x is non-Archimedean, $\left\{\overline{h_n}\right\}_n$ is a Cauchy sequence. Indeed, it follows from

$$\left|\overline{h_n} - \overline{h_m}\right|_x \leq \max\left\{ \left|\iota\left(\overline{h_n}\right) - \overline{h}\right|_y, \left|\overline{h} - \iota\left(\overline{h_m}\right)\right|_y \right\}.$$

It implies that

$$\overline{\mathscr{B}} \subset \widehat{\mathrm{Frac}(\mathscr{A}/}\mathfrak{p}_x),$$

where the right-hand side is the completion of $\mathrm{Frac}(\mathscr{A}/\mathfrak{p}_x)$ with respect to the norm induced by x. This follows from $\iota^\sharp(y) = x$.

Hence, it holds that

$$\widehat{\mathrm{Frac}(\mathscr{A}/}\mathfrak{p}_x) = \widehat{\mathrm{Frac}(\mathscr{B}/}\mathfrak{p}_y),$$

where the left-hand side is the completion of $\mathrm{Frac}(\mathscr{A}/\mathfrak{p}_x)$ with respect to the norm induced by x and the right-hand side is the completion of $\mathrm{Frac}(\mathscr{B}/\mathfrak{p}_y)$ with respect to the norm induced by y. Therefore, the assertion follows. $\square$

3.49. By Fact 3.32, this means $\mathscr{H}(x)$ is a local object which depends only on a point x of (good) K-analytic space. In particular, Proposition 3.41 and 3.48 imply that $\mathscr{H}(x)$ is a local object of X^{an}.

Next, we consider the Berkovich analytification X of some K-variety.

Lemma 3.50. *Let K be a non-Archimedean field, and let A be K-algebra of finite type. For each point $x \in (\mathrm{Spec} A)^{\mathrm{an}} =: X$, we set* $\ker |\cdot|_x = \mathfrak{p}_x$. *Then it holds that*

$$\mathscr{H}(x) = \widehat{\mathrm{Frac}(A/\mathfrak{p}_x)},$$

where the right-hand side is the completion of $\mathrm{Frac}(A/\mathfrak{p}_x)$ *with respect to the norm induced by* $|\cdot|_x$.

Proof. We may assume that $A = K[T_1, \dots, T_n]$. Now we can take some $E(0, r) = \mathscr{M}(K\{r^{-1}T\})$ such that $x \in E(0, r)$. Set $\mathscr{A} := K\{r^{-1}T\}$. Here, $A \hookrightarrow \mathscr{A}$ induces $\iota : A/\mathfrak{p}_x \hookrightarrow \mathscr{A}/(\mathfrak{p}_x\mathscr{A})$. For brevity, $\mathscr{A}/(\mathfrak{p}_x\mathscr{A})$ is denoted by $\mathscr{A}/\mathfrak{p}_x$. Note that the quotient ring $\mathscr{A}/\mathfrak{p}_x$ coincides with what is appeared in (3.18). For $\overline{f} \in \mathscr{A}/\mathfrak{p}_x$, we set $f \in \mathscr{A} = K\{r^{-1}T\}$ as

$$f = \sum_I a_I T^I.$$

Then we set $f_n \in A$ as

$$f_n = \sum_{|I| \le n} a_I T^I.$$

Now x induces a bounded multiplicative norm on $\mathscr{A}/\mathfrak{p}_x$ naturally. It is also denoted by x. Then it holds that

$$\left|\overline{f} - \overline{f_n}\right|_x = \left|\sum_{|I|>n} a_I \overline{T}^I\right|_x \le \left\|\sum_{|I|>n} a_I \overline{T}^I\right\|_{\mathscr{A}/\mathfrak{p}_x} \le \left\|\sum_{|I|>n} a_I T^I\right\|_{\mathscr{A}} \to 0$$

as $n \to \infty$. This convergence follows from the definition of $K\{r^{-1}T\}$. Similarly to the discussion of Proposition 3.48, it implies that $\{\overline{f_n}\}$ is the Cauchy sequence whose limit is $\overline{f}$. Hence, the assertion follows. □

Proposition 3.51. *Let X be a variety over a non-Archimedean field K. We set $\pi_X : X^{\mathrm{an}} \to X$ as in* (3.43). *Then, for any $x \in X^{\mathrm{an}}$, it holds that*

$$\mathscr{H}(x) = \widehat{\kappa(\pi_X(x))},$$

where the right-hand side is the completion of the residue field $\kappa(\pi_X(x))$ at the point $\pi_X(x) \in X$ with respect to x.

Proof. It directly follows from Lemma 3.50. □

3.2. **Raynaud generic fiber.**

This section is mainly based on [Got20, §3] and [Got22, §2]. Recall the notation in (2.30). For R-algebra $\mathscr{A}$, we write $\mathscr{A}_K$ (resp., $\mathscr{A}_k$) instead of $\mathscr{A} \otimes_R K$ (resp., $\mathscr{A} \otimes_R k$).

3.52. Let $K\{T\}$ be the Tate algebra as in (3.21). Then we set

$$R\{T\} := \{f \in K\{T\} \mid ||f|| \leq 1\},$$

where $||\cdot||$ is the Gauss norm on $K\{T\}$. It is obvious that $R\{T\}$ is R-algebra and has a norm induced from the Gauss norm. Actually, the induced norm is complete. An R-algebra $\mathscr{A}$ is called an *admissible R-algebra* if $\mathscr{A}$ is a flat R- algebra isomorphic to $R\{T\}/I$ for some ideal $I \subset R\{T\}$. Then $\mathscr{A}$ has a norm induced from the norm on $R\{T\}$. Further, the norm on $\mathscr{A}$ is complete. That is, $\mathscr{A}$ is a Banach ring.

3.53. Recall the functor called *the Raynaud generic fiber*

$-_{\mathbf{rig}}$: {flat formal R-schemes locally of finite type} $\to$ {rigid K-spaces}.

The functor is constructed by sending an affine admissible formal R-scheme $\mathrm{Spf}\mathscr{A}$ to the K-affinoid space $\mathrm{Sp}\mathscr{A}_K$, where the underlying space of $\mathrm{Sp}\mathscr{A}_K$ is the set $\mathrm{Max}\mathscr{A}_K$ of all maximal ideals of $\mathscr{A}_K$ equipped with the weak topology with respect to $\mathscr{A}_K$ and the G-topology (cf. [BGR84, 9.1.4]). This functor first appeared in [Ray74]. It is known that this functor preserves fiber products. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism between formal R-schemes locally of finite type. If $f : \mathfrak{X} \to \mathfrak{Y}$ is a finite morphism (resp., closed immersion, open immersion, immersion, separated morphism), then $f_{\mathbf{rig}} : \mathfrak{X}_{\mathbf{rig}} \to \mathfrak{Y}_{\mathbf{rig}}$ is a finite morphism (resp., closed immersion, open immersion, immersion, separated morphism).

3.54. Berkovich gave the fully faithful functor

$-_0$: {separated strictly K-analytic spaces } $\to$ {rigid K-spaces}

in the process of basing his analytic spaces (cf. [Ber90, §3.3]). This functor preserves fiber products. In addition, the following also holds.

Proposition 3.55 ([Ber90, Proposition 3.3.2])**.** *Let $f : X \to Y$ be a morphism between separated strictly K-analytic spaces, $f : X \to Y$ is a finite morphism (resp., closed immersion, open immersion, immersion, separated morphism) if and only if $f_0 : X_0 \to Y_0$ is a finite morphism (resp., closed immersion, open immersion, immersion, separated morphism).*

For a flat formal R-scheme $\mathfrak{X}$ locally of finite type, there is a unique strictly K-analytic space X such that $\mathfrak{X}_{\mathbf{rig}} \cong X_0$. For simplicity of notation, we use the letter $\mathfrak{X}_{\mathbf{ber}}$ for this X. In particular, a K-affinoid space $\mathrm{Sp}\mathscr{A}_K$ corresponds to the Berkovich Spectrum $\mathscr{M}(\mathscr{A}_K)$, where $\mathscr{M}(\mathscr{A}_K)$ is the set of all bounded multiplicative seminorm on $\mathscr{A}_K$ equipped with the weak topology with respect to $\mathscr{A}_K$ and the G-topology (cf. [Ber90, §2, §3]). We note that $\mathscr{M}(\mathscr{A}_K) \subset X$ is closed but not necessarily open, although $\mathrm{Sp}\mathscr{A}_K \subset X_0$ is a closed and open set. That is, we regard the Raynaud generic fiber as the functor from the category of flat formal R-schemes locally of finite type to the category of separated strictly K-analytic spaces. By abuse of notation, we write $\mathscr{X}_{\mathbf{ber}}$ for $(\mathscr{X}_{\mathbf{for}})_{\mathbf{ber}}$ for a flat R-scheme $\mathscr{X}$ locally of finite type.

Definition 3.56. Let $\mathfrak{X}$ be a flat formal R-scheme locally of finite type. Then we can consider *the reduction map* $\mathrm{red}_{\mathfrak{X}} : \mathfrak{X}_{\mathbf{ber}} \to \mathfrak{X}$. Locally this map $\mathrm{red}_{\mathfrak{X}}|_{\mathscr{M}(\mathscr{A}_K)} : \mathscr{M}(\mathscr{A}_K) \to \mathrm{Spf}\mathscr{A} = \mathrm{Spec}\mathscr{A}_k$ is defined as follows: A point $x \in \mathscr{M}(\mathscr{A}_K)$ can be seen as a multiplicative seminorm on $\mathscr{A}_K$ that is bounded by the equipped norm on $\mathscr{A}_K$. Since $\mathscr{A}$ is an admissible R-algebra, the restriction of the equipped norm on $\mathscr{A}_K$ to $\mathscr{A}$ is bounded by 1. Hence, the restriction of x to $\mathscr{A}$ is also bounded by 1. Then,

$$\mathfrak{p}_x := \{f \in \mathscr{A} \mid |f(x)| < 1\} \subset \mathscr{A}$$

is a prime ideal of $\mathscr{A}$. It is clear that $\mathfrak{p}_x \in \mathrm{Spec}\mathscr{A}_k = \mathrm{Spf}\mathscr{A}$. Then we denote by $\mathrm{red}_{\mathfrak{X}}(x)$ the point corresponding to this prime ideal $\mathfrak{p}_x$. If $\mathfrak{X} = \mathscr{X}_{\mathbf{for}}$ for some flat R-scheme $\mathscr{X}$ locally of finite type, we write $\mathrm{red}_{\mathscr{X}}$ instead of $\mathrm{red}_{\mathfrak{X}}$. We sometimes call the image of $x \in \mathscr{X}$ via the reduction map $\mathrm{red}_{\mathscr{X}}$ the *center* of x.

From now on, we consider some properties of the reduction map.

3.57. Let $\mathfrak{X}$ be a flat formal R-scheme locally of finite type. Then the reduction map $\mathrm{red}_{\mathfrak{X}} : \mathfrak{X}_{\mathbf{ber}} \to \mathfrak{X}$ is anti-continuous and surjective. Please refer to [Ber90, §2.4] for details.

3.58. Recall the notation in (2.30) and (2.32). Let X be a smooth K-variety, and $\mathscr{X}$ a model of X. Then the reduction map $\mathrm{red}_{\mathscr{X}} : \mathscr{X}_{\mathbf{ber}} \to \mathscr{X}_{\mathbf{for}}$ is characterized as follows: Consider the canonical map $\pi_X : X^{\mathrm{an}} \to X$ as in (3.43). For any $x \in X^{\mathrm{an}}$, the canonical homomorphism

$$\kappa(\pi_X(x)) \hookrightarrow \mathscr{H}(x)$$

induces a morphism $\mathrm{Spec}\mathscr{H}(x) \to X$. Then, we obtain the morphism

$$\chi_x : \mathrm{Spec}\mathscr{H}(x) \to X \to \mathscr{X}.$$

This morphism gives the following diagram.

$$\begin{array}{ccc} \mathrm{Spec}\mathscr{H}(x) & \xrightarrow{\chi_x} & \mathscr{X} \\ \downarrow & \overset{\varphi_{x,\mathscr{X}}}{\nearrow} & \downarrow \\ \mathrm{Spec}\mathscr{H}(x)^{\circ} & \longrightarrow & \mathrm{Spec}R \end{array}$$

Here, this dotted arrow $\varphi_{x,\mathscr{X}}$ does not always exist. By the valuative criterion of separatedness (cf. [Har77, Theorem 4.3]), the dotted arrow $\varphi_{x,\mathscr{X}}$ is uniquely determined if it exists. Then we can easily see that $x \in \mathscr{X}_{\mathbf{ber}}$ if and only if $\varphi_{x,\mathscr{X}}$ exists. Further, if it holds, then $\mathrm{red}_{\mathscr{X}}(x)$ coincides with the image of the closed point of $\mathrm{Spec}\mathscr{H}(x)^{\circ}$ via the dotted arrow $\varphi_{x,\mathscr{X}}$. It implies that there is a canonical inclusion

$$\kappa(\mathrm{red}_{\mathscr{X}}(x)) \subset \widetilde{\mathscr{H}(x)}$$

induced by $\varphi_{x,\mathscr{X}}$ for each $x \in \mathscr{X}_{\mathbf{ber}}$. We also see that, if $\mathscr{X}$ is proper over R, then $\mathscr{X}_{\mathbf{ber}} = X^{\mathrm{an}}$. Indeed, it follows from the valuative criterion of properness (cf. [Har77, Theorem 4.7]).

3.59. Keep the notation in (3.58). For each point $x \in \mathscr{X}_{\mathbf{ber}}$, the image $\mathrm{red}_{\mathscr{X}}(x)$ of x via the reduction map is more closely related to the double residue field $\widetilde{\mathscr{H}(x)}$ than seen in (3.58). Let us show you that now.

Set $X^{\mathrm{val}} := \pi_X^{-1}(\xi_X)$, where $\pi_X : X^{\mathrm{an}} \to X$ is the canonical map as in (3.43) and ξ_X is the generic point of X. Let $\mathscr{X}$ and $\mathscr{Y}$ be models of X, and $f : \mathscr{Y} \to \mathscr{X}$ a proper birational morphism. For each $x \in X^{\mathrm{val}}$, the morphism $\chi_x : \mathrm{Spec}\mathscr{H}(x) \to \mathscr{X}$ factors through f. In addition, for each $x \in \mathscr{X}_{\mathbf{ber}}$, there is an unique morphism $\varphi_{x,\mathscr{X}} : \mathrm{Spec}\mathscr{H}(x)^\circ \to \mathscr{X}$ as in (3.58). Hence, for each $x \in X^{\mathrm{val}} \cap \mathscr{X}_{\mathbf{ber}}$, we can apply the valuative criterion of properness to the following diagram.

$$\begin{array}{ccc} \mathrm{Spec}\mathscr{H}(x) & \longrightarrow & \mathscr{Y} \\ \downarrow & \overset{\varphi}{\nearrow} & \downarrow f \\ \mathrm{Spec}\mathscr{H}(x)^\circ & \xrightarrow[\varphi_{x,\mathscr{X}}]{} & \mathscr{X} \end{array}$$

Then we obtain the unique morphism $\varphi : \mathrm{Spec}\mathscr{H}(x)^\circ \to \mathcal{Y}$ It is obvious that the morphism φ coincides with the dotted arrow $\varphi_{x,\mathscr{Y}}$ as appeared in (3.58) for $\mathscr{Y}$, instead of $\mathscr{X}$. It implies that

$$X^{\mathrm{val}} \cap \mathscr{X}_{\mathbf{ber}} \subset X^{\mathrm{val}} \cap \mathscr{Y}_{\mathbf{ber}}$$

and $f(\mathrm{red}_{\mathscr{Y}}(x)) = \mathrm{red}_{\mathscr{X}}(x)$. In particular, there exists an inclusion

$$\kappa(\mathrm{red}_{\mathscr{X}}(x)) \hookrightarrow \kappa(\mathrm{red}_{\mathscr{Y}}(x)).$$

It means that the lifting of the center induces the extension of the residue field of the center. As we saw in (3.58), it holds that

$$\kappa(\mathrm{red}_{\mathscr{X}}(x)) \subset \widetilde{\mathscr{H}(x)}$$

for each $x \in \mathscr{X}_{\mathbf{ber}}$. Then, it holds that

$$\kappa(\mathrm{red}_{\mathscr{X}}(x)) \subset \kappa(\mathrm{red}_{\mathscr{Y}}(x)) \subset \widetilde{\mathscr{H}(x)}$$

for each $x \in X^{\mathrm{val}} \cap \mathscr{X}_{\mathbf{ber}}$. Note that $\kappa(\mathrm{red}_{\mathscr{X}}(x))$ and $\widetilde{\mathscr{H}(x)}$ are not expected to coincide in general.

Here, we now describe $\widetilde{\mathscr{H}(x)}$ in terms of X or models of X.

Lemma 3.60. *Let X be a smooth K-variety, and $\pi_X : X^{\mathrm{an}} \to X$ the canonical map as in* (3.43)*. Then, for each $x \in X^{\mathrm{an}}$, it holds that*

$$\widetilde{\mathscr{H}(x)} = \kappa(\widetilde{\pi_X(x)}),$$

where the right-hand side is the residue field of $\kappa(\pi_X(x))$ with respect to the induced valuation $|\cdot|_x$ on $\kappa(\pi_X(x))$.

Proof. By Proposition 3.51, for any $f \in \mathscr{H}(x)^\circ \setminus \mathscr{H}(x)^{\circ\circ}$ and any $\varepsilon > 0$, there exists $g \in \kappa(\pi_X(x))$ such that $|f - g| < \varepsilon$. Now we take $\varepsilon < 1$. Then, we find that $|g| = |f| = 1$ and $\overline{f} = \overline{g} \in \widetilde{\kappa(\pi_X(x))}$. In conclusion, we obtain $\widetilde{\mathscr{H}(x)} = \widetilde{\kappa(\pi_X(x))}$. □

Theorem 3.61 (cf. [Got20, §3])**.** *Let X be a proper smooth K-variety. For $x \in X^{\mathrm{val}}$, it holds that*

$$\widetilde{\mathscr{H}(x)} = \varinjlim_{\mathscr{X}} \kappa(\mathrm{red}_{\mathscr{X}}(x)) = \bigcup_{\mathscr{X}} \kappa(\mathrm{red}_{\mathscr{X}}(x)),$$

where $\mathscr{X}$ runs over all proper models of X.

Remark 3.62. Note that there is a proper models of X. Indeed, given an embedding X into a suitable projective space $\mathbb{P}^m_K$, we can take a model $\mathcal{X}$ of X as the normalization of the closure of X in $\mathbb{P}^m_S$, where $S = \mathrm{Spec} R$. In [Got20, §3], we considered Theorem 3.61 in the case where K is a general non-Archimedean field, and in the case with fewer assumptions about X.

Proof of Theorem 3.61. First, we show that

$$\widetilde{\mathscr{H}(x)} = \bigcup_{\mathscr{X}} \kappa(\mathrm{red}_{\mathscr{X}}(x)).$$

Here, the right-hand side just means the set-theoretic union of the images of canonical inclusions $\kappa(\mathrm{red}_{\mathscr{X}}(x)) \hookrightarrow \widetilde{\mathscr{H}(x)}$. As we saw in Remark 3.62, we may assume that there is a proper model $\mathscr{X}$ of X. Now we take an affine open subset $U = \mathrm{Spec} A$ of $\mathscr{X}$ so that $\mathrm{red}_{\mathscr{X}}(x) \in U$. We want to show that for this model $\mathscr{X}$, the following holds.

$$\widetilde{\mathscr{H}(x)} = \bigcup_{\pi: \mathscr{X}' \to \mathscr{X}} \kappa(\mathrm{red}_{\mathscr{X}'}(x)),$$

where π ranges over all vertical blow-ups. By Lemma 3.60, we obtain

$$\widetilde{\mathscr{H}(x)} = \widetilde{K(X)},$$

where $K(X)$ is the function field of X. For any element

$$f = \overline{g/h} \in \widetilde{\mathscr{H}(x)} \setminus \kappa\left(\mathrm{red}_{\mathscr{X}}(x)\right),$$

where $g, h \in A$, we may assume that $|g|_x = |h|_x > 0$. Now we can take $l \in \mathbb{Z}_{>0}$ so that $|t^l|_x \leq |g|_x = |h|_x$, where t is an uniformizing parameter of R. For the ideal $I = (g, h, t^l) \subset A$, we can extend the corresponding ideal sheaf $\tilde{I}$ on U to an ideal sheaf $\mathscr{I}$ on $\mathscr{X}$. See [Har77, II, Exercises 5.15]. Further, we may assume that the defining ideal sheaf $\mathscr{I}$ contains the defining ideal sheaf of $l\mathscr{X}_k$ on $\mathscr{X}$. Now we consider the blow-up $\pi : \mathscr{X}' \to \mathscr{X}$ along the closed subscheme $V(\mathscr{I})$. Then $\mathscr{X}'$ is a model of X. Moreover, the proper model $\mathscr{X}'$ has an open affine scheme $U' = \mathrm{Spec}A[g/h, t^l/h]$. Since $|t^l/h|_x \leq 1$, we obtain a canonical morphism

$$A[g/h, t^l/h] \to \mathscr{H}(x)^\circ.$$

It means that $\mathrm{red}_{\mathscr{X}'}(x) \in \mathrm{Spec}A[g/h, t^l/h]$. Therefore, it holds that

$$\widetilde{\mathscr{H}(x)} = \bigcup_{\mathscr{X}} \kappa(\mathrm{red}_{\mathscr{X}}(x)).$$

On the other hand, we can easily see that the set of all proper models $\mathscr{X}$ of X is a directed set with respect to an opposite category of the category consisting of vertical blow-ups. In addition, for any vertical blow-up $\mathscr{X}' \to \mathscr{X}$, we obtain a canonical inclusion $\kappa(\mathrm{red}_{\mathscr{X}}(x)) \hookrightarrow \kappa(\mathrm{red}_{\mathscr{X}'}(x))$. Therefore, it holds that

$$\widetilde{\mathscr{H}(x)} = \varinjlim_{\mathscr{X}} \kappa(\mathrm{red}_{\mathscr{X}}(x)) = \bigcup_{\mathscr{X}} \kappa(\mathrm{red}_{\mathscr{X}}(x)).$$

□

3.3. **Berkovich retractions and the Essential skelton.**

This section is mainly based on [Got20, §6] and [Got22, §4].

3.63. Keep the notation in (2.30). In addition, we assume that the characteristic of the residue field k is 0. Then, Cohen's structure theorem implies an isomorphism $R \cong k[[t]]$. In particular, we obtain an injection $k \hookrightarrow R$.

Recall that $S = \mathrm{Spec}R$.

Definition 3.64. Let $\mathcal{X}$ be an S-variety. An ideal sheaf $\mathscr{I}$ on $\mathcal{X}$ is *vertical* if it is co-supported on the central fiber $\mathcal{X}_k$. A *vertical blow-up* $\mathcal{X}' \to \mathcal{X}$ means the normalized blow-up along a vertical ideal sheaf. The group $\mathrm{Div}_0(\mathcal{X})$ is defined as *the group consisting of vertical Cartier divisors* on $\mathcal{X}$.

3.65. When an S-variety $\mathcal{X}$ is normal, we can see that $\mathrm{Div}_0(\mathcal{X})$ is a free $\mathbb{Z}$-module of finite rank.

Definition 3.66. An S-variety $\mathcal{X}$ is called to be *SNC* if the central fiber $\mathcal{X}_k$ is a divisor with strict normal crossings and every intersection of the form D_J as in Definition 2.31 is connected.

Remark 3.67. An snc model of X as in (2.32) is not always SNC. However, we can obtain an SNC S-variety from an snc model of X by further blow-up along components of the possibly non-connected D_J's. In particular, the resulting SNC S-variety is also a model of X.

In general, the following is known.

Fact 3.68 (cf. [Tem08, Theorem 1.1])**.** *For any S-variety $\mathcal{X}$ with smooth generic fiber, there exists a vertical blow-up $\mathcal{X}' \to \mathcal{X}$ such that $\mathcal{X}'$ is SNC.*

3.69. Let X be a smooth connected projective K-analytic space. In other words, we can identify X with Y^{an} for some smooth projective K-variety Y. S-variety $\mathcal{X}$ is called a *model* of X if $\mathcal{X}$ is a normal and projective S-variety together with the datum of an isomorphism $\mathcal{X}_K^{\mathrm{an}} \cong X$. Here, note that $\mathcal{X}$ is a model of $\mathcal{X}_K$ in the sense of Definition 2.32. We denote by $\mathcal{M}_X$ *the set of all models of* X. It follows from the same argument in Remark 3.62 that $\mathcal{M}_X$ is nonempty. Note that $\mathcal{M}_X$ becomes a directed set by declaring $\mathcal{X}' \geq \mathcal{X}$ if there exists a vertical blow-up $\mathcal{X}' \to \mathcal{X}$.

3.70. Let $\mathcal{X}$ be an SNC model of X. We can write the central fiber as

$$\mathcal{X}_k = \sum_{i \in I} m_i E_i,$$

where $(E_i)_{i \in I}$ are irreducible components. Then, it follows that

$$\mathrm{Div}_0(\mathcal{X}) = \bigoplus_{i \in I} \mathbb{Z} E_i.$$

Set $\mathrm{Div}_0(\mathcal{X})^*_{\mathbb{R}} := \mathrm{Hom}(\mathrm{Div}_0(\mathcal{X}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Here, we denote by E_i^* the dual element of E_i and we set

$$e_i := \frac{1}{m_i} E_i^* \in \mathrm{Div}_0(\mathcal{X})^*_{\mathbb{R}}.$$

For each $J \subset I$ such that $E_J := \cap_{j \in J} E_j \neq \emptyset$, let $\hat{\sigma}_J \subset \mathrm{Div}_0(\mathcal{X})^*_{\mathbb{R}}$ be the simplicial cone defined by

$$\hat{\sigma}_J := \sum_{j \in J} \mathbb{R}_{\geq 0} e_j.$$

By definition, the pair $\{e_i\}$ is a basis of $\mathrm{Div}_0(\mathcal{X})^*_{\mathbb{R}}$. Hence, these cones naturally defines a fan $\hat{\Delta}(\mathcal{X})$ in $\mathrm{Div}_0(\mathcal{X})^*_{\mathbb{R}}$. We define the *dual complex* of $\mathcal{X}$ by

$$\Delta(\mathcal{X}) := \hat{\Delta}(\mathcal{X}) \cap \{\langle \mathcal{X}_k, \cdot \rangle = 1\},$$

where $\langle \cdot, \cdot \rangle$ is the natural bilinear form on $\mathrm{Div}_0(\mathcal{X})^*_{\mathbb{R}}$. Each $J \subset I$ such that $E_J \neq \emptyset$ corresponds to a simplicial face

$$\sigma_J := \hat{\sigma}_J \cap \{\langle \mathcal{X}_0, \cdot \rangle = 1\} = \mathrm{Conv}\{e_j \mid j \in J\}$$

of dimension $|J| - 1$ in $\Delta(\mathcal{X})$, where Conv denotes *convex hull*. This endows $\Delta(\mathcal{X})$ with the structure of a simplicial complex, such that σ_J is a face of σ_L if and only if $J \supset L$.

3.71. Under the same situation in (3.69), we denote by $\mathcal{M}_X^{\mathrm{SNC}}$ *the set of all SNC models of* X. By Fact 3.68, $\mathcal{M}_X \neq \emptyset$ implies $\mathcal{M}_X^{\mathrm{SNC}} \neq \emptyset$. We can easily see that $\mathcal{M}_X^{\mathrm{SNC}} \subset \mathcal{M}_X$ and $\mathcal{M}_X^{\mathrm{SNC}}$ is also a directed set by vertical blow-ups. If $\mathcal{X}' \geq \mathcal{X}$, where $\mathcal{X}', \mathcal{X} \in \mathcal{M}_X^{\mathrm{SNC}}$, then we obtain the natural map $\mathrm{Div}_0(\mathcal{X}')^*_{\mathbb{R}} \to \mathrm{Div}_0(\mathcal{X})^*_{\mathbb{R}}$ induced by the pull back $\mathrm{Div}_0(\mathcal{X}) \to \mathrm{Div}_0(\mathcal{X}')$ via the vertical blow-up. Further, it implies the natural map $\Delta(\mathcal{X}') \to \Delta(\mathcal{X})$. Hence, the following projective limit is well-defined.

$$\varprojlim_{\mathcal{X} \in \mathcal{M}_X^{\mathrm{SNC}}} \Delta(\mathcal{X}).$$

The following, which is a highly suggestive result, is first stated by [KS06]. After that, the proof is written by [BFJ16].

Fact 3.72 (Corollary 3.2 of [BFJ16])**.** *Under the situation in* (3.69)*, we obtain a canonical immersion* $\Delta(\mathcal{X}) \hookrightarrow X$ *(cf. Fact 3.78). In addition, for the canonical immersion* $\Delta(\mathcal{X}) \hookrightarrow X$*, there is a canonical retraction* $X \to \Delta(\mathcal{X})$*. Further, these retractions induce a canonical homeomorphism*

$$X \cong \varprojlim_{\mathcal{X} \in \mathcal{M}_X^{\mathrm{SNC}}} \Delta(\mathcal{X}).$$

Remark 3.73. To see a simplicial complex structure of $\Delta(\mathcal{X})$, we have considered the case when $\mathcal{X}$ is SNC. Actually, the dual intersection complex $\Delta(\mathcal{X})$ can also be defined as a topological space when $\mathcal{X}$ is an snc-model of X, as we stated in (2.36). In addition, for any snc-model $\mathcal{X}$, we also obtain a canonical immersion $\Delta(\mathcal{X}) \hookrightarrow X$ and a canonical retraction $X \to \Delta(\mathcal{X})$ as we will see in Definition 3.85. In Definition 3.85, the canonical retraction $X \to \Delta(\mathcal{X})$ is called a *Berkovioch retraction*. Note, however, that the dual intersection complex $\Delta(\mathcal{X})$ for an snc-model $\mathcal{X}$ does not always have a simplicial complex structure unlike the case as we stated in (3.70). Here, we denote by $\mathcal{M}_X^{\mathrm{snc}}$ *the set of all snc models of* X. It is obvious that $\mathcal{M}_X^{\mathrm{SNC}} \subset \mathcal{M}_X^{\mathrm{snc}} \subset \mathcal{M}_X$ and $\mathcal{M}_X^{\mathrm{snc}}$ is also a directed set. Since $\mathcal{M}_X^{\mathrm{SNC}}$ is cofinal in $\mathcal{M}_X$, Fact 3.72 can be rephrased as

$$X \cong \varprojlim_{\mathcal{X} \in \mathcal{M}_X^{\mathrm{snc}}} \Delta(\mathcal{X}).$$

3.74. Now consider the image of the canonical immersion $\Delta(\mathcal{X}) \hookrightarrow X$ as appeared in Fact 3.72. Use the same notation in (3.70). Recall that the central fiber of an SNC-model $\mathcal{X}$ is of the form $\mathcal{X}_k = \sum_{i \in I} m_i E_i$ as a divisor. Here, for each $J \subset I$, the intersection $E_J := \cap_{j \in J} E_j$ is either empty or a smooth irreducible k-variety. Let ξ_J be a generic point of E_J if $E_J \neq \emptyset$. For each $j \in J$, we can choose a local equation $f_j \in \mathscr{O}_{\mathcal{X}, \xi_J}$ of the prime divisor E_j, so that $(f_j)_{j \in J}$ is a regular system of parameters of the local ring $\mathscr{O}_{\mathcal{X}, \xi_J}$ because of the SNC condition. For this $\mathcal{X}$ and any (nonsingular) point $p \in \mathcal{X}$, we obtain the sequence $k \hookrightarrow R \to \mathscr{O}_{\mathcal{X},p}$. It implies $k \hookrightarrow \mathscr{O}_{\mathcal{X},p}$. Hence, by Cohen's structure theorem, after taking a field of representatives of $\kappa(\xi_J)$, we obtain an isomorphism

$$\iota : \widehat{\mathscr{O}}_{\mathcal{X},\xi_J} \cong \kappa(\xi_J)[[t_j, j \in J]]$$

defined by $\iota(f_j) = t_j$. That is, any element $f \in \mathscr{O}_{\mathcal{X},p}$ has a power series expansion of the form $f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{|J|}} c_\alpha f^\alpha \in \widehat{\mathscr{O}}_{\mathcal{X},\xi_J}$ via the isomorphism $\iota : \widehat{\mathscr{O}}_{\mathcal{X},\xi_J} \cong \kappa(\xi_J)[[t_j]]$, where $\iota(c_\alpha) \in \kappa(\xi_J)$ and $f^\alpha := \prod f_j^{\alpha_j}$.

Definition 3.75. Let X be a smooth connected projective K-analytic space. $x \in X$ is called a *quasi monomial valuation* if there exists an SNC model $\mathcal{X}$ of X and $s = \sum s_j e_j \in \sigma_J \subset \Delta(\mathcal{X})$ such that x is a valuation on $\mathscr{O}_{\mathcal{X},\xi_J}$ defined by the restriction of the following valuation on $\widehat{\mathscr{O}}_{\mathcal{X},\xi_J}$.

$$f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{|J|}} c_\alpha f^\alpha \in \widehat{\mathscr{O}}_{\mathcal{X},\xi_J} \mapsto |f| = \max_{c_\alpha \neq 0} r^\alpha,$$

where $r_j := \exp(-s_j) < 1$, and the above expansion of f is given by the isomorphism $\iota : \widehat{\mathscr{O}}_{\mathcal{X},\xi_J} \cong \kappa(\xi_J)[[t_j]]$ as in (3.74). Such a valuation on $\mathscr{O}_{\mathcal{X},\xi_J}$ is called a *monomial valuation* on $\mathcal{X}$.

Remark 3.76. It follows from the proof of [JM13, Proposition 3.1] that the construction in Definition 3.75 does not depend on the choice of the isomorphism $\iota : \widehat{\mathscr{O}}_{\mathcal{X},\xi_J} \cong \kappa(\xi_J)[[t_j, j \in J]]$. Definition 3.75 is slightly a priori different from the original one in [BFJ16] (See Remark 3.79) though it is still equivalent. We denote by X^{qm} *the set of all quasi monomial valuations of* X.

We now list a few properties of quasi monomial valuations.

Fact 3.77 (Corollary 3.9 of [BFJ16])**.** X^{qm} *is dense in* X.

Fact 3.78 (cf. Definition 3.7 and §3.3 of [BFJ16])**.** *We denote by* $\Delta'(\mathcal{X})$ *the image of the canonical immersion* $\Delta(\mathcal{X}) \to X$ *as appeared in Fact 3.72. Then, it holds that*

$$X^{\mathrm{qm}} = \bigcup_{\mathcal{X} \in \mathcal{M}_X^{\mathrm{SNC}}} \Delta'(\mathcal{X}).$$

Remark 3.79. In [BFJ16], quasi monomial valuations are defined by the right-hand side of the equation in Fact 3.78.

Theorem 3.80 (cf. [Got20, Theorem 6.11])**.** *Let* X *be a smooth connected projective* K*-analytic space. If* $x \in X^{\mathrm{qm}}$*, then there exists an SNC model* $\mathcal{X}$ *of* X *such that*

$$\widetilde{\mathscr{H}(x)} = \kappa(\mathrm{red}_{\mathcal{X}}(x)).$$

Proof. By the assumption, we can take an SNC model $\mathcal{X}$ such that x gives a monomial valuation on $\widehat{\mathscr{O}}_{\mathcal{X},\mathrm{red}_{\mathcal{X}}(x)}$ as Definition 3.75. Now we construct desirable vertical blow-up $\pi : \mathcal{X}' \to \mathcal{X}$ so that $\widetilde{\mathscr{H}(x)} = \kappa(\mathrm{red}_{\mathcal{X}'}(x))$.

By [Got20, Theorem 4.6], we see

$$\widetilde{\mathscr{H}(x)} = \kappa(c_{\mathcal{X}}(x))(f_1, \ldots, f_n) \text{ for some } f_1, \ldots, f_n \in \widetilde{\mathscr{H}(x)}.$$

For each $f_i = \overline{g_i/h_i}$, we can obtain a vertical blow-up $\mathcal{X}_i \to \mathcal{X}$ so that $f_i \in \kappa(\mathrm{red}_{\mathcal{X}_i}(x))$ in the same way as the discussion of Theorem 3.61. Theorem 3.61 implies that there exists a vertical blow-up $\mathcal{X}' \to \mathcal{X}$ of X such that

$$\widetilde{\mathscr{H}(x)} = \kappa(\mathrm{red}_{\mathcal{X}'}(x)).$$

Then we obtain an SNC model $\mathcal{X}''$ after taking further blow-up $\mathcal{X}'' \to \mathcal{X}'$ by [Tem08]. Such a vertical blow-up $\mathcal{X}'' \to \mathcal{X}' \to \mathcal{X}$ is the desired one. □

Remark 3.81. In [Got20], we defined what is called a quasi monomial valuation for several cases, and proved that variants of Theorem 3.80 also hold for these variants of quasi monomial valuations (cf. [Got20, Theorem 3.5, Theorem 4.6, Theorem 5.7]).

Next, we consider the canonical retraction $X \to \Delta(\mathcal{X})$ for an snc-model $\mathcal{X}$ of X in Fact 3.72 concretely.

3.82. A *Calabi-Yau variety* X over a field F is a smooth, proper, geometrically connected scheme X over F such that the canonical bundle K_X is trivial. In particular, abelian varieties are also Calabi-Yau varieties. A *volume form* ω is a non-zero global section of K_X. In the same way, a (complex) *Calabi-Yau manifold* X is a compact complex manifold X such that the canonical bundle K_X is trivial. Further, a volume form ω is a non-zero global section of K_X. In particular, the volume form ω on X is often called a *holomorphic volume form*.

Definition 3.83. Let X be a Calabi-Yau variety over K and ω be a volume form on X. Then we can define the *weight function*

$$\mathrm{wt}_\omega : X^{\mathrm{an}} \to \mathbb{R} \cup \{\infty\}.$$

Please refer to [MN12, §4.5] for details. The *essential skelton* $\mathrm{Sk}(X)$ of X is the subset of X^{an} consisting of points where wt_ω reaches its minimal value. Since X is Calabi-Yau, ω is uniquely determined up to a scalar multiple. Multiplying ω with a scalar changes the weight function by a constant. Therefore, $\mathrm{Sk}(X)$ depends only on X not on ω.

3.84. Let X be a smooth connected K-variety and let $\mathscr{X}$ be an snc-model of X over S. Fact 3.72 states that the dual intersection complex $\Delta(\mathscr{X})$ of $\mathscr{X}_k$ is canonically embedded into X^{an}. We denote by $\mathrm{Sk}(\mathscr{X})$ its image of

$\Delta(\mathscr{X})$. $\mathrm{Sk}(\mathscr{X})$ is called the *Berkovich skelton* of $\mathscr{X}$ and has the simplicial structure induced by $\Delta(\mathscr{X})$ if $\mathscr{X}$ is SNC. If X is a Calabi-Yau variety over K, then the essential skelton $\mathrm{Sk}(X)$ as in Definition 3.83 is canonically homeomorphic to the subcomplex of $\mathrm{Sk}(\mathscr{X})$. If the snc-model $\mathscr{X}$ is *good minimal dlt-model with a technical assumption* as in [NXY19, (1.11)], then it follows from [NX16, 3.3.3] that the image of this embedding is exactly the essential skeleton $\mathrm{Sk}(X)$. In particular, we give a simplicial complex structure to $\mathrm{Sk}(X)$ by the one of $\mathrm{Sk}(\mathscr{X})$ if $\mathscr{X}$ is SNC. Note that the technical assumption is satisfied when $\mathscr{X}$ is an snc-model. Please refer to [NXY19, (2.3)] for details.

Definition 3.85. Let X be a smooth connected K-variety and let $\mathscr{X}$ be an snc-model of X over S. We assume that $X^{\mathrm{an}} = \mathscr{X}_{\mathbf{ber}}$. In particular, if $\mathscr{X}$ is projective over S, then X is projective over K and this assumption holds. Here, we now construct the *Berkovich retraction* associated with an snc-model $\mathscr{X}$ of X in accordance with [NXY19, (2.4)] (or [BFJ16, §3]).

Let x be a point in X^{an} and let $\mathrm{red}_{\mathscr{X}}(x)$ be its reduction on $\mathscr{X}_k$ as we saw in Definition 3.56. We denote by Z the closure of $\mathrm{red}_{\mathscr{X}}(x) = \xi$. Then Z is a non-empty stratum of $\mathscr{X}_k$. Thus, it determines a unique face σ of the dual intersection complex $\Delta(\mathscr{X})$. Let $D_1, ..., D_r$ be the irreducible components of $\mathscr{X}_k$ that contain Z, and let $N_1, ..., N_r$ be their multiplicities in $\mathscr{X}_k$. Then $D_1, ..., D_r$ correspond to the vertices $v_1, ..., v_r$ of σ. Note that, for any snc-model $\mathscr{X}$, each irreducible component of $\mathscr{X}_k$ is Cartier since $\mathscr{X}$ is regular. We choose a local equation $f_i = 0$ for each D_i at $\mathrm{red}_{\mathscr{X}}(x)$. Then $\rho_{\mathscr{X}}(x)$ is defined as the point of the simplex σ with barycentric coordinates

$$\alpha = (-N_1 \log |f_1(x)|, \ldots, -N_r \log |f_r(x)|)$$

with respect to the vertices $(v_1, ..., v_r)$. The image $\rho_{\mathscr{X}}(x)$ of x corresponds to the monomial point represented by $(\mathscr{X}, (D_1, ..., D_r), \xi)$ and the tuple

$$(-\log |f_1(x)|, \ldots, -\log |f_r(x)|),$$

in the terminology of [MN12, 2.4.5] via the embedding of $\Delta(\mathscr{X})$ into X^{an}. We can easily verify that this definition does not depend on the local equations f_i and check that $\rho_{\mathscr{X}}$ is continuous, and that it is a retraction onto the skelton $\mathrm{Sk}(\mathscr{X}) = \Delta(\mathscr{X})$. Refer to [NXY19, (2.4)] for good minimal dlt-models with the technical assumption as introduced in *loc.cit.*.

Definition 3.86. Let X be a Calabi-Yau variety over K. If an snc-model $\mathscr{X}$ of X is a good minimal dlt-model of X with a technical assumption as in [NXY19, (1.11)]. Then we call the map $\rho_{\mathscr{X}} \colon X^{\mathrm{an}} \to \mathrm{Sk}(X)$ constructed in Definition 3.85 the *non-Archimedean SYZ fibration* associated with $\mathscr{X}$.

3.87. We note that, even though the subspace $\mathrm{Sk}(X)$ of X^{an} only depends on X, the simplicial complex structure on $\mathrm{Sk}(X)$ and the non-Archimedean

SYZ fibration $\rho_{\mathscr{X}} : X^{\mathrm{an}} \to \mathrm{Sk}(X)$ depend on the choice of the good minimal dlt-model $\mathscr{X}$. In [MN12, §3.2], the authors discussed the canonical piecewise integral affine structure of $\mathrm{Sk}(X)$ and revealed that this piecewise integral affine structure coincides with the one induced by $\Delta(\mathscr{X})$. In other words, the piecewise integral affine structure induced by $\Delta(\mathscr{X})$ does not depend on the choice of the good minimal dlt-model $\mathscr{X}$. However, this is closer to the topological structure than to the integral affine structure. In this thesis, we focus on the integral affine structure (more precisely, IAMS structure).

4. SYZ FIBRATION

In this chapter, we discuss SYZ fibrations for each maximally degenerating family of abelian varieties and their quotients. This chapter is mainly based on [GO22, §2].

4.1. **Preliminaries.**

Definition 4.1 (Special Lagrangian submanifold)**.** Let X be a Calabi-Yau manifold of dimension n over $\mathbb{C}$, ω a Kähler form of Ricci flat metric on X, and Ω a holomorphic volume form as in (3.82). Then a submanifold M in X is called *special Lagrangian submanifold* if M is a real manifold of dimension n such that $\omega|_M = 0$ and $\mathrm{Im}\Omega|_M = 0$.

Definition 4.2 (SYZ fibration)**.** A fibration $f : X \to B$ from a complex Calabi-Yau manifold to a topological space B is called a *special Lagrangian fibration* (or *SYZ fibration*) if each fiber $X_b := f^{-1}(b)$ is a special Lagrangian submanifold in X.

To state our main theorems, we introduce the following.

Definition 4.3. Let B be an real n-dimensional manifold. An *affine structure* (resp., *tropical affine structure*, *integral affine structure*) on B is an atlas $\{(U_i, \psi_i)\}$ of B consisting of coordinate charts $\psi_i : U_i \to \mathbb{R}^n$, whose transition functions $\psi_i \circ \psi_j^{-1}$ lie in $\mathrm{Aff}(\mathbb{R}^n) := \mathbb{R}^n \rtimes \mathrm{GL}(\mathbb{R}^n)$ (resp., $\mathrm{Aff}(\mathbb{Z}^n) := \mathbb{Z}^n \rtimes \mathrm{GL}(\mathbb{Z}^n)$). A pair of B and an affine structure (resp., tropical affine structure, integral affine structure) on B is called an *affine manifold* (resp., a *tropical affine manifold*, an *integral affine manifold*). Further, B is called an *integral* (resp., *tropical*) *affine manifold with singularities* if B is a C^0-manifold with an open set $B^{\mathrm{sm}} \subset B$ that has an integral (resp., tropical) affine structure, and such that $Z := B \setminus B^{\mathrm{sm}}$ is a locally finite union of locally closed submanifolds of codimension ≥ 2. Here, an integral affine manifold with singularities is often called an *IAMS* for short. In addition, we call this integral affine manifold B^{sm} *IAMS structure* of B.

Remark 4.4. The convention of Definition 4.3 follows that of e.g. [KS06, Gro13].

4.5. Let $f : X \to B$ be an SYZ fibration such that each fiber is a torus. Then the fibration gives two affine structures ∇_A, ∇_B and one metric g on the base B. Use the same notation as Definition 4.1. The affine structure ∇_A (resp., ∇_B) comes from the Kähler form ω (resp., the imaginary part of the holomorphic volume form $\mathrm{Im}\Omega$). The metric g is called the *Mclean metric*. It is known that two of the triple (∇_A, ∇_B, g) determine the rest. Refer to [Gro13, §1] for details.

4.6. We recall the *Gromov-Hausdorff limit* (cf. [BBI01]). We can define the *Gromov-Hausdorff distance* $d_{\mathrm{GH}}(X, Y)$ between two metric spaces X and Y. Further, it is known that this distance d_{GH} is a metric function on the set $\mathbb{M}$ consisting of the isometry classes of compact metric spaces. In Gromov's celebrated paper [Gro81], he proved that the subset $\mathcal{M}$ of $\mathbb{M}$ consisting of the isometry classes of compact Riemannian manifolds with Ricci curvature bounded below and diameter bounded above is relatively compact with respect to the Gromov-Hausdorff distance. It is known as Gromov's compactness theorem. That is, the closure $\overline{\mathcal{M}}^{\mathrm{GH}}$ of $\mathcal{M}$ in $\mathbb{M}$ is compact. Hence we can define a notion of convergence for sequences in $\mathcal{M}$, called Gromov-Hausdorff convergence. In particular, for any sequence of Ricci flat manifolds, we can take a convergent subsequence by rescaling the diameters to be 1. A compact metric space to which such a sequence converges is called *the Gromov-Hausdorff limit* of the sequence.

4.7. From now on, we use the following notation.

- Δ denotes the open disk $\{t \in \mathbb{C} \mid |t| < 1\}$.
- Δ^* denotes the punctured disk $\{t \in \mathbb{C} \mid 0 < |t| < 1\}$.
- $\mathbb{C}((t))^{\mathrm{mero}}$ denotes the field of the germs of meromorphic functions with possible pole only at the origin.
- The t-adic discrete valuation of $\mathbb{C}((t))^{\mathrm{mero}}$ is denoted as
$$\mathrm{val}_t \colon (\mathbb{C}((t))^{\mathrm{mero}} \setminus \{0\}) \to \mathbb{Z}.$$
- $(\mathcal{X}, \mathcal{L}) \to \Delta$ (resp., $(\mathcal{X}^*, \mathcal{L}^*) \to \Delta^*$) denotes a proper holomorphic map from a complex analytic space $\mathcal{X}$ (resp., $\mathcal{X}^*$) together with an ample line bundle $\mathcal{L}$ (resp., $\mathcal{L}^*$). We also denote as $(\mathcal{X}, \mathcal{L})/\Delta$ (resp., $(\mathcal{X}^*, \mathcal{L}^*)/\Delta^*$).
- Each fiber of $\mathcal{X}$ over $t \in \Delta$ is denoted as $\mathcal{X}_t$. The restriction of $\mathcal{L}$ to $\mathcal{X}_t$ is denoted as $\mathcal{L}_t$.
- For the above pair $(\mathcal{X}^*, \mathcal{L}^*)$, we associate the smooth projective variety X over $\mathbb{C}((t))^{\mathrm{mero}}$ with ample line bundle L.

- When $\mathcal{X}_t$s are abelian varieties of dimension g, the type of polarization $\mathcal{L}_t$ (cf. (2.28)) is

$$(e_1, \cdots, e_g) \in \mathbb{Z}^g_{>0}$$

such that $e_i \mid e_{i+1}$ for any $(0 <)i(< g)$. We associate a $g \times g$ diagonal matrix $E := \mathrm{diag}(e_1, \cdots, e_g)$.

4.2. **Abelian surfaces case.**

4.8. This section explains one method for partially proving Conjecture 1.4 by using hyperKähler rotation, following [OO21, §4]. The point is that, although *loc.cit* focused on the case of K3 surfaces, the same method applies to the case of abelian surfaces. In the next section, we generalize to higher dimensional abelian varieties by a more explicit different method.

Theorem 4.9 ([GO22, Theorem 2.1])**.** *Take any maximally degenerating family of polarized abelian surfaces (resp., polarized K3 surfaces possibly with ADE singularities) $(\mathcal{X}|_{\Delta^*}, \mathcal{L}|_{\Delta^*})$ over Δ^* with a fiber-preserving symplectic action of a finite group H on $\mathcal{X}|_{\Delta^*}$ together with linearization on $\mathcal{L}|_{\Delta^*}$ (e.g., H can be even trivial or simple $\{\pm 1\}$-multiplication in the case of abelian varieties). We denote the quotient by H as $(\mathcal{X}'|_{\Delta^*}, \mathcal{L}'|_{\Delta^*}) \to \Delta^*$. Then, the following hold:*

(i) *For any $t \in \Delta^*$ with $|t| \ll 1$, there is a special Lagrangian fibration $f_t\colon \mathcal{X}_t \to \mathcal{B}_t$ with respect to the Kähler form ω_t of the flat Kähler metric $g_{\mathrm{KE}}(\mathcal{X}_t)$ with $[\omega_t] = c_1(\mathcal{L}_t)$ and the imaginary part $\mathrm{Im}(\Omega_t)$ of a certain holomorphic volume form $(0 \neq)\Omega_t \in H^0(\mathcal{X}_t, \omega_{\mathcal{X}_t})$. Here, $\mathcal{B}_t$ is a 2-torus (resp., S^2) and so are all fibers of f_t. As in* (4.5)*, we denote by $\nabla_A(t)$ and $\nabla_B(t)$ the induced tropical affine structures on $\mathcal{B}_t$ by ω_t and $\mathrm{Im}(\Omega_t)$ respectively, as well as its McLean metric g_t. Below, we continue to assume $|t| \ll 1$. In the next section §6, we discuss how these f_t glue to a family.*

(ii) *Consider the obtained base associated with a tropical affine structure and a flat metric $(\mathcal{B}_t, \nabla_A(t), \nabla_B(t), g_t)$ for $t \neq 0$. They converge to another a 2-torus (resp., S^2) with the same additional structures $(\mathcal{B}_0, \nabla_A(0), \nabla_B(0), g_0)$ in the natural sense, when $t \to 0$. In this terminology, the Gromov-Hausdorff limit of*

$$(\mathcal{X}_t, g_{\mathrm{KE}}(\mathcal{X}_t)/\mathrm{diam}(g_{\mathrm{KE}}(\mathcal{X}_t))^2)$$

for $t \to 0$ coincides with $(\mathcal{B}_0, g_0)$. Here, $\mathrm{diam}(g_{\mathrm{KE}}(\mathcal{X}_t))$ refers to the diameter which is used to rescale the metric to that of diameter 1 (as in [KS06]*).*

(iii) *The H-action on $\mathcal{X}_t$ preserves the fibers of f_t. Thus, there is a natural induced action of H on $\mathcal{B}_0$, which preserves the three structures $\nabla_A(0), \nabla_B(0)$ and g_0. The natural quotient of f_t by H denoted as $f'_t\colon \mathcal{X}'_t \to \mathcal{B}'_t$ is again a special Lagrangian fibration with respect to the descents of ω_t the holomorphic volume form and $(0 \neq)\Omega_t \in H^0(\mathcal{X}_t, \omega_{\mathcal{X}_t})$ we chose above.*

(iv) *If $(\mathcal{X}|_{\Delta^*}, \mathcal{L}|_{\Delta^*})$ is a family of principally polarized abelian surfaces and H is trivial, the tropical affine structure $\nabla_A(0)$ on $\mathcal{B}_0$ is integral (cf. Definition 4.3) and its integral points consist of only 1 point, which automatically determines $\nabla_A(0)$. The corresponding Gram matrix of g_0 is the same $(cB(l_i, l_j))$ as appeared in* (2.39). *Also the transition function of the integral basis of $\nabla_A(0)$ to that of $\nabla_B(0)$ is given by the same matrix $(cB(l_i, l_j))$.*

Proof. Our assertions above (i) and (ii) for K3 surfaces case are proven in [OO21, Chapter 4 (and 5,6 partially)].

The proof of (i), (ii) for the case for polarized abelian surfaces and essentially easier and here we follow the method of *loc.cit*. (In the next section, we give another proof.) Hence, we below only sketch (review) the proof as a review and explain the differences with the original K3 surfaces case in [OO21, §4]. More precisely speaking, we use the arguments of the proofs of [OO21, Theorems 4.11, 4.20] and other claims on which they depend.

The (almost verbatim) change of basic setup is as follows. For an abelian surface $X \simeq \mathbb{C}^2/\Lambda$ with a lattice $\Lambda(\simeq \mathbb{Z}^4)$, set $\Lambda_{CT} := \bigwedge_{\mathbb{Z}}^2 \Lambda$. (Here, the subscript CT stands for complex torus.) By the orientation on G induced by the complex structure, we identify $\bigwedge_{\mathbb{Z}}^4 \Lambda \simeq \mathbb{Z}$ which induces a lattice structure on Λ_{CT} as isomorphic to $\mathrm{II}_{3,3} = U^{\oplus 3}$. Take a marking α of $H^2(\mathcal{X}_t, \mathbb{Z})$ for some t, and put $\lambda := \alpha(c_1(\mathcal{L}_t))$, and replace Λ_{2d} of [OO21, Chapter 4] by $\Lambda_{pas} := \lambda^{\perp} \subset \Lambda_{CT}$ which has signature $(2,3)$. The period domain for $(\mathcal{X}_t, \mathcal{L}_t)$ with weight 2 is

$$\Omega(\Lambda_{pas}) := \{[w] \in \mathbb{P}(\Lambda_{pas} \otimes \mathbb{C}) \mid (w, w) = 0, (w, \bar{w}) > 0\},$$

which replaces $\Omega(\Lambda_{\mathrm{K3}})$ of [OO21, §4.1]. By the accidental isomorphism $\mathrm{Sp}(4, \mathbb{R})/\{\pm 1\} \simeq SO_0(2,3)$, we can also identify the connected component of $\Omega(\Lambda_{pas})$ as the Siegel upper half space $\mathfrak{S}_2$ of degree 2. Similarly, we also define and use the union of Kähler cones $K\Omega$ (resp., $K\Omega^0$, $K\Omega^{e \geq 0}$) as [OO21, p.45-46]. Since the Kähler cones of complex tori are just connected components of positive cones, the definition of *loc.cit* works verbatim if we put $\Delta(\Lambda_{CT}) := \emptyset$, where $\Delta(\Lambda_{CT})$ is the set of simple roots.

Now we are ready to prove the above theorem. The desired special Lagrangian fibration (i) for polarized K3 surfaces case is obtained at the bottom of [OO21, p.47] (if we see π_i as a map from X_i) during the proof of [OO21, 4.20]. The proof uses (*loc.cit* Claim 6.12 which in turn follows from) Fact 4.14 and Claim 4.18 of *loc.cit*. It applies a hyperKähler rotation, which also works for flat complex tori because their holonomy groups are even trivial.

Proof of Claim 4.18 becomes much simpler for 2-dimensional complex torus case because we do not need to deal with the roots as "$\Delta(X)^+$".

The only nontrivial difference exists for Fact 4.14 of [OO21]. Indeed, it does *not* literally holds for complex tori just because the morphism from complex 2-tori to elliptic curves are not obtained as pencils.

Fact 4.10. *Let X be a complex 2-torus and let $e \in H^2(X, \mathbb{Z})$ a (primitive) isotropic. If e belongs to the closure of its Kähler cone, there exists a holomorphic fibration to an elliptic curve B as $X \twoheadrightarrow B$ whose fiber class is e.*

proof of Fact 4.10. We take a line bundle L with $c_1(L) = e$. From [BL04, §3.3], L is effective and represented by an elliptic curve E by the isotropicity and primitivity condition. If we replace E by a translation of E which contains the origin of X, then $X \twoheadrightarrow B := X/E$ is the desired morphism. □

Note that the fibers are only homologically identified as e, rather than linear equivalence classes (as in [OO21, 4.14] for K3 surfaces).

The proof of (ii) is similar to the construction of $X \to B$ of [OO21, bottom of p.34-p.35]. The only difference again is that we use above Fact 4.10 instead of Fact 4.14 of *loc.cit*. Then the proof is reduced to that of 4.22 of *loc.cit*, hence that of Chapter 5. The arguments there work verbatim and is even greatly simplifiable. Indeed, the main analytic difficulties in K3 surfaces case come from the presence of (varying) ADE singularities, which destroy the simplest uniform C^2-estimates. (In our case, it is even possible to confirm 4.22 of *loc.cit* explicitly.)

From here, we give the proofs of (iii), (iv). To show (iii), we proceed to analyze the effect of the action of H. Since the H-action on $\mathcal{X}_t$ is assumed to be symplectic i.e., the induced action on $H^0(\mathcal{X}_t, \Omega^2_{\mathcal{X}_t})$ is trivial, in particular $\mathrm{Im}(\Omega_t)$ is preserved by the action of H. Therefore, the complex structure of the hyperKähler rotation of $\mathcal{X}_t$ as constructed in [OO21, p.47] is also preserved by the action. The action also preserves the isotropic cohomology class e. Hence, from the construction of f_t, the action preserves its fibers and the claims of (iii) are proved.

For the proof of (iv), note that under an appropriate symplectic basis v_1, w_1, v_2, w_2 of $H^1(\mathcal{X}_t, \mathbb{Z})$ for the principal polarization $\langle -, - \rangle$ i.e., $\langle v_i, w_j \rangle = \delta_{i,j}$ and $\langle v_i, v_j \rangle = \langle w_i, w_j \rangle = 0$, e is written as $v_1 \wedge w_1$, $H^2(\mathcal{B}_t, \mathbb{Z})$ is written as $\mathbb{Z}(v_2 \wedge w_2)$, $\lambda = v_1 \wedge w_1 + v_2 \wedge w_2$. Then the direct computation shows that the integral affine structure $\nabla_A(0)$ on $\mathcal{B}_0 \simeq \mathbb{R}^2/\mathbb{Z}^2$ is the desired one.

The last claim then follows from the definition of Legendre transform (cf., [Hit97, §3], [Gro13, §1]). □

Remark 4.11. To construct the tuple $(\mathcal{B}_0, \nabla_A(0), \nabla_B(0), g_0)$ as Theorem 4.9 is called *SYZ picture*. Similarly, to construct the Gromov-Hausdorff limit of $(\mathcal{X}_t, g_{\mathrm{KE}}(\mathcal{X}_t)/\mathrm{diam}(g_{\mathrm{KE}}(\mathcal{X}_t))^2)$ for $t \to 0$ is called *Gromov-Hausdorff limit picture*. In [KS06], SYZ picture under the assumption that Gromov-Hausdorff limit picture holds is called *Collapse picture*. Note that SYZ picture gives two affine structures and one metric on the base, however, Gromov-Hausdorff limit picture gives only a metric space. Theorem 4.9 (ii) implies that SYZ picture and Gromov-Hausdorff picture for any maximally degenerating family of polarized abelian surfaces (resp., polarized K3 surfaces possibly with ADE singularities) give the same metric space.

Remark 4.12. The choice of Ω_t for each t is a priori up to constant multiplication but in [OO21, §4, especially (4.11)] and the above proof, we implicitly chose it carefully.

Remark 4.13 (Effect of monodromy on the hyperKahler rotation)**.** Both the constructions of $f_t\colon \mathcal{X}_t \to \mathcal{B}_t$ in [OO21, §4] and above theorem 4.9, are as holomorphic Lagrangian fibrations of the hyperKähler rotation $\mathcal{X}_t^\vee$. Let us consider the effect of monodromy $T \in \mathrm{GL}(H^2(\mathcal{X}_t, \mathbb{Z}))$ i.e., when t goes around the origin. Since T preserves the isotropic fiber class e of f_t, which is associated to the maximal degeneration $\mathcal{X} \to \Delta^*$, T only changes the marking and hence preserves f_t by the theory of harmonic integrals. However, note that the induced change of marking does change the hyperKähler rotation $\mathcal{X}_t^\vee$, hence $\mathcal{X}_t^\vee$ does not form a family. Indeed, as t goes around 0, then $\mathcal{X}_t^\vee$ will be twisted i.e., have different complex structures, while preserving their Jacobian fibrations.

Remark 4.14. In the context of Theorem 4.9, note that each family

$$(\mathcal{X}'|_{\Delta^*}, \mathcal{L}'|_{\Delta^*}) \to \Delta^*$$

may have several description as a quotient of K-trivial surfaces by finite group H, as $(\mathcal{X}|_{\Delta^*}, \mathcal{L}|_{\Delta^*}) \to (\mathcal{X}'|_{\Delta^*}, \mathcal{L}'|_{\Delta^*})$. Indeed, for some polarized family $(\mathcal{X}'|_{\Delta^*}, \mathcal{L}'|_{\Delta^*})$, one of covering family $(\mathcal{X}|_{\Delta^*}, \mathcal{L}|_{\Delta^*})$ is a family of polarized abelian surfaces while another is a family of polarized K3 surfaces.

Also note that if the fibers $\mathcal{X}_t$ are K3 surfaces, so are $\mathcal{X}'_t$. However, if the fibers $\mathcal{X}_t$ are abelian surfaces, $\mathcal{X}'_t$ becomes abelian surfaces if H only consists of translations, while X'_t becomes K3 surfaces otherwise.

Corollary 4.15 ([Got22, Corollary 5.28])**.** *Use the same notation as in Theorem 4.9. We assume that $(\mathcal{X}|_{\Delta^*}, \mathcal{L}|_{\Delta^*})$ is principally polarized. Then the Gromov-Hausdorff limit of $(\mathcal{X}'_t, g'_{\mathrm{KE}}(\mathcal{X}'_t))$ for $t \to 0$ coincides with the*

Gromov-Hausdorff limit $(\mathcal{B}'_0, g'_0)$ *of* $(\mathcal{B}'_t, g'_t)$ *for* $t \to 0$*, where the metric* $g'_{\mathrm{KE}}(\mathcal{X}'_t)$ *(resp.,* g'_t*) on* $\mathcal{X}'_t$ *(resp.,* $\mathcal{B}'$*) is induced by* $g_{\mathrm{KE}}(\mathcal{X}_t)$ *(resp.,* g_t*). Furthermore, the affine manifold* $(\mathcal{B}'_0, \nabla'_B(0))$ *with singularities coincides with the quotient of the affine manifold* $(\mathcal{B}_0, \nabla_B(0))$ *by the group* H*, where the affine structure* $\nabla'_B(0)$ *with singularities is induced by* $\nabla_B(0)$*. In particular, we can regard the affine structure* $\nabla'_B(0)$ *with singularities as an IAMS structure by rescaling.*

Proof. It follows from Theorem 4.9 (iii). In addition, the affine structure $\nabla_B(0)$ of B_0 is detemined by the matrix $B(l_i, l_j)$ up to scaling, by Theorem 4.9 (iv). Hence, the last assertion holds. □

4.3. **Higher dimensional abelian varieties case.**

4.16. In this section, we prove Conjecture 1.4 for maximally degenerating abelian varieties in any dimension. The main tool is the following explicit description of the degenerations, which itself may be of interest. It is a complex analytic version of the well-known non-archimedean uniformization of abelian variety over a non-archimedean valued field by Mumford [Mum72] and Faltings-Chai [FC90]. Although [Oda19] included the statement with a rough sketchy proof, since we could not find a precise discussion in the literature, we include it here of the somewhat more global statement, with a more topological or analytic proof.

Lemma 4.17 ([GO22, Lemma 2.6], General description of maximal degenerating abelian varieties)**.** *Take any maximally degenerating family of polarized abelian varieties of any dimension* g*, which we denote again as* $(\mathcal{X}|_{\Delta^*}, \mathcal{L}|_{\Delta^*})$ *over* Δ^**. Suppose that the polarization is of type* $(e_1, \cdots, e_g)$ *where* $e_i (1 \leq i \leq g)$ *are all positive integers which satisfy* $e_i \mid e_{i+1}$ *for any* i*. (For instance,* e_i *are all* 1 *for principal polarization.) Then, these families are characterized explicitly as*

$$((\mathbb{C}^*)^g \times \Delta^*)/\mathbb{Z}^g \to \Delta^*, \tag{1}$$

where $\mathbb{Z}^g \ni {}^t(m_1, \cdots, m_g)$ *acts on* $(\mathbb{C}^*)^g \times \Delta^* \ni (z_1, \cdots, z_g, t)$ *by*

$$(z_1, \cdots, z_g, t) \mapsto \left((z_1, \cdots, z_g) \cdot {}^t\left(\prod_{1 \leq j \leq g} p_{i,j}(t)^{m_j} \right)_{1 \leq i \leq g}, t \right)$$

for some meromorphic functions $p_{i,j}(t) \in \mathbb{C}((t))$*. Here, meromorphic functions* $q_{i,j}(t) := p_{i,j}(t)^{e_i} \in \mathbb{C}((t))$ *satisfies the following:*

(i) *possible pole only at* $t = 0$*.*
(ii) *each* $q_{i,j}(t)$ *converges at* $t \in \Delta^*$*.*
(iii) $(q_{i,j}(t))_{1 \leq i,j \leq g}$ *is a symmetric matrix.*

From here, we often describe such family (1) *simply as either*

$$\bigsqcup_t (\mathbb{C}^*)^g/\langle p_{i,j}(t)\rangle \textit{ or } ((\mathbb{C}^*)^g \times \Delta^*)/\langle p_{i,j}(t)\rangle.$$

Proof. Note the polarized family is induced from a map $\tilde{\varphi}\colon \mathbb{H} \to \mathbb{H}_g$ where $\mathbb{H}(=\mathbb{H}_1)$ is the upper half plane, $\mathbb{H}_g$ is the (higher dimensional) Siegel upper half space of dimension $\frac{g(g+1)}{2}$, which descends to $\varphi\colon \Delta^* \to \mathrm{Sp}(E,\mathbb{Z})\backslash\mathbb{H}_g$. Here, we set

$$E := \begin{pmatrix} e_1 & 0 & \cdots & 0 \\ 0 & e_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & e_g \end{pmatrix}, \quad \tilde{E} := \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},$$

and

$$\mathrm{Sp}(E,\mathbb{Z}) := \{h \in \mathrm{GL}(2g,\mathbb{Z}) \mid h\tilde{E}{}^t h = \tilde{E}\}.$$

We denote $\mathrm{Sp}(E,\mathbb{Z})$ also as Γ. Consider the (rational) 0-cusp F of $\mathbb{H}_g$ in the Satake-Baily-Borel compactification (before dividing by Γ). Then the monodromy lies in the unipotent radical (cf., [ADMY75, Chapter III])

$$\begin{aligned} U(F)_{\mathbb{Z}} &= \left\{ \begin{pmatrix} I_g & B \\ 0 & I_g \end{pmatrix} \mid B \in \mathrm{GL}(g,\mathbb{R}) \right\} \cap \Gamma \\ &= \left\{ \begin{pmatrix} I_g & B \\ 0 & I_g \end{pmatrix} \mid B \in \mathrm{GL}(g,\mathbb{Z}), BE = {}^t(BE) \right\}. \end{aligned}$$

Therefore, if we denote the fiber $\mathcal{X}_t (t \in \mathbb{H} \text{ or } \Delta^*)$ as $\mathbb{C}^g/\begin{pmatrix} E \\ \Lambda \end{pmatrix}\mathbb{Z}^{2g}$ with $\Lambda \in \mathbb{H}_g$, for each $i \in \{1,\cdots,g\}$, $\Lambda(0,\cdots,0,\overbrace{1}^{i-\text{th}},0,\cdots,0)$ is invariant modulo $E\mathbb{Z}^g$. Hence, we can write $\mathcal{X}^*(=\mathcal{X}|_{\Delta^*})$ as $\sqcup_t((\mathbb{C}^*)^g)/\langle p_{i,j}(t)\rangle$ for some holomorphic functions $p_{i,j}$s over Δ^*. Note that the natural quotient of $\mathcal{X}^*$ by $\mu_{e_1} \times \cdots \times \mu_{e_g}$ is a family of principally polarized abelian varieties over Δ^*, which can be also written as $\sqcup_t((\mathbb{C}^*)^g)/\langle q_{i,j}(t)\rangle$ for some holomorphic functions $q_{i,j}$s over Δ^*. Here, since $(\frac{\mathrm{Im}(\log q_{i,j}(t))}{2\pi\sqrt{-1}})_{1\le i,j\le g}$ lie inside $\mathbb{H}_g$ for any $t \in \Delta^*$, $q_{i,j} = q_{j,i}$ and $p_{i,j}(t)^{e_i} = q_{i,j}(t)$. Again, since $(\frac{\mathrm{Im}(\log q_{i,j}(t))}{2\pi\sqrt{-1}})_{1\le i,j\le g}$ lie inside $\mathbb{H}_g$ for any $t \in \Delta^*$, it follows that all $q_{i,j}(t)$ and $p_{i,j}(t)$ are meromorphic. □

Below we give an additive description as well and fix notation.

4.18. If we describe the above abelian varieties in Lemma 4.17 in additive manner,

$$\mathcal{X}_t = \mathbb{C}^g / \begin{pmatrix} e_1 & 0 & 0 & \cdots & 0 & \\ 0 & e_2 & 0 & \cdots & 0 & \\ 0 & 0 & e_3 & \cdots & 0 & \frac{e_i \log p_{i,j}(t)}{2\pi\sqrt{-1}} \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & e_g & \end{pmatrix} \mathbb{Z}^{2g} \tag{2}$$

$$= (\mathbb{C}^*)^g / \langle p_{i,j}(t) \rangle_{i,j} \tag{3}$$

where the first identification is through the exponential map

$$z_i \mapsto e^{\frac{2\pi\sqrt{-1} z_i}{e_i}} =: Z_i (i = 1, \cdots, g).$$

If we set

$$E := \mathrm{diag}(e_1, \cdots, e_g) \tag{4}$$

$$= \begin{pmatrix} e_1 & 0 & 0 & \cdots & 0 \\ 0 & e_2 & 0 & \cdots & 0 \\ 0 & 0 & e_3 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & e_g \end{pmatrix}, \tag{5}$$

$$\Omega(t) := \left(\frac{1}{2\pi\sqrt{-1}} \log p_{i,j}(t) \right)_{i,j}, \tag{6}$$

and

$$\Omega'(t) := \left(\frac{e_i}{2\pi\sqrt{-1}} \log p_{i,j}(t) \right)_{i,j}, \tag{7}$$

then the dividing lattice of the above (2) splits as $E\mathbb{Z}^g \oplus \Omega'(t)\mathbb{Z}^g$ i.e.,

$$\Lambda_t := \begin{pmatrix} e_1 & 0 & 0 & \cdots & 0 & \\ 0 & e_2 & 0 & \cdots & 0 & \\ 0 & 0 & e_3 & \cdots & 0 & \frac{e_i \log p_{i,j}(t)}{2\pi\sqrt{-1}} \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & e_g & \end{pmatrix} \mathbb{Z}^{2g} \tag{8}$$

$$= E\mathbb{Z}^g \oplus \Omega'(t)\mathbb{Z}^g. \tag{9}$$

Note that, $\Omega'(t) = E\Omega(t)$ is symmetric, and $\Omega(t)$ (resp., $\Omega'(t)$) is determined up to $\mathbb{Z}^g$ (resp., $E\mathbb{Z}^g$).

We continue to use this notation for the coordinates z_is and Z_is, and set $x_i := \mathrm{Re}(z_i), y_i := \mathrm{Im}(z_i)$. We write

$$\frac{\log p_{i,j}(t)}{2\pi\sqrt{-1}} = \alpha_{i,j}(t) + \sqrt{-1}\beta_{i,j}(t), \tag{10}$$

with an ambiguity of $\alpha_{i,j}(t)$ modulo $\mathbb{Z}$. Indeed, if t goes around the origin 0, the monodromy effect is nontrivial only on $\alpha_{i,j}$ but trivial on $\beta_{i,j}(t)$. Similarly, for $q_{i,j}(t) = p_{i,j}(t)^{e_i}$, we write

$$\frac{\log q_{i,j}(t)}{2\pi\sqrt{-1}} = \alpha'_{i,j}(t) + \sqrt{-1}\beta'_{i,j}(t). \tag{11}$$

In particular, $(\beta'_{i,j}(t)) = \mathrm{Im}\Omega'(t)$.

Below, we prepare a basic lemma on the symmetry of maximally degenerating family, for the next theorem 4.20 (iv) which discusses finite quotients of abelian varieties. We explore more details in later section 6.3.

Lemma 4.19 ([GO22, Lemma 2.7])**.** *For the family of above Lemma 4.17, i.e.,*

$$\mathcal{X}^* = \sqcup_{t\in\Delta^*}((\mathbb{C}^*)^g/\langle p_{i,j}(t)\rangle_{i,j})_t \to \Delta^*_t,$$

we continue to use the notation in (4.18)*.*

Suppose there is an action of a group H holomorphically on $\mathcal{X}^*$ preserving the fibers $\mathcal{X}_t$ and $c_1(\mathcal{L}_t)$. We do not assume it preserves the 0*-section. For each $t \in \Delta^*$, recall that*

$$\Lambda_t = E\mathbb{Z}^g \oplus \Omega'(t)\mathbb{Z}^g \tag{12}$$

as a sublattice of $\mathbb{C}^g$. Then, for any $h \in H$, the induced $h_*\colon H_1(\mathcal{X}_t,\mathbb{Z}) \to H_1(\mathcal{X}_t,\mathbb{Z})$ preserves the first direct summand $\mathbb{Z}^g$ of the above (12)*. Restriction of h_* to it is denoted as $l_h \in \mathrm{GL}(g,\mathbb{Z})$. Further,*

$$\begin{array}{rccc} l\colon & H & \longrightarrow & \mathrm{GL}(g,\mathbb{Z}) \\ & \cup & & \cup \\ & h & \longmapsto & l_h \end{array},$$

is a group homomorphism.

Proof. The H-action on $\mathcal{X}_t$ induces a complex linear transformation $f_h \in \mathrm{GL}(g,\mathbb{C})$, i.e., at the level of universal covers of $\mathcal{X}_t$s, and what we want to show is that the image lies inside $\mathrm{GL}(g,\mathbb{Z})$. If we take only real $t \in \mathbb{R}_{>0}$, we can take a (continuous) branch of $\beta_{i,j}(t)$ to avoid possible ambiguity due to the monodromy. Then, each Λ_t for $t \in \mathbb{R}_{>0}$ can be canonically identified with $\mathbb{Z}^{2g}$. Take $e_1,\cdots,e_g,f_1,\cdots,f_g$ as the standard basis of $\Lambda_t = \mathbb{Z}^{2g}$, i.e., set

$$e_i := (0,\cdots,0,\overbrace{1}^{i-\mathrm{th}},0,\cdots,0)$$

and

$$f_i := (0, \cdots, 0, \overbrace{1}^{(i+g)-\text{th}}, 0, \cdots, 0)$$

for each $i \in \{1, \cdots, g\}$.

Now we prove the assertion by contradiction. Assume the contrary i.e., there is some $h \in H$ and $1 \leq i \leq g$ such that $l_h(e_i)$ does not sit inside $\sum_{1 \leq i \leq g} \mathbb{R}e_i$. Since $\mathrm{GL}(\Lambda_t) = \mathrm{GL}(2g, \mathbb{Z})$ is a discrete group, the l_h is induced by a particular isomorphism $g_h \colon \Lambda_t \to \Lambda_t$ for *any* t. That is, for any h and $t \in \Delta^*$, $l_h(\Lambda_t) = \Lambda_t$ so that g_h is obtained as a restriction of l_h. From the assumption that the imaginary part of $g_h(e_i) = l_h(e_i)$ is not the zero vector, if we write

$$g_h(e_i) = \sum_{1 \leq j \leq g} a_j e_j + \sum_{1 \leq j \leq g} b_j f_j.$$

where all a_j, b_js are integers, one can assume $b_j \neq 0$ for some j. Note

$$(\beta'_{i,j}(t))_{1 \leq i,j \leq g} \sim \frac{-\log |t|}{2\pi} (B'_{i,j})_{1 \leq i,j \leq g},$$

with positive definite symmetric matrix $(B'_{i,j})$, a part of Faltings-Chai degeneration data. That is, $B'_{i,j} = b(l_i, \phi(l_j))$ under the notation of (2.39). Here, $\sim$ means the ratio converges to 1 when $t \to 0$. Also, the flat metric's Gram matrix is, if we set $X(t) = (\alpha'_{i,j}(t))_{i,j}$, $Y(t) = (\beta'_{i,j}(t))_{i,j}$, then

$$\begin{pmatrix} E & 0 \\ X(t) & Y(t) \end{pmatrix} \begin{pmatrix} Y(t)^{-1} & 0 \\ 0 & Y(t)^{-1} \end{pmatrix} \begin{pmatrix} E & X(t) \\ 0 & Y(t) \end{pmatrix} \tag{13}$$

$$= \begin{pmatrix} EY(t)^{-1}E & EY(t)^{-1}X(t) \\ X(t)Y(t)^{-1}E & X(t)Y(t)^{-1}X(t) + Y(t) \end{pmatrix} \tag{14}$$

(cf., e.g. [Oda19, (3) during the proof of Theorem 2.1]) so that in particular $|e_i| \to 0$ while $|f_h(e_i)| \to \infty$ for $t \to 0$. Here, $|\cdot|$ denotes the length with respect to the canonical flat Kähler metric g_{KE} on $\mathcal{X}_t$. This contradicts the fact that H-action on $\mathcal{X}_t$ is an isometry for all $t \neq 0$. Hence the assertion is proven by contradiction. □

Now, we are ready to show the existence and description of special Lagrangian fibrations for maximally degenerating abelian varieties.

Theorem 4.20 ([GO22, Theorem 2.8])**.** *For any maximally degenerating family $(\mathcal{X}|_{\Delta^*}, \mathcal{L}|_{\Delta^*})/\Delta^*$ of polarized abelian varieties of dimension g, we again denote by ω_t the Kähler form of the flat metric $g_{\mathrm{KE}}(\mathcal{X}_t)$ with $[\omega_t] = c_1(\mathcal{L}_t)$. We also take a certain non-zero element $(0 \neq)\Omega_t \in H^0(\mathcal{X}_t, \omega_{\mathcal{X}_t} = \wedge^g \Omega_{\mathcal{X}_t})$. (See the proof for the actual choice of Ω_t.) Recall that $\mathcal{X}_t$ can be described as $\mathbb{C}^g/\Lambda_t$ for the lattice Λ_t of the form* (8) *of Lemma 4.19.*

Then, the following holds:

(i) *For each $t \in \Delta^*$, special Lagrangian fibration $f_t \colon \mathcal{X}_t \to \mathcal{B}_t$ with respect to ω_t and Ω_t exists ($g = 2$ case is proven in Theorem 4.9). $\mathcal{B}_t$ is a g-dimensional real torus described as*

$$\mathbb{R}^g/(\beta_{i,j}(t))_{1\le i,j\le g}\mathbb{Z}^g$$

under the notation (10)*. Furthermore, they fit into a family i.e., there is a continuous map $f \colon \mathcal{X}|_{\Delta^*} \to \cup_{t\neq 0}\mathcal{B}_t$ for a certain extended topology on $\cup_{t\neq 0}\mathcal{B}_t$, which restricts to f_t for each $t \neq 0$.*

(ii) *The tropical affine structure $\nabla_A(t)$ on $\mathcal{B}_t = \mathbb{R}^g/(\beta_{i,j}(t))_{1\le i,j\le g}\mathbb{Z}^g$ is descended from the integral affine structure on $\mathbb{R}^g$ whose integral points are $(\gamma_{i,j}(t))_{1\le i,j\le g}\mathbb{Z}^g$, where*

$$(\gamma_{i,j}(t))_{i,j} := E^{-1} \cdot (\beta'_{i,j}(t))_{i,j} \cdot E^{-1}$$

under the notation (11)*.*

(iii) *The tropical affine structure $\nabla_B(t)$ on $\mathcal{B}_t = \mathbb{R}^g/(\beta_{i,j}(t))_{1\le i,j\le g}\mathbb{Z}^g$ is descended from the integral affine structure on $\mathbb{R}^g$ whose integral points are $\frac{-\log|t|}{2\pi}\mathbb{Z}^g$.*

(iv) *Suppose there is a fiber-preserving holomorphic action of a group H on $(\mathcal{X}, c_1(\mathcal{L}))$ (as in Lemma 4.19). Then for each $t \neq 0$, the H-action on $\mathcal{X}_t$ descends to that on $\mathcal{B}_t$ which is unified into a continuous H-action on $\cup_{t\neq 0}\mathcal{B}_t$ with respect to which $\mathcal{X}^* := \mathcal{X}|_{\Delta^*} \to \cup_{t\neq 0}\mathcal{B}_t$ is H-equivariant.*

Further, if the group H also fixes the isomorphic class of $\mathcal{L}_t$ for each $t \neq 0$ (not only their c_1), then H must be a finite group. In that case, we can take the quotient of the map f by H. We denote the obtained map as $f/H \colon \mathcal{X}^*/H \to \cup_{t\neq 0}(\mathcal{B}_t/H)$.

(v) *Under the situation above* (iv)*, suppose further that the H-action on $\mathcal{X}^*$ preserves relative holomorphic n-forms i.e., acts trivially on $\omega_{\mathcal{X}^*/\Delta^*} = \mathcal{O}(K_{\mathcal{X}^*/\Delta^*})$. Then, for each $t \neq 0$, f/H is restricted to a special Lagrangian fibration with respect to the descent of ω_t and Ω_t. Further, they are of the form $g_t \colon \mathcal{X}_t/H \to \mathcal{B}_t/H$, where the additional structures on $\mathcal{B}_t/H$ is those descended from the ones described above* (ii), (iii).

Remark 4.21. A weaker version of the above statements (i) i.e., existence of special Lagrangian fibrations in *large open subset of* $\mathcal{X}_t$ are essentially proven also as the simple combination of [Li20, Theorem 1.3] and [Liu11]. (i) of the above result refines it, even in an explicit way.

Proof. We set $B'_{i,j} := \mathrm{val}_t(q_{i,j}(t))$. Recall that [Oda19, OO21, Got22] repeatedly proved that the Gromov-Hausdorff limit of $\mathcal{X}_t$ is identified with the g-dimensional torus with the Gram matrix $(B'_{i,j})_{1\le i,j\le g}$. Indeed, as we saw in (2.26), the standard description of Riemann forms tells us that

$$\omega_t = \frac{\sqrt{-1}}{2} \sum_{1\le i,j\le g} \gamma_{i,j}(t) dz_i \wedge d\overline{z_j}, \tag{15}$$

where $(\gamma_{i,j}(t))_{i,j} = E \cdot (\beta'_{i,j}(t))^{-1}_{i,j} \cdot E$ for the the basis $\{z_i\}$ of $\mathbb{C}^g$ such that the period matrix is of the form $(I\ \Omega(t))$ under the notation (6).

Remark 4.22. In particular, the conjectural asymptotic formula of Ricci-flat Kähler forms by [Li20, (11) also cf., §4.2, §4.5] holds even globally in this abelian varieties case.

From below, we use

$$\Omega_t = \frac{-2\pi}{\log|t|} dz_1 \wedge \cdots \wedge dz_g \in H^0(\mathcal{X}_t, K_{\mathcal{X}_t}).$$

This choice is carefully made as multiplication by constant does change the associated special Lagrangian fibrations.

Now we prove (i). We use Lemma 4.17 to set $\mathcal{X}_t = ((\mathbb{C}^*)^g/\langle p_{i,j}(t)\rangle_{i,j})$, and use the description and the notations (3), (10) and (15) above. Then, for each fiber $\mathcal{X}_t$, maps defined by

$$\begin{array}{cccc} \tilde{f}_t & : \mathbb{C}^g & \longrightarrow & \mathbb{R}^g \\ & \Psi & & \Psi \\ & (z_1, \cdots, z_g) & \longmapsto & (y_1, \cdots, y_g), \end{array}$$

where $y_i = \mathrm{Im} z_i$, descend to

$$f_t \colon \mathcal{X}_t \to \mathbb{R}^g/(\beta_{i,j})_{1\le i,j\le g}\mathbb{Z}^g = \mathbb{R}^g/\mathrm{Im}\Omega(t)\mathbb{Z}^g$$

under the notations (6) and (10).

We denotes an arbitrary fiber of f_t as $F(\cong \mathbb{R}^g/\mathbb{Z}^g)$ whose coordinates are defined by $x_i = \mathrm{Re} z_i$. From (15), $\omega_t|_F = 0$. Also, it is easy to see that

$$\mathrm{Im}\left(\Omega_t = \frac{-2\pi}{\log|t|} dz_1 \wedge \cdots \wedge dz_g\right)\Bigg|_F = 0.$$

Hence (i) follows. Next, (ii) and (iii) follows from the definition in [Hit97, Gro13] and standard calculation below. Since the fibers of f_t are the integral manifolds along the “real" directions x_is, the affine coordinates along $\nabla_A(t)$

(resp., $\nabla_B(t)$) of $\frac{\partial}{\partial y_i}$ is determined as

$$\Bigg(\int_{0\le x_1\le 1}\iota\left(\frac{\partial}{\partial y_i}\right)\sum_{1\le i,j\le g}\frac{\sqrt{-1}}{2}\gamma_{i,j}(dx_i+\sqrt{-1}dy_i)\wedge(dx_j-\sqrt{-1}dy_j),$$
$$\cdots\cdots,$$
$$\int_{0\le x_g\le 1}\iota\left(\frac{\partial}{\partial y_i}\right)\sum_{1\le i,j\le g}\frac{\sqrt{-1}}{2}\gamma_{i,j}(dx_i+\sqrt{-1}dy_i)\wedge(dx_j-\sqrt{-1}dy_j)\Bigg)$$
$$=(\gamma_{1,i},\cdots,\gamma_{g,i})$$

(resp.,

$$\left(\int_{\Gamma_1}\iota\left(\frac{\partial}{\partial y_i}\right)\mathrm{Im}\left(\frac{-2\pi}{\log|t|}(dx_1+\sqrt{-1}dy_1)\wedge\cdots\wedge(dx_g+\sqrt{-1}dy_g)\right),\right.$$
$$\cdots\cdots,$$
$$\left.\int_{\Gamma_g}\iota\left(\frac{\partial}{\partial y_i}\right)\mathrm{Im}\left(\frac{-2\pi}{\log|t|}(dx_1+\sqrt{-1}dy_1)\wedge\cdots\wedge(dx_g+\sqrt{-1}dy_g)\right)\right)$$
$$=\frac{-2\pi}{\log|t|}(0,\cdots,0,\overbrace{1}^{i\text{-th}},0,\cdots,0),$$

where Γ_i is the $g-1$-cycle on $f_t^{-1}(y)=F_y$ of the form

$$[0,1]\times\cdots\times\{x_i\}\times\cdots\times[0,1]\subset\mathbb{R}^g/\mathbb{Z}^g\cong F_y).$$

Set $\{w_i\}$ (resp., $\{\check{w}_i\}$) as the integral affine coordinates induced by $\nabla_A(t)$ (resp., $\nabla_B(t)$). The above forms are rephrased as

$${}^t(w_1,\ldots,w_g)=\left(E\left(\mathrm{Im}\Omega'(t)\right)^{-1}E\right){}^t(y_1,\ldots,y_g)$$

and

$${}^t(\check{w}_1,\ldots,\check{w}_g)=\left(\frac{-2\pi}{\log|t|}\right){}^t(y_1,\ldots,y_g).$$

Hence, by taking the inverse matrices, we can see that $\nabla_A(t)$ and $\nabla_B(t)$ give the integral affine structures on $\mathbb{R}^g/\mathrm{Im}\Omega(t)\mathbb{Z}^g$ defined by $E^{-1}\mathrm{Im}\Omega'(t)E^{-1}\mathbb{Z}^g$ and $\frac{-\log|t|}{2\pi}\mathbb{Z}^g$, respectively. Note that $E^{-1}\mathrm{Im}\Omega'(t)E^{-1}=\mathrm{Im}\Omega(t)E^{-1}$. In particular, the Mclean metric g_t with respect to $\{y_i\}$ is of the form

$$\frac{-2\pi}{\log|t|}E(\mathrm{Im}\Omega(t))^{-1}.$$

Indeed, by definition, the Mclean metric g_t is defined as follows:

$$\check{w}_i=g_t\left(\frac{\partial}{\partial w_i},\frac{\partial}{\partial w_j}\right)w_j.$$

See also [Gro13, Definition 1.1]. Hence,

$$\left(g_t\left(\frac{\partial}{\partial w_i},\frac{\partial}{\partial w_j}\right)\right)_{i,j}=\frac{-2\pi}{\log|t|}\left(\mathrm{Im}\Omega(t)\right)E^{-1}.$$

Since $E\left(\mathrm{Im}\Omega(t)\right)^{-1}={}^t\left(E\left(\mathrm{Im}\Omega(t)\right)^{-1}\right)$, we obtain the following:

$$\begin{aligned}\left(g_t\left(\frac{\partial}{\partial y_i},\frac{\partial}{\partial y_j}\right)\right)_{i,j}&=\frac{-2\pi}{\log|t|}\cdot{}^t\left(E\left(\mathrm{Im}\Omega(t)\right)^{-1}\right)\mathrm{Im}\Omega(t)E^{-1}E\left(\mathrm{Im}\Omega(t)\right)^{-1}\\&=\frac{-2\pi}{\log|t|}E\left(\mathrm{Im}\Omega(t)\right)^{-1}.\end{aligned}$$

Next, we show (iv). For the former statement, i.e., to descend the H-action to $\mathcal{B}|_{\Delta^*}$, it is enough to show that each $h\in H$ sends any fiber of f_t for $t\neq 0$ to one of its fibers. Note that they are written in the form

$$\overline{\mathbb{R}^g+\sqrt{-1}(c_1,\cdots,c_g)}\subset\mathbb{C}/(E\mathbb{Z}^g\oplus\Omega(t)\mathbb{Z}^g).$$

Therefore the claim follows Lemma 4.19 as the coefficients of the linear part of h-action are real numbers (actually integers).

When the H-action preserves the isomorphic class of $\mathcal{L}_t$ for each $t\neq 0$, H is regarded as a subgroup of the automorphism group of polarized abelian variety of $(\mathcal{X}_t,\mathcal{L}_t)$ for a fixed $t\neq 0$, which is well-known to be finite. Hence, H itself is finite. Thus, combined with the former statements of (iv), f/H exists.

The remained proof of (v) is immediate from the definition of special Lagrangian fibrations. □

Remark 4.23. Here we confirm that the special Lagrangian fibrations we constructed in Theorem 4.9 and Theorem 4.20 are compatible. That is, for the case of principally polarized abelian surfaces which are common to both setup, the maps f_t in Theorem 4.20 are examples of that of Theorem 4.9 although the latter are not completely unique (due to the small ambiguity of applying Siegel reduction as in [OO21, §4, (4.11)]), to be precise.

Indeed, if we put $(y_1',y_2'):=(\beta_{i,j})_{1\le i,j\le 2}\cdot{}^t(y_1,y_2)$, we have

(16) $\mathrm{Re}(\Omega_t)=dx_1\wedge dx_2-(\det(\beta_{i,j}))^{-1}dy_1'\wedge dy_2',$

(17) $\mathrm{Im}(\Omega_t)=dx_1\wedge(\beta_{2,1}'dy_1'+\beta_{2,2}'dy_2')-dx_2\wedge(\beta_{1,1}'dy_1'+\beta_{1,2}'dy_2').$

Topologically all the fibers of f_t (of Theorem 4.20) are identified with the 4-torus T with coordinates x_1,x_2,y_1',y_2'. In $H^2(T,\mathbb{R})$, if we put

$e:=[dx_1\wedge dx_2],$

$f:=[dy_1'\wedge dy_2']$ and

$v:=[dx_1\wedge(B_{2,1}'dy_1'+B_{2,2}'dy_2')-dx_2\wedge(B_{1,1}'dy_1'+B_{1,2}'dy_2')],$

where $(B_{i,j} := \mathrm{val}_t p_{i,j}(t))$, $(B'_{i,j})_{i,j} := (B_{i,j})^{-1}_{i,j}$ as a matrix, the above gives

$$\log|t| \cdot \mathrm{Re}(\Omega_t) = \log|t| e - O\left(\frac{1}{\log|t|}\right) f \tag{18}$$

$$\log|t| \cdot \mathrm{Im}(\Omega_t) = (v + o(1)), \tag{19}$$

for $t \to 0$. As these (18), (19) fit to the reduction condition [OO21, p.47, (4.11) and l.10-12] (set $N_i := \log|t|, \epsilon_i := \frac{1}{N_i}$ in the notation of *loc.cit*), it means the desired compatibility with the construction of f_t in Theorem 4.9.

Corollary 4.24 (SYZ picture for maximal degenerating family of polarized abelian varieties)**.** *Under the same setting as Theorem 4.20, we set the matrix*

$$B := \left(\mathrm{val}_t(p_{i,j}(t))\right)_{i,j}.$$

Then SYZ picture with respect to the SYZ fibrations f_t for each $t \in \Delta^*$ as in the proof of Theorem 4.20 gives the tuple

$$\begin{aligned}(\mathcal{B}_0, \nabla_A(0), \nabla_B(0), g_0) &\cong (\mathbb{R}^g/B\mathbb{Z}^g, BE^{-1}\mathbb{Z}^g, \mathbb{Z}^g, EB^{-1}) \\ &\cong (\mathbb{R}^g/\mathbb{Z}^g, E^{-1}\mathbb{Z}^g, B^{-1}\mathbb{Z}^g, BE),\end{aligned}$$

where the second (resp., third) terms of the right-hand sides represent the integral points induced by $\nabla_A(0)$ (resp., $\nabla_B(0)$), and the fourth terms of the right-hand sides represent the flat metrics induced by g_0 with respect to the standard basis of $\mathbb{R}^g$, respectively.

Proof. Theorem 4.20 implies that the each SYZ fibration f_t gives the tuple

$$\begin{aligned}&(\mathcal{B}_t, \nabla_A(t), \nabla_B(t), g_t) \\ &= \left(\mathbb{R}^g/\mathrm{Im}\Omega(t)\mathbb{Z}^g, \mathrm{Im}\Omega(t)E^{-1}\mathbb{Z}^g, \frac{-\log|t|}{2\pi}\mathbb{Z}^g, \frac{-2\pi}{\log|t|}E\left(\mathrm{Im}\Omega(t)\right)^{-1}\right)\end{aligned}$$

under the convention as explained above. Note that the Mclean metric g_t is determined by the transition function of the integral basis of $\nabla_A(t)$ to that of $\nabla_B(t)$. By rescaling, we obtain the following:

$$\begin{aligned}&\left(\mathbb{R}^g/\mathrm{Im}\Omega(t)\mathbb{Z}^g, \mathrm{Im}\Omega(t)E^{-1}\mathbb{Z}^g, \frac{-\log|t|}{2\pi}\mathbb{Z}^g, \frac{-2\pi}{\log|t|}E\left(\mathrm{Im}\Omega(t)\right)^{-1}\right) \\ &\cong \left(\mathbb{R}^g/\mathbb{Z}^g, E^{-1}\mathbb{Z}^g, \left(\frac{-2\pi}{\log|t|}\mathrm{Im}\Omega(t)\right)^{-1}\mathbb{Z}^g, \left(\frac{-2\pi}{\log|t|}\mathrm{Im}\Omega(t)\right)E\right).\end{aligned}$$

Here, as we will see in Lemma 6.13, it holds that

$$B = \lim_{t \to 0} \frac{-2\pi}{\log|t|}\mathrm{Im}(\Omega(t)).$$

Hence, the tuple $(\mathcal{B}_t, \nabla_A(t), \nabla_B(t), g_t)$ is convergent to a tuple

$$(\mathbb{R}^g/\mathbb{Z}^g, E^{-1}\mathbb{Z}^g, B^{-1}\mathbb{Z}^g, BE)$$

as $t \to 0$. That is,

$$(\mathcal{B}_0, \nabla_A(0), \nabla_B(0), g_0) \cong (\mathbb{R}^g/\mathbb{Z}^g, E^{-1}\mathbb{Z}^g, B^{-1}\mathbb{Z}^g, BE).$$

By rescaling, it is also described as $(\mathbb{R}^g/B\mathbb{Z}^g, BE^{-1}\mathbb{Z}^g, \mathbb{Z}^g, EB^{-1})$. □

Remark 4.25. Note that the tuple $(\mathcal{B}_0, \nabla_A(0), \nabla_B(0), g_0)$ given by SYZ picture is determined by only B and E. In addition, B and E are independent of each other.

If $(\mathcal{X}|_{\Delta^*}, \mathcal{L}|_{\Delta^*})/\Delta^*$ is principally polarized, then we obtain the tuple

$$(\mathcal{B}_0, \nabla_A(0), \nabla_B(0), g_0) \cong (\mathbb{R}^g/B\mathbb{Z}^g, B\mathbb{Z}^g, \mathbb{Z}^g, B^{-1})$$
$$\cong (\mathbb{R}^g/\mathbb{Z}^g, \mathbb{Z}^g, B^{-1}\mathbb{Z}^g, B).$$

It implies Theorem 4.9 (iv).

Corollary 4.26 (SYZ picture = Gromov-Hausdorff limit picture for K-trivial finite quotients of abelian varieties)**.** *Use the same notation as in Theorem 4.20. Then the Gromov-Hausdorff limit of $(\mathcal{X}'_t, g'_{\mathrm{KE}}(\mathcal{X}'_t))$ for $t \to 0$ coincides with the Gromov-Hausdorff limit $(\mathcal{B}'_0, g'_0)$ of $(\mathcal{B}'_t, g'_t)$ for $t \to 0$, where the metric $g'_{\mathrm{KE}}(\mathcal{X}'_t)$ (resp., g'_t) on $\mathcal{X}'_t$ (resp., $\mathcal{B}'$) is induced by $g_{\mathrm{KE}}(\mathcal{X}_t)$ (resp., g_t). Furthermore, the affine manifold $(\mathcal{B}'_0, \nabla'_B(0))$ with singularities coincides with the quotient of the affine manifold $(\mathcal{B}_0, \nabla_B(0))$ by the group H, where the affine structure $\nabla'_B(0)$ with singularities is induced by $\nabla_B(0)$. In particular, we can regard the affine structure $\nabla'_B(0)$ with singularities as an IAMS structure by rescaling.*

Proof. It follows from Theorem 4.20 (v). In addition, the affine structure $\nabla_B(0)$ of B_0 is detemined by the matrix $\mathrm{val}_t(p_{i,j}(t))$ up to scaling, by Theorem 4.20 (iii). Here, $p_{i,j}(t)$ is the meromorphic function as appeared in Lemma 4.17. Hence, the last assertion holds. □

5. Non-Archimedean SYZ Fibration

This chapter is mainly based on [Got22, §4 and §5].

5.1. **Preliminaries.**

In this section, we introduce some important results from [NXY19].

5.1. Let T be a split algebraic K-torus of dimension n with its character group M. We denote by $N = M^\vee$ the dual module of M. We define the *tropicalization map* $\rho_T : T^{\mathrm{an}} \to N_{\mathbb{R}}$ of T by

$$T^{\mathrm{an}} \ni x \mapsto (m \mapsto -\log|m(x)|) \in M^\vee_{\mathbb{R}} = N_{\mathbb{R}}.$$

Then ρ_T is continuous, and its fibers are (not necessarily strictly) K-affinoid tori. Further, the tropicalization map ρ_T has a canonical continuous section $s\colon N_{\mathbb{R}} \to T^{\mathrm{an}}$ that sends each $n \in N_{\mathbb{R}}$ to the Gauss point of the affinoid torus $\rho_T^{-1}(n)$. The image of s is called the *canonical skeleton* of T, and denoted by $\Delta(T)$. The map s induces a homeomorphism $N_{\mathbb{R}} \to \Delta(T)$. We identify $\Delta(T)$ with $N_{\mathbb{R}}$ via this homeomorphism.

Definition 5.2. Let Y be a K-analytic space, let B be a topological space and let $f\colon Y \to B$ be a continuous map. Then f is called an n-dimensional *affinoid torus fibration* if there is a open covering $\{U_i\}$ of B such that, for each U_i, there is an open subset V_i of $N_{\mathbb{R}} \cong \mathbb{R}^n$ and a commutative diagram

$$\begin{array}{ccc} f^{-1}(U_i) & \longrightarrow & \rho_T^{-1}(V_i) \\ {\scriptstyle f}\downarrow & \circlearrowright & \downarrow{\scriptstyle \rho_T} \\ U_i & \longrightarrow & V_i \end{array}$$

where the upper horizontal map is an isomorphism of K-analytic spaces and the lower horizontal map is a homeomorphism.

5.3. If $f\colon Y \to B$ is an affinoid torus fibration, then f induces an integral affine structure on the base B as follows: For each open set U in B as in Definition 5.2, we consider an invertible analytic function h on $f^{-1}(U)$. Then the absolute value of h is constant along the fibers of f [KS06, §4.1, Lemma 1]. Hence h implies a continuous function $|h|\colon U \to \mathbb{R}_{>0}$ by taking $|h(b)|$ as $|h(y)|$ for some $y \in f^{-1}(b)$. We can define the integral affine functions on U as the functions of the form $-\log|h|$. If U is connected, then we can identify the ring of integral affine functions on U with the ring of polynomial functions of degree 1 with $\mathbb{Z}$-coefficients on $V \subset N_{\mathbb{R}}$ so that this construction indeed defines an integral affine structure on B via the homeomorphism $U \to V$ [KS06, §4.1, Theorem 1]. More precisely, in *loc.cit.*, they considered affine functions whose coefficients are in $\mathbb{R}$, rather than $\mathbb{Z}$. However, that's because they allowed the base field K to be a general nontrivial valued field. Under the condition that K is a discrete-valued field as in our setting, we can obtain affine functions whose coefficients are in $\mathbb{Z}$ as above. That is, we can give the integral affine structure to B. To construct an integral affine manifold in this way is often called *non-Archimedean SYZ picture* (NA SYZ picture, for short).

5.2. **K-trivial finite quotients of abelian varieties case.**

5.4. First, we prepare two settings, one for general use and one for Kummer surfaces, following [Got22]. If it is too complicated, it is enough to just consider the latter setting (5.6), which is a special case of the former (5.5). In the latter half of this section, we consider a setting (5.26) for K-trivial finite quotients of abelian varieties, beyond [Got22].

Recall the notation in §2.3.

5.5 (general setting)**.** Let A be an Abelian variety over K and $\mathscr{A}$ be the Néron model of A. After taking a base change along $f : S' \to S$ as in (2.40) if necessary, there is a triple $(G, \mathscr{L}, \mathscr{M}) \in \mathbf{DEG}^{\mathrm{split}}_{\mathrm{ample}}$ such that $A = G_\eta$ and $G = \mathscr{A}^0$ by the semiabelian reduction [Del72, Exposé I, Théorème 6.1]. In addition, we may assume that a finite group H acts on $(G, \mathscr{L}, \mathscr{M})$ such that the fixed locus of H on A is constant K-group scheme by taking a further base change along $f : S' \to S$ as above, without loss of generality. We assume that G is maximally degenerated. For the tuple $(M, L, \phi, a, b) = \mathbf{For}(F((G, \mathscr{L}, \mathscr{M})))$, there is a decomposition Σ as Lem 2.55 after taking a base change along $f : S' \to S$ as above. In particular, the decomposition Σ is $\Gamma = L \rtimes H$-admissible.

Let $\tilde{\mathscr{P}}$ be the toroidal compactification of $T = \mathrm{Spec} K[M]$ over R associated with Σ as constructed in (2.45) and $\mathscr{P}$ be the projective model of A as Theorem 2.44. This $\tilde{\mathscr{P}}$ is an SNC model of T. $\mathscr{P}$ is a Kulikov model of A as we see in (2.46). By definition, this Kulikov model $\mathscr{P}$ is a good minimal dlt model with a technical assumption as in [NXY19, (2.3)]. Hence, it follows that $\mathrm{Sk}(A) = \mathrm{Sk}(\mathscr{P})$. Further, we replace $\mathscr{L}$ by $\mathscr{L}^{\otimes\kappa}$ so that $\mathscr{L}$ extends to the ample line bundle $\mathscr{L}_{\mathscr{P}}$ on $\mathscr{P}$. Since $\mathscr{M}$ is trivial in our setting, there is no need to consider $\mathscr{M}$ in particular. Since it holds that $T^{\mathrm{an}} = \tilde{\mathscr{P}}_{\mathbf{ber}}$ and $A^{\mathrm{an}} = \mathscr{P}_{\mathbf{ber}}$, we can define the Berkovich retractions for these SNC-models $\tilde{\mathscr{P}}$ and $\mathscr{P}$. We denote by $\rho_{\tilde{\mathscr{P}}}$ (resp., $\rho_{\mathscr{P}}$) the Berkovich retraction associated with $\tilde{\mathscr{P}}$ (resp., $\mathscr{P}$) as in Definition 3.85. In particular, $\rho_{\mathscr{P}}$ is a non-Archimedean SYZ fibration. Let ρ_T be the tropicalization map of T.

5.6 (setting for Kummer surfaces)**.** Let A be an Abelian surface over K and X be the Kummer surface associated with A. We denote by $\mathscr{A}$ the Néron model of A. After taking a base change along $f : S' \to S$ as in (2.40) if necessary, there is a $(G, \mathscr{L}, \mathscr{M}) \in \mathbf{DEG}^{\mathrm{split}}_{\mathrm{ample}}$ such that $A = G_\eta$ and $G = \mathscr{A}^0$ by the semiabelian reduction [Del72, Exposé I, Théorème 6.1]. In addition, we may assume that the group $H = \{\pm 1\}$ acts on $(G, \mathscr{L}, \mathscr{M})$ so that the K-group scheme $A[2]$ is constant by taking a further base change along $f : S' \to S$ as above, without loss of generality. We assume that G is maximally degenerated. For the tuple $(M, L, \phi, a, b) = \mathbf{For}(F((G, \mathscr{L}, \mathscr{M})))$, there is a decomposition Σ as Lem 2.55 after taking a base change along $f : S' \to S$ as above. In particular, the decomposition Σ is $\Gamma = L \rtimes H$-admissible.

Let $\tilde{\mathscr{P}}$ be the toroidal compactification of $T = \mathrm{Spec} K[M]$ over R associated with Σ as constructed in (2.45) and $\mathscr{P}$ be the projective model of A as Theorem 2.44. This $\tilde{\mathscr{P}}$ is an SNC model of T. $\mathscr{P}$ is a Kulikov model of A as we see in (2.46). Further, we replace $\mathscr{L}$ by $\mathscr{L}^{\otimes\kappa}$ so that $\mathscr{L}$ extends to the ample line bundle $\mathscr{L}_{\mathscr{P}}$ on $\mathscr{P}$. Since $\mathscr{M}$ is trivial in our setting, there is no need to consider $\mathscr{M}$ in particular. We denote by $\mathscr{X}$ the Kulikov model of X associated with Σ as in (2.58). By definition, these Kulikov models $\mathscr{P}$ and $\mathscr{X}$ are good minimal dlt models with a technical assumption as in [NXY19, (2.3)]. Hence, it holds that $\mathrm{Sk}(A) = \mathrm{Sk}(\mathscr{P})$ and $\mathrm{Sk}(X) = \mathrm{Sk}(\mathscr{X})$. In addition, we note that $T^{\mathrm{an}} = \tilde{\mathscr{P}}_{\mathbf{ber}}$, $A^{\mathrm{an}} = \mathscr{P}_{\mathbf{ber}}$ and $X^{\mathrm{an}} = \mathscr{X}_{\mathbf{ber}}$. Hence, we can define the Berkovich retractions for these SNC-models $\tilde{\mathscr{P}}$, $\mathscr{P}$ and $\mathscr{X}$. We denote by $\rho_{\tilde{\mathscr{P}}}$ (resp., $\rho_{\mathscr{P}}$, $\rho_{\mathscr{X}}$) the Berkovich retraction associated with $\tilde{\mathscr{P}}$ (resp., $\mathscr{P}$, $\mathscr{X}$) as in Definition 3.85. In particular, $\rho_{\mathscr{P}}$ and $\rho_{\mathscr{X}}$ are non-Archimedean SYZ fibrations. Let ρ_T be the tropicalization map of T.

Remark 5.7. As we can see, the setting (5.5) is a generalization of (5.6). Under the setting (5.5), we consider a general dimensional abelian variety with an action of a general finite group. However, we did not deal with K-trivial finite quotients of abelian varieties under this setting (5.5) in [Got22]. At the end of this section, we consider K-trivial finite quotients of abelian varieties beyond [Got22].

Proposition 5.8 ([Got22, Proposition 5.5])**.** *Under the setting as in* (5.5)*, the Berkovich retraction $\rho_{\tilde{\mathscr{P}}}$ of $\tilde{\mathscr{P}}$ is equal to the tropicalization map ρ_T. In particular, $\rho_{\tilde{\mathscr{P}}}$ is an affinoid torus fibration.*

Proof. We set $d := \dim N$. The decomposition Σ gives the smooth rational polyhedral decomposition $\overline{\Sigma}$ in $N_{\mathbb{R}}$ obtained by intersectiong the cones in Σ with $N_{\mathbb{R}} \times \{1\}$. As we saw in (2.52), the Berkovich skelton $\mathrm{Sk}(\tilde{\mathscr{P}})$ coincides with $N_{\mathbb{R}}$. Moreover, simplicial structure of $\mathrm{Sk}(\tilde{\mathscr{P}})$ coincides with $\overline{\Sigma}$. Let $\sigma \in \Sigma$ be the smallest cone containing $\rho_T(x) \in N_{\mathbb{R}} \cong N_{\mathbb{R}} \times \{1\}$.

We set $\sigma = \mathbb{R}_{\geq 0}\tilde{n}_0 + \cdots + \mathbb{R}_{\geq 0}\tilde{n}_s$, where $\tilde{n}_i = (n_i, 1)$. We extend these elements to a $\mathbb{Z}$-basis $\tilde{n}_0, ..., \tilde{n}_d$ of $\tilde{N}_{\mathbb{R}}$. Let $\tilde{m}_i = (m_i, r_i) \in \tilde{M}$ be the dual basis of $\tilde{M}$. We may assume that

$$\rho_T(x) = \sum_{i=0}^{s} a_i \tilde{n}_i =: \tilde{n} = (n, 1) \in N_{\mathbb{R}} \times \{1\} \cong N_{\mathbb{R}},$$

where $\sum a_i = 1$ and $a_i > 0$ for all $0 \leq i \leq s$.

We set $A_\sigma = R[\tilde{M} \cap \sigma^\vee] \cong R[Y_0, ..., Y_s, Y_{s+1}^{\pm}, ..., Y_d^{\pm}]/(Y_0 \cdots Y_s - t)$, where $Y_i := t^{r_i} X^{m_i}$. Then $U_\sigma := \mathrm{Spec} A_\sigma \subset \tilde{\mathscr{P}}$.

It follows that $-\log|Y_j(x)| = \langle \tilde{m}_j, \tilde{n} \rangle = \langle \tilde{m}_j, \sum a_i \tilde{n}_i \rangle = a_j$ for $x \in \rho_T^{-1}(n)$ and for all $0 \leq j \leq d$. Therefore $\mathrm{red}_{\mathscr{X}}(x)$ coincides with the generic point ξ_σ of the toric stratum D_σ corresponding to σ. Moreover, each irreducible component D_i of $\tilde{\mathscr{P}}_0$ that contains D_σ corresponds to each one dimensional face $\tau_i = \mathbb{R}_{\geq 0}\tilde{n}_i$ of σ. Therefore, it follows that

$$\rho_{\tilde{\mathscr{P}}}(x) = \sum_{i=0}^{s} a_i \tilde{n}_i = \rho_T(x).$$

□

Corollary 5.9. *Under the setting as in* (5.6)*, the Berkovich retraction* $\rho_{\tilde{\mathscr{P}}}$ *of* $\tilde{\mathscr{P}}$ *is equal to the tropicalization map* ρ_T*. In particular,* $\rho_{\tilde{\mathscr{P}}}$ *is a* 2 *dimensional affinoid torus fibration.*

Proof. It follows by exactly the same argument as above Proposition 5.8. □

5.10. Under the setting as in Definition 3.85, let $\rho_{\mathscr{X}} : X^{\mathrm{an}} \twoheadrightarrow \mathrm{Sk}(\mathscr{X}) \subset X^{\mathrm{an}}$ be the Berkovich retraction, where the simplicial structure of $\mathrm{Sk}(\mathscr{X})$ is given by a decomposition $\Sigma = \{\sigma_\alpha\}_{\alpha \in I}$. Since the retraction $\rho_{\mathscr{X}}$ is continuous, the inverse image $\rho_{\mathscr{X}}^{-1}(\mathrm{Star}(\sigma_\alpha))$ is an open set. In particular, it holds that

$$X^{\mathrm{an}} = \bigcup_{\alpha \in I} \rho_{\mathscr{X}}^{-1}(\mathrm{Star}(\sigma_\alpha)).$$

We call this covering *the retraction covering of* X^{an} *associated with* $\mathscr{X}$. In other words, we can regard taking an snc-model of X as taking a retraction covering of X^{an}. To be precise, the stratification of the formal completion $\mathscr{X}_{\mathbf{for}}$ gives the retraction covering. We note that $\rho_{\mathscr{X}}^{-1}(\mathrm{Star}(\sigma_\alpha)) = \mathrm{red}_{\mathscr{X}}^{-1}(D_\alpha)$, where D_α is the scheme-theoretic intersection of the irreducible components corresponding to 1-dimensional faces of σ_α. Let ξ_α be a stratum of $\mathscr{X}_k$ corresponding to σ_α. Then $D_\alpha = \overline{\{\xi_\alpha\}}$.

5.11. For the decomposition $\Sigma = \{\sigma_\alpha\}_{\alpha \in I}$ as in (5.5), the Berkovich skelton $\mathrm{Sk}(\tilde{\mathscr{P}})$ is described as follows:

$$\mathrm{Sk}(\tilde{\mathscr{P}}) = \bigcup_{\alpha \in I^+} \overline{\sigma}_\alpha \cong N_{\mathbb{R}} \cong N_{\mathbb{R}} \times \{1\},$$

where $\overline{\sigma}_\alpha := \sigma_\alpha \cap (N_{\mathbb{R}} \times \{1\})$ as in (2.52). Theorem 3.3 implies that $\Gamma = L \rtimes H$ acts on $\mathrm{Sk}(\tilde{\mathscr{P}})$ as follows:

$$S_{(l,h)}((n,1)) = (n \circ h + \tilde{b}(l), 1).$$

Moreover, $\mathrm{Sk}(\mathscr{P}) = \bigcup_{\alpha \in I_L^+} \overline{\sigma}_\alpha$ (resp., $B := \bigcup_{\alpha \in I_\Gamma^+} \overline{\sigma}_\alpha$) is isomorphic to $\mathrm{Sk}(\tilde{\mathscr{P}})/L$ (resp., $\mathrm{Sk}(\tilde{\mathscr{P}})/\Gamma$) as simplicial complex. By Lemma 2.55, the morphism $\mathrm{Sk}(\tilde{\mathscr{P}}) \to \mathrm{Sk}(\mathscr{P})$ is an unbranched cover such that its fundamental group is isomorphic to L, and the morphism $\mathrm{Sk}(\mathscr{P}) \to B$ is a branched double cover. Under the more concrete condition (5.6), the ramification locus of this morphism $\mathrm{Sk}(\mathscr{P}) \to B = \mathrm{Sk}(\mathscr{X})$ is $Z := \frac{1}{2}L/\Gamma$. In particular, Z consists of 4 points.

5.12. Under the setting as in (5.5), the action of Γ on $\tilde{\mathscr{P}}$ induces Γ-action on T^{an} via the Raynaud generic fiber. In particular, the reduction map $\mathrm{red}_{\tilde{\mathscr{P}}}$ and the Berkovich retraction map $\rho_{\tilde{\mathscr{P}}}$ are Γ-equivariant. That is, it holds that

$\rho_{\tilde{\mathscr{P}}}(\gamma \cdot x) = S_\gamma(\rho_{\tilde{\mathscr{P}}}(x))$ for all $x \in T^{\mathrm{an}}$ and $\gamma \in \Gamma$. Further, we can also verify that the Berkovich retraction $\rho_{\mathscr{P}}$ of $\mathscr{P}$ is H-equivaliant, similarly.

Lemma 5.13. *Under the setting* (5.5)*, the following diagram commutes.*

$$\begin{array}{ccc} T^{\mathrm{an}} & \xrightarrow{/L} & A^{\mathrm{an}} \\ {\scriptstyle \rho_{\tilde{\mathscr{P}}}}\big\downarrow & \circlearrowright & {\scriptstyle \rho_{\mathscr{P}}}\big\downarrow \\ \mathrm{Sk}(\tilde{\mathscr{P}}) & \xrightarrow{/L} & \mathrm{Sk}(\mathscr{P}) \end{array}$$

Proof. Since G is maximally degenerated, it holds that $\mathscr{P}_{\mathbf{for}} \cong \tilde{\mathscr{P}}_{\mathbf{for}}/L$ as in (2.45). In particular, we obtain the morphism $f : \tilde{\mathscr{P}}_{\mathbf{for}} \to \mathscr{P}_{\mathbf{for}}$. Then $f_{\mathbf{ber}} : T^{\mathrm{an}} \to A^{\mathrm{an}}$ is the morphism appearing in the above diagram. Let $g : \mathrm{Sk}(\tilde{\mathscr{P}}) \to \mathrm{Sk}(\mathscr{P})$ be the morphism appearing in the above diagram, similarly. Here, the proof is completed by showing the commutativity $\rho_{\mathscr{P}} \circ f_{\mathbf{ber}} = g \circ \rho_{\tilde{\mathscr{P}}}$. By definition, the image $\rho_{\tilde{\mathscr{P}}}(x)$ of $x \in T^{\mathrm{an}}$ is determined by the point $\xi = \mathrm{red}_{\tilde{\mathscr{P}}}(x)$ coresponding to the cone $\sigma_\xi \in \Sigma$, the irreducible components $D_1, ..., D_r$ containing ξ and the barycentric coordinates $(v_1, ..., v_r)$ with respect to the vertices corresponding to these D_i, where each D_i corresponds to the 1-dimensional face σ_{α_i} of the cone σ for some $\alpha_i \in I^1$. Then the image $\rho_{\mathscr{P}}(f_{\mathbf{ber}}(x))$ is determined by the point $f(\xi) = \mathrm{red}_{\mathscr{P}}(f_{\mathbf{ber}}(x))$, the irreducible components $f(D_1), ..., f(D_r)$ and the barycentric coordinates $(v_1, ..., v_r)$ with respect to the vertices corresponding to these $f(D_i)$, where each $f(D_i)$ corresponds to 1-dimensional cone $\sigma_{\overline{\alpha}_i}$ for some $\overline{\alpha}_i \in I_L^+$ as in Theorem 2.44, where $\alpha_i \in I^1$ is the one above. On the other hand, $g(\rho_{\tilde{\mathscr{P}}}(x))$ is determined by the simplex $g(\overline{\sigma}) \in \overline{\Sigma}/L$ and the barycentric coordinates $(v_1, ..., v_r)$ with respect to the vertices $g(\overline{\sigma}_{\alpha_i})$, where $\alpha_i \in I^1$ is the one above. Here, the retraction $\rho_{\tilde{\mathscr{P}}} : T^{\mathrm{an}} \to \mathrm{Sk}(\tilde{\mathscr{P}})$ is L-equivaliant as we see in (5.12). Hence we obtain $\rho_{\mathscr{P}}(f_{\mathbf{ber}}(x)) = g(\rho_{\tilde{\mathscr{P}}}(x))$. That is, the above diagram commutes. $\square$

Proposition 5.14. *Under the setting* (5.6)*, let π be the blow up $\pi : \mathrm{Bl}_{A[2]}A \to A$. The following diagram commutes.*

$$\begin{array}{ccccc} & & (\mathrm{Bl}_{A[2]}A)^{\mathrm{an}} & \xrightarrow{H\backslash} & \\ & & {\scriptstyle \pi^{\mathrm{an}}}\big\downarrow & & \searrow \\ T^{\mathrm{an}} & \xrightarrow{/L} & A^{\mathrm{an}} & \circlearrowright & X^{\mathrm{an}} \\ {\scriptstyle \rho_{\tilde{\mathscr{P}}}}\big\downarrow & \circlearrowright & {\scriptstyle \rho_{\mathscr{P}}}\big\downarrow & & {\scriptstyle \rho_{\mathscr{X}}}\big\downarrow \\ \mathrm{Sk}(\tilde{\mathscr{P}}) & \xrightarrow{/L} & \mathrm{Sk}(\mathscr{P}) & \xrightarrow{H\backslash} & \mathrm{Sk}(\mathscr{X}) \\ & & \xrightarrow[\quad\quad\quad\quad\quad]{/\Gamma} & & \end{array}$$

Proof. It follows by the same argument as above Lemma 5.13 that the left part of the above diagram commutes. Hence it is enough to show that the right part of the above diagram commutes. We set $\tilde{\mathscr{X}} := \mathrm{Bl}_{\mathscr{A}[2]}\mathscr{P}$ as in (2.58). This $\tilde{\mathscr{X}}$ is an snc model of $\mathrm{Bl}_{A[2]}A$. We denote by $\rho_{\tilde{\mathscr{X}}}$ the Berkovich retraction. Since $\mathrm{Sk}(\mathscr{P}) = \mathrm{Sk}(\tilde{\mathscr{X}})$ as we see in (2.59), it holds that $\rho_{\tilde{\mathscr{X}}} = \rho_{\mathscr{P}} \circ \pi^{\mathrm{an}}$. Since π is the blow-up along the fixed locus of H, the blow-up π is H-equivaliant. In particular, H-equivaliant retraction $\rho_{\mathscr{P}}$ implies that $\rho_{\tilde{\mathscr{X}}}$ is H-equivaliant. After that, we can check the commutativity directly by representing the two images concretely as in the proof of Lemma 5.13. Hence, the right part of the above diagram commutes. □

Proposition 5.15 (cf.[NXY19, Proposition 3.8])**.** *Under the setting* (5.5)*, the morphism* $T^{\mathrm{an}} \to A^{\mathrm{an}}$ *is an unbranched cover. Moreover the open sets of the form* $\rho_{\mathscr{P}}^{-1}(\mathrm{Star}(\overline{\sigma}_\alpha))$ *for any* $\alpha \in I^+$ *are evenly covered neighborhoods. In particular,* $\rho_{\mathscr{P}}$ *is an affinoid torus fibration.*

Proof. By the property (e) of Lemma 2.55, $\mathrm{Star}(\overline{\sigma}_\alpha) \subset \mathrm{Sk}(\mathscr{P})$ is an evenly covered neighborhood with respect to $\mathrm{Sk}(\tilde{\mathscr{P}}) \to \mathrm{Sk}(\mathscr{P})$, where we identify $\mathrm{Star}(\overline{\sigma}_\alpha) \subset \mathrm{Sk}(\mathscr{P})$ with one of the sheets $\mathrm{Star}(\overline{\sigma}_\alpha) \subset \mathrm{Sk}(\tilde{\mathscr{P}})$. For each $l \in L \setminus \{0\}$, the following diagram holds.

$$\begin{array}{ccc} \rho_{\tilde{\mathscr{P}}}^{-1}(\mathrm{Star}(\overline{\sigma}_\alpha)) & \xrightarrow[l]{\simeq} & l \cdot \rho_{\tilde{\mathscr{P}}}^{-1}(\mathrm{Star}(\overline{\sigma}_\alpha)) \\ \downarrow \rho_{\tilde{\mathscr{P}}} & & \downarrow \rho_{\tilde{\mathscr{P}}} \\ \mathrm{Star}(\overline{\sigma}_\alpha) & \xrightarrow[S_l]{\simeq} & S_l(\mathrm{Star}(\overline{\sigma}_\alpha)) \end{array}$$

In particular, the upper horizontal map is an isomorphism of K-analytic spaces and the lower horizontal map is a homeomorphism. The property (e) of Lemma 2.55 says that $\mathrm{Star}(\overline{\sigma}_\alpha) \cap S_l(\mathrm{Star}(\overline{\sigma}_\alpha)) = \emptyset$. It implies that $\rho_{\tilde{\mathscr{P}}}^{-1}(\mathrm{Star}(\overline{\sigma}_\alpha)) \cap l \cdot \rho_{\tilde{\mathscr{P}}}^{-1}(\mathrm{Star}(\overline{\sigma}_\alpha)) = \emptyset$. By Lemma 5.13, we obtain $\rho_{\tilde{\mathscr{P}}}^{-1}(\mathrm{Star}(\overline{\sigma}_\alpha)) \cong \rho_{\mathscr{P}}^{-1}(\mathrm{Star}(\overline{\sigma}_\alpha))$. That is, we can identify $\rho_{\mathscr{P}}^{-1}(\mathrm{Star}(\overline{\sigma}_\alpha))$ with one of the sheets $\rho_{\tilde{\mathscr{P}}}^{-1}(\mathrm{Star}(\overline{\sigma}_\alpha))$. Hence, $\rho_{\mathscr{P}}^{-1}(\mathrm{Star}(\overline{\sigma}_\alpha))$ is an evenly covered neighborhoods. By Proposition 5.8, $\rho_{\tilde{\mathscr{P}}} = \rho_T$ follows. It implies the last assertion. □

Corollary 5.16. *Under the setting* (5.6)*, the morphism* $T^{\mathrm{an}} \to A^{\mathrm{an}}$ *is an unbranched cover. Moreover the open sets of the form* $\rho_{\mathscr{P}}^{-1}(\mathrm{Star}(\overline{\sigma}_\alpha))$ *for any* $\alpha \in I^+$ *are evenly covered neighborhoods. In particular,* $\rho_{\mathscr{P}}$ *is a 2-dimensional affinoid torus fibration.*

Proof. It follows by the same argument as above Proposition 5.15. □

5.17. In [NXY19, Proposition 3.8], they used the decomposition Σ which is constructed in Proposition 2.44 and proved that the Berkovich retraction $\rho_{\mathscr{P}}$

does not depend on the choice of such decomposition. On the other hand, the reason why we adopted the decomposition which is constructed in Lemma 2.55 is to show directly that $\rho_{\mathscr{P}}$ is an affinoid torus fibration by looking at the covering map concretely.

Corollary 5.18. *Under the setting* (5.6)*, the morphism* $T^{\mathrm{an}} \setminus \rho_T^{-1}(\frac{1}{2}L) \to X^{\mathrm{an}} \setminus \rho_{\mathscr{X}}^{-1}(Z)$ *is an unbranched cover. Moreover the open sets of the form* $\rho_{\mathscr{X}}^{-1}(\mathrm{Star}(\overline{\sigma}_\alpha))$ *for any* $\alpha \in I^+ \setminus I_{\mathrm{sing}}$ *are evenly covered neighborhoods. In particular, the restriction of* $\rho_{\mathscr{X}}$ *to the open set* $X^{\mathrm{an}} \setminus \rho_{\mathscr{X}}^{-1}(Z)$ *is a 2-dimensional affinoid torus fibration.*

Proof. The morphism $(\mathrm{Bl}_{A[2]}A)^{\mathrm{an}} \to X^{\mathrm{an}}$ as in Proposition 5.14 induces the morphism

$$A^{\mathrm{an}} \setminus \rho_{\mathscr{P}}^{-1}(\frac{1}{2}L/L) \to X^{\mathrm{an}} \setminus \rho_{\mathscr{X}}^{-1}(Z)$$

by restricting to the open set which is isomorphic to $A^{\mathrm{an}} \setminus \rho_{\mathscr{P}}^{-1}(\frac{1}{2}L/L)$. By composing with $T^{\mathrm{an}} \to A^{\mathrm{an}}$, we consider the morphism

$$T^{\mathrm{an}} \setminus \rho_T^{-1}(\frac{1}{2}L) \to X^{\mathrm{an}} \setminus \rho_{\mathscr{X}}^{-1}(Z).$$

By the property (f) of Lemma 2.55, the above exceptional part $\frac{1}{2}L$ corresponds to I_{sing}. By the property (g) of Lemma 2.55, $\mathrm{Star}(\overline{\sigma}_\alpha) \subset \mathrm{Sk}(\mathscr{X})$ is an evenly covered neighborhood with respect to $\mathrm{Sk}(\tilde{\mathscr{P}}) \to \mathrm{Sk}(\mathscr{X})$ for all $\alpha \in I^+ \setminus I_{\mathrm{sing}}$. Hence, this morphism $T^{\mathrm{an}} \setminus \rho_T^{-1}(\frac{1}{2}L) \to X^{\mathrm{an}} \setminus \rho_{\mathscr{X}}^{-1}(Z)$ is an unbranched cover. Moreover, we obtain the latter assertion by using Proposition 5.15. □

Proposition 5.19 (cf.[NXY19, (3.6), Proposition 3.8])**.** *Under the setting* (5.5)*, the induced integral affine structure on* $\mathrm{Sk}(A)$ *by* $\rho_{\mathscr{P}}$ *coincides with the quotient structure on* $N_{\mathbb{R}}/L$*.*

Proof. It follows from (2.46) that $\mathrm{Sk}(A) = \mathrm{Sk}(\mathscr{P})$. By Proposition 5.15, the non-Archimedean SYZ fibration $\rho_{\mathscr{P}}$ is an affinoid torus fibration. Hence this fibration $\rho_{\mathscr{P}}$ induces the integral affine structure on $\mathrm{Sk}(A)$. Then the following commutative diagram

$$\begin{array}{ccc}
T^{\mathrm{an}} & \xrightarrow{/L} & A^{\mathrm{an}} \\
{\scriptstyle \rho_T}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \rho_{\mathscr{P}}} \\
N_{\mathbb{R}} & \xrightarrow[/L]{} & \mathrm{Sk}(A)
\end{array}$$

gives the morphism $N_{\mathbb{R}} \to \mathrm{Sk}(A)$ between integral affine manifolds. In particular, this morphism is defined by taking the quotient of $N_{\mathbb{R}}$ by the lattice $\tilde{b} : L \hookrightarrow N_{\mathbb{R}}$. Hence, this finishes the proof. □

Corollary 5.20. *Let $T^2 = N_{\mathbb{R}}/L$ be the integral affine manifold constructed in Proposition 5.19, and let $\mathcal{T}_{T^2}$ be the local system on T^2 of lattices of tangent vectors. Then, the radiance obstruction $c_{T^2} \in \mathrm{H}^1(T^2, \mathcal{T}_{T^2})$ (cf.*[GH84], [GS06]*) coincides with $\tilde{b} \in \mathrm{Hom}(L, N) \subset \mathrm{Hom}(L, N_{\mathbb{R}})$ via the canonical isomorphism $\mathrm{H}^1(T^2, \mathcal{T}_{T^2}) \cong \mathrm{Hom}(L, N_{\mathbb{R}})$.*

Proof. It directly follows from Proposition 5.19. □

5.21. In [NXY19, Theorem 6.1], they proved that for each maximally degenerating projective Calabi-Yau variety X over K and any good minimal dlt-model $\mathscr{X}$ over S, the singular locus Z of the essential skeleton $\mathrm{Sk}(X)$ with the IAMS structure induced by $\mathrm{Sk}(\mathscr{X})$ is contained in the union of the faces of codimension ≥ 2 in $\mathrm{Sk}(\mathscr{X})$. In particular, the singular locus is of codimension ≥ 2. Further, in *loc.cit.*, they proved that the *piecewise* integral affine structure of $\mathrm{Sk}(X)$ induced by this IAMS structure of $\mathrm{Sk}(X)$ does not depend on the choice of such dlt-models.

As we state in (3.87), however, what is called a piecewise integral structure is closer to the topological structure than to the integral affine structure. In other words, the IAMS structure of $\mathrm{Sk}(X)$ induced by $\mathrm{Sk}(\mathscr{X})$ *does* depend on the choice of such dlt-models. In general, it is difficult to describe its IAMS structure explicitly, but in the case of Kummer surfaces, it can be described as follows:

Theorem 5.22 ([Got22, Theorem 5.19])**.** *Under the setting* (5.6)*, the restriction of the non-Archimedean SYZ fibration $\rho_{\mathscr{X}} : X^{\mathrm{an}} \to \mathrm{Sk}(X)$ to the open set $X^{\mathrm{an}} \setminus \rho_{\mathscr{X}}^{-1}(Z)$ is a 2-dimensional affinoid torus fibration. Moreover, the integral affine structure on $\mathrm{Sk}(X) \setminus Z$ induced by $\rho_{\mathscr{X}}$ coincides with the restriction of the quotient structure on $N_{\mathbb{R}}/\Gamma$, where $\Gamma = L \rtimes H$.*

Proof. It follows from (2.58) that $\mathrm{Sk}(X) = \mathrm{Sk}(\mathscr{X})$. By Corollary 5.18, $\rho_{\mathscr{X}}|_{X^{\mathrm{an}} \setminus \rho_{\mathscr{X}}^{-1}(Z)}$ is an affinoid torus fibration. The following commutative diagram

$$\begin{array}{ccc} T^{\mathrm{an}} \setminus \rho_T^{-1}(\frac{1}{2}L) & \longrightarrow & X^{\mathrm{an}} \setminus \rho_{\mathscr{X}}^{-1}(Z) \\ {\scriptstyle \rho_T}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \rho_{\mathscr{X}}} \\ N_{\mathbb{R}} \setminus \frac{1}{2}L & \xrightarrow[/\Gamma]{} & \mathrm{Sk}(X) \setminus Z \end{array}$$

gives the unbranched cover $N_{\mathbb{R}} \setminus \frac{1}{2}L \to \mathrm{Sk}(X) \setminus Z$. In the same manner as above Proposition 5.19, we obtain the isomorphism

$$\mathrm{Sk}(X) \setminus Z \cong (N_{\mathbb{R}} \setminus \frac{1}{2}L)/\Gamma = (N_{\mathbb{R}}/\Gamma) \setminus \{4pts\}$$

as an integral affine manifold. □

Corollary 5.23. *Let $S^2 = N_{\mathbb{R}}/\Gamma$ be the IAMS constructed in Theorem 5.22 and let $\mathcal{T}_{S^2\setminus Z}$ be the local system on $S^2 \setminus Z$ of lattices of tangent vectors. We denote by $\iota : S^2 \setminus Z \to S^2$ the natural inclusion. Then the radiance obstruction $c_{S^2} \in \mathrm{H}^1(S^2, \iota_* \mathcal{T}_{S^2\setminus Z})$ coincides with $\frac{1}{2}\tilde{b} \in \mathrm{Hom}(L, N_{\mathbb{R}})$ via the isomorphism $\mathrm{Hom}(L, N_{\mathbb{R}}) \cong \mathrm{H}^1(T^2, \mathcal{T}_{T^2}) \cong \mathrm{H}^1(S^2, \iota_* \mathcal{T}_{S^2\setminus Z})$ induced by the quotient morphism $T^2 \to S^2$ between these IAMS. Further, the radiance obstruction c_{S^2} is contained in $\mathrm{Hom}(L, N)$.*

Proof. Tsutsui proved that the quotient morphism $q : T^2 \to S^2$ induces the isomorphism $q_* : \mathrm{H}^1(T^2, \mathcal{T}_{T^2}) \cong \mathrm{H}^1(S^2, \iota_* \mathcal{T}_{S^2\setminus Z})$ such that $c_{S^2} = \frac{1}{2} c_{T^2}$ holds in his unpublished work[Tsu20]. Hence, the first assertion directly follows from Theorem 5.22, Corollary 5.20 and the above Tsutsui's work.

On the other hand, Overkamp proved that the map $b : L \times M \to \mathbb{Z}$ as in (5.6) takes only even values [Ove21, Proposition 3.5]. Hence, $\tilde{b} : L \to N$ also takes only even values. It implies that $c_{S^2} = \frac{1}{2}\tilde{b} \in \mathrm{Hom}(L, N)$. □

These results are proved in [Got22]. After that, we extend these results to more general K-trivial finite quotients of abelian varieties.

Theorem 5.24 ([GO22, Proof of Corollary 3.3])**.** *Consider an arbitrary abelian variety A over K of dimension g with an action of a finite group H as appeared in* (5.5)*. Assume that H acts trivially on the canonical bundle ω_A on A so that the canonical bundle $\omega_{A/H}$ on A/H is trivial. Then, for the H-equivariant SNC model $\mathscr{P}$ as appeared in Theorem 2.44, the pair $(\mathscr{P}/H, (\mathscr{P}/H)_k)$ is qdlt in the sense of* [dFKX17]*. Further, we assume that for any nontrivial $h \in H$, the fixed locus of its action on* $\mathrm{Sk}(A)$ *is 0-dimensional. Then $(\mathscr{P}/H, (\mathscr{P}/H)_k)$ is dlt.*

Proof. We write the irreducible decomposition of $\mathscr{P}_k$ as $\cup_i E_i$. We want to show that for any h which is not the identity e, h does not fix any E_i pointwise. Suppose the contrary and take a general point of $x \in E_i$. $\mathscr{P}$ is smooth over R at x, where R is the DVR of K. We take local coordinates $(x_1, \cdots, x_g)$ of $x \in E_i$ which we extend to h-invariant coordinates around $x \in \mathscr{P}$. Then $(x_1, \cdots, x_g, t)$ is a H-invariant local coordinates of $x \in \mathscr{P}$, which contradicts with nontriviality of h. Hence $(\mathscr{P}/H)_k$ is reduced. In particular, it implies that the quotient $(\mathscr{P}/H, (\mathscr{P}/H)_k)$ is qdlt.

Suppose there is $h(\neq e) \in H$ which preserves a strata Z of $\mathscr{P}_k$ (a log canonical center of $(\mathscr{P}, \mathscr{P}_k)$) pointwise. Then the strata of the dual complex $\mathrm{Sk}(A)$ which corresponds to Z is fixed by h, hence contradicts with our last assumption. It implies that the quotient $(\mathscr{P}/H, (\mathscr{P}/H)_k)$ is dlt. □

Remark 5.25. Theorem 5.24 partially extends a result by Overkamp in the Kummer surfaces case cf., [Ove21, §2, §3]).

We are now in a position to generalize Theorem 5.22.

5.26 (setting for K-trivial finite quotients of abelian varieties)**.** Let A be a g-dimensional abelian variety over K, H be a group satisfying the whole conditions as appeared in Theorem 5.24, and X be the quotient of the abelian variety A by the group H.

We denote by $\mathscr{A}$ the Néron model of A. After taking a base change along $f : S' \to S$ as in (2.40) if necessary, there is a $(G, \mathscr{L}, \mathscr{M}) \in \mathbf{DEG}^{\mathrm{split}}_{\mathrm{ample}}$ such that $A = G_\eta$ and $G = \mathscr{A}^0$ by the semiabelian reduction [Del72, Exposé I, Théorème 6.1]. We assume that G is maximally degenerated. For the tuple $(M, L, \phi, a, b) = \mathbf{For}(F((G, \mathscr{L}, \mathscr{M})))$, there is a decomposition Σ as Lem 2.55 after taking a base change along $f : S' \to S$ as above. In particular, the decomposition Σ is $\Gamma = L \rtimes H$-admissible.

Let $\tilde{\mathscr{P}}$ be the toroidal compactification of $T = \mathrm{Spec} K[M]$ over R associated with Σ as constructed in (2.45) and $\mathscr{P}$ be the projective model of A as Theorem 2.44. This $\tilde{\mathscr{P}}$ is an SNC model of T. $\mathscr{P}$ is a Kulikov model of A as we see in (2.46). Further, we replace $\mathscr{L}$ by $\mathscr{L}^{\otimes \kappa}$ so that $\mathscr{L}$ extends to the ample line bundle $\mathscr{L}_{\mathscr{P}}$ on $\mathscr{P}$. Since $\mathscr{M}$ is trivial in our setting, there is no need to consider $\mathscr{M}$ in particular. We denote by $\mathscr{X}$ the dlt model of X associated with Σ as in Theorem 5.24. That is, $\mathscr{X} := \mathscr{P}/H$. By definition, these Kulikov models $\mathscr{P}$ and $\mathscr{X}$ are good minimal dlt models with a technical assumption as in [NXY19, (2.3)]. Hence, it holds that $\mathrm{Sk}(A) = \mathrm{Sk}(\mathscr{P})$ and $\mathrm{Sk}(X) = \mathrm{Sk}(\mathscr{X})$. In addition, we note that $T^{\mathrm{an}} = \tilde{\mathscr{P}}_{\mathbf{ber}}$, $A^{\mathrm{an}} = \mathscr{P}_{\mathbf{ber}}$, and $X^{\mathrm{an}} = \mathscr{X}_{\mathbf{ber}}$. Hence, we can define the Berkovich retractions for these models $\tilde{\mathscr{P}}$, $\mathscr{P}$, and $\mathscr{X}$. We denote by $\rho_{\tilde{\mathscr{P}}}$ (resp., $\rho_{\mathscr{P}}$, $\rho_{\mathscr{X}}$) the Berkovich retraction associated with $\tilde{\mathscr{P}}$ (resp., $\mathscr{P}$, $\mathscr{X}$) as in Definition 3.85. In particular, $\rho_{\mathscr{P}}$ and $\rho_{\mathscr{X}}$ are non-Archimedean SYZ fibrations. Let ρ_T be the tropicalization map of T. Here, we denote by $Z \subset \mathrm{Sk}(\mathscr{X})$ the ramification locus of $\mathrm{Sk}(\mathscr{P}) \to \mathrm{Sk}(\mathscr{X})$.

Theorem 5.27 (NA SYZ picture for K-trivial finite quotients of abelian varieties)**.** *Under the setting* (5.26)*, the restriction of the non-Archimedean SYZ fibration* $\rho_{\mathscr{X}} : X^{\mathrm{an}} \to \mathrm{Sk}(X)$ *to the open set* $X^{\mathrm{an}} \setminus \rho_{\mathscr{X}}^{-1}(Z)$ *is an affinoid torus fibration. In partucular, the integral affine structure on* $\mathrm{Sk}(X) \setminus Z$ *induced by* $\rho_{\mathscr{X}}$ *coincides with the restriction of the quotient structure on* $N_{\mathbb{R}}/\Gamma$*, where* $\Gamma = L \rtimes H$*. Moreover, the skelton* $\mathrm{Sk}(X)$ *is an IAMS, that is* $\mathrm{codim} Z \geq 2$.

Proof. In a similar way as Proposition 5.14, the following diagram commutes.

$$\begin{array}{ccccc}
T^{\mathrm{an}} & \xrightarrow{/L} & A^{\mathrm{an}} & \xrightarrow{H\backslash} & X^{\mathrm{an}} \\
\downarrow{\scriptstyle \rho_{\tilde{\mathscr{P}}}} & \circlearrowleft & \downarrow{\scriptstyle \rho_{\mathscr{P}}} & \circlearrowleft & \downarrow{\scriptstyle \rho_{\mathscr{X}}} \\
\mathrm{Sk}(\tilde{\mathscr{P}}) & \xrightarrow{/L} & \mathrm{Sk}(\mathscr{P}) & \xrightarrow{H\backslash} & \mathrm{Sk}(\mathscr{X}) \\
 & & \underbrace{\qquad\qquad}_{/\Gamma} & &
\end{array}$$

Hence, it induces an isomorphism $\mathrm{Sk}(X)\backslash Z \cong (N_{\mathbb{R}}/\Gamma)\backslash Z$ as an integral affine manifold by the same discussion as Theorem 5.22.

To finish the proof, we show $\mathrm{codim} Z \geq 2$. Since X is K-trivial, the ramification divisor R of the finite morphism $f : A \to X$ vanishes. If D is a fixed prime divisor on A for some $h \in H$, then D is a ramification divisor on A. Indeed, when we set ξ and ξ' as the generic point of D and $f(D)$, the dimension of the finite morphism $\mathcal{O}_{X,\xi'} \to \mathcal{O}_{A,\xi}$ between two DVR's is given by the stabilizer of H. In particular, $\dim_{\mathcal{O}_{X,\xi'}}(\mathcal{O}_{A,\xi}) \geq 2$. On the other hand, $\dim_{\mathcal{O}_{X,\xi'}}(\mathcal{O}_{A,\xi})$ is equal to the value of an uniformizing parameter of $\mathcal{O}_{X,\xi'}$ for the discrete valuation on $\mathcal{O}_{A,\xi}$. This is nothing but the ramification index of D. Hence, D is a ramification divisor. From now on, we show that A has a fixed divisor for some $h \in H$ if $\mathrm{codim} Z = 1$. As in (2.42), any action h on A can lift to an action on the split torus $T (= \mathrm{Spec} K[M])$. In particular, we may assume $h \in \mathrm{GL}(N) \cap \mathrm{GL}(L)$, where $N = M^{\vee}$. Then this action descends to the skeleton $\mathrm{Sk}(\mathscr{P}) \cong N_{\mathbb{R}}/L$ via the canonical projection $N_{\mathbb{R}} \to N_{\mathbb{R}}/L$. By construction, any action $h \in H$ on $\mathrm{Sk}(\mathscr{P})$ is given in this way. If $\mathrm{codim} Z = 1$, then some h fixes some 1 codimensional subspace in $N_{\mathbb{R}}$. Fix such an h. Here, for the simplicial decomposition of $N_{\mathbb{R}}$ as in Lemma 2.55, the stabilizer of H on each simplex is trivial. It implies that h has $g-1$ linear independent eigenvectors with eigenvalues 1. In particular, h is diagonalizable. Further, $h \in \mathrm{GL}(N)$ implies $\det h = \pm 1$. If $\det h = 1$, then h must be trivial. Hence, $\det h = -1$. That is, h is diagonalizable with eigenvalues $(-1, 1, \dots, 1)$. Since $N = M^{\vee}$, the same holds for the action of h on M. Then we can take eigenvectors of h in M since $h \in \mathrm{GL}(M)$. In particular, we take a primitive eigenvector $m \in M$ of h with eigenvalue -1. Here, $z^m - 1 = z_1^{m_1} \cdots z_g^{m_g} - 1$ is a prime element of $K[M]$. Indeed, we can take a basis of M such that an element of the basis is $m \in M$ since $M/m\mathbb{Z}$ is a free $\mathbb{Z}$-module and $0 \to m\mathbb{Z} \to M \to M/m\mathbb{Z} \to 0$. That is, we may assume that $m = (1, 0, \dots, 0)$ after taking some $B \in \mathrm{GL}(M)$. Then $z^m - 1 = z_1 - 1$ is a prime element of $K[M]$ since

$$K[M]/(z_1 - 1) \cong K[z_2^{\pm}, \dots, z_g^{\pm}].$$

Note that m is an eigenvector of h with eigenvalue -1. It implies that the prime divisor on T defined by $z^m - 1 = 0$ is invariant for h. Note that we can take an affinoid domain V of T^{an} such that the restriction of $T^{\mathrm{an}} \to A^{\mathrm{an}}$ to V is an isomorphism and the interior of V intersects the closed analytic space defined by $z^m - 1 = 0$. It implies that A has a prime divisor locally defined by $z^m - 1 = 0$. In particular, the prime divisor on A is invariant for h. It is a contradiction. Hence $\mathrm{codim} Z \geq 2$ follows. □

Remark 5.28. Under the setting (5.26), these IAMS are uniquely determined by M, L and b. Hence, these IAMS do not depend on the polarization ϕ. Under the notation in Corollary 4.24, $\tilde{b}(L)$ (resp. $\phi(L)$) corresponds to B (resp., E). It means that the tuple $(M, L, \phi, a, b) = \mathbf{For}(F((G, \mathscr{L}, \mathscr{M})))$ has enough information to construct an IAMS with a metric given by SYZ picture as in Corollary 4.24. In our definition of NA SYZ picture, we could not reflect the polarization to the IAMS. Later, Pille-Schneider proposed more polished NA SYZ picture that also reflects the polarization [PS22, Conjectute 3.21].

5.3. **Equivalence between SYZ picture and NA SYZ picture.**

5.29. In this section, we prove the coincidence between SYZ picture and NA SYZ picture for K-trivial finite quotients of polarized abelian varieties.

5.30. Consider the same situation as (5.26). For the maximally degenerating polarized abelian variety $(G_\eta, \mathscr{L}_\eta)$ of dimension g, we assume that $k = \mathbb{C}$ and H-action fixes $\mathcal{L}_\eta$. It gives a situation as in Theorem 4.20.

Lemma 5.31. *Under the setting* (5.30)*, we use the same notation as Corollary 4.26. Then the integral affine manifold induced by the non-Archimedean SYZ fibration coincides with the integral affine manifold induced by the family of SYZ fibrations. That is,* $\mathrm{Sk}(\mathscr{P})$ *and* $\nabla_B(0)$ *give the same integral affine structure (up to scaling) to the g-torus* $T^g \cong \mathbb{R}^g/\mathbb{Z}^g$.

Proof. It follows from Proposition 5.19 and Corollary 4.26. Indeed, we obtain the integral affine structures on $\mathbb{R}^g/\mathbb{Z}^g$ as follows: In NA SYZ picture, the integral affine structure of $\mathbb{R}^g/\mathbb{Z}^g \cong N_{\mathbb{R}}/L$ is given by the inclusion $\tilde{b} : L \to N$. In SYZ picture, it follows from Corollary 4.26 that the integral affine structure of $\mathbb{R}^g/\mathbb{Z}^g$ is given by the matrix $B(l_i, l_j)$ as in (2.39) up to scaling. Hence, these two pictures give the same integral affine structure to $N_{\mathbb{R}}/L$ up to scaling. □

Theorem 5.32 (cf. [Got22, Theorem 5.31], [KS06, Conjecture 3] for K-trivial finite quotients of abelian varieties)**.** *Under the setting* (5.30)*, we use the same notation as Corollary 4.26. Then the smooth locus of the IAMS induced by the non-Archimedean SYZ fibration coincides with that of the IAMS* *induced by the family of SYZ fibrations up to scaling. That is,* $\mathrm{Sk}(\mathscr{X})$ *and* $\nabla'_B(0)$ *give the same IAMS structure (up to scaling) to the topological space* $N_{\mathbb{R}}/\Gamma$.

Proof. It follows from Theorem 5.27, Corollary 4.26 and Lemma 5.31. Indeed, those two IAMS structures are the quotient of $N_{\mathbb{R}}/L$ by H. Hence, these two pictures give the same integral affine structure to $N_{\mathbb{R}}/\Gamma$ up to scaling. □

Remark 5.33. Note that, in NA SYZ picture, we were implicitly rescaling the affine structure by taking a base change $f : S' \to S$ as in (5.30). That is, the rescaling is inevitable.

We only treat the B-side integral affine structures such as $\nabla_B(0)$ given by SYZ picture in Theorem 5.32. It is because the A-side affine structures depend on polarizations by definition although NA SYZ picture does not depend on polarizations as we see in Remark 5.28. Actually, the B-side affine structures do not depend on polarizations as we see in (4.18).

The result of [Got22, Theorem 5.31] is contained in Theorem 5.32. Indeed, under the setting (5.6), the IAMS induced by the non-Archimedean SYZ fibration for the Kummer surface X that is given in Theorem 5.22, is the same as the IAMS induced by the non-Archimedean SYZ fibration for the singular Kummer surface A/H that is given in Theorem 5.27. Further, the B-side affine structures given by SYZ picture for the Kummer surface X and the singular Kummer surface A/H are the same too, while the A-side affine structures might be different.

Theorem 5.32 states that NA SYZ picture is equivalent to SYZ picture for finite quotients of abelian varieties somehow. In §6.2, we give another proof of Theorem 5.32 and the reason why these two pictures coincide.

Remark 5.34. In [KS06, Conjecture 3], they predicted an equivalence between Collapse picture (or SYZ Picture assuming that Gromov-Hausdorff limit picture holds) and NA SYZ picture. Note that, in our definitions, SYZ picture gives two affine structures and one metric, Gromov-Hausdorff limit picture gives a metric space, and NA SYZ picture gives an affine structure. That is, the resulting objects of these pictures are different. Corollary 4.26 and Theorem 5.32 imply that, for K-trivial finite quotients of abelian varieties, SYZ picture and Gromov-Hausdorff limit picture give the same metric space, and (the B-side affine structure of) SYZ picture and NA SYZ picture are the same as an IAMS.

6. Hybrid SYZ fibration

As briefly discussed in the introduction, this section discusses gluing two kinds of fibrations in originally very different natures:

- special Lagrangian fibrations (SYZ fibrations) and
- non-archimedean SYZ fibration,

which is a sort of enhanced answer to [KS06, Conjecture 3]. Indeed, we use it to show the conjecture for certain finite quotients of abelian varieties, generalizing [Got22, §5]. This chapter is mainly based on [GO22, §3 and §4].

6.1. **Preliminaries.**

6.1. For the gluing, we use the recent technology of the hybrid norm originated in [Ber10] and re-explored in [BJ17, §2], [Oda19, Appendix]. It is also closely related to earlier Morgan-Shalen's partial compactification technique [MS84]. More precisely speaking, [BJ17] shows the following by two different constructions i.e., as the projective limit of Morgan-Shalen type extension and as a variant of Berkovich analytification.

6.2. As we saw in §3, Berkovich originally considered the Berkovich analytifications of locally algebraic schemes over a non-Archimedean field K. Later, in [Ber10, §1], he presents the Berkovich analytifications of locally algebraic schemes over a Banach ring K. In *loc.cit.*, he only considered their topological aspects. In this thesis, we also consider only topological aspects. Later, Poineau and Lemanissier study their analytic aspects, that is, their structure sheaves. See [Poi10], [Poi13] and [LP20].

Here, we now introduce the Berkovich analytifications of locally algebraic schemes over a Banach ring K, following [Ber10]. Let X be a locally algebraic scheme over a Banach ring K. For any affine open subset $U = \mathrm{Spec} A$ of X, let U^{An} be the set consisting of all multiplicative seminorms on A whose restrictions to K are bounded be the equipped norm on K. The topology on U^{An} is the weakest topology such that, for any $f \in A$, the corresponding map $U^{\mathrm{An}} \ni |\cdot| \mapsto |f| \in \mathbb{R}$ is continuous. By gluing together the spaces U^{An}, we obtain a topological space X^{An}.

We can easily see that this construction generalizes the one over a non-Archimedean field. Compare with (3.39). Here, X^{An} is Hausdorff, locally compact and countable at infinity. In a similar way to (3.43), we obtain a canonical map $\varphi : X^{\mathrm{An}} \to X$ defined by $x \mapsto \ker|\cdot|_x$. Further, the construction $X \mapsto X^{\mathrm{An}}$ is functorial and satisfies the following:

- A morphism $f : X \to Y$ between locally algebraic schemes over a Banach ring is an open (resp., closed) embedding, then so is the corresponding morphism $f^{\mathrm{An}} : X^{\mathrm{An}} \to Y^{\mathrm{An}}$.
- A morphism $f : X \to Y$ between locally algebraic schemes over a Banach ring is surjective, then so is $f^{\mathrm{An}} : X^{\mathrm{An}} \to Y^{\mathrm{An}}$.

In particular, we obtain the morphism $\lambda : X^{\mathrm{An}} \to \mathscr{M}(K)$ corresponding to the structure morphism $X \to \mathrm{Spec} K$.

6.3. We now introduce some results about the hybrid norm. As in (3.6), the hybrid norm $|\cdot|_{\mathrm{hyb}}$ on $\mathbb{C}$ is determined by $|z|_{\mathrm{hyb}} := \max\{|z|_0, |z|_\infty\}$. We denote by $\mathbb{C}_{\mathrm{hyb}}$ the Banach field $(\mathbb{C}, |\cdot|_{\mathrm{hyb}})$. Then

$$\mathscr{M}(\mathbb{C}_{\mathrm{hyb}}) \cong [0,1].$$

Indeed, we can easily see that any element of $\mathscr{M}(\mathbb{C}_{\mathrm{hyb}})$ is of the form $|\cdot|_\infty^\rho$, where $\rho \in [0,1]$. Consider the analytification of a locally algebraic scheme X over $\mathbb{C}_{\mathrm{hyb}}$. As in (6.2), we obtain the structure morphism $\lambda : X^{\mathrm{An}} \to [0,1]$. It follows from [Jon16, Theorem C] that $\lambda : X^{\mathrm{An}} \to [0,1]$ is open map. Further, the Gelfand-Mazur Theorem (cf. [Gel41]) implies a canonical homeomorphism $X(\mathbb{C}) \cong \lambda^{-1}(\rho)$, where $\rho \in (0,1]$. On the other hand, it holds that $\lambda^{-1}(0) \cong X^{\mathrm{an}}$, where the right-hand side means the Berkovich analytification of X over $(\mathbb{C}, |\cdot|_0)$.

6.4. Fix $\epsilon \ll 1$. Now consider the Banach ring over $\mathbb{C}_{\mathrm{hyb}}$ defined as

$$\mathcal{A}_\epsilon := \left\{ f = \sum_{i\in\mathbb{Z}} a_i t^i \in \mathbb{C}((t)) \;\middle|\; a_i \in \mathbb{C}, ||f|| := \sum_{i\in\mathbb{Z}} |a_i|_{\mathrm{hyb}} \epsilon^i < +\infty \right\}.$$

Note that $f \in \mathcal{A}_\epsilon$ is holomorphic on $\overline{\Delta}_\epsilon^{\,*} := \overline{\Delta}_\epsilon \setminus \{0\}$ and meromorphic at $0 \in \overline{\Delta}_\epsilon$. Then, it follows from [Poi10, Proposition 2.1.1] that $\mathscr{M}(\mathcal{A}_\epsilon) \cong \overline{\Delta}_\epsilon$, where $\Delta_\epsilon := \{t \in \mathbb{C} \mid |t|_\infty < \epsilon\}$. In particular, the multiplicative seminorm on $\mathcal{A}_\epsilon$ corresponding to $z \in \overline{\Delta}_\epsilon$ is defined by

$$|f| = \begin{cases} \epsilon^{-\frac{\log|f(t)|_\infty}{\log|t|_\infty}} & (t \neq 0) \\ \epsilon^{-\mathrm{val}_t(f)} & (t = 0), \end{cases}$$

where val_t is the t-adic (additive) valuation on $\mathbb{C}((t))$ as in (4.7). Here, the map $p : \mathscr{M}(\mathcal{A}_\epsilon) \to \mathscr{M}(\mathbb{C}_{\mathrm{hyb}})$ corresponding to $\mathbb{C}_{\mathrm{hyb}} \to \mathcal{A}_\epsilon$ is given by

$$p(z) = \begin{cases} \frac{\log\epsilon}{\log|t|_\infty} & (t \neq 0) \\ 0 & (t = 0) \end{cases}$$

under the indentifications $\mathscr{M}(\mathcal{A}_\epsilon) \cong \overline{\Delta}_\epsilon$ and $\mathscr{M}(\mathbb{C}_{\mathrm{hyb}}) \cong [0,1]$. It implies that each fiber $p^{-1}(\rho)$ is of the form $\{|t|_\infty = \epsilon^{\frac{1}{\rho}}\} \subset \overline{\Delta}_\epsilon$, where $\rho \in (0,1]$, and $p^{-1}(0)$ consists only one point corresponding to the valuation $\epsilon^{-\mathrm{val}_t}$ on $\mathbb{C}((t))$. We denote the Berkovich analytification of a locally algebraic scheme X over the Banach ring $\mathcal{A}_\epsilon$ by X^{hyb}. Such an analytification is called a *hybrid analytification* after [BJ17]. Similarly, we set $\pi^{\mathrm{hyb}} : X^{\mathrm{hyb}} \to \overline{\Delta}_\epsilon$ as the map corresponding to the structure morphism $X \to \mathrm{Spec}\mathcal{A}_\epsilon$.

Now consider $\mathcal{X}/\Delta$ as in (4.7). Then $\mathcal{X}$ can be considered as a locally algebraic scheme over the Banach ring $\mathcal{A}_\epsilon$ for $0 < \epsilon < 1$. Indeed, the morphism $\pi : \mathcal{X} \to \Delta$ induces a morphism $\mathcal{X} \to \mathrm{Spec}\mathcal{A}_\epsilon$. Then we can consider the hybrid analytification $\mathcal{X}^{\mathrm{hyb}}$. Here, we may assume $\Delta_\epsilon = \Delta$ by rescaling. Now it is known that the structure morphism $\pi^{\mathrm{hyb}} : \mathcal{X}^{\mathrm{hyb}} \to \overline{\Delta}_\epsilon$ coincides with the morphism $\pi : \mathcal{X} \to \Delta$ away from the central fiber. More precisely, the following fact is known.

Fact 6.5 ([BJ17, §4 and Appendix])**.** *For each smooth projective family $\pi^* \colon \mathcal{X}^* \to \Delta^*$ and associated smooth projective variety X over $\mathbb{C}((t))^{\mathrm{mero}}$, there is a topological space $\pi^{\mathrm{hyb}} \colon \mathcal{X}^{\mathrm{hyb}} \to \Delta$ such that*

- *(i) $\pi^{\mathrm{hyb}}|_{\Delta^*} = \pi^* \colon \mathcal{X}^* \to \Delta^*$ with the complex analytic topology,*
- *(ii) $(\pi^{\mathrm{hyb}})^{-1}(0) = X^{\mathrm{an}}$, where X^{an} means the Berkovich analytification of $X/\mathbb{C}((t))^{\mathrm{mero}}$.*

Remark 6.6. Fact 6.5 implies the topology of each fiber depends only on the generic fiber $\mathcal{X}^*$ (or X). To obtain the continuous map $\pi^{\mathrm{hyb}} \colon \mathcal{X}^{\mathrm{hyb}} \to \Delta$ as appeared in Fact 6.5, we have to choose a model $\mathcal{X}$ of X. These choices affect how horizontal sections converge.

6.2. **Hybrid SYZ fibration.**

We are now in a position to define *hybrid SYZ fibration*. The following is one of our main theorems. We have not found literature which conjectured the statements.

Theorem 6.7 (Hybrid SYZ fibration, [GO22, Theorem 3.2])**.** *For any maximally degenerating family $(\mathcal{X}|_{\Delta^*}, \mathcal{L}|_{\Delta^*})/\Delta^*$ of polarized abelian varieties of dimension g we use the same notation as Theorem 4.20. Then, the following hold.*

- *(i) There is a family of tropical affine manifolds (real tori) i.e., a topological space $\mathcal{B} := \sqcup_{t\in\Delta}\mathcal{B}_t$ with a natural continuous map to $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$ and continuous family of tropical affine structures on $\mathcal{B}_t$ such that for $t \neq 0$, they coincide with $(\mathcal{B}_t, \nabla_B(t))$ of Theorem 4.20.*

  Further, it comes with a continuous proper map

$$f^{\mathrm{hyb}} \colon \mathcal{X}^{\mathrm{hyb}} \to \mathcal{B}, \tag{20}$$

  such that
  - *(a) for $t \neq 0$, $f^{\mathrm{hyb}}|_{\mathcal{X}_t} = f_t$ i.e., coincides with the special Lagrangian fibrations in Theorem 4.20.*
  - *(b) $f^{\mathrm{hyb}}|_{t=0} = f^{\mathrm{hyb}}|_{X^{\mathrm{an}}}$ is the non-archimedean SYZ fibration as appeared in Proposition 5.15.*

  We call this map f^{hyb} a hybrid SYZ fibration.
- *(ii) If a group H acts holomorphically in a fiber-preserving manner on $\mathcal{X}|_{\Delta^*}$, which fixes $c_1(\mathcal{L}_t)$ for $t \neq 0$, it induces continuous actions on both $\mathcal{X}^{\mathrm{hyb}}$ and $\mathcal{B}$. (See also Theorem 6.23 for more details.) Furthermore, f^{hyb} is H-equivariant with respect to the induced actions of H. If the group H also fixes $\mathcal{L}_t$, then H must be a finite group*

and it descends to the quotient $\mathcal{X}^{\mathrm{hyb}}/H \to \mathcal{B}/H$, which we denote as f^{hyb}/H.

(iii) *Under the setup of above* (ii)*, we further assume that H is given by actions as considered in* (5.26)*. Then, $f^{\mathrm{hyb}}/H\colon \mathcal{X}^{\mathrm{hyb}}/H \to \mathcal{B}/H$ is again fiberwise special Lagrangian fibrations for $t \neq 0$ and a non-archimedean SYZ fibration for $t = 0$.*

Proof. First, we prove (i). Recall from Fact 6.5 and [BJ17] that $\mathcal{X}^{\mathrm{hyb}}$ is the projective limit of Morgan-Shalen type space $\mathcal{X}^{\mathrm{hyb}}(\mathcal{X}) := \mathcal{X}^* \sqcup \Delta(\mathcal{X}_0)$ with the hybrid topology of [BJ17, §2], where $\mathcal{X}^* \subset \mathcal{X} \to \Delta$ runs over SNC models and $\Delta(\mathcal{X}_0)$ denotes the dual complex of central fibers.

Here we take the Mumford construction by [Mum72, Kün98] (also cf., [Got22, MN22]) and construct a smooth minimal model $\mathcal{X}^* \subset \mathcal{X}$ which is projective over Δ and the central fiber $\mathcal{X}_0$ is simple normal crossing. Then, recall from [BJ17, §2] that the Morgan-Shalen type partial compactification $(\mathcal{X}^* \subset)\mathcal{X}^{\mathrm{hyb}}(\mathcal{X})$ satisfies

- π^{hyb} extends to a continuous proper map $\overline{\pi^{\mathrm{hyb}}}\colon \mathcal{X}^{\mathrm{hyb}}(\mathcal{X}) \to \Delta$,
- $(\overline{\pi^{\mathrm{hyb}}})^{-1}(0)$ is the dual complex of $\mathcal{X}_0$.

See [BJ17, §2] (also cf., original [MS84]) for the details including the description of the topology. Below, we will show that there is a map

$$f(\mathcal{X})\colon \mathcal{X}^{\mathrm{hyb}}(\mathcal{X}) \to \mathcal{B}$$

such that the composite of $f(\mathcal{X})$ together with the natural retraction

$$\mathcal{X}^{\mathrm{hyb}} \to \mathcal{X}^{\mathrm{hyb}}(\mathcal{X})$$

([Ber99], see also [KS06, BJ17]) gives the desired f^{hyb}.

We take an arbitrary sequence

$$P_k = (t_k, (Z_i)_k) \in \Delta^* \times (\mathbb{C}^*)^g \ \ (k = 1, 2, \cdots)$$

which satisfies that for each $i \in \{1, \cdots, g\}$, there is a real constant c_i such that

$$\frac{\log|(Z_i)_k|}{\log|t_k|} \to c_i \tag{21}$$

for $k \to \infty$. This condition means that, the points P_k $(k = 1, 2, \cdots)$ converge to $(c_1, \cdots, c_g) \in \mathbb{R}^g$ modulo $(B_{i,j})\mathbb{Z}^g$ in the Morgan-Shalen type partial compactification $\mathcal{X}^{*,\mathrm{hyb}}(\mathcal{X})$. By the definition of the topology put on the Morgan-Shalen type compactification (cf., [MS84], [BJ17, §2]), to prove the desired assertion, it is enough to show that $f_{t_k}(P_k)$ converges to the same $\overline{(c_1, \cdots, c_g)} \in \mathcal{B}_0 \subset \mathcal{B}$ for $k \to \infty$.

So we consider the sequence $\{f_{t_k}(P_k)\} \in \mathcal{B}$. By Theorem 4.20 (ii) and (iii), it is represented by

$$\mathrm{Im}\left(\frac{\log(Z_i)_k}{2\pi\sqrt{-1}}\right) \text{ mod. } (\beta_{i,j}(t_k))_{1\le i,j\le g}\mathbb{Z}^g, \tag{22}$$

hence as a point in $\mathcal{B}_{t_k} \subset \mathcal{B}$,

$$f_{t_k}(P_k) = \left(\frac{\log|(Z_i)_k|}{\log|t_k|}\right) \text{ mod. } \left(\frac{2\pi\beta_{i,j}(t_k)}{-\log|t_k|}\right)_{1\le i,j\le g}\mathbb{Z}^g. \tag{23}$$

Since $\overline{\mathrm{Im}\frac{t}{\sqrt{-1}}} = -\log|t|$ in general, applying to (23), $f_{t_k}(P_k)$ converge to $(c_1,\cdots,c_g) \in \mathcal{B}_0 \subset \mathcal{B}$ for $k \to \infty$ by the assumption (21) on the sequence P_k. To finish the proof of (i), we confirm that the obtained f^{hyb} is proper. It follows from the fact that $\mathcal{X}^{\mathrm{hyb}}(\mathcal{X}) \to \Delta$ for each SNC model $\mathcal{X}$ is always a proper map, together with the description of $\mathcal{X}^{\mathrm{hyb}}$ as the projective limit of $\mathcal{X}^{\mathrm{hyb}}(\mathcal{X})$ for a certain projective system of the model $\mathcal{X}$s ([KS06, Appendix A, Theorem 10], [BJ17, 4.12]) thanks to Tychonoff's theorem.

Next, we prove (ii). It follows from the functoriality of the construction of hybrid analytification ([Ber10]) that the H-action induces a natural continuous H-action on the whole $\mathcal{X}^{\mathrm{hyb}}$. Theorem 4.20 (iv) states the existence of continuous H-action $\mathcal{B}|_{t\neq 0}$ with which $f^{\mathrm{hyb}}|_{t\neq 0}$ is H-equivariant, which we want to extend to the whole base $\mathcal{B}$. Recall there is a H-equivariant SNC minimal model of $\mathcal{X}^*$ by [Kün98, 3.5, esp. (v)], applied over the DVR of holomorphic germs $\mathcal{O}^{\mathrm{hol}}_{\mathbb{C},0}$ at $0 \in \mathbb{C}$. Hence, there is a natural H-action on $\mathcal{B}_{t=0}$ so that $f^{\mathrm{hyb}}|_{t=0}$ is H-equivariant. Hence, summing up above, we have a H-action on $\mathcal{B}$ with which f^{hyb} is H-equivariant. What remains to complete the proof of (ii) is the continuity of the H-action on $\mathcal{B}$.

Suppose the contrary. Then there is a sequence x_i $(i = 1,2,\cdots) \in \mathcal{B}$ and $h \in H$ such that while $\lim_{i\to\infty} x_i$ exists in $\mathcal{B}_0$, but

$$\lim_{i\to\infty} h\cdot x_i \neq h\cdot\left(\lim_{i\to\infty} x_i\right). \tag{24}$$

We can lift each x_i to $\tilde{x}_i \in \mathcal{X}^{\mathrm{hyb}}(\mathcal{X})$ whose image in $\mathcal{B}$ is x_i. From the properness of f^{hyb} proven in (i) above and locally compactness of $\mathcal{B}$, we can and do assume that that $\lim_{i\to\infty}\tilde{x}_i$ exists. Then, $\lim_{i\to\infty} h\cdot\tilde{x}_i$ maps down to $h\cdot(\lim_{i\to\infty} x_i)$ which contradicts with (24). Therefore, we can take the finite quotient $f^{\mathrm{hyb}}/H \colon \mathcal{X}^{\mathrm{hyb}}/H \to \mathcal{B}/H$ in the category of topological spaces.

Finally, we prove (iii). From Theorem 4.20 (v), the assertions hold away from $t = 0$. Here, take a H-equivariant maximal degeneration $\mathcal{X}$ over Δ corresponding to a maximal degeneration $\mathscr{P}$ of the abelian variety A as in (5.26). Then the quotient $\mathscr{P}/H$ is again a relatively minimal dlt model of A/H by Theorem 5.24. Further, it holds that $f^{\mathrm{hyb}}/H|_{t=0} = \rho_{\mathscr{P}/H}$ by

Theorem 5.27. Hence, the assertion holds. See [BM19, 6.1.9] for a related result. □

Now, we are ready to rigorously formulate and prove [KS06, Conjecture 3] for K-trivial finite quotients of abelian varieties of any dimension, under a slight assumption.

Corollary 6.8 ([GO22, Corollary 3.3],[KS06, Conjecture 3] for K-trivial finite quotients of abelian varieties)**.** *Consider an arbitrary maximally degenerating family* $(\mathcal{X}^*, \mathcal{L}^*)/\Delta^*$ *of polarized abelian varieties of dimension* g *and a holomorphic action of a group* H *(which must be a finite group) on* $\mathcal{X}^*$ *in a fiber-preserving manner such that*

- *the* H*-action fixes* $\mathcal{L}_t$ *for* $t \neq 0$*,*
- *the* H*-action acts trivially on* $\omega_{\mathcal{X}^*/\Delta^*}$ *so that* $\mathcal{X}^*/H \to \Delta^*$ *is again a relatively K-trivial family,*
- *the* H*-action causes a toric action on the toroidal compactification* $\tilde{\mathscr{P}}$ *of the split torus as appeared in* (5.26).

Then the following two tropical affine manifolds with singularities are isomorphic (see Theorem 6.7 for the precise meaning):

(i) *(complex side) the limit of the base* $\mathcal{B}_t/H$ *of special Lagrangian fibrations for* $t \to 0$ *with* $\nabla_B(t)$

(ii) *(non-archimedean side) the base* $\mathcal{B}_0/H$ *which underlies the non-archimedean SYZ fibration.*

Proof. It follows from Theorem 6.7 (ii), (iii) and their proofs. □

Remark 6.9. The complimentary “prediction"s in [KS06, Conjecture 3] are the existence of some “natural" interpretation of the non-archimedean SYZ fibration via Gromov-Hausdorff limit, together with a coincidence of critical locus. Note that original claim (the way to define π_{mer}) itself was not rigorous and the formulation itself was a nontrivial problem. Nevertheless, our construction and study of $\mathcal{X}^{\mathrm{hyb}}(\mathcal{X})$ and identification of its central fiber with the Gromov-Hausdorff limit of $\mathcal{X}_t$s in the proof of Theorem 6.7 rigorously formulated their predictions and, at the same time, affirmatively proven them.

Remark 6.10 (K3 surfaces case)**.** It is harder to consider the same problem for polarized K3 surfaces and currently we are not sure if it works. Indeed, a subtlety exists here, as pointed out to the first author by V.Alexeev in April 2019. That is, in the construction of $\mathcal{B}_0$ in [OO21], the appearing singularities of affine structures are always of Kodaira type i.e., which underlies singular fibers of minimal elliptic surfaces as well-known classical classification by Kodaira. On the other hand, the essential skeleta [AET19] constructed as dual complexes of Kulikov degenerations of degree 2 polarized

K3 surfaces, are not necessarily of Kodaira type. Hence, careful choice of Kulikov degenerations seems necessary (even if the analogous hybrid SYZ fibrations exist).

6.3. **Degenerating abelian varieties and crystallographic groups.**

6.11. In this section, we reveal, in an explicit manner, the relation between the automorphism group of families of abelian varieties and the automorphism group of their Gromov-Hausdorff collapses, which can be regarded as a crystallographic group.

6.12 (Set up)**.** Consider the family of abelian varieties with polarizations of the type $E = \mathrm{diag}(e_1, \cdots, e_g)$, following Lemma 4.19, i.e.,

$$\mathcal{X}^* = ((\mathbb{C}^*)^g \times \Delta^*)/\langle p_{i,j}(t)\rangle_{1\le i,j\le g} \to \Delta^*,$$

for $p_{i,j}(t) \in \mathbb{C}((t))^{\mathrm{mero}}$, where each $p_{i,j}(t)$ converges at any $t \in \Delta^*$. Recall that the fibers are

$$\mathbb{C}^g/(\mathbb{Z}^g \oplus \Omega(t)\mathbb{Z}^g) \cong \mathbb{C}^g/(E\mathbb{Z}^g \oplus \Omega'(t)\mathbb{Z}^g)$$

where $\Omega(t) := \left(\frac{1}{2\pi\sqrt{-1}} \log p_{i,j}(t)\right)_{i,j}$ and $\Omega'(t) = \left(\frac{e_i}{2\pi\sqrt{-1}} \log p_{i,j}(t)\right)_{i,j}$. Of course, $\Omega(t)$ and $\Omega'(t)$ have the ambiguities caused by the monodromy effect. However, the lattices $(I\ \Omega(t))\mathbb{Z}^{2g} \subset \mathbb{C}^g$ and $(E\ \Omega'(t))\mathbb{Z}^{2g} \subset \mathbb{C}^g$ are well-defined. In addition, the imaginary part of $\Omega(t)$ (resp., $\Omega'(t)$) is also well-defined (see the proof of Lemma 4.17).

Note that

$$\mathrm{Im}(\Omega'(t)) = \left(\frac{-1}{2\pi} \log |q_{i,j}(t)|\right)_{1\le i,j\le g},$$

where $q_{i,j}(t) := p_{i,j}(t)^{e_i}$.

In the following way, we re-describe the matrices $B = (\mathrm{val}_t p_{i,j}(t))_{i,j} \in \mathrm{Mat}_{g\times g}(\mathbb{Z})$ and $B' = (\mathrm{val}_t q_{i,j}(t))_{i,j} \in \mathrm{Mat}_{g\times g}(\mathbb{Z})$, which appeared in [FC90, Got22, Oda19] and the proof of Theorem 4.20.

Lemma 6.13. *Under the above setup 6.12, the following hold:*

$$(B_{i,j})_{1\le i,j\le g} = \lim_{t\to 0} \frac{-2\pi}{\log|t|} \mathrm{Im}(\Omega(t)),$$

$$(B'_{i,j})_{1\le i,j\le g} = \lim_{t\to 0} \frac{-2\pi}{\log|t|} \mathrm{Im}(\Omega'(t)).$$

Proof. The assertion simply follows from

$$\lim_{t\to 0} \frac{\log|p_{i,j}(t)|}{\log|t|} = \mathrm{val}_t p_{i,j}(t) \text{ and } \lim_{t\to 0} \frac{\log|q_{i,j}(t)|}{\log|t|} = \mathrm{val}_t q_{i,j}(t).$$

□

Below, we often denote $B_{i,j}$ simply as B. Now we define and study the following two automorphism groups of X in our context.

Definition 6.14. Under the above setup 6.12,

(i) $\mathrm{Aut}(X, c_1(L))$ consists of automorphisms of X which preserve the first Chern class $c_1(L)$. In other words, $f \in \mathrm{Aut}(X, c_1(L))$ preserves the flat Kähler metric on each $\mathcal{X}_t$ with its Kähler class $c_1(\mathcal{L}_t)$.

(ii) $\mathrm{Aut}(X, L)$ consists of automorphisms of X which preserve L. In other words, $f \in \mathrm{Aut}(X, L)$ preserves the isomorphic class of the principal polarization $\mathcal{L}_t$ on each $\mathcal{X}_t$.

It is clear that $\mathrm{Aut}(X, L)$ is a subgroup of $\mathrm{Aut}(X, c_1(L))$.

Firstly, we refine Lemma 4.19.

Lemma 6.15. *Under the above setup 6.12, the image of*

$$l\colon \ \mathrm{Aut}(X, c_1(L)) \to \mathrm{GL}(g, \mathbb{Z})$$

in Lemma 4.19 lies in $\mathrm{GL}(g, \mathbb{Z}) \cap \mathrm{O}(g, EB^{-1}) \cap B(\mathrm{GL}(g, \mathbb{Z}))B^{-1}$, *where* $\mathrm{O}(g, EB^{-1}) := \{M \in \mathrm{GL}_g(\mathbb{R}) \mid {}^tMEB^{-1}M = EB^{-1}\}$.

Proof. Note that $f \in \mathrm{Aut}(X, c_1(L))$ induces an automorphism of $\mathcal{X}^*$ over Δ^*, possibly after shrinking the radius of Δ^* by rescale, which we denote by the same letter $f \in \mathrm{Aut}(\mathcal{X}^*)$. The restriction $f_t := f|_{\mathcal{X}_t}$ gives an automorphism of $(\mathcal{X}_t, c_1(\mathcal{L}_t))$. By the argument of Lemma 4.19, the automorphism f_t of $\mathcal{X}_t = \mathbb{C}^g/(I\ \Omega(t))\mathbb{Z}^{2g}$ is induced from a linear transformation $l(f) = M \in \mathrm{GL}(g, \mathbb{Z})$ which is independ of $t \in \Delta^*$. Consider the polarized abelian variety $(\mathcal{X}_t, \mathcal{L}_t)$ defined by $\mathbb{C}^g/(I\ \Omega(t))\mathbb{Z}^{2g}$ as (6.12). The metric on $\mathcal{X}_t$ induced by $c_1(\mathcal{L}_t)$ is given by the Hermite matrix $(\mathrm{Im}\Omega(t))^{-1}$ on $\mathbb{C}^g$ as we see in (2.26). Since the automorphism f_t of $\mathcal{X}_t = \mathbb{C}^g/(I\ \Omega(t))\mathbb{Z}^{2g}$ preserving the metric $c_1(\mathcal{L}_t)$, the corresponding $M \in \mathrm{GL}(g, \mathbb{Z})$ satisfies the equation ${}^tME(\mathrm{Im}\Omega(t))^{-1}M = E(\mathrm{Im}\Omega(t))^{-1}$ for any $t \in \Delta^*$. By Proposition 6.13, the above equation gives an equation ${}^tMEB^{-1}M = EB^{-1}$ by taking the limit for $t \to 0$ after multiplying by $\frac{-\log|t|}{2\pi}$.

To finish the proof, we now show $M \in B(\mathrm{GL}(g, \mathbb{Z}))B^{-1}$, that is,

$$MB\mathbb{Z}^g = B\mathbb{Z}^g$$

for the above $M = l(f)$. Since $M \in \mathrm{O}(g, B^{-1})$, we obtain

$$MB = BE^{-1t}M^{-1}E.$$

It follows from the proof of Lemma 4.19 that $ME\mathbb{Z}^g = E\mathbb{Z}^g$. That is, $E^{-1}ME \in \mathrm{GL}(g, \mathbb{Z})$. It is equivalent to $E^{-1t}M^{-1}E \in \mathrm{GL}(g, \mathbb{Z})$. Hence, $MB\mathbb{Z}^g = BE^{-1t}M^{-1}E\mathbb{Z}^g = B(E^{-1t}M^{-1}E)\mathbb{Z}^g = B\mathbb{Z}^g$. □

Corollary 6.16. *Under the same setting as Lemma 6.15, when* $e_1 = \cdots = e_g = 1$ *i.e., principally polarized case, the restriction* $l|_{\mathrm{Aut}(X,L)}$ *is injective.*

Proof. An automorphism $f \in \mathrm{Aut}(X, c_1(L))$ induces an automorphism $\tilde{f}\colon (\mathbb{C}^*)^g \times \Delta^* \to (\mathbb{C}^*)^g \times \Delta^*$, possibly after shrinking the radius of Δ^* by rescale. The automorphism $\tilde{f}$ can be described as follows.

$$\begin{array}{cccc} \tilde{f}\colon & (\mathbb{C}^*)^g \times \Delta^* & \longrightarrow & (\mathbb{C}^*)^g \times \Delta^* \\ & \Psi & & \Psi \\ & (z,t) & \longmapsto & (\tilde{f}(z,t), t) \end{array}$$

By Lemma 6.15, we can write

$$\tilde{f}(z,t) = \tilde{f}(1,t) z^{l(f)},$$

where

$$l(f) = (a_{i,j})_{1 \le i,j \le g} \in \mathrm{GL}(g, \mathbb{Z}),$$
$$\tilde{f}(1,t) := \tilde{f}((1, \dots, 1), t),$$

and

$$\tilde{f}(1,t) z^M := \left(\tilde{f}_i(1,t) \prod_j z_j^{a_{i,j}} \right)_i \in (\mathbb{C}^*)^g,$$

where $\tilde{f}_i(1,t) \in \mathbb{C}((t))^{\mathrm{mero}}$ are meromorphic functions on Δ. Here, we set $T_{\tilde{f}}(z,t) := \tilde{f}(1,t) z$. It induces a translation on each fiber $\mathcal{X}_t$. In particular, $T_{\tilde{f}}^{-1} \circ \tilde{f}(z,t) = z^{l(f)}$ holds. If $l(f) = l(f')$, then it holds that

$$T_{\tilde{f}}^{-1} \circ \tilde{f}(z,t) = T_{\tilde{f}'}^{-1} \circ \tilde{f}'(z,t).$$

That is, the equation $\tilde{f}'(z,t) = T_{f'\tilde{f}^{-1}} \circ \tilde{f}(z,t)$ holds. The argument so far does not need the assumption of principal polarization. In other words, it holds for general polarizations.

The morphisms f and f' preserve the ample line bundle L. That is, f_t and f'_t preserve $\mathcal{L}_t$ for each fiber $\mathcal{X}_t$. Hence, the morphism $T_{f'\tilde{f}^{-1}}|_t$ preserves the principal polarization $\mathcal{L}_t$. Since $\mathcal{L}_t$ is principal polarization, i.e., the morphism

$$\begin{array}{cccc} \phi_{\mathcal{L}_t}\colon & \mathcal{X}_t & \longrightarrow & \mathrm{Pic}^0 \mathcal{X}_t \\ & \Psi & & \Psi \\ & x & \longmapsto & T_x^*(\mathcal{L}_t) \otimes \mathcal{L}_t^{-1} \end{array}$$

is an isomorphism as we saw in (2.28). Hence, there is no (nontrivial) translation that preserves the principal polarization $\mathcal{L}_t$. Therefore, the equation $\tilde{f}'(1,t) = \tilde{f}(1,t)$ holds. It implies that $T_{\tilde{f}} = T_{\tilde{f}'}$. Hence the equation

$$\tilde{f}(z,t) = T_{\tilde{f}}(z^{l(f)}) = T_{\tilde{f}'}(z^{l(f')}) = \tilde{f}'(z,t)$$

holds. That is, the homomorphism $l|_{\mathrm{Aut}(X,L)}$ is injective. □

Note that the homomorphism

$$l : \mathrm{Aut}(X, c_1(L)) \to \mathrm{GL}(g,\mathbb{Z}) \cap \mathrm{O}(g, EB^{-1}) \cap B(\mathrm{GL}(g,\mathbb{Z}))B^{-1}$$

ignores the effect of translations of the abelian varieties. Now we consider a homomorphism that also reflects translations.

Theorem 6.17 (cf. [GO22, Theorem 4.6])**.** *Under the same situation as Lemma 6.15, there is a natural exact sequence*

$$\begin{aligned} 1 &\longrightarrow \mathrm{Hol}(\Delta, (\mathbb{C}^*)^g) \hookrightarrow \mathrm{Aut}(X, c_1(L)) \\ &\xrightarrow{p} (\mathrm{GL}(g,\mathbb{Z}) \cap \mathrm{O}(g, EB^{-1}) \cap B(\mathrm{GL}(g,\mathbb{Z}))B^{-1}) \ltimes \mathbb{Z}^g/B\mathbb{Z}^g, \end{aligned}$$

where $\mathrm{Hol}(\Delta, (\mathbb{C}^*)^g)$ *denotes the group of the germs of holomorphic maps from the neighborhood of* $0 \in \mathbb{C}$ *to* $(\mathbb{C}^*)^g$.

6.18. Note that, via a surjective homomorphism

$$\begin{aligned} s\colon &(\mathrm{GL}(g,\mathbb{Z}) \cap \mathrm{O}(g, EB^{-1}) \cap B(\mathrm{GL}(g,\mathbb{Z}))B^{-1}) \ltimes \mathbb{Z}^g \\ &\to (\mathrm{GL}(g,\mathbb{Z}) \cap \mathrm{O}(g, EB^{-1}) \cap B(\mathrm{GL}(g,\mathbb{Z}))B^{-1}) \ltimes \mathbb{Z}^g/B\mathbb{Z}^g, \end{aligned}$$

$s^{-1}(\mathrm{Im}(p))$ is a *crystallographic group*. Later, in Theorem 6.23, we provide a geometric meaning to the above map p via “tropicalization" i.e., passing to the base of special Lagrangian fibrations or non-archimedean SYZ fibrations.

Proof of Theorem 6.17. In the proof of Corollary 6.16, we construct a pair $(l(f), T_{\tilde{f}})$ for any automorphism $f \in \mathrm{Aut}(X, c_1(L))$. As we saw in the proof of Corollary 6.16, for a generally polarized case, we can also construct a pair $(l(f), T_{\tilde{f}})$ from any automorphism $f \in \mathrm{Aut}(X, c_1(L))$ in the same way.

By the definition of $\tilde{f}(1,t)$, the lift $\tilde{f}(1,t)$ of $f(1,t)$ is uniquely determined up to $\langle p_{i,j}(t)\rangle$. Hence,

$$\mathrm{val}_t(f(1,t)) := \overline{\mathrm{val}_t(\tilde{f}(1,t))} \in \mathbb{Z}^g/B\mathbb{Z}^g$$

is well-defined. Then we define $p(f) := (l(f), \mathrm{val}_t(f(1,t)))$. It is clear that the map p is a homomorphism.

To end the proof, we verify that $\ker p = \mathrm{Hol}(\Delta, (\mathbb{C}^*)^g)$. Take a lift $\tilde{f}(z,t)$ of $f(z,t) \in \ker p$. By definition of $\ker p$, the lift $\tilde{f}(z,t)$ can be described as $\tilde{f}(1,t)z$, where $\mathrm{val}_t\tilde{f}(1,t) = \left(\mathrm{val}_t\tilde{f}_i(1,t)\right)_i = Bv \in B\mathbb{Z}^g$ for some $v \in \mathbb{Z}^g$. Now we consider the morphism $F_{\tilde{f}} : \Delta^* \to (\mathbb{C}^*)^g$ defined by

$$F_{\tilde{f}}(t) := \tilde{f}(1,t)(p_{i,j}(t))^{-v} := \left(\tilde{f}_i(1,t)\prod_j p_{i,j}(t)^{-v_j}\right)_i,$$

where $v = (v_i) \in \mathbb{Z}^g$. It is clear that $\mathrm{val}_t F(t) = 0 \in \mathbb{Z}^g$. In other words, $F_{\tilde{f}}(t) \in \mathrm{Hol}(\Delta, (\mathbb{C}^*)^g)$. Then $F_{\tilde{f}}(t)$ does not depend on how $\tilde{f}$ is taken.

Hence F_f denotes $F_{\tilde{f}}$. Indeed, the lift $\tilde{f}(1,t)$ of $f(1,t)$ is uniquely determined up to $\langle p_{i,j}(t)\rangle$. Any other lift $\tilde{f}'(z,t)$ of $f(z,t)$ can be described as $\tilde{f}(z,t)(p_{i,j}(t))^w$ for some $w \in \mathbb{Z}^g$. Consider $F_{\tilde{f}'}(t)$ in the same manner with respect to $\tilde{f}'$. Since $\mathrm{val}_t \tilde{f}'(1,t) = Bv + Bw = B(v+w)$, it holds that

$$\begin{aligned} F_{\tilde{f}'}(t) &= \tilde{f}'(1,t)(p_{i,j}(t))^{-(v+w)} = \tilde{f}(1,t)(p_{i,j}(t))^w (p_{i,j}(t))^{-(v+w)} \\ &= \tilde{f}(1,t)(p_{i,j}(t))^{-v} = F_{\tilde{f}}(t). \end{aligned}$$

Hence, we obtain the homomorphism $\varphi : \ker p \to \mathrm{Hol}(\Delta, (\mathbb{C}^*)^g)$ defined by $f \mapsto F_f(t)$. On the other hand, for any $F(t) \in \mathrm{Hol}(\Delta, (\mathbb{C}^*)^g)$, the morphism $F(t)z : (\mathbb{C}^*)^g \times \Delta^* \to (\mathbb{C}^*)^g \times \Delta^*$ descends to an automorphism $f_F \in \ker p$. That is, we obtain the homomorphism $\psi : \mathrm{Hol}(\Delta, (\mathbb{C}^*)^g) \to \ker p$ defined by $F \mapsto f_F$. It is obvious that $\varphi \circ \psi = \mathrm{id}$. By construction of φ, for any $f \in \ker p$, we can take $F_f(t)z$ as a lift of f. It implies that $\psi \circ \varphi = \mathrm{id}$ also holds. Therefore, $\ker p \cong \mathrm{Hol}(\Delta, (\mathbb{C}^*)^g)$ holds. □

6.19. Now we consider more geometric meaning of

$$(\mathrm{GL}(g,\mathbb{Z}) \cap \mathrm{O}(g, EB^{-1}) \cap B(\mathrm{GL}(g,\mathbb{Z}))B^{-1}) \ltimes \mathbb{Z}^g/B\mathbb{Z}^g.$$

For the family $(\mathcal{X}^*, c_1(\mathcal{L}^*))$ under discussion, set

$$\mathcal{B}_0 := (\mathcal{B}_0, \nabla_A(0), \nabla_B(0), g_0)$$

as Theorem 4.20. Then Theorem 4.20 implies that

$$\mathcal{B}_0 = (\mathcal{B}_0, \nabla_A(0), \nabla_B(0), g_0) = (\mathbb{R}^g/B\mathbb{Z}^g, BE^{-1}\mathbb{Z}^g, \mathbb{Z}^g, EB^{-1})$$

in the convention of Corollary 4.24. Then the pair $(\nabla_B(0), g_0)$ induces the other affine structure $\nabla_A(0)$ as a Legendre dual by [Hit97]. Hence, $\mathcal{B}_0$ is abbreviated as $(\mathcal{B}_0, \nabla_B(0), g_0)$ for simplicity. In summary, we see that

$$\mathcal{B}_0 = (\mathcal{B}_0, \nabla_B(0), g_0) = (\mathbb{R}^g/B\mathbb{Z}^g, \mathbb{Z}^g, EB^{-1})$$

for the family $(\mathcal{X}^*, c_1(\mathcal{L}^*))$ under discussion. See [GH84] or [GS06] for a series of definitions on (integral) affine manifolds.

Definition 6.20. For the flat integral affine manifold $(\mathbb{R}^g/B\mathbb{Z}^g, \mathbb{Z}^g, EB^{-1})$ as above, define the *automorphic group*

$$\mathrm{Aut}(\mathbb{R}^g/B\mathbb{Z}^g, \mathbb{Z}^g, EB^{-1})$$

to be the group consisting of automorphisms that preserve the structure of $(\mathbb{R}^g/B\mathbb{Z}^g, \mathbb{Z}^g, EB^{-1})$. That is, an element $f \in \mathrm{Aut}(\mathbb{R}^g/B\mathbb{Z}^g, \mathbb{Z}^g, EB^{-1})$ is an integral affine map $f : \mathbb{R}^g/B\mathbb{Z}^g \xrightarrow{\sim} \mathbb{R}^g/B\mathbb{Z}^g$ that preserves the integral points $\mathbb{Z}^g/B\mathbb{Z}^g$ and the flat metric induced by EB^{-1}.

Proposition 6.21. *Under the notation of Definition 6.20, the following holds:*

$$\begin{aligned}&(\mathrm{GL}(g,\mathbb{Z})\cap \mathrm{O}(g,EB^{-1})\cap B(\mathrm{GL}(g,\mathbb{Z}))B^{-1})\ltimes \mathbb{Z}^g/B\mathbb{Z}^g\\ &\cong \mathrm{Aut}(\mathbb{R}^g/B\mathbb{Z}^g,\mathbb{Z}^g,EB^{-1}).\end{aligned}$$

Proof. Consider the universal covering $u:\mathbb{R}^g\twoheadrightarrow \mathbb{R}^g/B\mathbb{Z}^g$ which we denote by $x\mapsto \overline{x}$. Note that the map u is an integral affine map. For $f\in \mathrm{Aut}(\mathbb{R}^g/B\mathbb{Z}^g,\mathbb{Z}^g,EB^{-1})$, the map $f\circ u$ is also an universal covering. The universality gives an homeomorphism $\tilde{f}:\mathbb{R}^g\xrightarrow{\sim}\mathbb{R}^g$ such that $u\circ\tilde{f}=f\circ u$. Here, the map $\tilde{f}:(\mathbb{R}^g,EB^{-1})\xrightarrow{\sim}(\mathbb{R}^g,EB^{-1})$ is an isometry since the map f is an isometry and the map u is locally trivial. Hence,

$$\tilde{f}(x)=Mx+v,$$

where $M\in O(g,EB^{-1})$ and $v\in\mathbb{R}^g$. Moreover, since the map f is an integral affine morphism and the map u is locally trivial, the map $\tilde{f}$ is also an integral affine morphism. Hence, the above pair (M,v) satisfies $M\in \mathrm{GL}(g,\mathbb{Z})\cap B(\mathrm{GL}(g,\mathbb{Z}))B^{-1}$ and $v\in\mathbb{Z}^g$. It implies that $f(\overline{x})=M\overline{x}+\overline{v}$, where $(M,\overline{v})\in(\mathrm{GL}(g,\mathbb{Z})\cap\mathrm{O}(g,EB^{-1})\cap B(\mathrm{GL}(g,\mathbb{Z}))B^{-1})\ltimes\mathbb{Z}^g/B\mathbb{Z}^g$. That is, we obtain the homomorphism

$$\begin{aligned}\varphi:&\mathrm{Aut}(\mathbb{R}^g/B\mathbb{Z}^g,\mathbb{Z}^g,EB^{-1})\\ &\to(\mathrm{GL}(g,\mathbb{Z})\cap\mathrm{O}(g,EB^{-1})\cap B(\mathrm{GL}(g,\mathbb{Z}))B^{-1})\ltimes\mathbb{Z}^g/B\mathbb{Z}^g.\end{aligned}$$

Since the existence of the inverse map is obvious, the assertion follows. □

6.22. In particular, $\mathrm{Aut}(X,c_1(L))$ induces at most finite group actions on $\mathcal{B}_0$. In Theorem 6.7 (ii), we have already seen that $\mathrm{Aut}(X,c_1(L))$ induces continuous actions on both $\mathcal{X}^{\mathrm{hyb}}$ and $\mathcal{B}$. Further, f^{hyb} is $\mathrm{Aut}(X,c_1(L))$-equivariant. Now we describe the actions of $\mathrm{Aut}(X,c_1(L))$ concretely, by relating the H-action of Theorem 6.7 (ii) and the map p in Theorem 6.17.

Theorem 6.23 (cf. [GO22, Theorem 4.9])**.** *Under the same situation as Theorem 6.17, for any subgroup H of $\mathrm{Aut}(X,c_1(L))$, consider the induced actions on $\mathcal{X}^{\mathrm{hyb}}$ and $\mathcal{B}$ of H as Theorem 6.7. We denote the restriction of the latter action on $\mathcal{B}_t$ as $p_t(h)$ for each $h\in H$ and $t\in\Delta$.*

Then $p_t(h)$ is explicitly described as

(i) *For $t\neq 0$,*

$$\left(l(h),\log_{|t|}|\tilde{h}(1,t)|\right)\in\mathrm{GL}(g,\mathbb{Z})\ltimes\left(\mathbb{R}^g/\left(\frac{-2\pi}{\log|t|}\mathrm{Im}\Omega(t)\right)\mathbb{Z}^g\right),$$

where $\tilde{h}(1,t)$ is a lift of $h(1,t)$. Note that the ambiguity of the choice of $\tilde{h}(1,t)$ vanishes on $\mathcal{B}_t$.

(ii) *$p_0(h)$ on $\mathcal{B}_0$ is equal to $p(h)$ as constructed in Theorem 6.17.*

Further, $f_t(hx)=p_t(h)f_t(x)$ for each $h\in H$ and $t\in\Delta$.

Proof. First, we show (i). By the proof of Theorem 6.17, $h \in H$ is lifted to

$$\tilde{h}(z,t) = \psi(t)t^v z^M : (\mathbb{C}^*)^g \times \Delta^* \to (\mathbb{C}^*)^g,$$

where $(M,v) = p(h)$ and $\psi(t) := \tilde{h}(1,t)/t^v \in \mathrm{Hol}(\Delta, (\mathbb{C}^*))$. By the argument of Lemma 6.15, $M = l(h)$ induces the automorphism of

$$\left(\left(\mathbb{R}^g / \left(\frac{-2\pi}{\log|t|}\mathrm{Im}\Omega(t)\right)\mathbb{Z}^g\right), \mathbb{Z}^g, E\left(\frac{-2\pi}{\log|t|}\mathrm{Im}(\Omega(t))\right)^{-1}\right)$$

for $t \neq 0$. Since

$$\mathcal{B}_t \cong \left(\left(\mathbb{R}^g / \left(\frac{-2\pi}{\log|t|}\mathrm{Im}\Omega(t)\right)\mathbb{Z}^g\right), \mathbb{Z}^g, E\left(\frac{-2\pi}{\log|t|}\mathrm{Im}(\Omega(t))\right)^{-1}\right),$$

the pair $\left(l(h), \log_{|t|}|\tilde{h}(1,t)|\right)$ also induces the automorphism of $\mathcal{B}_0$. Note that the lift $\tilde{h}(1,t)$ of $h(1,t)$ is uniquely determined up to $\langle p_{i,j}(t)\rangle$. It implies that $\log_{|t|}|\tilde{h}(1,t)|$ is well-defined on

$$\mathbb{R}^g / \left(\frac{-2\pi}{\log|t|}\mathrm{Im}\Omega(t)\right) = \mathbb{R}^g / \left(\log_{|t|}|p_{i,j}(t)|\right).$$

Now consider $f_t(hx)$. By the argument of the proof of Theorem 6.7, special Lagrangian fibration $f_t(x = (x_1, \cdots, x_g))$ is described for $t \neq 0$ as

$$f_t(x) = \left(-\log_{|t|}|x_i|\right)_{i=1,\cdots,g} \in \mathbb{R}^g / \left(\frac{-2\pi}{\log|t|}\mathrm{Im}\Omega(t)\right)\mathbb{Z}^g.$$

Hence, it is clear that

$$f_t(hx) = l(h)f_t(x) + \log_{|t|}|\tilde{h}(1,t)| = \left(l(h), \log_{|t|}|\tilde{h}(1,t)|\right) \cdot f_t(x).$$

By Theorem 6.7, f_t is H-equivalent. It implies that $\left(l(h), \log_{|t|}|\tilde{h}(1,t)|\right)$ is equal to the induced action $p_t(h)$ on $\mathcal{B}_t$ for $t \neq 0$.

Now we prove (ii) by considering $f_0(hx)$. Now, the map $f_0(x)$ is defined as $(-\log|Z|_x) \in \mathbb{R}^g / B\mathbb{Z}^g$, where $x \in X^{\mathrm{an}}$, $|\cdot|_x$ means the corresponding multiplicative seminorm on X and $Z = (Z_i)$ is the coordinates of the split algebraic torus as appeared in [Got22]. Then, it follows from $\log|\psi(t)|_x = 0$ that $f_0(hx) = p(h)f_0(x)$. Thus, $p_0(h) = p(h)$ holds similarly. Finally, the H-equivalence of f^{hyb} implies the last claim of the theorem. □

6.24. In general, we cannot expect that the symmetry of $\mathcal{B}_0$ lifts on the symmetry of $(X, c_1(L))$ or $\mathcal{X}^*$. However, we shall see that such lifting exists for the following special situation.

We start with a $B = (b_{i,j})_{i,j} \in \mathrm{Mat}_{g\times g}(\mathbb{Z})$ such that EB is a positive definite symmetric matrix for some $E = \mathrm{diag}(e_1, \dots e_g)$ as appeared in (6.12). Then we obtain the following family

$$\mathcal{X}_B^* = (\Delta^* \times (\mathbb{C}^*)^g)/\langle t^B \rangle_{i,j} \to \Delta^*,$$

where t^B means the $g \times g$ matrix $\left(t^{b_{i,j}}\right)_{i,j}$. We denote the associated polarized smooth variety to $(\mathcal{X}_B^*, \mathcal{L}_B^*)$ by (X_B, L_B).

In this situation, we can strengthen Theorem 6.17 as a split exact sequence.

Proposition 6.25. *Under the above setup 6.24, we have*

$$\mathrm{Aut}(X_B, c_1(L_B)) \cong \mathrm{Aut}(\mathcal{B}_0) \ltimes \mathrm{Hol}(\Delta, (\mathbb{C}^*)^g).$$

Proof. By Theorem 6.17, it suffices to see that there is a homomorphism $\iota : \mathrm{Aut}(\mathcal{B}_0) \to \mathrm{Aut}(X_B, c_1(L_B))$ such that $p \circ \iota = \mathrm{id}$. For $(M, \overline{v}) \in \mathrm{Aut}(\mathcal{B}_0) \cong (\mathrm{GL}(g,\mathbb{Z}) \cap \mathrm{O}(g, EB^{-1}) \cap B(\mathrm{GL}(g,\mathbb{Z}))B^{-1}) \ltimes \mathbb{Z}^g/B\mathbb{Z}^g$, we consider the map $\tilde{f} : (\mathbb{C}^*)^g \times \Delta^* \to (\mathbb{C}^*)^g \times \Delta^*$ defined by $\tilde{f}(z,t) := t^v z^M$ in the same manner of the proof of Corollary 6.15. Here, we see that the map $\tilde{f}$ descends to the map $f : (\mathcal{X}_B^*, c_1(\mathcal{L}_B^*)) \to (\mathcal{X}_B^*, c_1(\mathcal{L}_B^*))$. Indeed, the fiber $\mathcal{X}_B^*|_t$ is given by $\mathbb{C}^g/(I\ \Omega(t))\mathbb{Z}^{2g}$ for each $t \in \Delta^*$, where $\Omega(t) = \frac{\log t}{2\pi\sqrt{-1}} B$. The above $M \in (\mathrm{GL}(g,\mathbb{Z}) \cap \mathrm{O}(g, EB^{-1}) \cap B(\mathrm{GL}(g,\mathbb{Z}))B^{-1})$ satisfies $MB\mathbb{Z}^g = B\mathbb{Z}^g$ and ${}^tMEB^{-1}M = EB^{-1}$. In particular, it implies that $M\mathrm{Im}\Omega(t)\mathbb{Z}^g = \mathrm{Im}\Omega(t)\mathbb{Z}^g$ and ${}^tME(\mathrm{Im}\Omega(t))^{-1}M = E(\mathrm{Im}\Omega(t))^{-1}$. Hence the induced automorphism $z^M : \mathcal{X}_B^* \to \mathcal{X}_B^*$ is well-defined and preserves the metric for each fiber. Since translations have no effects on the period matrix and the metric, $\tilde{f}$ descends to the automorphism f of $(X_B, c_1(L_B))$. Note that the morphism f does not depend on how $\tilde{f}$ is taken. Indeed, the ambiguity of v vanishes in the process of obtaining f from $\tilde{f}$. Hence we obtain the homomorphism $\iota : \mathrm{Aut}(\mathcal{B}_0) \to \mathrm{Aut}(X_B, c_1(L_B))$. By construction, it is obvious that $p \circ \iota = \mathrm{id}$. Therefore, the assertion holds. □

Remark 6.26. We only considered maximal degenerations case in this thesis but degenerations of g-dimensional polarized abelian varieties $(\mathcal{X}^*, \mathcal{L}^*)/\Delta^*$ with other torus rank i should work similarly. Firstly, the same method as [Mat16, proof of 1.1, cf., also Remark 5.3] by studying actions on weight filtration or weight spectral sequences, implies that for any polarized endomorphism f of $(\mathcal{X}^*, \mathcal{L}^*)/\Delta^*$, any eigenvalue e of $f^*|_{H^1(\mathcal{X}_t,\mathbb{C})}$, the degree d of the minimal polynomial of rational coefficients satisfies $d \geq \max\{i, 2g-2i\}$. We may explore more details in future.

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