博　士　論　文

Birational geometry and compactifications of modular varieties and arithmetic of modular forms

（モジュラー多様体の双有理幾何学とコンパクト化及び
モジュラー形式の数論について）

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Birational geometry and compactifications of modular varieties and arithmetic of modular forms

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To my family.

Abstract

In this thesis, we study the birational types of modular varieties, their several compactifications, and the modularity of the generating series of special cycles on Shimura varieties. In particular, we focus on ball quotients and orthogonal modular varieties in terms of modular forms.

First, we consider when ball quotients are of general type. To prove that they are of general type, there are three types of obstructions: reflective, cusp and elliptic obstructions. We give a tool, which is a criterion called low slope cusp form trick, to study cusp obstructions. Moreover, we prove that reflective obstructions are small enough in higher dimensions and as a byproduct, the finiteness of reflective modular forms. We remark that elliptic obstructions were already resolved by Behrens. These results are the unitary analog of the work by Gritsenko-Hulek-Sankaran and Ma on orthogonal modular varieties.

Second, we work on the birational classification of modular varieties in terms of reflective modular form. As a consequence, we show that the Baily-Borel compactification of certain modular varieties are Fano varieties, Calabi-Yau varieties or have ample canonical divisors with mild singularities. This includes important examples in algebraic geometry, for instance, the moduli space of (log) Enriques surfaces.

Third, we consider a particular ball quotient, which is the moduli space of 8 points on $\mathbb{P}^1$, a so-called ancestral Deligne-Mostow space. We prove that the Deligne-Mostow isomorphism does not lift to a morphism between the Kirwan blow-up of the GIT quotient and the unique toroidal compactification of the corresponding ball quotient. In addition, we show that these spaces are not K-equivalent, even though they are natural blow-ups at the unique cusps and have the same cohomology. This is analogous to the work of Casalaina-Martin-Grushevsky-Hulek-Laza on the moduli space of cubic surfaces.

Finally, we prove that the generating series of special cycles on orthogonal or unitary Shimura varieties has certain modularity under the Beilinson-Bloch conjecture. Historically, Hirzebruch-Zagier observed that the intersection numbers of Heegner divisors on a Hilbert modular surface generate a certain weight 2 elliptic modular form, and Kudla-Millson generalized this to orthogonal or unitary Shimura varieties with the cohomological coefficients. We work on the Chow group coefficients, hence our results are the generalization of *Kudla's modularity conjecture* treated by Borcherds, Bruinier, Kudla, Liu, Millson, Raum and Yuan-Zhang-Zhang.

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Contents

CHAPTER 1

Introduction

1.1. Outline of this thesis

Modular varieties play an important role in a wide range. On the one side, they have aspects such as moduli spaces. In this thesis, for example, the modular varieties realized as moduli spaces of K3 surfaces, (log) Enriques surfaces, and points on $\mathbb{P}^1$ are treated. On the other hand, their birational properties are closely related to modular forms. We can apply the number theoretic methods such as the Borcherds lift to describe their geometry based on the celebrated work of Baily-Borel [**7**] and Mumford [**120**].

In this thesis, we study the birational geometric properties and several compactifications of ball quotients and orthogonal modular varieties in terms of modular forms, and modularity of the generating series of special cycles on Shimura varieties. The contents are as follows.

(1) To prove that a modular variety is of general type, there are three types of obstructions: reflective, cusp and elliptic obstructions. We show that low slope cusp form trick, a criterion to show that modular varieties are of general type in terms of cusp forms, holds for ball quotients. This gives a tool to study cusp obstructions. This is proved by classifying irregular cusps. In addition, we determine the relationship between irregular cusps of ball quotients and ones of orthogonal modular varieties, studied by Ma [**109**]. See Chapter 2 for details. This result is based on [**112**].
(2) We prove that reflective obstructions are small enough in higher dimensions in the case of ball quotients. Our result reduces the study of the Kodaira dimension of unitary modular varieties to the construction of a cusp form of small weight in a quantitative manner. As a byproduct, we formulate and partially prove the finiteness of Hermitian lattices admitting reflective modular forms, which is a unitary analog of the conjecture by Gritsenko-Nikulin in the orthogonal case. See Chapter 3 for details. This result is based on the preprint [**114**].
(3) We prove that the Baily-Borel compactification of certain modular varieties are Fano varieties, Calabi-Yau varieties or have ample canonical divisors with mild singularities. We also prove some variants statements, give applications and discuss various examples including new ones, for instance, the moduli spaces of unpolarized (log) Enriques surfaces. See Chapter 4 for details. This result is based on the joint work [**115**] with Yuji Odaka.
(4) The moduli space of 8 points on $\mathbb{P}^1$, a so-called ancestral Deligne-Mostow space, is, by work of Kondō, also a moduli space of K3 surfaces. We prove that the Deligne-Mostow isomorphism does not lift to a morphism between the Kirwan blow-up of the GIT quotient and the unique toroidal compactification of the corresponding ball quotient. Moreover, we show that these spaces are not K-equivalent, even though they are natural blow-ups at the unique cusps and have the same cohomology. This is analogous to the work of Casalaina-Martin-Grushevsky-Hulek-Laza

on the moduli space of cubic surfaces. We further briefly discuss other cases of moduli space of points in $\mathbb{P}^1$ where a similar behavior can be observed, hinting at a more general, but not yet fully understood phenomenon. The moduli spaces of ordinary stable maps, that is the Fulton-MacPherson compactification of the configuration space of points in $\mathbb{P}^1$, play an important role in the proof. See Chapter 5 for details. This result is based on the joint work [**69**] with Klaus Hulek.

(5) We prove the modularity of the generating series of special cycles on orthogonal and unitary Shimura varieties under the Beilinson-Bloch conjecture. We work on the Chow group coefficients, hence our results are the generalization of *Kudla's modularity conjecture* treated by Borcherds, Bruinier, Kudla, Liu, Millson, Raum and Yuan-Zhang-Zhang. See Chapter 6 and 7 for details. These are based on [**111, 113**].

The outline of this thesis is as follows. In Chapter 2, we study irregular cusps of ball quotients. This is used to measure the order of modular forms at the boundary of toroidal compactifications and prove low slope cusp form trick, a criterion asserting that ball quotients are of general type. In Chapter 3, we estimate the dimension of modular forms vanishing on branch divisors. This implies that reflective obstructions of ball quotients do not affect to prove that they are of general type if their dimension is sufficiently large, say greater than 138. In Chapter 4, we introduce a certain class of modular forms, which is called "special reflective modular forms". By using this notion, we prove a criterion, claiming some modular varieties are Fano varieties, Calabi-Yau varieties or have ample canonical divisors. In Chapter 5, we study the moduli space of 8 points on $\mathbb{P}^1$, classically treated in the Deligne-Mostow theory [**32**] and by Kondo [**88**]. We show the Deligne-Mostow isomorphism does not lift between the Kirwan blow-up and the toroidal compactification, compute their cohomology and prove that they are not K-equivalent. In Chapter 6 (resp. Chapter 7), we introduce the notion of the generating series, constructed geometrically in terms of Shimura varieties, and prove that they are certain modular forms for orthogonal Shimura varieties over totally real fields (resp. unitary Shimura varieties over CM-fields) under the Beilinson-Bloch conjecture. Our main results show their modularity with Chow group coefficients, which is a generalization of Kudla-Millson [**96**] and Yuan-Zhang-Zhang [**151**].

In the rest of this chapter, we shall introduce the notion of modular varieties and give precise statements of our results, although a more detailed introduction will be provided at the beginning of each chapter.

1.2. Notation

1.2.1. Modular varieties (Chapter 2, 3, 4, 5). In Chapter 2, 3, 4, 5, we study the birational geometry of ball quotients. Let us introduce their notion. The following notation in this section will be used in the above Chapters.

Let $F := \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field where d is a square-free negative integer. Let $-D$ be its discriminant and $\mathscr{O}_F$ ring of integers. Let $(L, \langle\ ,\ \rangle)$ be a Hermitian lattice over $\mathscr{O}_F$ of signature $(1, n)$ with $n > 1$. Here, Hermitian forms are complex linear in the first argument and complex conjugate linear in the second argument, and take value in a finite free $\mathscr{O}_F$-module M of rank 1. Below, we take

$$M = \begin{cases} \frac{1}{\sqrt{D}}\mathscr{O}_F & \text{(Chapter } 2, 4) \\ \mathscr{O}_F & \text{(Chapter } 3, 5). \end{cases}$$

Accordingly, we have the unitary group $\mathrm{U}(L)$ over $\mathbb{Z}$. Let D_L be the Hermitian symmetric domain associated with $\mathrm{U}(L\otimes_{\mathbb{Z}}\mathbb{R})$:

$$D_L := \{v \in V\otimes_F \mathbb{C}\setminus\{0\} \mid \langle v, v\rangle > 0\}/\mathbb{C}^{\times}.$$

Then, for a finite index subgroup $\Gamma \subset \mathrm{U}(L)(\mathbb{Z})$, we define

$$\mathscr{F}_L(\Gamma) := D_L/\Gamma.$$

This is a quasi-projective variety over $\mathbb{C}$ and called a *unitary modular variety* or a *ball quotient.*

We call L *primitive* if there does not exist Hermitian lattice $L' \subset L$ of the same rank as L so that the quotient L/L' is a non-trivial torsion $\mathscr{O}_F$-module. We also define the dual lattice $L^\vee$ of L:

$$L^\vee := \{v \in L\otimes_{\mathscr{O}_F} F \mid \langle v, w\rangle \in M \text{ for any } w \in L\}.$$

This lattice contains L as a finite index lattice, so the *discriminant group* $A_L := L^\vee/L$ is a finite $\mathscr{O}_F$-module. We call L is *unimodular* if $L = L^\vee$. As an important example of an arithmetic group, the *discriminant kernel* $\widetilde{\mathrm{U}}(L)$ is defined by

$$\widetilde{\mathrm{U}}(L) := \{g \in \mathrm{U}(L)(\mathbb{Z}) \mid g|_{A_L} = \mathrm{id}\}.$$

On the other hand, in Chapter 2, 4, we also study orthogonal modular varieties. Now, as above, let us prepare some notions. For a quadratic form $(\Lambda, (\ ,\ ))$ of signature $(2, m)$ over $\mathbb{Z}$ with $m > 1$, we realize the Hermitian symmetric domain associated to $\mathrm{O}^+(\Lambda)(\mathbb{R})$ as $\mathscr{D}_\Lambda$ which is defined as one of the connected components of

$$\{v \in \mathbb{P}(\Lambda\otimes\mathbb{C}) \mid (v, v) = 0,\ (v, \overline{v}) > 0\}.$$

Throughout this thesis, we denote by $\mathscr{L}$ the automorphic line bundle of weight 1 on unitary or orthogonal modular varieties.

In this thesis, we usually study the relationship between ball quotients and orthogonal modular varieties in terms of the following embedding, studied in [**68**]. For a Hermitian lattice $(L, \langle\ ,\ \rangle)$ of signature $(1, n)$, we define the associated quadratic lattice $(L_Q, (\ ,\ ))$ over $\mathbb{Z}$ of signature $(2, 2n)$, where $L_Q := L$ as a $\mathbb{Z}$-module and $(\ ,\ ) := \mathrm{Tr}_{F/\mathbb{Q}}\langle\ ,\ \rangle$. Then, we obtain embeddings

$$\iota : \mathrm{U}(L) \hookrightarrow \mathrm{O}^+(L_Q), \tag{1.2.1}$$

and $D_L \hookrightarrow \mathscr{D}_{L_Q}$. In this embedding, we identify the unitary group $\mathrm{U}(L)$ with a subgroup of $\mathrm{O}^+(L_Q)$.

1.2.2. Orthogonal Shimura varieties (Chapter 6). Let us recall the setting of Kudla [**92**], [**95**] and Rosu-Yott [**131**] to define orthogonal Shimura varieties and special cycles.

Let d and e be positive integers satisfying $1 \le e < d$. Let E_0 be a totally real field of degree d with real embeddings $\sigma_1, \dots, \sigma_d$. Let V be a non-degenerate quadratic space of dimension $n+2$ over E_0 whose signature is $(n, 2)$ at $\sigma_1, \dots, \sigma_e$ and $(n+2, 0)$ at $\sigma_{e+1}, \dots, \sigma_d$. We put $V_{\sigma_i,\mathbb{C}} := V\otimes_{F,\sigma_i}\mathbb{C}$ and $\mathbb{P}(V_{\sigma_i,\mathbb{C}}) := (V_{\sigma_i,\mathbb{C}}\setminus\{0\})/\mathbb{C}^\times$. Let $D_i \subset \mathbb{P}(V_{\sigma_i,\mathbb{C}})$ be the Hermitian symmetric domain defined as follows:

$$D_i := \{v \in V_{\sigma_i,\mathbb{C}}\setminus\{0\} \mid \langle v, v\rangle = 0,\ \langle v, \bar{v}\rangle < 0\}/\mathbb{C}^\times \quad (1 \le i \le e).$$

We put $D := D_1 \times \cdots \times D_e$. Let $\mathrm{GSpin}(V)$ be the general spin group of V over E_0, which is a connected reductive group over E_0. We put $\mathcal{G} := \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSpin}(V)$ and consider the Shimura varieties associated with $(\mathcal{G}, D)$. Then, for any open compact subgroup $K_f \subset$

$\mathcal{G}(\mathbb{A}_f)$, the Shimura datum $(\mathcal{G},D)$ gives a *orthogonal Shimura variety* M_{K_f} over $\mathbb{C}$, whose $\mathbb{C}$-valued points are given as follows:

$$M_{K_f}(\mathbb{C}) = \mathcal{G}(\mathbb{Q})\backslash(D \times \mathcal{G}(\mathbb{A}_f))/K_f.$$

Here $\mathbb{A}_f$ is the ring of finite adèles of $\mathbb{Q}$. We remark that M_{K_f} has a canonical model over a number field called the reflex field. Hence M_{K_f} is canonically defined over $\overline{\mathbb{Q}}$. In this subsection, $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$ embedded in $\mathbb{C}$. By abuse of notation, in this chapter, the canonical model of M_{K_f} over $\overline{\mathbb{Q}}$ is also denoted by the same symbol M_{K_f}. Then the Shimura variety M_{K_f} is a projective variety over $\overline{\mathbb{Q}}$ since $1 \leq e < d$. It is a smooth variety over $\overline{\mathbb{Q}}$ if K_f is sufficiently small.

For $i = 1, \dots, e$, let $\mathscr{L}_i \in \mathrm{Pic}(D_i)$ be the line bundle which is the restriction of $\mathscr{O}_{\mathbb{P}(V_{\sigma_i,\mathbb{C}})}(-1)$ to D_i. By pulling back to D, we get $p_i^*\mathscr{L}_i \in \mathrm{Pic}(D)$, where $p_i \colon D \to D_i$ are the projection maps. These line bundles descend to $\mathscr{L}_{K_f,i} \in \mathrm{Pic}(M_{K_f}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and thus we obtain $\mathscr{L} := \mathscr{L}_{K_f,1} \otimes \cdots \otimes \mathscr{L}_{K_f,e}$ on M_{K_f}.

We shall define special cycles following Kudla [**92**], [**95**] and Rosu-Yott [**131**]. Let $W \subset V$ be a totally positive subspace over E_0. We denote $\mathcal{G}_W := \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GSpin}(W^\perp)$. Let $D_W := D_{W,1} \times \cdots \times D_{W,e}$ be the Hermitian symmetric domain associated with $\mathcal{G}_W$, where

$$D_{W,i} := \{w \in D_i \mid \forall v \in W_{\sigma_i},\ \langle v, w\rangle = 0\} \quad (1 \leq i \leq e).$$

Then we have an embedding of Shimura data $(\mathcal{G}_W, D_W) \hookrightarrow (\mathcal{G}, D)$. For any open compact subgroup $K_f \subset \mathcal{G}(\mathbb{A}_f)$ and $g \in \mathcal{G}(\mathbb{A}_f)$, we have an associated Shimura variety $M_{gK_fg^{-1},W}$ over $\mathbb{C}$:

$$M_{gK_fg^{-1},W}(\mathbb{C}) = \mathcal{G}_W(\mathbb{Q})\backslash(D_W \times \mathcal{G}_W(\mathbb{A}_f))/(gK_fg^{-1} \cap \mathcal{G}_W(\mathbb{A}_f)).$$

Assume that K_f is neat so that the following morphism

$$\begin{aligned} M_{gK_fg^{-1},W}(\mathbb{C}) &\to M_{K_f}(\mathbb{C}) \\ [\tau, h] &\mapsto [\tau, hg] \end{aligned}$$

is a closed embedding [**95**, Lemma 4.3]. Let $Z(W, g)_{K_f}$ be the image of this morphism. We consider $Z(W, g)_{K_f}$ as an algebraic cycle of codimension $e \dim_{E_0} W$ on M_{K_f} defined over $\overline{\mathbb{Q}}$.

For any positive integer r and $x = (x_1, \dots, x_r) \in V^r$, let $U(x)$ be the E_0-subspace of V spanned by $x_1, \dots, x_r$. We define the *special cycle* in the Chow group

$$Z(x, g)_{K_f} \in \mathrm{CH}^{er}(M_{K_f})_{\mathbb{C}} := \mathrm{CH}^{er}(M_{K_f}) \otimes_{\mathbb{Z}} \mathbb{C}$$

by

$$Z(x, g)_{K_f} := Z(U(x), g)_{K_f} (\mathrm{c}_1(\mathscr{L}^\vee_{K_f,1}) \cdots \mathrm{c}_1(\mathscr{L}^\vee_{K_f,e}))^{r - \dim U(x)}$$

if $U(x)$ is totally positive. Otherwise, we put $Z(x, g)_{K_f} := 0$.

For a Bruhat-Schwartz function $\phi_f \in \mathbf{S}(V(\mathbb{A}_f)^r)^{K_f}$, *Kudla's generating function* is defined to be the following formal power series with coefficients in $\mathrm{CH}^{er}(M_{K_f})_{\mathbb{C}}$ in the variable $\tau = (\tau_1, \dots, \tau_d) \in (\mathscr{H}_r)^d$:

$$Z_{\phi_f}(\tau) := \sum_{x \in \mathcal{G}(\mathbb{Q})\backslash V^r} \sum_{g \in \mathcal{G}_x(\mathbb{A}_f)\backslash \mathcal{G}(\mathbb{A}_f)/K_f} \phi_f(g^{-1}x) Z(x, g)_{K_f} q^{T(x)}.$$

Here $\mathcal{G}_x \subset \mathcal{G}$ is the stabilizer of x, $\mathscr{H}_r$ is the Siegel upper half plane of genus r, $T(x)$ is the moment matrix $\frac{1}{2}((x_i, x_j))_{i,j}$, and

$$q^{T(x)} := \exp(2\pi\sqrt{-1}\sum_{i=1}^{d} \operatorname{Tr} \tau_i T(x)^{\sigma_i}).$$

For a $\mathbb{C}$-linear map $\ell\colon \operatorname{CH}^{er}(M_{K_f})_{\mathbb{C}} \to \mathbb{C}$, we put

$$\ell(Z_{\phi_f})(\tau) := \sum_{x\in\mathcal{G}(\mathbb{Q})\backslash V^r} \sum_{g\in\mathcal{G}_x(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f)/K_f} \phi_f(g^{-1}x)\ell(Z(x,g)_{K_f})q^{T(x)},$$

which is a formal power series with complex coefficients in the variable $\tau \in (\mathscr{H}_r)^d$.

1.2.3. Unitary Shimura varieties (Chapter 7). Let d, e, and n be positive integers such that $e < d$. Let E_0 be a totally real field of degree d with real embeddings $\sigma_1, \ldots, \sigma_d$ and E be a CM extension of E_0. We write ∂_{E_0} for the different ideal of E_0. Let $(V_E, \langle\ ,\ \rangle)$ be a non-degenerate Hermitian space of dimension $n+1$ over E whose signature is $(n,1)$ at $\sigma_1, \ldots, \sigma_e$ and $(n+1, 0)$ at $\sigma_{e+1}, \ldots, \sigma_d$.

For $i = 1, \ldots, e$, let $V_{E,\sigma_i,\mathbb{C}} := V_E \otimes_{E_0,\sigma_i} \mathbb{C}$ and $D_i^E \subset \mathbb{P}(V_{E,\sigma_i,\mathbb{C}})$ be the Hermitian symmetric domain defined as:

$$D_i^E := \{v \in V_{E,\sigma_i,\mathbb{C}}\backslash\{0\} \mid \langle v, v\rangle > 0\}/\mathbb{C}^\times.$$

We use

$$D_E := D_1^E \times \cdots \times D_e^E.$$

Let $\mathrm{U}(V_E)$ be the unitary group of V_E over E_0, which is also a reductive group over E_0. We put $\mathcal{H} := \operatorname{Res}_{F/\mathbb{Q}} \mathrm{U}(V_E)$ and consider the Shimura varieties associated with the Shimura datum $(\mathcal{H}, D_E)$. Then, for any open compact subgroup $K_f^{\mathcal{H}} \subset \mathcal{H}(\mathbb{A}_f)$, we obtain a *unitary Shimura variety* $M_{K_f^{\mathcal{G}}}$ over $\mathbb{C}$, which is a projective variety over $\overline{\mathbb{Q}}$ as in Section 1.2.2. In this thesis, as above, we assume that $K_f^{\mathcal{H}}$ is sufficiently small.

In Chapter 7, we solve a modularity problem on unitary Shimura varieties by using orthogonal Shimura varieties defined in Section 1.2.2. Here, we use a slightly different notation (but objects are the same) from Chapter 6, thus let us introduce them.

We define $V_{E_0} := V_E$, considered as an E_0-vector space and $(\ ,\ ) := \operatorname{Tr}_{E/E_0}\langle\ ,\ \rangle$. Then, $(V_{E_0}, (\ ,\ ))$ is a quadratic space of dimension $2n+2$ over E_0 whose signature is $(2n, 2)$ at $\sigma_1, \ldots, \sigma_e$ and $(2n+2, 0)$ at $\sigma_{e+1}, \ldots, \sigma_d$. We define D_{E_0} similarly. We put $\mathcal{G} := \operatorname{Res}_{E_0/\mathbb{Q}} \operatorname{GSpin}(V_{E_0})$ and define $N_{K_f^{\mathcal{G}}}$ similarly for an open compact subgroup $K_f^{\mathcal{G}} \subset \mathcal{G}(\mathbb{A}_f)$. Let $L \subset V_{E_0}$ be a lattice, and L' denotes the dual lattice. Now, we get a group embedding, $\mathcal{H} \hookrightarrow \mathcal{G}$. From here, we assume that $K_f^{\mathcal{H}} = \mathcal{G}(\mathbb{A}_f) \cap K_f^{\mathcal{G}}$ so that

$$\iota\colon M_{K_f^{\mathcal{H}}} \hookrightarrow N_{K_f^{\mathcal{G}}}. \tag{1.2.2}$$

In this thesis, we also assume that $K_f^{\mathcal{G}}$ is sufficiently small.

We will also work on the modularity constructed by unitary Shimura varieties as above. However, their notation of special cycles is similar to Subsection 1.2.2, hence we omit the details. For any positive integer r, we define the *special cycle* in the Chow group

$$Z^{\mathcal{H}}(x,g)_{K_f^{\mathcal{H}}} \in \operatorname{CH}^{er}(M_{K_f^{\mathcal{H}}})_{\mathbb{C}} := \operatorname{CH}^{er}(M_{K_f^{\mathcal{H}}}) \otimes_{\mathbb{Z}} \mathbb{C}$$

as the orthogonal case in Section 1.2.2. Then, for a Bruhat-Schwartz function $\phi_f \in \mathbf{S}(V_E(\mathbb{A}_f)^r)^{K_f^{\mathcal{H}}}$, *Kudla's generating function* is defined to be the following formal power series with coefficients in $\mathrm{CH}^{er}(M_{K_f^{\mathcal{H}}})_{\mathbb{C}}$ in the variable $\tau = (\tau_1, \ldots, \tau_d) \in (\mathscr{H}_r)^d$:

$$Z^{\mathcal{H}}_{\phi_f}(\tau) := \sum_{x \in \mathcal{H}(\mathbb{Q}) \backslash V_E^r} \sum_{g \in \mathcal{H}_x(\mathbb{A}_f) \backslash \mathcal{H}(\mathbb{A}_f)/K_f^{\mathcal{H}}} \phi_f(g^{-1}x) Z^{\mathcal{H}}(x, g)_{K_f^{\mathcal{H}}} q^{T(x)}.$$

We define $Z^{\mathcal{G}}_{\phi_f}(\tau)$ similarly.

Remark 1.2.1. We explain that $Z^{\mathcal{G}}_{\phi_f}(\tau)$ is an analog of a theta function. For a totally real definite matrix $\beta \in M_r(F)$, let $\Omega_\beta := \{x \in V_{E_0}^r \mid T(x) = \beta\}$, and we consider the Fourier expansion with respect to β. Now we choose β such that $\Omega_\beta \neq \varnothing$ and fix $x_0 \in \Omega_\beta(E_0)$. For $\xi_j \in \mathcal{G}(\mathbb{A}_f)$, we have

$$\mathrm{Supp}(\phi_f) \cap \Omega_\beta(\mathbb{A}_f) = \coprod_{j=1}^{\ell} K_f^{\mathcal{G}} \cdot \xi_j \cdot x_0,$$

and we put

$$Z^{\mathcal{G}}(\beta, \phi_f)_{K_f^{\mathcal{G}}} := \sum_{j=1}^{\ell} \phi_f(\xi_j^{-1} \cdot x_0) Z^{\mathcal{G}}(x_0, \xi_j)_{K_f^{\mathcal{G}}}.$$

Then, $Z^{H}_{\phi_f}(\tau)$ becomes

$$Z^{\mathcal{G}}_{\phi_f}(\tau) = \sum_{\beta \geq 0} Z^{\mathcal{G}}(\beta, \phi_f)_{K_f^H} q^{\beta}$$

and by adding Kudla-Millson forms and Gaussian functions, this is exactly a theta function in the cohomology group. For details, see [**92**].

In Chapter 6 and 7, we will prove that the generating series is a certain modular form with its coefficients in the Chow groups. To clarify the notion of "modular", we introduce it.

Definition 1.2.2. Let V be a vector space over $\mathbb{C}$ and f be a formal power series with coefficients in V. We say f is modular if for any $\mathbb{C}$-linear map $\ell \colon V \to \mathbb{C}$ such that $\ell(f)$ is absolutely convergent, $\ell(f)$ is modular.

1.3. Low slope cusp form trick

We will introduce the notion "irregular cusps" of ball quotients and study them in Chapter 2. Here, let us state the main application of the study of irregular cusps. The following is a unitary analog of [**56**, Theorem 1.1] or [**109**, Theorem 8.9]. For the definition of (semi-)irregular cusps, see Chapter 2.

Theorem 1.3.1 (Low slope cusp form trick, Theorem 2.6.3). *Let F be an imaginary quadratic field and L be a Hermitian lattice of signature $(1, n)$ over $\mathscr{O}_F$. For a finite index subgroup $\Gamma \subset \mathrm{U}(L)(\mathbb{Z})$, we assume that there is a non-zero cusp form Ψ of weight k with respect to Γ on D_L. In addition, we make the following assumptions.*

1. *$v_R(\Psi)/k > (r_i - 1)/(n + 1)$ for every irreducible component R_i of the ramification divisors $D_L \to \mathscr{F}_L(\Gamma)$ with ramification index r_i.*
2. *$v_I(\Psi)/k > 1/(n + 1)$ for every regular isotropic sublattice $I \subset L$.*
3. *$v_I(\Psi)/k > m_I/(n + 1)$ for every (semi-)irregular isotropic sublattice $I \subset L$ with index m_I.*
4. *$n \geq \max_{i,I}\{r_i - 2, m_I - 1\}$.*
5. *$\overline{\mathscr{F}_L(\Gamma)}$ has at worst canonical singularities.*

Then the ball quotient $\mathscr{F}_L(\Gamma)$ is of general type.

The last condition on canonical singularities has been solved by Behrens when the dimension is sufficiently large, thus the question comes down to whether there exists a cusp form with a low slope that vanishes on the branch divisors. In addition, this observation suggests that if the dimension of ball quotients is sufficiently large, they are of general type. In Chapter 3, we will show the existence of modular forms, not necessarily cusp forms, vanishing on branch divisors for higher dimensional ball quotients. More strongly, we will prove that there exist many enough of them.

1.4. Reflective obstructions of ball quotients

1.4.1. The toroidal compactification of ball quotients. The canonical bundle of $K_{\overline{\mathscr{F}_L(\Gamma)}}$ is described as

$$K_{\overline{\mathscr{F}_L(\Gamma)}} \sim_{\mathbb{Q}} (n+1)\mathscr{L} - \sum_i \frac{d_i - 1}{d_i} B_i - \Delta \tag{1.4.1}$$

in $\mathrm{Pic}(\overline{\mathscr{F}_L(\Gamma)}) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\mathscr{L}$ is the Hodge bundle and $B_i \subset \overline{\mathscr{F}_L(\Gamma)}$ is the branch divisor of the map $D_L \to \mathscr{F}_L(\Gamma)$ with branch index d_i and Δ is the boundary. Note the difference of the notation of B_i in Section 1.5.

One strategy to prove that $\mathscr{F}_L(\Gamma)$ is of general type is to rewrite (1.4.1) as

$$K_{\overline{\mathscr{F}_L(\Gamma)}} \sim_{\mathbb{Q}} \mathcal{M}_\Gamma(a) + \Big\{(n+1-a)\mathscr{L} - \Delta\Big\},$$

for some positive integer $a > 0$, where

$$\mathcal{M}_\Gamma(a) := a\mathscr{L} - \sum_i \frac{d_i - 1}{d_i} B_i,$$

and show that

(A) (Reflective obstructions) $\mathcal{M}_\Gamma(a)$ is big,

(B) (Cusp obstructions) $(n+1-a)\mathscr{L} - \Delta$ is effective.

Combined with the result of [**9**], this would imply that $\overline{\mathscr{F}_L(\Gamma)}$ is of general type. In this chapter, we give a solution to (A) in a quantitative manner with respect to a. Note that if (A) and (B) hold, then $K_{\overline{\mathscr{F}_L(\Gamma)}}$ is big. The remaining problem, namely the effectiveness of $(n+1-a)\mathscr{L} - \Delta$, is the same as the construction of a non-zero cusp form on D_L of weight $n+1-a < n+1$; we do not consider this (see Remark 3.1.5 and Subsection 3.1.2).

1.4.2. Main results. Let $X_L := \mathscr{F}_L(\mathrm{U}(L)(\mathbb{Z}))$, $\mathcal{M}(a) := \mathcal{M}_{\mathrm{U}(L)}(a)$ and $S := \prod_p p$ where p runs over any prime number which divides D and $\det(L)$. Let us introduce an important assumption.

($\heartsuit$) $\mathrm{SU}(L')$ and $\mathrm{SU}(\ell^\perp \cap L)$ are principal for any $[\ell] \in \mathcal{R}_L(F)$, where $L' := \ell\mathscr{O}_F \oplus (\ell^\perp \cap L) \subset L$.

The definition of "principal" is given in Subsection 3.1.4. A vector $[\ell] \in \mathcal{R}_L(F)$ defines a branch divisor; for the definition of the set $\mathcal{R}_L(F)$, see Section 3.3. The main theorem in Chapter 3 is as follows.

Theorem 1.4.1 (Theorem 3.8.1). *Let L be a primitive Hermitian lattice over $\mathscr{O}_F$ of signature $(1,n)$ with $n > 2$. Assume ($\heartsuit$). Then, for a positive integer a, the line bundle $\mathcal{M}(a)$ is big if $\dim X_L = n$ or S is sufficiently large.*

It follows that the reflective obstructions can be resolved for $\mathscr{F}_L(\Gamma)$ with sufficiently large n or S. We will prove that specific lattices, called "unramified square-free" lattices below, satisfy (♡). Note that a lower bound for n and S in Theorem 3.1.1 can be easily computed.

1.4.3. Application I: Kodaira dimension. In this subsection, we assume $n \geq 13$ and $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3})$. These assumptions come from [**9**, Theorem 4], which asserts that $\mathscr{F}_L(\Gamma)$ has at worst canonical singularities and branch divisors of the map $D_L \to \mathscr{F}_L(\Gamma)$ do not exist at the boundary. Note that $\overline{X_L}$ contains no irregular cusps [**112**]. Under (B), we state an application to the birational type of X_L.

Theorem 1.4.2 (Corollary 3.8.2, Theorem 3.8.3). *Assume that* (♡) *holds and there exists a non-zero cusp form of weight lower than* $n+1$ *with respect to* $\mathrm{U}(L)$*. Then,* X_L *is of general type if* $\dim X_L = n$ *or* S *is sufficiently large.*

1.4.4. Application II: Reflective modular forms. Next, let us consider reflective modular forms. Let f be a modular form of some weight and character with respect to Γ on D_L. We say that f is *reflective* if the divisor of L is set-theoretically contained in the ramification divisors of $D_L \to \mathscr{F}_L(\Gamma)$. Reflective modular forms appear in many fields of mathematics; see [**52, 54, 61**]. Gritsenko-Nikulin [**61**, Conjecture 2.5.5] conjectured finiteness of quadratic lattices admitting a non-zero reflective modular form, and Ma [**107**, Corollary 1.9] proved it. Here, we consider an analogous problem for Hermitian lattices. We say that L is *reflective with slope* r for $r > 0$ if there exists a reflective modular form on D_L with its slope r; for the definition of the *slope* of a modular form, see [**107**, Subsection 1.3].

Conjecture 1.4.3 (Finiteness of Hermitian lattices admitting reflective modular forms). *For an* $r > 0$ *and a fixed* F*,*

$$\{\text{Hermitian reflective lattices with slope less than } r\}/\sim$$

is a finite set.

We can partially prove Conjecture 1.4.3 from a computation of the Hirzebruch-Mumford volumes.

Corollary 1.4.4 (Corollary 3.8.4). *For an* $r > 0$ *and a fixed* F_0*,*

$$\{\text{Unramified square-free reflective lattices with slope less than } r \mid n > 2\}/\sim$$

is a finite set.

For the definition of unramified square-free lattices and F_0, see Chapter 3.

1.5. Fano modular varieties with mostly branched cusps

In this section, let us introduce a general theorem. In the later Section 4.3, we apply them to various concrete examples. First, we introduce some notations.

1.5.1. Convention and Notation. In Chapter 4, we discuss the linear equivalence class of a Cartier divisor and the corresponding holomorphic line bundle interchangeably. Similarly, we do not distinguish the $\mathbb{Q}$-linear equivalence class of a $\mathbb{Q}$-Cartier divisor and the corresponding $\mathbb{Q}$-line bundle. We use the following notations throughout.

- $\mathbb{G}$ is a simple algebraic group over $\mathbb{Q}$, not isogenous to $\mathrm{SL}(2)$.
- G is the identity component of $\mathbb{G}(\mathbb{R})$, which we assume to be a simple Lie group.

- K is a maximal compact subgroup of G.
- The corresponding Hermitian symmetric domain is G/K.
- Take an arithmetic subgroup $\Gamma \subset \mathbb{G}(\mathbb{Q})$ i.e., commensurable to $\mathbb{G}(\mathbb{Z})$. By abuse of notation, we omit the notation of $\mathbb{Z}$-valued points in this chapter.
- $X := \Gamma\backslash G/K$ and its Baily-Borel compactification $\overline{X}^{\mathrm{BB}}$ ([**133, 7**]).
- $\mathbb{H}$ denotes the upper half plane (which is an example of X).
- $\partial\overline{X}^{\mathrm{BB}}$ denotes the boundary of the Baily-Borel compactification, i.e., $\overline{X}^{\mathrm{BB}} \setminus X$.
- Denote a toroidal compactification of X in the sense of [**6**], with an arbitrary fixed cone decomposition, simply as $\overline{X}^{\mathrm{tor}}$. (The choice of cone decompositions do not affect the following discussions.)
- Denote the boundary divisor $\overline{X}^{\mathrm{tor}} \setminus X$ as Δ (with coefficients 1).
- Denote the branch divisor of $G/K \to \Gamma\backslash G/K$ to be $\cup_i B_i (\subset X)$ with prime divisors B_i and branch (or ramification) degree d_i. By abuse of notation, we denote by $\overline{B_i}^{\mathrm{tor}}$ (resp. $\overline{B_i}^{\mathrm{BB}}$) the closure of B_i in $\overline{X}^{\mathrm{tor}}$ or $\overline{X}^{\mathrm{BB}}$.
- $X^o := X \setminus \cup_i B_i$.
- $\mathscr{L} := K_{\overline{X}^{\mathrm{tor}}} + \Delta + \sum_i \frac{d_i-1}{d_i}\overline{B}_i^{\mathrm{tor}} \in \mathrm{Pic}(\overline{X}^{\mathrm{tor}}) \otimes \mathbb{Q}$ and its descended (automorphic) $\mathbb{Q}$-line bundle on $\overline{X}^{\mathrm{BB}}$, i.e., $K_{\overline{X}^{\mathrm{BB}}} + \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{BB}}$.
- Recall from [**7**] and [**120**, 3.4, 4.2 (also see 1.3)] that $\mathscr{L}$ is ample (resp. semiample) on $\overline{X}^{\mathrm{BB}}$ (resp. $\overline{X}^{\mathrm{tor}}$) and a meromorphic section of $\mathscr{L}^{\otimes t}$ for $t \in \mathbb{Z}_{>0}$ corresponds to a meromorphic automorphic form of arithmetic weight ct for some $c \in \mathbb{Z}$. Throughout this thesis, a weight always simply refers to the arithmetic weight (in the sense of e.g., [**58**]) and call c the *canonical weight*, following e.g., [**58**]. See also Lemma 4.2.4 for the calculation of c.

1.5.2. Special reflective modular forms. Recall that reflective modular forms are the concept originally formulated in [**52**] for orthogonal case, which means that their divisor is defined by reflections. In this chapter, we consider the following stronger properties, or proper subclass of reflective modular forms. The upshot of our general observation is that the existence of such special reflective modular forms give strong implications on the birational properties of modular varieties (see Theorem 1.5.3). These modular forms are rare, but luckily still various interesting examples are known (cf. [**52**], Section 4.3). We also construct new examples in Section 4.3.

Assumption 1.5.1 (Special reflective modular forms - General case). Consider the following subclasses of reflective modular forms.

(1) A non-vanishing holomorphic section f of
$$\mathscr{O}_X(N(s(X)\mathscr{L} - \sum_i \frac{d_i-1}{d_i}B_i))\left(:= \mathscr{L}^{\otimes aN}\left(-\sum_i \frac{N(d_i-1)}{d_i}B_i\right)\right)$$
for some $N \in \mathbb{Z}_{>0}$, $s(X) \in \mathbb{Q}_{>0}$ with $s(X)N, \frac{N}{d_i} \in \mathbb{Z}_{>0}$.

(2) A non-vanishing holomorphic section f of $\mathscr{O}_X(N(s(X)\mathscr{L} - \sum_i c_iB_i))$ for some $N \in \mathbb{Z}_{>0}$, $s(X) \in \mathbb{Q}_{>0}$, and $c_i \in \mathbb{Q}$ with $0 \leq c_i \leq \frac{d_i-1}{d_i}$ for all i, such that $s(X)N, Nc_i \in \mathbb{Z}$.

We follow the same convention below.

Assumption 1.5.2 (Special reflective modular forms - orthogonal case). For $n > 2$, assume that there is a quadratic lattice Λ of signature $(2, n)$ such that $\mathbb{G} =$

$O^+(\Lambda \otimes \mathbb{Q})$ with $\Gamma \subset O^+(\Lambda)$. In this situation, we consider the following subclasses of reflective modular forms.

(1) A non-vanishing holomorphic section f of $\mathcal{O}_X(N(s(X)\mathscr{L} - \frac{1}{2}\sum_i B_i))$ for some $N \in \mathbb{Z}_{>0}$, $s(X) \in \mathbb{Q}_{>0}$ with $s(X)N, \frac{N}{2} \in \mathbb{Z}_{>0}$.

Indeed, for the above $\mathbb{G}$ and Γ, Gritsenko-Hulek-Sankaran showed that every branch divisor arises from reflections (of order 2) [**58**, 2.12, 2.13], i.e., the ramification degrees d_i are all 2. Note that N is unessential as it gets multiplied when replacing f by its power, while the quantity $s(X)$ is more essential and sometimes called a *slope* in the literature.

Below, we discuss various modular varieties X which can be roughly divided into two types, i.e., those with modular forms satisfying Assumption 1.5.1 (1), and those with modular forms satisfying Assumption 1.5.1 (2). The former is discussed in Subsection 4.2.1, with examples given in Section 4.3, and the latter is discussed in Subsection 4.2.2 while some examples are given in [**55, 110**].

1.5.3. Main general results.

Here is our first general theorem.

Theorem 1.5.3 (Birational properties). *We follow the notation as above. If there is a reflective modular form that satisfies Assumption 1.5.1* (1) *with some* $s(X) \in \mathbb{Q}_{>0}$, *then the Baily-Borel compactification* $\overline{X}^{\mathrm{BB}}$ *of* $X = \Gamma\backslash D$ *only has log canonical singularities and* X^o *is quasi-affine. In addition,*

(1) if $s(X) > 1$, *then* $\overline{X}^{\mathrm{BB}}$ *is a Fano variety i.e.,* $-K_{\overline{X}^{\mathrm{BB}}}$ *is ample (*$\mathbb{Q}$*-Cartier),*
(2) if $s(X) = 1$, *then* $\overline{X}^{\mathrm{BB}}$ *is a Calabi-Yau variety i.e.,* $K_{\overline{X}^{\mathrm{BB}}} \sim_{\mathbb{Q}} 0$, *or*
(3) if $s(X) < 1$, *then* $K_{\overline{X}^{\mathrm{BB}}}$ *is ample.*

The quantity $s(X)$ in Theorem 1.5.3 is the (arithmetic) weight of the modular form s divided by such canonical weight c and some constant; see Remark 4.3.8 and 4.3.27. As an application of Theorem 1.5.3, we will prove that the moduli space of (log-)Enriques surfaces are Fano; see Example 4.3.13, 4.3.17.

1.6. Revisiting the moduli space of 8 points on $\mathbb{P}^1$

The Deligne-Mostow theory [**32**] gives us an isomorphism between $\mathcal{M}^{\mathrm{GIT}}$, which is the moduli space of unordered 8 points on $\mathbb{P}^1$ and the Baily-Borel compactification of an appropriate 5-dimensional ball quotient $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}$. We are interested in the lifting of the Deligne-Mostow isomorphism to the unique toroidal compactification. There exist two natural blow-ups, playing important roles here: the Kirwan blow-up $f : \mathcal{M}^{\mathrm{K}} \to \mathcal{M}^{\mathrm{GIT}}$ and the toroidal compactification $\pi : \overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}} \to \overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}$. Here, the Kirwan blow-up $\mathcal{M}^{\mathrm{K}}$ is the partial desingularization of $\mathcal{M}^{\mathrm{GIT}}$ whose center is located in the polystable orbits (which is a unique point $\{c_{4,4}\}$ in our case). The toroidal compactification $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ is a blow-up of $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}$ at the point $\{\xi\}$, which is the unique cusp, i.e., the Baily-Borel boundary. The above Deligne-Mostow isomoropshim sends $c_{4,4}$ to ξ, thus restricting to an isomorphism $\mathcal{M}^{\mathrm{K}} \setminus f^{-1}(c_{4,4}) \cong \overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}} \setminus \pi^{-1}(\xi)$. In this setting, our first main result asserts that the birational map $g : \mathcal{M}^{\mathrm{K}} \dashrightarrow \overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ does not extend to a morphism.

$$\begin{array}{ccc} \mathcal{M}^{\mathrm{K}} & \overset{g}{\dashrightarrow} & \overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}} \\ \downarrow f & & \downarrow \pi \\ \mathcal{M}^{\mathrm{GIT}} & \xrightarrow{\phi} & \overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}. \end{array}$$

Theorem 1.6.1 (Theorem 5.3.15). *Neither the Deligne-Mostow isomorphism $\phi : \mathcal{M}^{\mathrm{GIT}} \to \overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}$ nor its inverse ϕ^{-1} lift to a morphism between the Kirwan blow-up $\mathcal{M}^{\mathrm{K}}$ and the unique toroidal compactification $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$.*

This result still leaves the possibility open that the Kirwan blow-up and the toroidal compactification are isomorphic as abstract varieties. One obstruction to this could be that the varieties are topologically different. Indeed, the topology of these varieties is of independent interest (and indeed this was the starting point of [**27**] and [**28**] in the case of cubic threefolds and cubic surfaces). We compute the cohomology of these varieties, according to the Kirwan method [**79, 76, 77**] and Casalaina-Martin-Grushevsky-Hulek-Laza [**27**]. Wherever a space X has at most finite quotient singularities, we work with singular cohomology with rational coefficients and denote this by $H^k(X)$. In the other cases, notably the GIT quotient and the Baily-Borel compactification of ball quotients, we work with intersection cohomology (of middle perversity) and denote this by $IH^k(X)$. Note that for spaces with finite quotient singularities singular cohomology and intersection cohomology coincide. The cohomology groups of the varieties under consideration are given as follows.

Theorem 1.6.2 (Theorem 5.5.1, 5.5.2, 5.5.6, 5.5.8). *All the odd degree cohomology of the following projective varieties vanishes. In even degrees, their Betti numbers are given by:*

j	0	2	4	6	8	10
$\dim H^j(\mathcal{M}^{\mathrm{K}})$	1	2	3	3	2	1
$\dim IH^j(\overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}})$	1	1	2	2	1	1
$\dim H^j(\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}})$	1	2	3	3	2	1
$\dim H^j(\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}})$	1	43	99	99	43	1
$\dim IH^j(\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}})$	1	8	29	29	8	1
$\dim IH^j(\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}})$	1	43	99	99	43	1

thus, all the Betti numbers of $\mathcal{M}^{\mathrm{K}}$ and $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ are the same.

Here, $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}$ denotes the Baily-Borel compactification of a 5-dimensional ball quotient, which is an $\mathfrak{S}_8$-cover of $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}$ and isomorphic to $\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}}$, the moduli space of ordered 8 points on $\mathbb{P}^1$. Also, we denote by $\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}}$ the Kirwan blow-up of $\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}}$ and by $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}}$ the toroidal blow-up of $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}$. For more precise descriptions of these varieties, as well as the bounded symmetric domain and arithmetic subgroups, see Section 5.2.

Again, this result leaves the possibility that $\mathcal{M}^{\mathrm{K}}$ and $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ are isomorphic as abstract varieties. We rule this out by showing that these spaces are not K-equivalent. Recall that two projective normal $\mathbb{Q}$-Gorenstein varieties X and Y are called *K-equivalent* if there is a common resolution of singularities Z dominating X and Y birationally

$$\begin{array}{ccc} & Z & \\ {}^{f_X}\swarrow & & \searrow^{f_Y} \\ X & \dashleftarrow\dashrightarrow & Y \end{array}$$

such that $f_X^* K_X \sim_{\mathbb{Q}} f_Y^* K_Y$. For K-equivalent varieties, the top intersection numbers are equal: $K_X^n = K_Y^n$, where n is the dimension of X and Y. We shall use this property to show that $\mathcal{M}^{\mathrm{K}}$ and $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ are not K-equivalent. Thus, these varieties are in particular not isomorphic as abstract varieties, even though they are the blow-ups at the same points of $\mathcal{M}^{\mathrm{GIT}} \cong \overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}$ and have the same Betti numbers.

Theorem 1.6.3 (Theorem 5.4.6). *The Kirwan blow-up $\mathcal{M}^{\mathrm{K}}$ and the toroidal compactification $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ are not K-equivalent and hence, in particular, not isomorphic as abstract varieties.*

As we shall see later, the situation is in contrast to the case of moduli of ordered points, where we have an isomorphism $\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}} \cong \overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}}$.

1.7. Modularity of the generating series: the case of orthogonal Shimura varieties

Let notation be as in Section 1.2.2. Our main result in Chapter 6 is below.

Theorem 1.7.1 (Theorem 6.1.5). *Assume $n \geq 3$ and Conjecture 6.1.3 for the Shimura variety M_{K_f} for $m = e$. Let $r \geq 1$ be a positive integer.*

(1) If $\ell\colon \mathrm{CH}^{er}(M_{K_f})_{\mathbb{C}} \to \mathbb{C}$ is a linear map over $\mathbb{C}$ such that $\ell(Z_{\phi_f})(\tau)$ is absolutely convergent, then $\ell(Z_{\phi_f})(\tau)$ defines a Hilbert-Siegel modular form of genus r and weight $1 + n/2$.

(2) If $r = 1$, for any linear map $\ell\colon \mathrm{CH}^{e}(M_{K_f})_{\mathbb{C}} \to \mathbb{C}$, the formal power series $\ell(Z_{\phi_f})(\tau)$ is absolutely convergent and we get a Hilbert modular form of weight $1 + n/2$.

For the case of $n \leq 2$, see Theorem 6.1.6. There we will use “an embedding trick”.

Remark 1.7.2. (1) If ℓ factors through a linear map $\ell'\colon H^{2er}(M_{K_f}, \mathbb{C}) \to \mathbb{C}$, Theorem 1.7.1 and Theorem 6.1.6 were proved unconditionally by Kudla [**95**, Section 5.3] and Rosu-Yott [**131**, Theorem 1.1].

(2) When $e = 1$, we recover the results of Yuan-Zhang-Zhang. (Note that Conjecture 6.1.3 is true when $m = 1$. See Remark 6.1.4.) This case is called *Kudla's modularity conjecture*, stated by Kudla in [**94**, Section 3.2, Problem 1] and proved unconditionally by Yuan-Zhang-Zhang in [**151**, Theorem 1.2]. However, they also assumed the absolute convergence of the generating series for $r > 1$.

(3) We do not know the absolute convergence of the generating series $\ell(Z_{\phi_f})(\tau)$. When $F = \mathbb{Q}$ and $d = e = 1$, Bruinier and Westerholt-Raum proved unconditionally that $\ell(Z_{\phi_f})(\tau)$ is absolutely convergent for any ℓ in [**26**, Corollary 1.4].

(4) Kudla [**95**] proved the absolute convergence and the modularity of generating series in the same setting as ours, assuming Conjecture 6.1.3 for Shimura varieties associated with quadratic spaces of sufficiently large rank.

1.8. Modularity of the generating series: the case of unitary Shimura varieties

We also work on the modularity of the generating series on unitary Shimura varieties. For notations, see Subsection 1.2.3. In the context of *Kudla's modularity conjecture*, our problem is as follows.

Conjecture 1.8.1. *The generating series $Z^{\mathcal{H}}_{\phi_f}(\tau)$ is a Hilbert-Hermitian modular form of weight $n+1$ and genus r.*

Here, the precise definition of the notion "modular", see Definition 1.2.2. We give two partial solutions to this problem in this chapter. See Corollary 1.8.3 and Theorem 1.8.4.

First, we can prove the modularity of the generating series of special divisors by using the regularized theta lift on orthogonal groups.

Theorem 1.8.2 (Theorem 7.3.1). *Assume that $e = 1$ and $r = 1$. Then, $Z^{\mathcal{H}}_{\phi_f}(\tau)$ is a Hermitian modular form for $\mathrm{SU}(1,1)$ of weight $n+1$ under the assumption that the series converges absolutely.*

This implies the case of higher codimensional cycles.

Corollary 1.8.3 (Corollary 7.3.2). *Assume $e = 1$. Then, $Z^{\mathcal{H}}_{\phi_f}(\tau)$ is a Hilbert-Hermitian modular form for $\mathrm{U}(r,r)$ of weight $n+1$ under the assumption that the series converges absolutely.*

This gives another proof of Theorem 1.8.4 for the $e = 1$ case and [**103**, Theorem 3.5]. This is shown unconditionally differently from Theorem 1.8.4. Now, we state the theorem for $e > 1$.

Theorem 1.8.4 (Theorem 7.4.1). *$Z^{\mathcal{H}}_{\phi_f}(\tau)$ is a Hilbert-Hermitian modular form for $\mathrm{U}(r,r)$ of weight $n+1$ under Conjecture 6.1.3 for $m = e$ with respect to orthogonal Shimura varieties and the assumption that the series converges absolutely for $e > 1$.*

Remark 1.8.5. Kudla [**95**] and the author [**111**] proved the modularity of the generating series associated with orthogonal Shimura varieties for $e > 1$. Their results are shown by using the Kudla-Millson's cohomological coefficient result [**97**] and reducing the problem to this cohomological case under the Beilinson-Bloch conjecture for orthogonal Shimura varieties. Therefore one might think that the modularity of the generating series associated with unitary Shimura varieties would also be proved in the same way, but the Hodge numbers appearing in the cohomology of unitary Shimura varieties do not seem to vanish [**95**, Remark 1.2].

Historically, for unitary Shimura varieties, Kudla-Millson [**97**] studied the cohomological coefficients case. In the Chow group, Hofmann [**68**] showed the $\mathrm{SL}_2(\cong \mathrm{SU}(1,1))$-modularity of the generating series over imaginary quadratic fields for the $r = 1, e = 1$ case, and Liu [**103**] showed Hermitian modularity for the $e = 1$ case, assuming the absolute convergence of the generating series. Therefore we give a generalization of their work. On the other hand, Xia [**148**] showed Liu's result, not assuming the absolute convergence of the generating series. He uses the formal Fourier-Jacobi series method similar to the work over $\mathbb{Q}$ of Bruinier-Westerholt-Raum [**26**].

Theorem 1.8.2 and Corollary 1.8.3 are included in Theorem 1.8.4 under the Beilinson-Bloch conjecture, but we give another proof only working for $r = 1$, using regularized theta lifts.

CHAPTER 2

Irregular cusps of ball quotients

2.1. Introduction

When calculating the order of modular forms on modular curves at cusps, we need to consider whether the cusp is regular or not. If it is irregular, then the order of the modular forms is defined as half the order determined by its Fourier expansion at the cusp. More precisely, irregular cusps of modular curves are cusps whose widths are strictly smaller than the period for Fourier expansion; this is explained in detail in [**33**]. In the case of orthogonal modular varieties, Ma [**109**] defined and studied irregular cusps. He classified the structures of discriminant groups for the case of discriminant kernel when irregular cusps may exist on the orthogonal modular varieties and constructed examples. Finally, he proved the low slope cusp form trick, which is a modification of the low weight cusp form trick [**56**, Theorem 1.1] when the irregular cusps arise, and used it to show that some orthogonal modular varieties are of general type.

In this chapter, we work on ball quotients. First, we define irregular cusps on them. Unlike the case of orthogonal modular varieties, in our situation, there may exist branch divisors with branch indices 2,3,4 or 6 as explained in Section 2.3. Considering the effects of these cusps, as a main result, we give a sufficient condition for a ball quotient to be of general type in terms of modular forms, called the low slope cusp form trick. On the other hand, we shall give an example of a ball quotient of non-negative Kodaira dimension in Section 2.7. This is done by constructing a cusp form, satisfying a weaker condition appearing in this trick. Second, we consider the relationship between regular/irregular cusps on ball quotients and regular/irregular cusps on orthogonal modular varieties when a Hermitian symmetric domain of type I is embedded into one of type IV. In this situation, we prove that regular cusps map to regular cusps and determine whether irregular cusps map to regular or irregular cusps. Third, we classify the structures of the discriminant group when the discriminant kernel may have irregular cusps in Section 2.4 and Appendix 2.A. Finally, we construct concrete examples of irregular cusps of any index for any imaginary quadratic field with class number 1 in Section 2.8. Before stating our results, we should summarize our settings.

Now, let us introduce the notion of irregular cusps. Let I be a rank 1 primitive isotropic sublattice of L and $\Gamma(I)_{\mathbb{Q}}$ be the stabilizer of $I \otimes_{\mathscr{O}_F} F$. We denote by $W(I)_{\mathbb{Q}}$ its unipotent part and $Z(I)_{\mathbb{Q}}$ the center of $W(I)_{\mathbb{Q}}$. We say I is *irregular with* (*at least*) *index* 2 if $Z(I)_{\mathbb{Q}} \cap \Gamma \neq Z(I)_{\mathbb{Q}} \cap \langle \Gamma, -\mathrm{id} \rangle$ holds. We have to consider whether the cusp corresponding to I branches with a higher index or not for $F = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, but for simplicity, we only concern ourselves with this case here. At irregular cusps, we have to pay attention to the vanishing order of modular forms and related pluricanonical forms.

Here, we shall state our main result, which is a unitary analog of [**56**, Theorem 1.1] or [**109**, Theorem 8.9].

Theorem 2.1.1 (Low slope cusp form trick, Theorem 2.6.3). *Let F be an imaginary quadratic field and L be a Hermitian lattice of signature $(1, n)$ over $\mathscr{O}_F$. For a finite* *index subgroup $\Gamma \subset \mathrm{U}(L)(\mathbb{Z})$, we assume that there is a non-zero cusp form Ψ of weight k with respect to Γ on D_L. In addition, we make the following assumptions.*

(1) *$v_R(\Psi)/k > (r_i - 1)/(n+1)$ for every irreducible component R_i of the ramification divisors $D_L \to \mathscr{F}_L(\Gamma)$ with ramification index r_i.*
(2) *$v_I(\Psi)/k > 1/(n+1)$ for every regular isotropic sublattice $I \subset L$.*
(3) *$v_I(\Psi)/k > m_I/(n+1)$ for every (semi-)irregular isotropic sublattice $I \subset L$ with index m_I.*
(4) *$n \geq \max_{i,I}\{r_i - 2, m_I - 1\}$.*
(5) *$\overline{\mathscr{F}_L(\Gamma)}$ has at worst canonical singularities.*

Then the ball quotient $\mathscr{F}_L(\Gamma)$ is of general type.

Remark 2.1.2. Assumptions (4) and (5) are satisfied if $n \geq 13$ and $d < -3$ by [**9**, Theorem 4].

Here, $\overline{\mathscr{F}_L(\Gamma)}$ is the canonical toroidal compactification of $\mathscr{F}_L(\Gamma)$. For the notion of "semi-irregular", see Section 2.3. We also consider the relationship between regular/irregular cusps on D_L and regular/irregular cusps on $\mathscr{D}_{L_Q}$. Note that irregular cusps on $\mathscr{D}_{L_Q}$ have been studied by Ma [**109**]. Let $\Gamma_O \subset \mathrm{O}^+(L_Q)(\mathbb{Z})$ be a finite index subgroup and $\Gamma_U \subset \mathrm{U}(L)(\mathbb{Z})$ be its restriction to the unitary group. In the following proposition, regular/irregular cusps on D_L (resp. $\mathscr{D}_{L_Q}$) mean regular/irregular cusps with respect to Γ_U (resp. Γ_O).

Proposition 2.1.3. (1) *For any imaginary quadratic field F, regular cusps on D_L map to regular cusps on $\mathscr{D}_{L_Q}$.*
(2) *For $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$, irregular cusps on D_L map to irregular cusps on $\mathscr{D}_{L_Q}$.*
(3) *For $F = \mathbb{Q}(\sqrt{-1})$, irregular cusps with index 2 or 4 on D_L map to irregular cusps with index 2 on $\mathscr{D}_{L_Q}$ and semi-irregular cusps with index 2 on D_L map to regular cusps on $\mathscr{D}_{L_Q}$.*
(4) *For $F = \mathbb{Q}(\sqrt{-3})$, irregular cusps with index 2 or 6 and semi-irregular cusps with index 2 on D_L map to irregular cusps with index 2 on $\mathscr{D}_{L_Q}$ and irregular cusps with index 3 and semi-irregular cusps with index 3 on D_L map to regular cusps on $\mathscr{D}_{L_Q}$.*

For the case of discriminant kernel, we completely classify discriminant groups when the lattice may have irregular cusps.

Proposition 2.1.4. *If F is an imaginary quadratic field of class number 1, and the discriminant kernel of a unitary group has an irregular cusp, then the discriminant group of an even Hermitian lattice is one of those listed in Appendix 2.A.*

2.2. 0-dimensional cusps

Now, let us recall the toroidal compactification of $\mathscr{F}_L(\Gamma)$ and its cusps. For a rank 1 primitive isotropic sublattice $I \subset L$, let $\Gamma(I)_{\mathbb{Q}} := \mathrm{Stab}_{\mathrm{U}(L)(\mathbb{Q})}(I_F)$ be the stabilizer of $I_F := I \otimes_{\mathscr{O}_F} F$. Here, we review the structure of $\Gamma(I)_{\mathbb{Q}}$; see [**9**] and [**105**] for details. Let

$$W(I)_{\mathbb{Q}} := \mathrm{Ker}(\Gamma(I)_{\mathbb{Q}} \to \mathrm{U}(I^{\perp}/I_F) \times \mathrm{GL}(I_F))$$

be the unipotent radical of $\Gamma(I)_{\mathbb{Q}}$ and

$$Z(I)_{\mathbb{Q}} := \mathrm{Ker}(\Gamma(I)_{\mathbb{Q}} \to \mathrm{GL}(I^{\perp}))$$

be its center. We fix a generator e of I. By [**105**], we define

$$T_{e\otimes v}(z) := z + \langle z, e\rangle v - \langle z, v\rangle e - \frac{1}{2}\langle v, v\rangle\langle z, e\rangle e$$

for $v \in I^{\perp}$ and $z \in V$. Then, the following properties hold:

$$\begin{cases} T_{e\otimes\mu v} & = T_{\overline{\mu}e\otimes v} \ (\mu \in F) \\ T_{e\otimes\lambda e} & = \mathrm{id}_V \ (\lambda \in \mathbb{Q}) \\ T_{e\otimes v}T_{e\otimes u} & = T_{e\otimes(v+u+\frac{1}{2}\langle v,u\rangle e)}. \end{cases}$$

Thus, it follows that $T_{e\otimes v}$ depends only on $\overline{I}_F \otimes I^{\perp}/(\overline{I}\otimes I)(\mathbb{Q})$. Here,

$$(\overline{I}\otimes I)(\mathbb{Q}) := \{\lambda(e\otimes e) \mid \lambda \in \mathbb{Q}\}.$$

From the definition of $T_{e\otimes v}$, it follows $T_{e\otimes v} = \mathrm{id}_{I^{\perp}}$ for $e\otimes v \in \overline{I}_F \otimes I_F$ so that

$$\begin{array}{ccc} \overline{I}_F \otimes I_F/(\overline{I}\otimes I)(\mathbb{Q}) = \sqrt{d}(\overline{I}\otimes I)(\mathbb{Q}) & \cong & Z(I)_{\mathbb{Q}} \\ \sqrt{d}\lambda(e\otimes e) & \mapsto & T_{\sqrt{d}\lambda(e\otimes e)}. \end{array} \tag{2.2.1}$$

More directly, by choosing a basis $\{e, b_1, \dots, b_{n-1}, e'\}$ of V such that $\{e, b_1, \dots, b_{n-1}\}$ is a basis of $I^{\perp}$ and $\langle e, e'\rangle = 1$, the Hermitian form is given by

$$\left(\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & B & 0 \\ \hline 1 & 0 & 0 \end{array}\right)$$

for some Hermitian matrix B, and the center of $W(I)_{\mathbb{Q}}$ is given by

$$Z(I)_{\mathbb{Q}} = \left\{ \left(\begin{array}{c|c|c} 1 & 0 & \lambda\sqrt{d} \\ \hline 0 & I_{n-1} & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \middle| \ \lambda \in \mathbb{Q} \right\}.$$

This gives the isomorphism (2.2.1) more explicitly,

$$\begin{array}{ccc} \overline{I}_F \otimes I_F/(\overline{I}\otimes I)(\mathbb{Q}) = \sqrt{d}(\overline{I}\otimes I)(\mathbb{Q}) & \cong & Z(I)_{\mathbb{Q}} \\ \sqrt{d}\lambda(e\otimes e) & \mapsto & \left(\begin{array}{c|c|c} 1 & 0 & 2\lambda\sqrt{d} \\ \hline 0 & I_{n-1} & 0 \\ \hline 0 & 0 & 1 \end{array}\right). \end{array}$$

(See [**9**, Lemma 12] for a description.) Now, $\Gamma(I)_{\mathbb{Q}}$ acts on both sides of the equation. The natural action on the left-hand side coincides with the adjoint action on the right-hand side.

$$T_{\sqrt{d}\lambda\gamma(e\otimes e)} = \gamma^{-1}T_{\sqrt{d}\lambda(e\otimes e)}\gamma \quad (\gamma \in \Gamma(I)_{\mathbb{Q}}).$$

We also have the following isomorphism,

$$V(I)_{\mathbb{Q}} \cong \overline{I}_F \otimes I^{\perp}/I_F$$

by [**105**]. Here, $V(I)_{\mathbb{Q}}$ is defined in (2.2.4). For a finite index subgroup $\Gamma \subset \mathrm{U}(L)(\mathbb{Z})$, we introduce the following notation from [**6**] and [**109**]:

$$\Gamma(I)_{\mathbb{Z}} := \Gamma(I)_{\mathbb{Q}} \cap \Gamma,\ W(I)_{\mathbb{Z}} := W(I)_{\mathbb{Q}} \cap \Gamma,\ Z(I)_{\mathbb{Z}} := Z(I)_{\mathbb{Q}} \cap \Gamma$$

$$\overline{\Gamma(I)}_{\mathbb{Z}} := \Gamma(I)_{\mathbb{Z}}/Z(I)_{\mathbb{Z}},\ V(I)_{\mathbb{Z}} := W(I)_{\mathbb{Z}}/Z(I)_{\mathbb{Z}},\ \Gamma_I := \Gamma(I)_{\mathbb{Z}}/W(I)_{\mathbb{Z}}$$

$$\overline{\Gamma(I)}_{\mathbb{Q}} := \Gamma(I)_{\mathbb{Q}}/Z(I)_{\mathbb{Z}},\ W(I)_{\mathbb{Q}/\mathbb{Z}} := W(I)_{\mathbb{Q}}/Z(I)_{\mathbb{Z}},\ Z(I)_{\mathbb{Q}/\mathbb{Z}} := Z(I)_{\mathbb{Q}}/Z(I)_{\mathbb{Z}}.$$

Now we have the following exact sequences:

$$0 \to V(I)_{\mathbb{Z}} \to \overline{\Gamma(I)}_{\mathbb{Z}} \to \Gamma_I \to 1 \tag{2.2.2}$$

$$0 \to W(I)_{\mathbb{Q}/\mathbb{Z}} \to \overline{\Gamma(I)}_{\mathbb{Q}} \to \mathrm{U}(I^{\perp}/I_F) \times \mathrm{GL}(I_F) \tag{2.2.3}$$

$$0 \to Z(I)_{\mathbb{Q}/\mathbb{Z}} \to W(I)_{\mathbb{Q}/\mathbb{Z}} \to V(I)_{\mathbb{Q}} \to 0. \tag{2.2.4}$$

Note that $Z(I)_{\mathbb{Q}/\mathbb{Z}}$ is a torsion subgroup of $T(I) := Z(I)_{\mathbb{C}}/Z(I)_{\mathbb{Z}}$. Let $c_I := \mathbb{P}(I \otimes_{\mathscr{O}_F} \mathbb{C})$ be the cusp corresponding to I. We need a representation of D_L as a Siegel domain of the third kind. We define $D(I) := Z(I)_{\mathbb{C}} D_L$. Then, we obtain the following fibration by [**6**]:

$$D(I) \cong Z(I)_{\mathbb{C}} \times V(I)_{\mathbb{C}} \times c_I \overset{\pi_1}{\to} D(I)' := D(I)/Z(I)_{\mathbb{C}} \overset{\pi_2}{\to} c_I.$$

Moreover, from this fibration, we have

$$D_L = \{(z,u) \in D(I) \mid \Im(z) - h\langle u,u\rangle \in C(I)\}$$

for a cone $C(I)$ in $Z(I)_{\mathbb{R}}$ and some real-bilinear quadratic form $h : \mathbb{C}^{n-1} \times \mathbb{C}^{n-1} \to Z(I)_{\mathbb{R}}$. Accordingly, we have

$$\mathcal{X}(I) := D/Z(I)_{\mathbb{Z}} \subset D(I)/Z(I)_{\mathbb{Z}} \overset{\overline{\pi}_1}{\to} D(I)'.$$

Here, the quotient fiber bundle $\overline{\pi}_1$ is a principal fiber bundle under the algebraic torus $T(I) := Z(I)_{\mathbb{C}}/Z(I)_{\mathbb{Z}}$. Since $\dim_{\mathbb{R}}(Z(I)_{\mathbb{R}}) = 1$, there exists a natural toric embedding $T(I) \hookrightarrow \overline{T(I)}$. In accordance with [**6**], we define $\overline{\mathcal{X}(I)}$ as the interior of closure of $\mathcal{X}(I)$ in $\mathcal{X}(I) \times_{T(I)} \overline{T(I)}$.

Finally, the toroidal compactification of $\mathscr{F}_L(\Gamma)$ is defined by taking the canonical cone decomposition:

$$\overline{\mathscr{F}_L(\Gamma)} := (D_L \cup \bigcup_{I \subset L} \overline{\mathcal{X}(I)})/\sim,$$

where I is a rank 1 primitive isotropic sublattice of L and the equivalence relation is defined in [**6**].

Remark 2.2.1. We can also construct the Baily-Borel compactification $\overline{\mathscr{F}_L(\Gamma)}^{\mathrm{BB}}$ of a ball quotient $\mathscr{F}_L(\Gamma)$ as follows. We define the rational completion D_L^{BB} as the union of D_L and 0-dimensional cusps:

$$D_L^{\mathrm{BB}} := D_L \cup \bigcup_{I \subset L} c_I.$$

Here, $I \subset L$ runs over the rank 1 primitive isotropic sublattices. Now we define $\overline{\mathscr{F}_L(\Gamma)}^{\mathrm{BB}} := D_L^{\mathrm{BB}}/\Gamma$.

2.3. Irregular cusps

2.3.1. Case of $\mathbb{Q}(\sqrt{-1})$. Throughout this subsection, we assume $F = \mathbb{Q}(\sqrt{-1})$. Let us define irregular cusps.

Proposition 2.3.1. *The following are equivalent.*

(1) $Z(I)_{\mathbb{Z}} = Z(I)_{\mathbb{Q}} \cap \langle \Gamma, -\mathrm{id}\rangle \neq Z(I)_{\mathbb{Q}} \cap \langle \Gamma, \sqrt{-1}\,\mathrm{id}\rangle$.

(2) $-\mathrm{id} \in \Gamma$, $\sqrt{-1}\,\mathrm{id} \notin \Gamma$, *and* $\sqrt{-1}T_{\sqrt{-1}\lambda(e\otimes e)} \in \Gamma(I)_{\mathbb{Z}}$ *for some* $\sqrt{-1}\lambda(e \otimes e) \in \sqrt{-1}(\overline{I} \otimes I)(\mathbb{Q})$.

(3) $-\mathrm{id} \in \Gamma$, $\sqrt{-1}\,\mathrm{id} \notin \Gamma$, *and there exists an element* $\gamma \in \overline{\Gamma(I)}_{\mathbb{Z}}$ *of order* 4, *acting on* $Z(I)_{\mathbb{Z}}$ *and* $V(I)_{\mathbb{C}}$ *trivially and* $\mathcal{X}(I)$ *non-trivially, and whose image in* $\mathrm{U}(I^{\perp}) \times \mathrm{GL}(I_F)$ *is* $(\sqrt{-1}\,\mathrm{id}_{I^{\perp}/I_F}, \sqrt{-1}\,\mathrm{id}_{I_F})$. *Moreover, the order of this non-trivial action on* $\mathcal{X}(I)$ *is* 2.

Proof. (1) $\Rightarrow$ (2) Since $\sqrt{-1}\,\mathrm{id} \notin \Gamma$, there exists an element $T_{\sqrt{-1}\lambda(e\otimes e)} \in Z(I)_{\mathbb{Q}} \cap \langle \Gamma, \sqrt{-1}\,\mathrm{id}\rangle \setminus Z(I)_{\mathbb{Z}}$ for some $\sqrt{-1}\lambda(e\otimes e) \in \sqrt{-1}(\overline{I}\otimes I)(\mathbb{Q})$. Now $\langle \Gamma, \sqrt{-1}\,\mathrm{id}\rangle = \Gamma \sqcup \sqrt{-1}\Gamma$ so that $T_{\sqrt{-1}\lambda(e\otimes e)} \in \sqrt{-1}\Gamma$. Combining this with the condition $-\mathrm{id} \in \Gamma$, it follows $\sqrt{-1}T_{\sqrt{-1}\lambda(e\otimes e)} \in \Gamma(I)_{\mathbb{Z}}$.

(2) $\Rightarrow$ (1) Since $-\mathrm{id} \in \Gamma$, we have $Z(I)_{\mathbb{Z}} = Z(I)_{\mathbb{Q}} \cap \langle \Gamma, -\mathrm{id}\rangle$. On the other hand, $\sqrt{-1}T_{\sqrt{-1}\lambda(e\otimes e)} \in \Gamma(I)_{\mathbb{Z}}$ and $\sqrt{-1}\,\mathrm{id} \notin \Gamma$ together shows that $T_{\sqrt{-1}\lambda(e\otimes e)} \in Z(I)_{\mathbb{Q}} \cap \langle \Gamma, \sqrt{-1}\,\mathrm{id}\rangle \setminus Z(I)_{\mathbb{Z}}$.

(2) $\Rightarrow$ (3) Let $\gamma := \sqrt{-1}T_{\sqrt{-1}\lambda e\otimes \lambda e}$ be an order 4 element in $\overline{\Gamma(I)_{\mathbb{Z}}}$. The element γ acts on $\overline{I}$ as $-\sqrt{-1}$-times and $I^{\perp}/I$ as $\sqrt{-1}$-times. Hence, γ acts on $V(I)_{\mathbb{C}}$ trivially. By definition, $\sqrt{-1}\,\mathrm{id}$ and $T_{\sqrt{-1}\lambda(e\otimes e)}$ act on $Z(I)_{\mathbb{Z}}$ trivially, so the same holds for γ. We also have the image of $\gamma \in \overline{\Gamma(I)_{\mathbb{Z}}}$ in $\mathrm{U}(I^{\perp}) \times \mathrm{GL}(I_F)$ is $(\sqrt{-1}\,\mathrm{id}_{I^{\perp}/I_F}, \sqrt{-1}\,\mathrm{id}_{I_F})$.

On the other hand, under the assumption $\sqrt{-1} \notin \Gamma$, it follows $T_{\sqrt{-1}\lambda(e\otimes e)} \notin Z(I)_{\mathbb{Z}}$. This means that γ acts on $\mathcal{X}(I)$ non-trivially. Note that $Z(I)_{\mathbb{Q}}$ acts on $\mathcal{X}(I) \subset T(I) := Z(I)_{\mathbb{C}}/Z(I)_{\mathbb{Z}}$ as a translation, so the above action is a non-trivial translation.

(3) $\Rightarrow$ (2) From (2.2.3), we have $\gamma = (\sqrt{-1}\,\mathrm{id}_{I^{\perp}/I_F}, \sqrt{-1}\,\mathrm{id}_{I_F}, \alpha)$ for some $\alpha \in W(I)_{\mathbb{Q}/\mathbb{Z}}$. Since γ acts on $V(I)_{\mathbb{C}}$ trivially, it follows that the image of α in $V(I)_{\mathbb{Q}}$ is 0 in (2.2.4), so $\alpha \in Z(I)_{\mathbb{Q}/\mathbb{Z}}$. Hence, $\gamma = (\sqrt{-1}\,\mathrm{id}_{I^{\perp}/I_F}, \sqrt{-1}\,\mathrm{id}_{I_F}, T_{\sqrt{-1}\lambda(e\otimes e)})$ for some $\sqrt{-1}\lambda(e\otimes e) \in \sqrt{-1}(\overline{I}\otimes I)(\mathbb{Q})$. Now, we have $\sqrt{-1}\,\mathrm{id}_L = (\sqrt{-1}\,\mathrm{id}_{I^{\perp}/I_F}, \sqrt{-1}\,\mathrm{id}_{I_F}, 0)$, so combining this with $\gamma = (\sqrt{-1}\,\mathrm{id}_{I^{\perp}/I_F}, \sqrt{-1}\,\mathrm{id}_{I_F}, T_{\sqrt{-1}\lambda(e\otimes e)})$, it follows $\sqrt{-1}\gamma = -T_{\sqrt{-1}\lambda(e\otimes e)} \in \Gamma$. Since we have assumed $-\mathrm{id} \in \Gamma$ so that $\sqrt{-1}T_{\sqrt{-1}\lambda(e\otimes e)} \in \Gamma$.

□

Geometrically, the existence of such a cusp corresponds to the existence of a branch divisor on the boundary of a ball quotient with branch index 2. We can show the following propositions in the same way as Proposition 2.3.1.

Definition 2.3.2. We say that I is *semi-irregular with index* 2 if the conditions in Proposition 2.3.1 are satisfied. Here, we define $Z(I)'_{\mathbb{Z}} := Z(I)_{\mathbb{Q}} \cap \langle \Gamma, \sqrt{-1}\,\mathrm{id}\rangle$ and $\Gamma(I)'_{\mathbb{Z}} := \langle \Gamma(I)_{\mathbb{Z}}, \sqrt{-1}\,\mathrm{id}\rangle / \langle \sqrt{-1}\,\mathrm{id}\rangle$.

Now, let us treat the index 4 case.

Proposition 2.3.3. *The following statements are equivalent.*

1. $Z(I)_{\mathbb{Z}} \neq Z(I)_{\mathbb{Q}} \cap \langle \Gamma, -\mathrm{id}\rangle \neq Z(I)_{\mathbb{Q}} \cap \langle \Gamma, \sqrt{-1}\,\mathrm{id}\rangle$, *that is, all three are different.*
2. $-\mathrm{id}, \sqrt{-1}\,\mathrm{id} \notin \Gamma$, *and* $-\sqrt{-1}T_{\sqrt{-1}\lambda(e\otimes e)} \in \Gamma(I)_{\mathbb{Z}}$ *for some* $\sqrt{-1}\lambda(e\otimes e) \in \sqrt{-1}(\overline{I}\otimes I)(\mathbb{Q})$.
3. $\sqrt{-1}\,\mathrm{id} \notin \Gamma$, *and there exists an element* $\gamma \in \overline{\Gamma(I)_{\mathbb{Z}}}$ *of order* 4 *acting on* $Z(I)_{\mathbb{Z}}$ *and* $V(I)_{\mathbb{C}}$ *trivially and* $\mathcal{X}(I)$ *non-trivially, and whose image in* $\mathrm{U}(I^{\perp}) \times \mathrm{GL}(I_F)$ *is* $(\sqrt{-1}\,\mathrm{id}_{I^{\perp}/I_F}, \sqrt{-1}\,\mathrm{id}_{I_F})$. *Moreover, the order of this non-trivial action on* $\mathcal{X}(I)$ *is* 4.

Proof. This can be proven in the same way as Proposition 2.3.1. □

Definition 2.3.4. We say that I is *irregular with index* 4 if the conditions in Proposition 2.3.3 are satisfied. Here, we define $Z(I)'_{\mathbb{Z}} := Z(I)_{\mathbb{Q}} \cap \langle \Gamma, \sqrt{-1}\,\mathrm{id}\rangle$ and $\Gamma(I)'_{\mathbb{Z}} := \langle \Gamma(I)_{\mathbb{Z}}, \sqrt{-1}\,\mathrm{id}\rangle / \langle \sqrt{-1}\,\mathrm{id}\rangle$.

2.3.2. Case of $\mathbb{Q}(\sqrt{-3})$. Throughout this subsection, we assume $F = \mathbb{Q}(\sqrt{-3})$. Let ω be a primitive root of unity.

Proposition 2.3.5. *The following statements are equivalent.*

(1) $Z(I)_{\mathbb{Z}} = Z(I)_{\mathbb{Q}} \cap \langle \Gamma, \omega \,\mathrm{id} \rangle \neq Z(I)_{\mathbb{Q}} \cap \langle \Gamma, -\mathrm{id} \rangle$.

(2) $\omega \,\mathrm{id} \in \Gamma$, $-\mathrm{id} \notin \Gamma$, *and* $-T_{\sqrt{-3}\lambda(e\otimes e)} \in \Gamma(I)_{\mathbb{Z}}$ *for some* $\sqrt{-3}\lambda(e \otimes e) \in \sqrt{-3}(\overline{I} \otimes I)(\mathbb{Q})$.

(3) $\omega \,\mathrm{id} \in \Gamma$, $-\mathrm{id} \notin \Gamma$, *and there exists an element* $\gamma \in \overline{\Gamma(I)}_{\mathbb{Z}}$ *of order* 6*, acting on* $Z(I)_{\mathbb{Z}}$ *and* $V(I)_{\mathbb{C}}$ *trivially and* $\mathcal{X}(I)$ *non-trivially, and whose image in* $\mathrm{U}(I^{\perp}) \times \mathrm{GL}(I_F)$ *is* $(-\mathrm{id}_{I^{\perp}/I_F}, -\mathrm{id}_{I_F})$*. Moreover, the order of this non-trivial action on* $\mathcal{X}(I)$ *is* 2.

Proof. This can be proven in the same way as Proposition 2.3.1. □

Definition 2.3.6. We say that I is *semi-irregular with index* 2 if the conditions in Proposition 2.3.5 are satisfied. Here, we define $Z(I)'_{\mathbb{Z}} := Z(I)_{\mathbb{Q}} \cap \langle \Gamma, \omega \,\mathrm{id} \rangle$ and $\Gamma(I)'_{\mathbb{Z}} := \langle \Gamma(I)_{\mathbb{Z}}, \omega \,\mathrm{id} \rangle / \langle \omega \,\mathrm{id} \rangle$.

Proposition 2.3.7. *The following statements are equivalent.*

(1) $Z(I)_{\mathbb{Z}} = Z(I)_{\mathbb{Q}} \cap \langle \Gamma, -\mathrm{id} \rangle \neq Z(I)_{\mathbb{Q}} \cap \langle \Gamma, -\omega \,\mathrm{id} \rangle$.

(2) $-\mathrm{id} \in \Gamma$, $\omega \,\mathrm{id} \notin \Gamma$, *and* $-T_{\sqrt{-3}\lambda(e\otimes e)} \in \Gamma(I)_{\mathbb{Z}}$ *for some* $\sqrt{-3}\lambda(e \otimes e) \in \sqrt{-3}(\overline{I} \otimes I)(\mathbb{Q})$.

(3) $-\mathrm{id} \in \Gamma$, $\omega \,\mathrm{id} \notin \Gamma$, *and there exists an element* $\gamma \in \overline{\Gamma(I)}_{\mathbb{Z}}$ *of order* 6*, acting on* $Z(I)_{\mathbb{Z}}$ *and* $V(I)_{\mathbb{C}}$ *trivially and* $\mathcal{X}(I)$ *non-trivially, and whose image in* $\mathrm{U}(I^{\perp}) \times \mathrm{GL}(I_F)$ *is* $(\omega \,\mathrm{id}_{I^{\perp}/I_F}, \omega \,\mathrm{id}_{I_F})$*. Moreover, the order of this non-trivial action on* $\mathcal{X}(I)$ *is* 3.

Proof. This can be proven in the same way as Proposition 2.3.1. □

Definition 2.3.8. We say that I is *semi-irregular with index* 3 if the conditions in Proposition 2.3.5 are satisfied. Here, we define $Z(I)'_{\mathbb{Z}} := Z(I)_{\mathbb{Q}} \cap \langle \Gamma, \omega \,\mathrm{id} \rangle$ and $\Gamma(I)'_{\mathbb{Z}} := \langle \Gamma(I)_{\mathbb{Z}}, \omega \,\mathrm{id} \rangle / \langle \omega \,\mathrm{id} \rangle$.

Proposition 2.3.9. *The following statements are equivalent.*

(1) $Z(I)_{\mathbb{Z}} = Z(I)_{\mathbb{Q}} \cap \langle \Gamma, -\mathrm{id} \rangle \neq Z(I)_{\mathbb{Q}} \cap \langle \Gamma, \omega \,\mathrm{id} \rangle$.

(2) $-\mathrm{id} \in \Gamma$, $\omega \,\mathrm{id} \notin \Gamma$, *and* $\omega T_{\sqrt{-3}\lambda(e\otimes e)} \in \Gamma(I)_{\mathbb{Z}}$ *for some* $\sqrt{-3}\lambda(e \otimes e) \in \sqrt{-3}(\overline{I} \otimes I)(\mathbb{Q})$.

(3) $-\mathrm{id} \in \Gamma$, $\omega \,\mathrm{id} \notin \Gamma$, *and there exists an element* $\gamma \in \overline{\Gamma(I)}_{\mathbb{Z}}$ *of order* 3*, acting on* $Z(I)_{\mathbb{Z}}$ *and* $V(I)_{\mathbb{C}}$ *trivially and* $\mathcal{X}(I)$ *non-trivially, and whose image in* $\mathrm{U}(I^{\perp}) \times \mathrm{GL}(I_F)$ *is* $(\omega \,\mathrm{id}_{I^{\perp}/I_F}, \omega \,\mathrm{id}_{I_F})$*. Moreover, the order of this non-trivial action on* $\mathcal{X}(I)$ *is* 3.

Proof. This can be proven in the same way as Proposition 2.3.1. □

Definition 2.3.10. We say that I is *irregular with index* 3 if the conditions in Proposition 2.3.9 are satisfied. Here, we define $Z(I)'_{\mathbb{Z}} := Z(I)_{\mathbb{Q}} \cap \langle \Gamma, \omega \,\mathrm{id} \rangle$ and $\Gamma(I)'_{\mathbb{Z}} := \langle \Gamma(I)_{\mathbb{Z}}, \omega \,\mathrm{id} \rangle / \langle \omega \,\mathrm{id} \rangle$.

Proposition 2.3.11. *The following statements are equivalent.*

(1) $Z(I)_{\mathbb{Z}} \neq Z(I)_{\mathbb{Q}} \cap \langle \Gamma, -\mathrm{id} \rangle \neq Z(I)_{\mathbb{Q}} \cap \langle \Gamma, \omega \rangle$*, that is, all three are different.*

(2) $-\mathrm{id}, \omega \,\mathrm{id} \notin \Gamma$, *and* $-\omega T_{\sqrt{-3}\lambda(e\otimes e)} \in \Gamma(I)_{\mathbb{Z}}$ *for some* $\sqrt{-3}\lambda(e \otimes e) \in \sqrt{-3}(\overline{I} \otimes I)(\mathbb{Q})$.

(3) $-\mathrm{id}, \omega\,\mathrm{id} \notin \Gamma$, *and there exists an element* $\gamma \in \overline{\Gamma(I)_{\mathbb{Z}}}$ *of order* 6, *acting on* $Z(I)_{\mathbb{Z}}$ *and* $V(I)_{\mathbb{C}}$ *trivially and* $\mathcal{X}(I)$ *non-trivially, and whose image in* $\mathrm{U}(I^{\perp}) \times \mathrm{GL}(I_F)$ *is* $(-\omega\,\mathrm{id}_{I^{\perp}/I_F}, -\omega\,\mathrm{id}_{I_F})$. *Moreover, the order of this non-trivial action on* $\mathcal{X}(I)$ *is* 6.

Proof. This can be proven in the same way as Proposition 2.3.1. □

Definition 2.3.12. We say that I is *irregular with index* 6 if the conditions in Proposition 2.3.11 are satisfied. Here, we define $Z(I)'_{\mathbb{Z}} := Z(I)_{\mathbb{Q}} \cap \langle \Gamma, -\mathrm{id}, \omega\,\mathrm{id}\rangle$ and $\Gamma(I)'_{\mathbb{Z}} := \langle \Gamma(I)_{\mathbb{Z}}, -\mathrm{id}, \omega\,\mathrm{id}\rangle / \langle -\mathrm{id}, \omega\,\mathrm{id}\rangle$.

2.3.3. Other cases. Let F be any imaginary quadratic field.

Proposition 2.3.13. *The following statements are equivalent.*

(1) $Z(I)_{\mathbb{Z}} \neq Z(I)_{\mathbb{Q}} \cap \langle \Gamma, -\mathrm{id}\rangle$.
(2) $-\mathrm{id} \notin \Gamma$, *and* $-T_{\sqrt{d}\lambda(e\otimes e)} \in \Gamma(I)_{\mathbb{Z}}$ *for some* $\sqrt{d}\lambda(e\otimes e) \in \sqrt{d}(\overline{I} \otimes I)(\mathbb{Q})$.
(3) $-\mathrm{id} \notin \Gamma$, *and there exists an element* $\gamma \in \overline{\Gamma(I)_{\mathbb{Z}}}$ *of order* 2, *acting on* $Z(I)_{\mathbb{Z}}$ *and* $V(I)_{\mathbb{C}}$ *trivially and* $\mathcal{X}(I)$ *non-trivially, and whose image in* $\mathrm{U}(I^{\perp}) \times \mathrm{GL}(I_F)$ *is* $(-\mathrm{id}_{I^{\perp}/I_F}, -\mathrm{id}_{I_F})$. *Moreover, the order of this non-trivial action on* $\mathcal{X}(I)$ *is* 2.

Proof. This can be proven in the same way as Proposition 2.3.1. □

Definition 2.3.14. We say that I is *irregular with index* 2 if the following holds. If $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$, then the conditions in Proposition 2.3.13 are satisfied. If $F = \mathbb{Q}(\sqrt{-1})$, then the conditions in Proposition 2.3.13 are satisfied and the conditions in Proposition 2.3.3 are not satisfied. If $F = \mathbb{Q}(\sqrt{-3})$, then the conditions in Proposition 2.3.13 are satisfied and the conditions in Proposition 2.3.5 and Proposition 2.3.11 are not satisfied. In these cases, we define $Z(I)'_{\mathbb{Z}} := Z(I)_{\mathbb{Q}} \cap \langle \Gamma, -\mathrm{id}\rangle$ and $\Gamma(I)'_{\mathbb{Z}} := \langle \Gamma(I)_{\mathbb{Z}}, -\mathrm{id}\rangle / \langle -\mathrm{id}\rangle$.

Definition 2.3.15. We say that I is *regular* if I is not irregular or semi-irregular in the sense of the above definitions.

2.3.4. Relation with irregular cusps on orthogonal modular varieties. Now, let us give another description of regular or irregular cusps. We define

$$Z(I)^{\star}_{\mathbb{Z}} := \begin{cases} (\{\pm 1, \pm\sqrt{-1}\} Z(I)_{\mathbb{Q}}) \cap \Gamma & (F = \mathbb{Q}(\sqrt{-1})) \\ (\{\pm 1, \pm\omega, \pm\omega^2\} Z(I)_{\mathbb{Q}}) \cap \Gamma & (F = \mathbb{Q}(\sqrt{-3})) \\ (\{\pm 1\} Z(I)_{\mathbb{Q}}) \cap \Gamma & (F \neq \mathbb{Q}(\sqrt{-1}, \mathbb{Q}(\sqrt{-3}))). \end{cases}$$

We can classify irregular cusps according to the structure of $Z(I)^{\star}_{\mathbb{Z}}/Z(I)_{\mathbb{Z}}$.

For $F = \mathbb{Q}(\sqrt{-1})$,

$$Z(I)^{\star}_{\mathbb{Z}}/Z(I)_{\mathbb{Z}} \cong \begin{cases} 1 & (\text{type } R_1) \\ \langle -\mathrm{id}\rangle \cong \mathbb{Z}/2\mathbb{Z} & (\text{type } R_2) \\ \langle \sqrt{-1}\,\mathrm{id}\rangle \cong \mathbb{Z}/4\mathbb{Z} & (\text{type } R_4) \\ \langle -T_{\sqrt{-1}\lambda(e\otimes e)}\rangle \cong \mathbb{Z}/2\mathbb{Z} & (\text{type } I_2) \\ \langle -\mathrm{id}, -\sqrt{-1}T_{\sqrt{-1}\lambda(e\otimes e)}\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & (\text{type } SI_2) \\ \langle -\sqrt{-1}T_{\sqrt{-1}\lambda(e\otimes e)}\rangle \cong \mathbb{Z}/4\mathbb{Z} & (\text{type } I_4). \end{cases}$$

For $F = \mathbb{Q}(\sqrt{-3})$,

$$Z(I)^{\star}_{\mathbb{Z}}/Z(I)_{\mathbb{Z}} \cong \begin{cases} 1 & \text{(type } R_1\text{)} \\ \langle -\mathrm{id}\rangle \cong \mathbb{Z}/2\mathbb{Z} & \text{(type } R_2\text{)} \\ \langle \omega\,\mathrm{id}\rangle \cong \mathbb{Z}/3\mathbb{Z} & \text{(type } R_3\text{)} \\ \langle -\omega\,\mathrm{id}\rangle \cong \mathbb{Z}/6\mathbb{Z} & \text{(type } R_6\text{)} \\ \langle -T_{\sqrt{-3}\lambda(e\otimes e)}\rangle \cong \mathbb{Z}/2\mathbb{Z} & \text{(type } I_2\text{)} \\ \langle -\omega, -T_{\sqrt{-3}\lambda(e\otimes e)}\rangle \cong \mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z} & \text{(type } SI_2\text{)} \\ \langle \omega T_{\sqrt{-3}\lambda(e\otimes e)}\rangle \cong \mathbb{Z}/3\mathbb{Z} & \text{(type } I_3\text{)} \\ \langle -\mathrm{id}, \omega T_{\sqrt{-3}\lambda(e\otimes e)}\rangle \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z} & \text{(type } SI_3\text{)} \\ \langle -\omega T_{\sqrt{-3}\lambda(e\otimes e)}\rangle \cong \mathbb{Z}/6\mathbb{Z} & \text{(type } I_6\text{)}. \end{cases}$$

For $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$,

$$Z(I)^{\star}_{\mathbb{Z}}/Z(I)_{\mathbb{Z}} \cong \begin{cases} 1 & \text{(type } R_1\text{)} \\ \langle -\mathrm{id}\rangle \cong \mathbb{Z}/2\mathbb{Z} & \text{(type } R_2\text{)} \\ \langle -T_{\sqrt{d}\lambda(e\otimes e)}\rangle \cong \mathbb{Z}/2\mathbb{Z} & \text{(type } I_2\text{)}. \end{cases}$$

Here, type $R_\star$ corresponds to regular cusps, and type $I_\star$ (resp. $SI_\star$) corresponds to irregular (resp. semi-irregular) cusps with index $\star$.

Now, we will explicitly show how the type of cusps varies when arithmetic subgroups change, and consider the relationship between unitary cusps and orthogonal cusps. Figures 2.3.1, 2.3.2, and 2.3.3 show whether the cusps with respect to finite index subgroups of $\mathrm{U}(L)(\mathbb{Z})$ are regular or irregular according to inclusions. We fix an irregular cusp I. For a finite index subgroup $\Gamma \subset \mathrm{U}(L)(\mathbb{Z})$, these figures represent the type candidates of another finite index subgroup $\Gamma' \subset \mathrm{U}(L)(\mathbb{Z})$ having the inclusion relationship with Γ. If $\Gamma \subset \Gamma'$ and Γ is type X, then Γ' is type located above X in the figures, and if $\Gamma' \subset \Gamma$, then Γ' is type located below X in the figures. For example, in Figure 2.3.2, for $F = \mathbb{Q}(\sqrt{-1})$, let Γ be type R_2. Then $\Gamma' \supset \Gamma$ is type R_2, SI_2 or R_4. On the other hand if $\Gamma' \subset \Gamma$, then Γ' is type R_2, I_2 or R_1. Circle nodes mean regular cusps and diamond nodes mean irregular cusps.

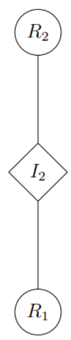

FIGURE 2.3.1. $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$

Next, let us discuss the relationship between regular/irregular cusps on ball quotients and regular/irregular cusps on orthogonal modular varieties, as studied in [**109**].

Specifically, we get

$$\mathrm{U}(V) = \{\gamma \in \mathrm{O}^+((L_Q)_{\mathbb{Q}}) \mid j_d\gamma j_d = d\gamma\},$$

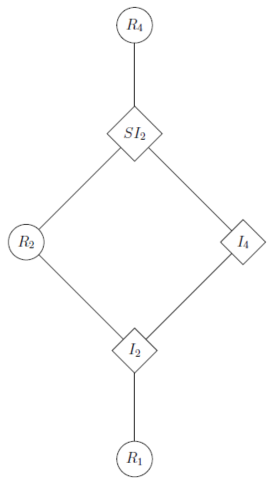

FIGURE 2.3.2. $F = \mathbb{Q}(\sqrt{-1})$

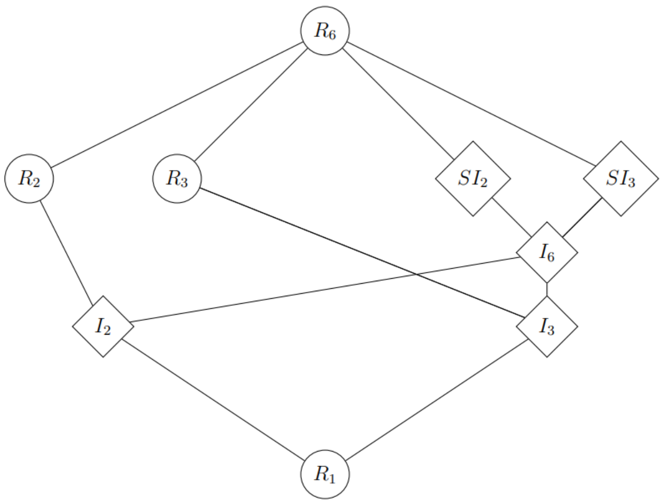

FIGURE 2.3.3. $F = \mathbb{Q}(\sqrt{-3})$

where $j_d \in \mathrm{O}^+((L_Q)_{\mathbb{Q}})$ satisfies $j_d^2 = d\,\mathrm{id}_{L_Q}$. Explicitly,

$$j_d := \begin{pmatrix} \begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix} \end{pmatrix}.$$

We are concerned whether the image of regular/irregular cusps on ball quotients by (1.2.1) are regular or irregular on orthogonal modular varieties. By [**68**, Proposition 2], a 0-dimensional cusp on D_L, corresponding to a rank 1 primitive isotropic sublattice $I \subset L$ maps to a 1-dimensional cusp on $\mathscr{D}_{L_Q}$, corresponding to the rank 2 primitive isotropic sublattice $I_Q \subset L_Q$ spanned by I and $\sqrt{d}I$ (or $(1+\sqrt{d})/2I$ for $d \equiv 1 \bmod 4$). Ma studied irregular cusps on orthogonal modular varieties; here, we will review some of his results. In orthogonal cases, only 2-ramifications may occur; they are classified as follows:

$$Z(I_Q)^{\star}_{\mathbb{Z}}/Z(I_Q)_{\mathbb{Z}} \cong \begin{cases} 1 & (\text{type } (R_1)_O) \\ \langle -\operatorname{id}\rangle \cong \mathbb{Z}/2\mathbb{Z} & (\text{type } (R_2)_O) \\ \langle -T_{\sqrt{d}\lambda(e\otimes e)}\rangle \cong \mathbb{Z}/2\mathbb{Z} & (\text{type } (I_2)_O) \end{cases}$$

where $Z(I_Q)_{\mathbb{Z}}$ is the intersection of the center of the unipotent part of the stabilizer of I_Q in $\mathrm{O}^+((L_Q)_{\mathbb{Q}})$ and a finite index subgroup $\Gamma_O \subset \mathrm{O}^+(L_Q)(\mathbb{Z})$ as in our unitary case. Type $(R_1)_O$ and $(R_2)_O$ (resp. $(I_2)_O$) means that I_Q is regular (resp. irregular with index 2) in $\mathscr{D}_{L_Q}/\Gamma_O$. Note that the image of $Z(I)_{\mathbb{Q}}$ is precisely $Z(I_Q)_{\mathbb{Q}}$ and the image of the discriminant kernel in the unitary group is a subgroup of the discriminant kernel in the orthogonal group. By [**109**, Corollary 3.6], we obtain Figure 2.3.4 in the orthogonal case.

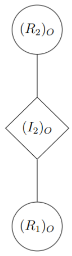

FIGURE 2.3.4. Orthogonal case

Now, let us study the image of regular/irregular cusps on orthogonal modular varieties. Refer to Figures 2.3.5, 2.3.6, and 2.3.7. By [**56**, Lemma 2.5], for a 1-dimensional cusp $J \subset L_Q$, the center of the unipotent part of its stabilizer in $\mathrm{O}^+((L_Q)_{\mathbb{Q}})$ is described as

$$Z(J)_{\mathbb{Q}} = \left\{ \left(\begin{array}{ccc} I_2 & 0 & \begin{pmatrix} 0 & e\lambda \\ \lambda & 0 \end{pmatrix} \\ 0 & I_{2n-2} & 0 \\ 0 & 0 & I_2 \end{array} \right) \middle|\ \lambda \in \mathbb{Q} \right\}$$

for some $e \in \mathbb{Q}$. For a 2-dimensional $\mathbb{Q}$-isotropic subspace $J_{\mathbb{Q}} \subset (L_Q)_{\mathbb{Q}}$, if we consider it to be a subset of V, it defines an F-subspace of V if and only if $e = d$. In that case, the corresponding subspace I_F is a 1-dimensional F-isotropic subspace of V and hence corresponds to a 0-dimensional cusp. This shows that when $e = d$, $\iota(Z(I)_{\mathbb{Q}}) = Z(J)_{\mathbb{Q}}$. We also have $\iota(-\operatorname{id}) = -\operatorname{id}$, $\iota(\sqrt{-1}\operatorname{id}) = j_{-1}$ and

$$\iota(\omega\operatorname{id}) = \left(\begin{array}{ccc} \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \end{array} \right).$$

In this situation, consider the following problem. Let $J \subset L_Q$ be a 1-dimensional cusp and $e = d$ as above. Let $I \subset L$ be the corresponding 0-dimensional cusp. Note that $\iota(Z(I)_{\mathbb{Q}}) = Z(J)_{\mathbb{Q}}$ holds. We assume J is a regular or an irregular cusp in the sense of [**109**, Definition 6.2] with respect to a finite index subgroup $\Gamma_O \subset \mathrm{O}^+(L_Q)(\mathbb{Z})$. We shall determine whether the corresponding cusp I is regular or irregular in the sense of the above definitions with respect to $\Gamma_U := \iota^{-1}(\Gamma_O)$.

If J is irregular, then Γ_O is type $(I_2)_O$. In this case, since $-\operatorname{id} \notin \Gamma_O$, we have $-\operatorname{id} \notin \Gamma_U$; moreover, from the fact $\iota(Z(I)_{\mathbb{Q}}) = Z(J)_{\mathbb{Q}}$, it follows that I is irregular and Γ_U is type I_2,

I_4, SI_2 or I_6. On the other hand, if J is irregular, then Γ_O is type $(R_1)_O$ or $(R_2)_O$. In the first case, since $-\mathrm{id} \in \Gamma_O$, it follows that $-\mathrm{id} \in \Gamma_U$, so we have that Γ_U is type R_1, R_3, or I_3. In the second case, since $-\mathrm{id} \notin \Gamma_U$, it follows that Γ_U is type R_2, R_4, SI_2 or SI_3.

In the following figures, star nodes mean that regular cusps in unitary groups become irregular cusps in orthogonal groups. These figures show what the type of $\Gamma_O \subset \mathrm{O}^+(L_Q)(\mathbb{Z})$ is when $\Gamma_U \subset \mathrm{U}(L)(\mathbb{Z})$ is a certain type. For example, for $F = \mathbb{Q}(\sqrt{-1})$, if $\Gamma_U \subset \mathrm{U}(L)(\mathbb{Z})$ is type R_4, then the corresponding 1-dimensional cusp is type $(R_2)_O$. Indeed, regular cusps on D_L map to regular cusps on $\mathscr{D}_{L_Q}$. On the other hand, for $F = \mathbb{Q}(\sqrt{-3})$, if $\Gamma \subset \mathrm{U}(L)(\mathbb{Z})$ is type SI_3, i.e, semi-irregular with index 3, then the corresponding 1-dimensional cusp is regular (type $(R_2)_O$).

From Figures 2.3.5, 2.3.6, and 2.3.7, we obtain the following proposition. Let $\Gamma_O \subset \mathrm{O}^+(L_Q)(\mathbb{Z})$ be a finite index subgroup and $\Gamma_U \subset \mathrm{U}(L)(\mathbb{Z})$ be its restriction. Here, regular/irregular cusps on D_L (resp. $\mathscr{D}_{L_Q}$) mean regular/irregular cusps with respect to Γ_U (resp. Γ_O).

Proposition 2.3.16. *(1) For any imaginary quadratic field F, regular cusps on D_L map to regular cusps on $\mathscr{D}_{L_Q}$.*

(2) For $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$, irregular cusps on D_L map to irregular cusps on $\mathscr{D}_{L_Q}$.

(3) For $F = \mathbb{Q}(\sqrt{-1})$, irregular cusps with index 2 or 4 on D_L map to irregular cusps with index 2 on $\mathscr{D}_{L_Q}$, and semi-irregular cusps with index 2 on D_L map to regular cusps on $\mathscr{D}_{L_Q}$.

(4) For $F = \mathbb{Q}(\sqrt{-3})$, irregular cusps with index 2 or 6 and semi-irregular cusps with index 2 on D_L map to irregular cusps with index 2 on $\mathscr{D}_{L_Q}$, and irregular cusps with index 3 and semi-irregular cusps with index 3 on D_L map to regular cusps on $\mathscr{D}_{L_Q}$.

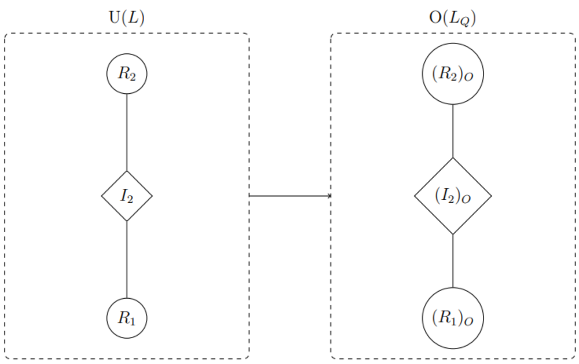

FIGURE 2.3.5. Relationship for $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$

2.4. Discriminant kernel case

Here, we shall show a structure theorem of the discriminant group when the discriminant kernel may have irregular cusps. In this section, we assume that the class number of F is 1. For a rank 1 primitive isotropic sublattice I of L and a generator e of I, the quantity $\mathrm{div}(I)$ denotes a generator of the principal ideal $\{\langle \ell, e\rangle \mid \ell \in L\}$. Note that, unlike the orthogonal case, there is no canonical choice of this quantity. Let $\Gamma \subset \mathrm{U}(L)(\mathbb{Z})$ be a finite index subgroup.

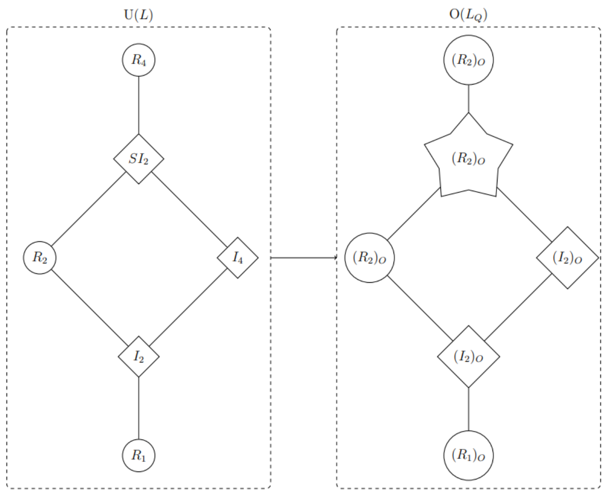

FIGURE 2.3.6. Relationship for $F = \mathbb{Q}(\sqrt{-1})$

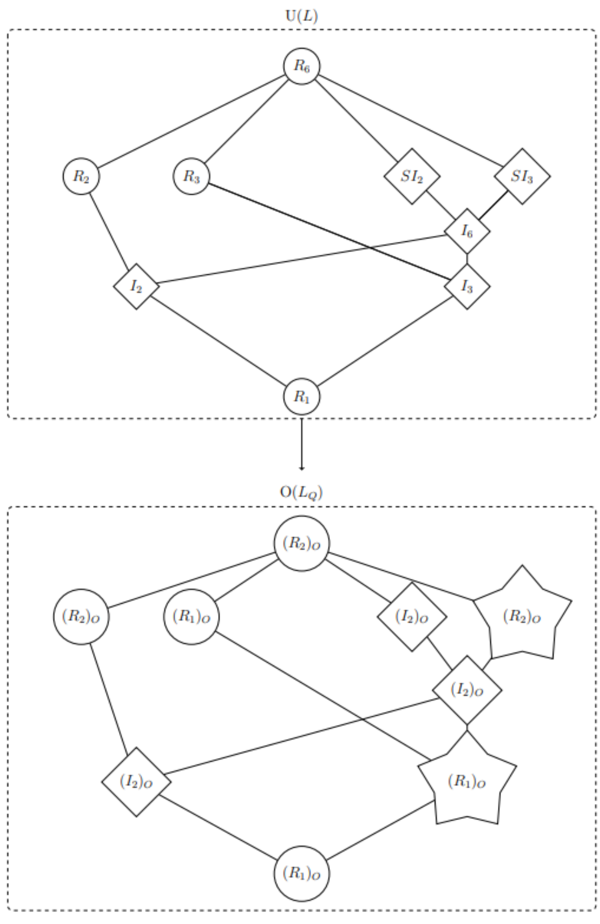

FIGURE 2.3.7. Relationship for $F = \mathbb{Q}(\sqrt{-3})$

In this section and Appendix 2.A, we assume that L is *even*, that is, $\langle \ell, \ell \rangle \in \mathbb{Z}$ for any $\ell \in L$ in the sense of [**68**]. Note that this implies that the associated quadratic lattice is even. This corresponds to the assumption in [**109**, Subsection 4.1]. Let $a, b \in \mathbb{Z}$ be integers with $a \neq 0$ or $b \neq 0$. This section uses the following notation:

$$\mathrm{div}(I) = \begin{cases} \frac{2a+(1+\sqrt{d})b}{2\sqrt{d}} & (d \equiv 1 \bmod 4) \\ \frac{a+b\sqrt{d}}{2\sqrt{d}} & (d \equiv 2, 3 \bmod 4). \end{cases}$$

2.4.1. Preparation.

Lemma 2.4.1 (cf. [109, Lemma 4.1]). *Assuming $\widetilde{\mathrm{U}}(L) \subset \Gamma$, we have $\sqrt{d}(\overline{I} \otimes I)(\mathbb{Z}) \subset Z(I)_{\mathbb{Z}}$. Here,*

$$\sqrt{d}(\overline{I} \otimes I)(\mathbb{Z}) := \{\sqrt{d}\lambda(e \otimes e) \mid \lambda \in \mathbb{Z}\}.$$

PROOF. For $\sqrt{d}\lambda(e \otimes e) \in \sqrt{d}(\overline{I} \otimes I)(\mathbb{Z})$, we can show that $T_{\sqrt{d}\lambda(e\otimes e)}$ preserves the discriminant group and this gives the inclusion $\sqrt{d}(\overline{I} \otimes I)(\mathbb{Z}) \subset Z(I)_{\mathbb{Z}}$. □

Lemma 2.4.2 (cf. [109, Lemma 4.3]). *Let $\Gamma = \widetilde{\mathrm{U}}(L)$.*

(1) For any imaginary quadratic field F with class number 1, if I is irregular with index 2, then $2/\mathrm{div}(I)$ is an element of $\mathscr{O}_F$.
(2) For $F = \mathbb{Q}(\sqrt{-1})$, if I is semi-irregular with index 2 (resp. irregular index 4), then $(1-\sqrt{-1})/\mathrm{div}(I)$ (resp. $(1+\sqrt{-1})/\mathrm{div}(I)$) is an element of $\mathscr{O}_F$.
(3) For $F = \mathbb{Q}(\sqrt{-3})$, if I is semi-irregular with index 2 (resp. (semi-)irregular with index 3, irregular with index 6), then $2/\mathrm{div}(I)$ (resp. $(1-\omega)/\mathrm{div}(I)$, $(1+\omega)/\mathrm{div}(I)$) is an element of $\mathscr{O}_F$.

PROOF. (1) Assume $-T_{\sqrt{d}\lambda(e\otimes e)} \in \Gamma = \widetilde{\mathrm{U}}(L)$ for some $\sqrt{d}\lambda(e \otimes e) \in \sqrt{d}(\overline{I} \otimes I)(\mathbb{Q})$. Then, for any $v \in I^{\perp} \cap I^{\vee}$, we have

$$-T_{\sqrt{d}\lambda(e\otimes e)}(v) = -v \in v + L$$

because $-T_{\sqrt{d}\lambda(e\otimes e)}$ acts on the discriminant group of L trivially. This implies that $2v \in L$. By substituting $v = e'/\mathrm{div}(I)$, we find that $2/\mathrm{div}(I) \in \mathscr{O}_F$. We can prove (2) and (3) similarly by calculating $\pm\sqrt{-1}T_{\sqrt{-1}\lambda(e\otimes e)}$ and $\pm\omega T_{\sqrt{-3}\lambda(e\otimes e)}$.

□

Lemma 2.4.3 (cf. [109, Lemma 4.2]). *Let $\widetilde{\mathrm{U}}(L) \subset \Gamma$. Assume that the following holds for any $\lambda \in F$; if $2\sqrt{d} \cdot \overline{\mathrm{div}(I)}\lambda$ is an element of $\mathscr{O}_F$, then λ is an element of $\mathbb{Z}$. Then, I is regular.*

PROOF. For a fixed $\mathrm{div}(I)$, we take an $e' \in L$ such that $\langle e, e' \rangle = \mathrm{div}(I)$. Now, we shall prove that we can take e' to be an isotropic vector.

For simplicity, we only consider the case of $d \equiv 2, 3 \bmod 4$. We assume $\langle e', e' \rangle \neq 0$. Let $f := (p + q\sqrt{d})e + e'$ for some integers $p, q \in \mathbb{Z}$. Note that $\langle e, f \rangle = \mathrm{div}(I)$. Then, since we have

$$\langle e, e' \rangle = \mathrm{div}(I) = \frac{a + b\sqrt{d}}{2\sqrt{d}},$$

it follows that $\langle f, f \rangle = 0$ holds if and only if

$$aq + bp = -\langle e', e' \rangle. \tag{2.4.1}$$

Here, $-\langle e', e' \rangle$ is in $\mathbb{Z}$ from the condition that L is even. On the other hand, by our assumption in lemma, the greatest common divisor of a and b is 1 so that there exist some integers p' and q' that make the equation (2.4.1) hold. Hence, it suffices to replace e' with

$(p' + q'\sqrt{d})e + e'$. The same discussion holds for the case of $d \equiv 1 \bmod 4$. Below, we take e' to be an isotropic vector.

First, suppose I is irregular with index 2. Equivalently, we can assume $-\mathrm{id} \notin \Gamma$ and $-T_{\sqrt{d}\lambda(e\otimes e)} \in \Gamma$. Since $T_{\sqrt{d}\lambda(e\otimes e)}$ preserves L, we have

$$T_{\sqrt{d}\lambda(e\otimes e)}(e') = e' + 2\sqrt{d}\langle e', e\rangle \lambda e \in L.$$

By assumption, $\lambda \in \mathscr{O}_F$ so that $\sqrt{d}\lambda(e\otimes e) \in \sqrt{d}(\overline{I}\otimes I)(\mathbb{Z})$. By Lemma 2.4.1, $T_{\sqrt{d}\lambda(e\otimes e)} \in U(I)_{\mathbb{Z}}$, so we obtain $T_{\sqrt{d}\lambda(e\otimes e)} \in \Gamma$. This implies $-\mathrm{id} \in \Gamma$, which is a contradiction.

We can give similar proofs for other irregular lattices I. □

For analysis of the structures of discriminant groups, we need some invariant decomposition theorem of finitely generated modules over a principal ideal domain.

Proposition 2.4.4. *Let $\mathscr{O}$ be a principal ideal domain, N be a finite module over $\mathscr{O}$ and $p \neq 0$ be a prime element in $\mathscr{O}$. We assume that an exact sequence*

$$0 \to \mathscr{O}/p^m \to N \to \bigoplus_{i=1}^{s} (\mathscr{O}/p^i)^{\oplus a_i} \to 0$$

exists for some non-negative integers $m, s, a_1, \ldots, a_s \in \mathbb{Z}$. Then, the isomorphism class of N satisfying the above exact sequence corresponds to the pair $(i_0, \ldots, i_k, m_0, \ldots, m_k)$ such that

$$\begin{cases} i_0 < \cdots < i_k & \\ a_{i_\ell} > 0 & (0 < \ell \leq k) \\ a_{i_0} > 0 & (\text{if } i_0 > 0) \\ m_0 + \cdots + m_k = m & (m_i > 0 \text{ for any } i) \\ 0 < m_\ell < i_{\ell+1} - i_\ell & (0 \leq \ell < k). \end{cases}$$

Moreover,

$$N \cong \begin{cases} \mathscr{O}/p^{m_0} \oplus \bigoplus_{\ell=1}^{k} \Big\{ (\mathscr{O}/p^{i_\ell})^{\oplus(a_{i_\ell}-1)} \oplus \mathscr{O}/p^{m_\ell + i_\ell} \Big\} \oplus \bigoplus_{\substack{j \neq i_t \\ \text{for any } t}} (\mathscr{O}/p^j)^{\oplus a_j} & (i_0 = 0) \\ \bigoplus_{\ell=0}^{k} \Big\{ (\mathscr{O}/p^{i_\ell})^{\oplus(a_{i_\ell}-1)} \oplus \mathscr{O}/p^{m_\ell + i_\ell} \Big\} \oplus \bigoplus_{\substack{j \neq i_t \\ \text{for any } t}} (\mathscr{O}/p^j)^{\oplus a_j} & (i_0 > 0). \end{cases}$$

Below, we especially compute the case of class number 1 and discriminant kernels. In the rest of this section, let $\Gamma = \widetilde{\mathrm{U}}(L)$. Combining these calculation, it will be possible to narrow down the list of candidates of discriminant groups; see Appendix 2.A for the classification of A_L.

2.4.2. Case of $\mathbb{Q}(\sqrt{-1})$.

Let $F = \mathbb{Q}(\sqrt{-1})$.

Proposition 2.4.5. *(1) If I is irregular with index 2, then $\mathrm{div}(I) = \pm 1, \pm\sqrt{-1}, \pm 1 \pm \sqrt{-1}, \pm 2, \pm 2\sqrt{-1}$.*

(2) If I is semi-irregular with index 2, then $\mathrm{div}(I) = \pm 1 \pm \sqrt{-1}$.

(3) If I is irregular with index 4, then $\mathrm{div}(I) = \pm 1 \pm \sqrt{-1}$.

PROOF. (1) We have

$$\frac{2}{\operatorname{div}(I)} = \frac{4(b + a\sqrt{-1})}{a^2 + b^2}.$$

Hence, $2/\operatorname{div}(I) \in \mathscr{O}_F$ implies $(a, b) = (\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1), (\pm 2, 0), (0, \pm 2), (\pm 2, \pm 2), (\pm 4, 0), (0, \pm 4)$ and these pairs are the candidates for irregular I with index 2 by Lemma 2.4.2. On the other hand,

$$2\sqrt{-1}\langle e', e\rangle\lambda = (a + b\sqrt{-1})\lambda.$$

If $(a + b\sqrt{-1})\lambda \in \mathscr{O}_F$ implies $\lambda \in \mathbb{Z}$, then I is regular by Lemma 2.4.3. In this case, the pairs $(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)$ satisfy the condition in Lemma 2.4.3; that is, if (a, b) is one of these pairs, then I is regular. Hence, from the above discussion, if I is irregular, then $(a, b) = (\pm 2, 0), (0, \pm 2), (\pm 2, \pm 2), (\pm 4, 0), (0, \pm 4)$ so that $\operatorname{div}(I) = \pm 1, \pm\sqrt{-1}, \pm 1 \pm \sqrt{-1}, \pm 2, \pm 2\sqrt{-1}$.

(2) We have

$$\frac{1 - \sqrt{-1}}{\operatorname{div}(I)} = \frac{2(a + b) + 2(a - b)\sqrt{-1}}{a^2 + b^2}.$$

Hence, $(1 - \sqrt{-1})/\operatorname{div}(I) \in \mathscr{O}_F$ implies $(a, b) = (\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1), (\pm 2, \pm 2)$ and these pairs are the candidates for semi-irregular I with index 2 by Lemma 2.4.2. By performing the same calculation, if I is irregular, then $(a, b) = (\pm 2, \pm 2)$ so that $\operatorname{div}(I) = (\pm 2 \pm 2\sqrt{-1})/2 = \pm 1 \pm \sqrt{-1}$.

We can prove (3) in the same way. □

2.4.3. Case of $\mathbb{Q}(\sqrt{-3})$. Let $F = \mathbb{Q}(\sqrt{-3})$. See Subsection 2.4.4 for the (semi-)index 2 case.

Proposition 2.4.6. *(1) If I is (semi-)irregular with index 3, then*

$$\operatorname{div}(I) = \frac{2a + (1 + \sqrt{-3})b}{2\sqrt{-3}}$$

has the candidates listed in Table 2.4.1.

(2) If I is irregular with index 6, then

$$\operatorname{div}(I) = \frac{2a + (1 + \sqrt{-3})b}{2\sqrt{-3}}$$

has the candidates listed in Table 2.4.2.

PROOF. These also follow from a direct calculation. □

a	-3	-3	-2	-1	-1	-1	-1	0	0	0	0	1	1	1	1	2	3	3
b	0	3	1	-1	0	1	2	-3	-1	1	3	-2	-1	0	1	-1	-3	0

TABLE 2.4.1. Candidates for $\operatorname{div}(I)$ for (semi-)irregular I with index 3

a	-2	-1	-1	-1	-1	0	0	1	1	1	1	2
b	1	-1	0	1	2	-1	1	-2	-1	0	1	-1

TABLE 2.4.2. Candidates for $\operatorname{div}(I)$ for irregular I with index 6

2.4.4. Other cases. Let $F \neq \mathbb{Q}(\sqrt{-1})$ be an imaginary quadratic field with class number 1, that is, $F = \mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-7})$, $\mathbb{Q}(\sqrt{-11})$, $\mathbb{Q}(\sqrt{-19})$, $\mathbb{Q}(\sqrt{-43})$, $\mathbb{Q}(\sqrt{-67})$ or $\mathbb{Q}(\sqrt{-163})$. Then, by performing a similar calculation to the one above, we can prove the following proposition by using a computer.

Proposition 2.4.7. *(1) Let* $d \equiv 1 \bmod 4$. *If* I *is irregular with index* 2, *then*

$$\operatorname{div}(I) = \frac{2a + (1+\sqrt{d})b}{2\sqrt{d}}$$

has the candidates listed in Table 2.4.3.

(2) Let $d = -2$. *If* I *is irregular with index* 2, *then*

$$\operatorname{div}(I) = \frac{a + b\sqrt{-2}}{2\sqrt{-2}}$$

has the candidates listed in Table 2.4.4.

Proof. These also follow from a direct calculation. □

d													
-3	a	-4	-2	-2	-2	-2	-2	-1	-1	-1	-1	0	0
	b	2	-2	0	1	2	4	-1	0	1	2	-2	-1
	a	0	0	1	1	1	1	2	2	2	2	2	4
	b	1	2	-2	-1	0	1	-4	-2	-1	0	2	-2
-7	a	-4	-3	-2	-2	-1	-1	-1	0	0	1	1	1
	b	1	-1	0	4	0	1	2	-1	1	-2	-1	0
	a	2	2	3	4								
	b	-4	0	1	-1								
$-11, -19, -43, -67, -163$	a	-2	-2	-1	-1	1	1	2	2				
	b	0	4	0	2	-2	0	-4	0				

Table 2.4.3. Candidates for $\operatorname{div}(I)$ for irregular I with index 2 and $d \equiv 1 \bmod 4$

a	2	-2	4	-4	0	0	0	0
b	0	0	0	0	2	-2	4	-4

Table 2.4.4. Candidates for $\operatorname{div}(I)$ for irregular I with index 2 and $d = -2$

2.5. Ramification divisors and canonical singularities

Now, we consider how irregular cusps affect the geometry of $\overline{\mathscr{F}_L(\Gamma)}$. The essence of this section is due to [**109**, Section 7].

Corollary 2.5.1. *Let* I *be a rank* 1 *primitive isotropic sublattice of* L. *Then,* I *is an irregular with index* m *if and only if the map* $\overline{\mathcal{X}(I)} \to \overline{\mathcal{X}(I)/\Gamma(I)_{\mathbb{Z}}}$ *ramifies along the unique boundary divisor with ramification index* m. *Moreover, if we take the quotient* $Z(I)^{\star}_{\mathbb{Z}}/Z(I)_{\mathbb{Z}}$, *then* $\overline{D_L/Z(I)^{\star}_{\mathbb{Z}}} \to \overline{\mathscr{F}_L(\Gamma)}$ *does not ramify along the unique boundary divisor.*

Proof. The first claim follows from Propositions 2.3.1, 2.3.3, 2.3.5, 2.3.7, 2.3.9 2.3.11 and 2.3.13, and the fact that the unique boundary divisor is $V(I)_{\mathbb{C}}$. The second claim follows in the same way as [**109**, Proposition 7.2 (2)]. □

Remark 2.5.2. Note that, in the adjoint case, Ma [**108**] proved there is no branch divisor on the boundary of any toroidal compactification of modular varieties.

Now, let us treat the canonical singularities on the boundary divisors on ball quotients.

Proposition 2.5.3. *If $n \geq 13$ and $d < -3$, then the canonical toroidal compactification $\overline{\mathscr{F}_L(\Gamma)}$ has canonical singularities at the boundary points.*

Proof. If there is no irregular primitive isotropic sublattice $I \subset L$, then the claim follows from [**9**]. Otherwise, in the same way as [**109**, Proposition 7.4], we have

$$\overline{(D_L/Z(I)_{\mathbb{Z}})}/\overline{\Gamma(I)_{\mathbb{Z}}} \cong \overline{(D_L/Z(I)'_{\mathbb{Z}})}/(\Gamma(I)'_{\mathbb{Z}}/Z(I)'_{\mathbb{Z}}).$$

The claim is proved combining this with [**9**]. □

2.6. Low slope cusp form trick

Let $\mathscr{L} := \mathscr{O}(-1)|_{D_L}$ and χ be a character of Γ. A Γ-invariant section Ψ of $\mathscr{L}^{\otimes k} \otimes \chi$ is called a modular form of weight k with character χ. We consider D_L as a Siegel domain of the third kind. In our setting, for any rank 1 primitive isotropic sublattice $I \subset L$, the corresponding cusp c_I is a point, so we will omit this in the Siegel domain of the third kind and consider $D_L \subset D(I) = Z(I)_{\mathbb{C}} \times V(I)_{\mathbb{C}}$. Here, z and $u = (u_1, \ldots, u_{n-1})$ denote the local coordinates of $Z(I)_{\mathbb{C}}$ and $V(I)_{\mathbb{C}}$, respectively. We take a nowhere vanishing section s_I of $\mathscr{L}$ with respect to I in the same way as in [**109**]. Then when we write $\Psi = f s_I^{\otimes k} \otimes 1$, the holomorphic function f on D_L satisfies the following modularity condition:

$$f(\gamma[v]) = \chi(\gamma) j(\gamma, [v])^{\otimes k} f([v]) \quad (\gamma \in \Gamma, [v] \in D_L)$$

where $j(\gamma, [v])$ is the automorphy factor. We assume $\chi|_{Z(I)_{\mathbb{Z}}} = 1$ so that f descends to a function on $D_L/Z(I)_{\mathbb{Z}}$. Then the Fourier expansion of f is

$$f(z, u) = \sum_{\rho \in Z(I)_{\mathbb{Z}}^{\vee}} \varphi_\rho(u) \exp(2\pi\sqrt{-1}\langle \rho, z \rangle).$$

For a generator w_I of $C(I)$, we define the vanishing order $v_I(\Psi)$ as

$$v_I(\Psi) := \min\{\langle \ell, w_I \rangle \mid \ell \in Z(I)_{\mathbb{Z}}^{\vee}, \varphi_\rho(\ell) \neq 0\}.$$

Moreover, we define the geometric vanishing order $v_{I,\mathrm{geom}}(\Psi)$ as

$$v_{I,\mathrm{geom}}(\Psi) := \begin{cases} v_I(\Psi) & (I : \text{regular}) \\ \frac{1}{m} v_I(\Psi) & (I : \text{(semi-)irregular with index } m). \end{cases}$$

Then, we can give these vanishing orders a geometrical interpretation.

Proposition 2.6.1 ([109, Proposition 8.4, 8.5 and 8.6]). *(1) $v_I(\Psi)$ is the vanishing order of Ψ over $\overline{\mathcal{X}(I)}$ along the unique boundary divisor $V(I)_{\mathbb{C}}$.*

(2) If $s_I^{\otimes k}|_{Z(I)_{\mathbb{Z}}^\star} = 1$, then $v_{I,\mathrm{geom}}(\Psi)$ is the vanishing order of Ψ over $\overline{\mathcal{X}(I)'}$ along the unique boundary divisor $V(I)'_{\mathbb{C}}$.

(3) $\mathscr{L}^{\otimes n+1} \otimes \det \cong K_{\overline{\mathcal{X}(I)'}} + V(I)'_{\mathbb{C}}$ over $\overline{\mathcal{X}(I)'}$.

The vanishing orders of canonical forms are measured in $\overline{\mathscr{F}_L(\Gamma)}$. Now, the projection $\overline{\mathcal{X}(I)'} \to \overline{\mathscr{F}_L(\Gamma)}$ does not ramify, so we can measure the order of canonical forms by pulling back to $\overline{\mathcal{X}(I)'}$, i.e., for a modular form Ψ of weight $(n+1)k$ and a corresponding k-canonical form ω_Ψ,

$$v_{V(I)_{\mathbb{C}}}(\omega_\Psi) = v_{V(I)'_{\mathbb{C}}}(\pi^\star(\omega_\Psi)) = v_{I,\mathrm{geom}}(\Psi) - k.$$

On the other hand, the projection $\overline{\mathcal{X}(I)} \to \overline{\mathcal{X}(I)'}$ ramifies with index m if I is (semi-)irregular with index m so that

$$v_{V(I)_{\mathbb{C}}}(\omega_\Psi) = \frac{1}{m} v_I(\Psi) - k.$$

Proposition 2.6.2. *The k-canonical form corresponding to a modular form Ψ of weight $(n+1)k$ extends holomorphically over the regular locus of $\overline{\mathscr{F}_L(\Gamma)}$ if and only if the following conditions hold:*

(1) $v_R(\Psi) \geq (r_i - 1)k$ for every irreducible component R_i of the ramification divisors $D_L \to \mathscr{F}_L(\Gamma)$ with ramification index r_i.
(2) $v_I(\Psi) \geq k$ for every regular isotropic sublattice $I \subset L$.
(3) $v_I(\Psi) \geq m_I k$ for every (semi-)irregular isotropic sublattice $I \subset L$ with index m_I.

Proof. To conclude the proof, combine the above discussion and [**109**, Corollary 8.8]. □

Theorem 2.6.3 (Low slope cusp form trick). *Let F be an imaginary quadratic field and L be a Hermitian lattice of signature $(1, n)$ over $\mathscr{O}_F$. For a finite index subgroup $\Gamma \subset \mathrm{U}(L)(\mathbb{Z})$, we assume that there is a non-zero cusp form Ψ of weight k with respect to Γ on D_L. In addition, we make the following assumptions.*

(1) $v_R(\Psi)/k > (r_i - 1)/(n+1)$ for every irreducible component R_i of the ramification divisors $D_L \to \mathscr{F}_L(\Gamma)$ with ramification index r_i.
(2) $v_I(\Psi)/k > 1/(n+1)$ for every regular isotropic sublattice $I \subset L$.
(3) $v_I(\Psi)/k > m_I/(n+1)$ for every (semi-)irregular isotropic sublattice $I \subset L$ with index m_I.
(4) $n \geq \max_{i,I}\{r_i - 2, m_I - 1\}$.
(5) $\overline{\mathscr{F}_L(\Gamma)}$ has at worst canonical singularities.

Then the ball quotient $\mathscr{F}_L(\Gamma)$ is of general type.

Remark 2.6.4. By [**9**, Theorem 4], assumptions (4) and (5) are satisfied if $n \geq 13$ and $d < -3$.

Proof. By taking some power of Ψ, we may assume that Ψ has trivial character. Note that r_i is at most 6 by [**9**, Corollary 3]. First, let us assume that k is not divisible by $n+1$. Let $m := \max_I\{m_I\} \leq 6$ and $r := \max_i\{r_i\} \leq 6$. By taking some power of F, since $n \geq \max\{r-2, m-1\}$, we may assume that

$$\frac{k}{n+1} \geq \left[\frac{k}{n+1}\right] + \frac{m-1}{m}, \quad \frac{k}{n+1} \geq \left[\frac{k}{n+1}\right] + \frac{r-2}{r-1}.$$

Then, for every ramification divisor with ramification index r_i and every (semi-)irregular isotropic sublattice I with index m_I, we have

$$\left[\frac{m_I k}{n+1}\right] = m_I\left[\frac{k}{n+1}\right] + 1, \quad \left[\frac{(r_i - 1)k}{n+1}\right] = (r_i - 1)\left[\frac{k}{n+1}\right] + 1.$$

Hence, for $N_0 := [\frac{k}{n+1}] + 1$, we have

(1) $v_R(\Psi) \geq (r_i - 1)N_0$ for every irreducible component R_i of the ramification divisors $D \to \mathscr{F}_L(\Gamma)$ with ramification index r_i.
(2) $v_I(\Psi) \geq N_0$ for every regular isotropic sublattice $I \subset L$.
(3) $v_I(\Psi) \geq m_I N_0$ for every (semi-)irregular isotropic sublattice $I \subset L$ with index m_I.

Now we have

$$V_\ell := \Psi^\ell M_{((n+1)N_0-k)\ell}(\Gamma) \hookrightarrow M_{(n+1)N_0\ell}(\Gamma).$$

From the above discussion, any element in V_ℓ holomorphically extends the ℓN_0-canonical form over the regular locus of $\overline{\mathscr{F}_L(\Gamma)}$. On the other hand, Behrens [**9**, Theorem 4] showed the canonical singularities of $\overline{\mathscr{F}_L(\Gamma)}$. Combining this result and Proposition 2.5.3, we find that ℓN_0-canonical forms holomorphically extend over the desingularization of $\overline{\mathscr{F}_L(\Gamma)}$; that is, we can calculate the Kodaira dimension of $\mathscr{F}_L(\Gamma)$ using some desingularization of $\mathscr{F}_L(\Gamma)$. By Hirzebruch's proportionality principle, the dimension of V_ℓ grows like ℓ^{n+1} and hence $\mathscr{F}_L(\Gamma)$ is of general type.

Second, we assume that k is divisible by $n+1$. In this case, we can take N_0 in the above discussion to be $k/(n+1)$. □

Remark 2.6.5. (1) One can construct a non-zero cusp form for $n < 13$, which satisfies (1)-(4) in Theorem 2.6.3, by using a restriction of quasi-pull back of the Borcherds form for $F = \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$.

(2) It is known that unitary groups of unimodular Hermitian lattices have no reflections for $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ [**115, 146**]. Hence, if there exists a cusp form of weight less than $n+1$ which vanishes on irregular cusps with higher order, then $\mathscr{F}_L(\Gamma)$ is of general type in this situation.

2.7. A ball quotient having non-negative Kodaira dimension

To prove that ball quotients are of general type, we need to construct a cusp form of low weight which vanishes on branch divisors with appropriate order by Theorem 2.6.3. For the orthogonal modular varieties case, this was done by using Borcherds lift [**56, 84, 109**]. For the unitary case, it seems to be difficult to construct a low slope cusp form satisfying Theorem 2.6.3 (5), by using unitary Borcherds lift [**68**] because the Borcherds form exists on a 13-dimensional ball. However, the existence of a cusp form with weaker conditions imposed implies that the Kodaira dimension is non-negative by Freitag's criterion [**40**]. In this section, we shall construct a cusp form of canonical weight on a ball quotient and conclude that it has non-negative Kodaira dimension. Note that in the notation of this chapter, the canonical weight is $n+1$.

Let $L_{U\oplus U}$ be an even unimodular Hermitian lattice of signature $(1,1)$ over $\mathscr{O}_{\mathbb{Q}(\sqrt{-2})}$ defined by the matrix

$$\frac{1}{2\sqrt{-2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then its associated quadratic lattice $(L_{U\oplus U})_Q$ is $U \oplus U$.

Let $L_{E_8(-1)}$ be an even unimodular Hermitian lattice of signature $(0,4)$ over $\mathscr{O}_{\mathbb{Q}(\sqrt{-2})}$ defined by the matrix

$$-\frac{1}{2}\begin{pmatrix} 2 & 0 & \sqrt{-2}+1 & \frac{1}{2}\sqrt{-2} \\ 0 & 2 & \frac{1}{2}\sqrt{-2} & 1-\sqrt{-2} \\ 1-\sqrt{-2} & -\frac{1}{2}\sqrt{-2} & 2 & 0 \\ -\frac{1}{2}\sqrt{-2} & \sqrt{-2}+1 & 0 & 2 \end{pmatrix}.$$

Then its associated quadratic lattice $(L_{E_8(-1)})_Q$ is $E_8(-1)$.

Let $L_{\langle -2\rangle\oplus\langle -4\rangle}$ be an even unimodular Hermitian lattice of signature $(0,1)$ over $\mathscr{O}_{\mathbb{Q}(\sqrt{-2})}$ defined by the matrix

$$(-1).$$

Then its associated quadratic lattice $(L_{\langle -2\rangle\oplus\langle -4\rangle})_Q$ is $\langle -2\rangle\oplus\langle -4\rangle$. We define $L_{(\langle -2\rangle\oplus\langle -4\rangle)^\perp}$ be the orthogonal complement of $L_{\langle -2\rangle\oplus\langle -4\rangle}$ in $L_{E_8(-1)}$. Let $L := L_{U\oplus U}\oplus L_{E_8(-1)}\oplus L_{E_8(-1)}\oplus L_{(\langle -2\rangle\oplus\langle -4\rangle)^\perp}$ be a Hermitian lattice of signature $(1,12)$ over $\mathscr{O}_{\mathbb{Q}(\sqrt{-2})}$ whose associated quadratic lattice is $U\oplus U\oplus E_8(-1)\oplus E_8(-1)\oplus(\langle -2\rangle\oplus\langle -4\rangle)^\perp$.

For $II_{2,26} := U\oplus U\oplus E_8(-1)\oplus E_8(-1)\oplus E_8(-1)$, we embed $L_Q\hookrightarrow II_{2,26}$ by Nikulin's theorem. On the Hermitian symmetric domain $\mathscr{D}_{II_{2,26}}$, there exists the Borcherds form Φ_{12}, a modular form of weight 12 with respect to $\mathrm{O}^+(II_{2,26})$ with character det. This is obtained by using the Borcherds lift of the inverse of Ramanujan's tau function.

Proposition 2.7.1. *There exists a non-zero cusp form* Ψ_{13} *of weight* 13 *with respect to* $\widetilde{\mathrm{U}}(L)$ *with character* det*.*

Proof. Since the complement of L_Q in $II_{2,26}$ has exactly two (-2)-vectors, by [**60**, Theorem 8.2], the quasi-pull back f_{13} of Φ_{12} is a cusp form of weight 12+2/2=13 with respect to $\widetilde{\mathrm{O}}^+(L_Q)$ with character det. Then by restricting f_{13} to D_L, we obtain a cusp form $\Psi_{13} := \iota^\star f_{13}$ of weight 13 with respect to $\widetilde{\mathrm{U}}(L)$ with character det on a 12-dimensional ball D_L. □

Therefore, since the canonical bundle on D_L is isomorphic to $\mathscr{O}(-13)$, by Freitag's criterion [**40**], we have the following.

Proposition 2.7.2. *The ball quotient* $\mathscr{F}_L(\widetilde{\mathrm{U}}(L))$ *has non-negative Kodaira dimension.*

2.8. Examples

In this section, we give, as examples, the irregular cusps with any branch indices for any imaginary quadratic fields with class number 1.

2.8.1. Case of $\mathbb{Q}(\sqrt{-1})$.

Let $\eta := 1+\sqrt{-1}$.

Example 2.8.1. Let $a = 2b+1$ be an integer with $b\geq 0$ and L be a Hermitian lattice of signature $(1, b+1)$ defined by

$$\langle -1\rangle^{\oplus b}\oplus\begin{pmatrix} 0 & \eta^a\\ \overline{\eta}^a & 0\end{pmatrix}.$$

Then, we have

$$A_L\cong(\mathscr{O}_{\mathbb{Q}(\sqrt{-1})}/\eta^2)^{\oplus b}\oplus(\mathscr{O}_{\mathbb{Q}(\sqrt{-1})}/\eta^{a+2})^{\oplus 2}.$$

We put

$$M := \begin{pmatrix} 0 & \eta^a\\ \overline{\eta}^a & 0\end{pmatrix}.$$

We take a generator $e_1,\dots,e_b$ of $\langle -1\rangle^b$ and v, w of M. In other words, $\langle e_i, e_j\rangle = -\delta_{ij}$ and $\langle v,v\rangle = \langle w,w\rangle = 0$, $\langle v,w\rangle = \eta^a$. We define A_v to be the subgroup of A_M generated by v/η^{a+2}.

Now we take an isotropic vector

$$\ell := e_1+\cdots+e_b+v+w.$$

Let

$$\Gamma := \widetilde{\mathrm{U}}(L)^v := \{g \in \mathrm{U}(L)(\mathbb{Z}) \mid g|_{A_v} = \mathrm{id}\}.$$

Then, we have

$$\begin{cases} -\mathrm{id} \in \Gamma, \sqrt{-1}\,\mathrm{id} \notin \Gamma & (a = -1) \\ -\mathrm{id}, \sqrt{-1}\,\mathrm{id} \notin \Gamma & (a \geq 0). \end{cases}$$

Now for $\lambda := 1/2^{b+1}$, we can show

$$-\sqrt{-1}T_{\lambda\sqrt{-1}(\ell\otimes\ell)} \in \Gamma$$

by our assumption on a and b, that is,

$$-\sqrt{-1}T_{\lambda\sqrt{-1}(\ell\otimes\ell)}(\frac{v}{\eta^{a+2}}) = \frac{v}{\eta^{a+2}} \in A_v,$$

$$-\sqrt{-1}T_{\lambda\sqrt{-1}(\ell\otimes\ell)}(e_i) \in L,\ -\sqrt{-1}T_{\lambda\sqrt{-1}(\ell\otimes\ell)}(w) \in L.$$

Hence, ℓ defines an irregular sublattice of L with index 4.

Example 2.8.2. Let L be a Hermitian lattice of signature $(1,3)$ defined by

$$\langle -\frac{1}{2}\rangle^{\oplus 2} \oplus \begin{pmatrix} 0 & \frac{\overline{\eta}}{2} \\ \frac{\eta}{2} & 0 \end{pmatrix}.$$

Then we have

$$A_L \cong (\mathscr{O}_{\mathbb{Q}(\sqrt{-1})}/\eta)^{\oplus 2}.$$

We put

$$M_1 := \langle -\frac{1}{2}\rangle^{\oplus 2},\ M_2 := \begin{pmatrix} 0 & \frac{\overline{\eta}}{2} \\ \frac{\eta}{2} & 0 \end{pmatrix}.$$

We take a generator e, f of M_1 and v, w of M_2. We define A_v to be the subgroup of A_L generated by v/η.

Now we take an isotropic vector

$$\ell := e + f + v + w.$$

Let

$$\Gamma := \widetilde{\mathrm{U}}(L)^v := \{g \in \mathrm{U}(L)(\mathbb{Z}) \mid g|_{A_v} = \mathrm{id}\}.$$

We put $\lambda := -1$. Then, we have

$$-\mathrm{id} \in \Gamma,\ \sqrt{-1}\,\mathrm{id} \notin \Gamma,\ \sqrt{-1}T_{-\sqrt{-1}(\ell\otimes\ell)} \in \Gamma.$$

Hence, ℓ defines an semi-irregular sublattice of L with index 2.

2.8.2. Case of $\mathbb{Q}(\sqrt{-3})$. Let $\omega := (-1+\sqrt{-3})/2$.

Example 2.8.3. Let L be a Hermitian lattice of signature $(1,2)$ defined by

$$\langle -1\rangle \oplus \begin{pmatrix} 0 & \omega \\ \overline{\omega} & 0 \end{pmatrix}.$$

Then we have

$$A_L \cong (\mathscr{O}_{\mathbb{Q}(\sqrt{-3})}/\sqrt{-3})^{\oplus 3}.$$

We take a generator e, v, w of L with $\langle e, e\rangle = -1$, $\langle v, v\rangle = \langle w, w\rangle = 0$ and $\langle v, w\rangle = \omega$. We define A_w to be the subgroup of A_L generated by $w/\sqrt{-3}$.

Now we take an isotropic vector

$$\ell := e + v + w.$$

Let

$$\Gamma := \widetilde{\mathrm{U}}(L)^w := \{g \in \mathrm{U}(L)(\mathbb{Z}) \mid g|_{A_w} = \mathrm{id}\}.$$

Then, we have

$$\omega\,\mathrm{id} \notin \Gamma.$$

Now for $\lambda := -1/2$, we can show

$$\omega T_{\lambda\sqrt{-3}(\ell\otimes\ell)} \in \Gamma,\ -\omega T_{\lambda\sqrt{-3}(\ell\otimes\ell)} \notin \Gamma.$$

Hence, ℓ defines an irregular sublattice of L with index 3.

Example 2.8.4. Let L be a Hermitian lattice of signature $(1,4)$ defined by

$$\langle -1\rangle^{\oplus 3} \oplus \begin{pmatrix} 0 & \frac{3+\sqrt{-3}}{2} \\ \frac{3-\sqrt{-3}}{2} & 0 \end{pmatrix}.$$

We have

$$A_L \cong (\mathscr{O}_{\mathbb{Q}(\sqrt{-3})}/\sqrt{-3})^{\oplus 3} \oplus (\mathscr{O}_{\mathbb{Q}(\sqrt{-3})}/3)^{\oplus 2}.$$

We take a generator e_1, e_2, e_3, v, w of L with $\langle e_i, e_j\rangle = -\delta_{ij}$, $\langle v, v\rangle = \langle w, w\rangle = 0$ and $\langle v, w\rangle = (3+\sqrt{-3})/2$. We define A_v to be the subgroup of A_L generated by $v/3$.

Now we take an isotropic vector

$$\ell := e_1 + e_2 + e_3 + f + v + w.$$

Let

$$\Gamma := \widetilde{\mathrm{U}}(L)^v := \{g \in \mathrm{U}(L)(\mathbb{Z}) \mid g|_{A_v} = \mathrm{id}\}.$$

Then, we have

$$-\mathrm{id}, \omega\,\mathrm{id} \notin \Gamma.$$

Now for $\lambda := -1/6$, we can show

$$-\omega T_{\lambda\sqrt{-3}(\ell\otimes\ell)} \in \Gamma.$$

Hence, ℓ defines an irregular sublattice of L with index 6.

2.8.3. General case. In this subsection, let $F = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field with $d \neq -1$ and $\eta := \sqrt{d}$.

Example 2.8.5. Let L be a Hermitian lattice of signature $(1,1)$ defined by

$$\begin{pmatrix} 0 & \eta \\ \overline{\eta} & 0 \end{pmatrix}.$$

We take a generator v, w of L. We define A_v to be the subgroup of A_L generated by

$$\begin{cases} \frac{v}{2\eta^2} & (d \equiv 2, 3 \bmod 4) \\ \frac{v}{\eta^2} & (d \equiv 1 \bmod 4). \end{cases}$$

Now we take an isotropic vector

$$\ell := e + f + v + w.$$

Let

$$\Gamma := \widetilde{\mathrm{U}}(L)^v := \{g \in \mathrm{U}(L)(\mathbb{Z}) \mid g|_{A_v} = \mathrm{id}\}.$$

Then, we have

$$-\mathrm{id} \notin \Gamma$$

if $d \neq -1$.

Now for $\lambda := -1/d$, we can show

$$-T_{\lambda\sqrt{d}(\ell\otimes\ell)} \in \Gamma.$$

Hence, ℓ defines an irregular sublattice of L with index 2.

2.A. Classification of discriminant groups

Below, for simplicity, we use the following concise notation for $\mathscr{O}_F$-modules. For $\eta_1, \eta_2 \in \mathscr{O}_F$ and $a, b, c, d \in \mathbb{Z}_{\geq 0}$, we write

$$a \cdot \eta^b \oplus c \cdot \eta^d$$

to denote the $\mathscr{O}_F$-module

$$(\mathscr{O}_F/\eta^b)^{\oplus a} \oplus (\mathscr{O}_F/\eta^d)^{\oplus c}.$$

Here, we give the candidates for discriminant groups when the discriminant kernel may have irregular cusps, over any imaginary quadratic fields with class number 1. We use the notations and assumptions in Section 2.4. Below, for each quantity $\mathrm{div}(I)$, we list possible candidates for A_L.

2.A.1. Case of $\mathbb{Q}(\sqrt{-1})$. Let $\eta := 1 + \sqrt{-1}$ and a, b be non-negative integers.

2.A.1.1. **Index 2 case**. Let I be an irregular isotropic sublattice of L with index 2 with respect to $\widetilde{\mathrm{U}}(L)$. Then, by Proposition 2.4.5, we have $\mathrm{div}(I) \equiv 1,\ 1+\sqrt{-1}$ or 2 modulo $\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}$.

If $\mathrm{div}(I) \equiv 1$, the candidates are

$$a \cdot \eta \oplus b \cdot \eta^2 \oplus \eta^c, \quad a \cdot \eta \oplus b \cdot \eta^2 \oplus \eta^{d_1} \oplus \eta^{d_2}$$

where $c = 3, 4, 5, 6$, $(d_1, d_2) = (3,3), (3,4), (3,5)$.

If $\mathrm{div}(I) \equiv 1 + \sqrt{-1}$, the candidates are

$$a \cdot \eta \oplus b \cdot \eta^2 \oplus \eta^c, \quad a \cdot \eta \oplus b \cdot \eta^2 \oplus \eta^{d_1} \oplus \eta^{d_2}$$

where $c = 4, 5, 6, 7, 8$, $(d_1, d_2) = (1,7), (3,3), (3,4), (3,5), (3,6), (3,7), (4,4), (4,5), (4,6), (5,5)$.

If $\mathrm{div}(I) \equiv 2$, the candidates are

$$a \cdot \eta \oplus b \cdot \eta^2 \oplus \eta^c, \quad a \cdot \eta \oplus b \cdot \eta^2 \oplus \eta^{d_1} \oplus \eta^{d_2}$$

where $c = 6, 7, 8, 9, 10$, $(d_1, d_2) = (3,5), (3,6), (3,7), (3,8), (4,4), (4,5), (4,6), (4,7), (4,8), (5,5), (5,6), (5,7), (6,6)$.

2.A.1.2. **Semi-irregular with index 2 or index 4 case**. Let I be a semi-irregular isotropic sublattice of L with index 2 or irregular with index 4 with respect to $\widetilde{\mathrm{U}}(L)$. Then, by Proposition 2.4.5, we have $\mathrm{div}(I) \equiv 1 + \sqrt{-1}$ modulo $\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}$.

If $\mathrm{div}(I) \equiv 1 + \sqrt{-1}$, the candidates are

$$a \cdot \eta \oplus \eta^c, \quad a \cdot \eta \oplus \eta^{d_1} \oplus \eta^{d_2}$$

where $c = 5, 6, 7$, $(d_1, d_2) = (1,6), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (4,4)$.

2.A.2. Case of $\mathbb{Q}(\sqrt{-2})$. Let $\eta := \sqrt{-2}$ and a, b be non-negative integers. Let I be an irregular isotropic sublattice of L with index 2 with respect to $\widetilde{\mathrm{U}}(L)$. Then, by Proposition 2.4.7 (2), we have $\mathrm{div}(I) \equiv 1/\sqrt{-2}$, 1, $\sqrt{-2}$ or 2 modulo $\mathscr{O}^\times_{\mathbb{Q}(\sqrt{-2})}$.

If $\mathrm{div}(I) \equiv 1/\sqrt{-2}$, the candidates are

$$a \cdot \eta \oplus b \cdot \eta^2 \oplus \eta^c, \quad a \cdot \eta \oplus b \cdot \eta^2 \oplus \eta^{d_1} \oplus \eta^{d_2}$$

where $c = 3, 4, 5, 6$, $(d_1, d_2) = (3, 3), (3, 4), (3, 5)$.

If $\mathrm{div}(I) \equiv 1$, the candidates are

$$a \cdot \eta \oplus b \cdot \eta^2 \oplus \eta^c, \quad a \cdot \eta \oplus b \cdot \eta^2 \oplus \eta^{d_1} \oplus \eta^{d_2}$$

where $c = 4, 5, 6, 7, 8$, $(d_1, d_2) = (2, 7), (3, 3), (3, 4), (3, 5), (3, 6), (3, 7), (4, 4), (4, 5), (4, 6), (5, 5)$.

If $\mathrm{div}(I) \equiv \sqrt{-2}$, the candidates are

$$a \cdot \eta \oplus b \cdot \eta^2 \oplus \eta^c, \quad a \cdot \eta \oplus b \cdot \eta^2 \oplus \eta^{d_1} \oplus \eta^{d_2}$$

where $c = 6, 7, 8, 9, 10$, $(d_1, d_2) = (3, 5), (3, 6), (3, 7), (3, 8), (4, 4), (4, 5), (4, 6), (4, 7), (4, 8), (5, 5), (5, 6), (5, 7), (6, 6)$.

If $\mathrm{div}(I) \equiv 2$, the candidates are

$$a \cdot \eta \oplus b \cdot \eta^2 \oplus \eta^c, \quad a \cdot \eta \oplus b \cdot \eta^2 \oplus \eta^{d_1} \oplus \eta^{d_2}$$

where $c = 6, 7, 8, 9, 10, 11, 12$, $(d_1, d_2) = (1, 11), (3, 7), (3, 8), (3, 9), (3, 10), (3, 11), (4, 6), (4, 7), (4, 8), (4, 9), (4, 10), (5, 5), (5, 6), (5, 7), (5, 8), (5, 9), (6, 6), (6, 7), (6, 8), (7, 7)$.

2.A.3. Case of $\mathbb{Q}(\sqrt{-3})$. Let $\eta := \sqrt{-3}$, $\delta := 2$ and a, b be non-negative integers.

2.A.3.1. **Index 2 case**. Let I be an irregular isotropic sublattice of L with index 2 with respect to $\widetilde{\mathrm{U}}(L)$. Then, by Proposition 2.4.7 (1), we have $\mathrm{div}(I) \equiv 1/\sqrt{-3}$, 1, $2/\sqrt{-3}$ or 2 modulo $\mathscr{O}^\times_{\mathbb{Q}(\sqrt{-3})}$.

If $\mathrm{div}(I) \equiv 1/\sqrt{-3}$, then A_L is isomorphic to $a \cdot \delta$ as $\mathscr{O}_{\mathbb{Q}(\sqrt{-3})}$-modules.

If $\mathrm{div}(I) \equiv 1$, the candidates are

$$a \cdot \delta \oplus \eta^2, \quad a \cdot \delta \oplus 2 \cdot \eta.$$

If $\mathrm{div}(I) \equiv 2/\sqrt{-3}$, the candidates are

$$a \cdot \delta \oplus \delta^c, \quad a \cdot \delta \oplus 2 \cdot \delta^2$$

where $c = 0, 2, 3$.

If $\mathrm{div}(I) \equiv 2$, the candidates are

$$a \cdot \delta \oplus \eta^2, \quad a \cdot \delta \oplus 2 \cdot \eta^2, \quad a \cdot \delta \oplus \delta^c \oplus \eta^2, \quad a \cdot \delta \oplus \delta^c \oplus 2 \cdot \eta, \quad a \cdot \delta \oplus 2 \cdot \delta^2 \oplus \eta^2, \quad a \cdot \delta \oplus 2 \cdot \delta^2 \oplus 2 \cdot \eta$$

where $c = 2, 3$.

2.A.3.2. **Index 3 case**. Let I be an irregular isotropic sublattice of L with index 3 with respect to $\widetilde{\mathrm{U}}(L)$. Then, by Proposition 2.4.6 (1), we have $\mathrm{div}(I) \equiv 1/\sqrt{-3}$, 1, $\sqrt{-3}$ modulo $\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-3})}$.

If $\mathrm{div}(I) \equiv 1/\sqrt{-3}$, then A_L is isomorphic to $a \cdot \delta$ as $\mathscr{O}_{\mathbb{Q}(\sqrt{-3})}$-modules.

If $\mathrm{div}(I) \equiv 1$, the candidates are

$$a \cdot \eta \oplus b \cdot \eta^2 \oplus \eta^c, \quad a \cdot \eta \oplus b \cdot \eta^2 \oplus 2 \cdot \eta^3$$

where $c = 0, 3, 4$.

If $\mathrm{div}(I) \equiv \sqrt{-3}$, the candidates are

$$a \cdot \eta \oplus b \cdot \eta^2 \oplus \eta^c, \quad a \cdot \eta \oplus b \cdot \eta^2 \oplus 2 \cdot \eta^3, \quad a \cdot \eta \oplus b \cdot \eta^2 \oplus \eta^3 \oplus \eta^d$$

where $c = 0, 3, 4, 5, 6$, $d = 4, 5$.

2.A.3.3. **Index 6 case**. Let I be an irregular isotropic sublattice of L with index 6 with respect to $\widetilde{\mathrm{U}}(L)$. Then, by Proposition 2.4.6 (2), we have $\mathrm{div}(I) \equiv 1/\sqrt{-3}$, 1 modulo $\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-3})}$.

If $\mathrm{div}(I) \equiv 1/\sqrt{-3}$, then A_L is trivial, that is, L is unimodular lattice.

If $\mathrm{div}(I) \equiv \sqrt{-3}$, the candidates are

$$\eta^2, \quad 2 \cdot \eta.$$

2.A.4. Case of $\mathbb{Q}(\sqrt{-7})$. Let $\eta_1 := (1 + \sqrt{-7})/2$, $\eta_2 := (-1 + \sqrt{-7})/2$, $\delta := \sqrt{-7}$ and a, b be non-negative integers. Let I be an irregular isotropic sublattice of L with index 2 with respect to $\widetilde{\mathrm{U}}(L)$. Then, by Proposition 2.4.7 (1), we have $\mathrm{div}(I) \equiv 1/\sqrt{-7}$, 1, $\eta_1/\sqrt{-7}$, $\eta_2/\sqrt{-7}$, $\eta_1\eta_2/\sqrt{-7}$, η_1, η_2 or $\eta_1\eta_2$ modulo $\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-7})}$.

If $\mathrm{div}(I) \equiv 1/\sqrt{-7}$, then A_L is isomorphic to $a \cdot \eta_1 \oplus b \cdot \eta_2$ as $\mathscr{O}_{\mathbb{Q}(\sqrt{-7})}$-modules.

If $\mathrm{div}(I) \equiv 1$, the candidates are

$$a \cdot \eta_1 \oplus b \cdot \eta_2 \oplus \delta^2, \quad a \cdot \eta_1 \oplus b \cdot \eta_2 \oplus 2 \cdot \delta.$$

If $\mathrm{div}(I) \equiv \eta_1/\sqrt{-7}$, the candidates are

$$(a-2) \cdot \eta_1 \oplus a \cdot \eta_2 \oplus 2 \cdot \eta_1^2, \quad (a-1) \cdot \eta_1 \oplus a \cdot \eta_2 \oplus \eta_1^3, \quad a \cdot \eta_1 \oplus a \cdot \eta_2 \oplus \eta_1^2, \quad (a+2) \cdot \eta_1 \oplus a \cdot \eta_2.$$

If $\mathrm{div}(I) \equiv \eta_2/\sqrt{-7}$, the candidates are

$$(a-2) \cdot \eta_2 \oplus a \cdot \eta_1 \oplus 2 \cdot \eta_2^2, \quad (a-1) \cdot \eta_2 \oplus a \cdot \eta_1 \oplus \eta_2^3, \quad a \cdot \eta_2 \oplus a \cdot \eta_1 \oplus \eta_2^2, \quad (a+2) \cdot \eta_2 \oplus a \cdot \eta_1.$$

If $\mathrm{div}(I) \equiv \eta_1$, the candidates are

$$(a-2) \cdot \eta_1 \oplus a \cdot \eta_2 \oplus 2 \cdot \eta_1^2 \oplus \delta^2, \quad (a-2) \cdot \eta_1 \oplus a \cdot \eta_2 \oplus 2 \cdot \eta_1^2 \oplus 2 \cdot \delta, \quad (a-1) \cdot \eta_1 \oplus a \cdot \eta_2 \oplus \eta_1^3 \oplus \delta^2,$$

$$(a-1)\cdot\eta_1\oplus a\cdot\eta_2\oplus\eta_1^3\oplus 2\cdot\delta,\quad a\cdot\eta_1\oplus a\cdot\eta_2\oplus\eta_1^2\oplus 2\cdot\delta,\quad a\cdot\eta_1\oplus a\cdot\eta_2\oplus\eta_1^2\oplus\delta^2,$$
$$(a+2)\cdot\eta_1\oplus a\cdot\eta_2\oplus\delta^2,\quad (a+2)\cdot\eta_1\oplus a\cdot\eta_2\oplus 2\cdot\delta.$$

If $\mathrm{div}(I)\equiv\eta_2$, the candidates are

$$(a-2)\cdot\eta_2\oplus a\cdot\eta_1\oplus 2\cdot\eta_2^2\oplus\delta^2,\quad (a-2)\cdot\eta_2\oplus a\cdot\eta_1\oplus 2\cdot\eta_2^2\oplus 2\cdot\delta,\quad (a-1)\cdot\eta_2\oplus a\cdot\eta_1\oplus\eta_2^3\oplus\delta^2,$$
$$(a-1)\cdot\eta_2\oplus a\cdot\eta_1\oplus\eta_2^3\oplus 2\cdot\delta,\quad a\cdot\eta_2\oplus a\cdot\eta_1\oplus\eta_2^2\oplus 2\cdot\delta,\quad a\cdot\eta_2\oplus a\cdot\eta_1\oplus\eta_2^2\oplus\delta^2,$$
$$(a+2)\cdot\eta_2\oplus a\cdot\eta_1\oplus\delta^2,\quad (a+2)\cdot\eta_2\oplus a\cdot\eta_1\oplus 2\cdot\delta.$$

If $\mathrm{div}(I)\equiv\eta_1\eta_2/\sqrt{-7}$, the candidates are

$$(a-2)\cdot\eta_1\oplus(a-2)\cdot\eta_2\oplus 2\cdot\eta_1^2\oplus 2\cdot\eta_2^2,\quad (a-2)\cdot\eta_1\oplus(a-1)\cdot\eta_2\oplus 2\cdot\eta_1^2\oplus\eta_2^3,\quad (a-2)\cdot\eta_1\oplus a\cdot\eta_2\oplus\eta_1^2\oplus 2\cdot\eta_2^2,$$
$$(a-2)\cdot\eta_1\oplus a\cdot\eta_2\oplus 2\cdot\eta_1^2\oplus\eta_2^2,\quad (a-2)\cdot\eta_1\oplus(a+1)\oplus\eta_1^2\oplus\eta_2^3,\quad (a-2)\cdot\eta_1\oplus(a+2)\oplus\eta_1^2\oplus\eta_2^2,$$
$$(a-1)\cdot\eta_1\oplus(a-2)\cdot\eta_2\oplus\eta_1^3\oplus 2\cdot\eta_2^2,\quad (a-1)\cdot\eta_1\oplus(a-1)\cdot\eta_2\oplus\eta_1^3\oplus\eta_2^3,\quad (a-1)\cdot\eta_1\oplus a\cdot\eta_2\oplus\eta_1^3\oplus\eta_2^2,$$
$$a\cdot\eta_1\oplus(a-2)\cdot\eta_2\oplus\eta_1^2\oplus 2\cdot\eta_2^2,\quad a\cdot\eta_1\oplus(a-2)\cdot\eta_2\oplus 2\cdot\eta_1^2\oplus\eta_2^2,\quad a\cdot\eta_1\oplus(a-1)\cdot\eta_2\oplus\eta_1^2\oplus\eta_2^3,$$
$$a\cdot\eta_1\oplus a\cdot\eta_2\oplus 2\cdot\eta_1^2,\quad a\cdot\eta_1\oplus a\cdot\eta_2\oplus 2\cdot\eta_2^2,\quad a\cdot\eta_1\oplus a\cdot\eta_2\oplus\eta_1^2\oplus\eta_2^2,$$
$$a\cdot\eta_1\oplus(a+1)\cdot\eta_2\oplus\eta_2^3,\quad a\cdot\eta_1\oplus(a+2)\cdot\eta_2\oplus\eta_1^2,\quad a\cdot\eta_1\oplus(a+2)\cdot\eta_2\oplus\eta_2^2,$$
$$(a+1)\cdot\eta_1\oplus(a-2)\cdot\eta_2\oplus\eta_1^3\oplus\eta_2^2,\quad (a+1)\cdot\eta_1\oplus a\cdot\eta_2\oplus\eta_1^3,\quad (a+2)\cdot\eta_1\oplus(a-2)\cdot\eta_2\oplus\eta_1^2\oplus\eta_2^2,$$
$$(a+2)\cdot\eta_1\oplus a\cdot\eta_2\oplus\eta_1^2,\quad (a+2)\cdot\eta_1\oplus a\cdot\eta_2\oplus\eta_2^2,\quad (a+2)\cdot\eta_1\oplus(a+2)\cdot\eta_2.$$

If $\mathrm{div}(I)\equiv\eta_1\eta_2$, the candidates are

$$(a-2)\cdot\eta_1\oplus(a-2)\cdot\eta_2\oplus 2\cdot\eta_1^2\oplus 2\cdot\eta_2^2\oplus\delta^2,\quad (a-2)\cdot\eta_1\oplus(a-2)\cdot\eta_2\oplus 2\cdot\eta_1^2\oplus 2\cdot\eta_2^2\oplus 2\cdot\delta,$$
$$(a-2)\cdot\eta_1\oplus(a-1)\cdot\eta_2\oplus 2\cdot\eta_1^2\oplus\eta_2^3\oplus\delta^2,\quad (a-2)\cdot\eta_1\oplus(a-1)\cdot\eta_2\oplus 2\cdot\eta_1^2\oplus\eta_2^3\oplus 2\cdot\delta,$$
$$(a-2)\cdot\eta_1\oplus a\cdot\eta_2\oplus 2\cdot\eta_1^2\oplus\eta_2^2\oplus\delta^2,\quad (a-2)\cdot\eta_1\oplus a\cdot\eta_2\oplus 2\cdot\eta_1^2\oplus\eta_2^2\oplus 2\cdot\delta,$$
$$(a-2)\cdot\eta_1\oplus(a+2)\cdot\eta_2\oplus 2\cdot\eta_1^2\oplus\delta^2,\quad (a-2)\cdot\eta_1\oplus(a+2)\cdot\eta_2\oplus 2\cdot\eta_1^2\oplus 2\cdot\delta,$$
$$(a-1)\cdot\eta_1\oplus(a-2)\cdot\eta_2\oplus\eta_1^3\oplus 2\cdot\eta_2^2\oplus\delta^2,\quad (a-1)\cdot\eta_1\oplus(a-2)\cdot\eta_2\oplus\eta_1^3\oplus 2\cdot\eta_2^2\oplus 2\cdot\delta,$$
$$(a-1)\cdot\eta_1\oplus(a-1)\cdot\eta_2\oplus\eta_1^3\oplus\eta_2^3\oplus\delta^2,\quad (a-1)\cdot\eta_1\oplus(a-1)\cdot\eta_2\oplus\eta_1^3\oplus\eta_2^3\oplus 2\cdot\delta,$$
$$(a-1)\cdot\eta_1\oplus a\cdot\eta_2\oplus\eta_1^3\oplus\eta_2^2\oplus\delta^2,\quad (a-1)\cdot\eta_1\oplus a\cdot\eta_2\oplus\eta_1^3\oplus\eta_2^2\oplus 2\cdot\delta,$$
$$(a-1)\cdot\eta_1\oplus(a+2)\cdot\eta_2\oplus\eta_1^3\oplus\delta^2,\quad (a-1)\cdot\eta_1\oplus(a+2)\cdot\eta_2\oplus\eta_1^3\oplus 2\cdot\delta,$$
$$a\cdot\eta_1\oplus(a-2)\cdot\eta_2\oplus\eta_1^2\oplus 2\cdot\eta_2^2\oplus\delta^2,\quad a\cdot\eta_1\oplus(a-2)\cdot\eta_2\oplus\eta_1^2\oplus 2\cdot\eta_2^2\oplus 2\cdot\delta,$$
$$a\cdot\eta_1\oplus(a-1)\cdot\eta_2\oplus\eta_1^2\oplus\eta_2^3\oplus\delta^2,\quad a\cdot\eta_1\oplus(a-1)\cdot\eta_2\oplus\eta_1^2\oplus\eta_2^3\oplus 2\cdot\delta,$$
$$a\cdot\eta_1\oplus a\cdot\eta_2\oplus\eta_1^2\oplus\eta_2^2\oplus\delta^2,\quad a\cdot\eta_1\oplus a\cdot\eta_2\oplus\eta_1^2\oplus\eta_2^2\oplus 2\cdot\delta,$$
$$a\cdot\eta_1\oplus(a+2)\cdot\eta_2\oplus\eta_1^2\oplus\delta^2,\quad a\cdot\eta_1\oplus(a+2)\cdot\eta_2\oplus\eta_1^2\oplus 2\cdot\delta,$$
$$(a+2)\cdot\eta_1\oplus(a-2)\cdot\eta_2\oplus 2\cdot\eta_2^2\oplus\delta^2,\quad (a+2)\cdot\eta_1\oplus(a-2)\cdot\eta_2\oplus 2\cdot\eta_2^2\oplus 2\cdot\delta,$$
$$(a+2)\cdot\eta_1\oplus(a-1)\cdot\eta_2\oplus\eta_2^3\oplus\delta^2,\quad (a+2)\cdot\eta_1\oplus(a-1)\cdot\eta_2\oplus\eta_2^3\oplus 2\cdot\delta,$$
$$(a+2)\cdot\eta_1\oplus a\cdot\eta_2\oplus\eta_2^2\oplus\delta^2,\quad (a+2)\cdot\eta_1\oplus a\cdot\eta_2\oplus\eta_2^2\oplus 2\cdot\delta,$$
$$(a+2)\cdot\eta_1\oplus(a+2)\cdot\eta_2\oplus\delta^2,\quad (a+2)\cdot\eta_1\oplus(a+2)\cdot\eta_2\oplus 2\cdot\delta.$$

2.A.5. Other cases. Let $F = \mathbb{Q}(\sqrt{d})$, where $d = -11$, -19, -43, -67 or -163, $\eta := \sqrt{d}$, $\delta := 2$ and a, b be non-negative integers. Let I be an irregular isotropic sublattice of L with index 2 with respect to $\widetilde{\mathrm{U}}(L)$. Then, by Proposition 2.4.7 (1), we have $\mathrm{div}(I) \equiv 1/\sqrt{d}$, $2/\sqrt{d}$, 1 or 2 modulo $\mathscr{O}_F^\times$.

If $\mathrm{div}(I) \equiv 1/\sqrt{d}$, then A_L is isomorphic to $a \cdot \delta \oplus \eta$ as $\mathscr{O}_F$-modules.

If $\mathrm{div}(I) \equiv 2/\sqrt{d}$, the candidates are

$$a \cdot \delta \oplus \delta^c, \quad a \cdot \delta \oplus 2 \cdot \delta^2, \quad a \cdot \delta \oplus \eta^2, \quad a \cdot \delta \oplus 2 \cdot \eta$$

where $c = 0, 2, 3$.

If $\mathrm{div}(I) \equiv 1$, the candidates are

$$a \cdot \delta \oplus \eta^2, \quad a \cdot \delta \oplus 2 \cdot \eta.$$

If $\mathrm{div}(I) \equiv 2$, the candidates are

$$a \cdot \delta \oplus \delta^c \oplus \eta^2, \quad a \cdot \delta \oplus \delta^c \oplus 2 \cdot \eta, \quad a \cdot \delta \oplus 2 \cdot \delta^2 \oplus \eta^2, \quad a \cdot \delta \oplus 2 \cdot \delta^2 \oplus 2 \cdot \eta$$

where $c = 2, 3$.

CHAPTER 3

Reflective obstructions of ball quotients

3.1. Introduction

The study of the birational type of modular varieties is an important problem. Tai [**137**], Freitag [**40**] and Mumford [**121**] showed that the Siegel modular varieties $\mathcal{A}_g$ are of general type if $g \geq 7$. Gritsenko-Hulek-Sankaran [**58**] showed that the moduli spaces of polarized K3 surfaces, which are 19-dimensional orthogonal modular varieties, are of general type if the polarization degree is sufficiently large. Moreover, Ma [**107**] proved that orthogonal modular varieties are of general type if their dimension is sufficiently large. A common theme in this series of works implies that if the data defining modular varieties is "sufficiently large", then the associated modular varieties are of general type.

Motivated by these work, we study an analogous problem for unitary modular varieties. There exist three types of obstructions to prove that they are of general type, as in the orthogonal case [**58**, Theorem 1.1]. They are *reflective obstructions*, arising from branch divisors, *cusp obstructions*, arising from divisors at infinity, and *elliptic obstructions*, arising from singularities. Note that the elliptic obstructions were resolved in [**9**]. In this chapter, we study the reflective obstructions and prove that they are sufficiently small in higher dimension. The key to the proof is to apply Prasad's volume formula to estimate the dimension of the space of modular forms on ball quotients.

3.1.1. Main results. The main theorem in this chapter is as follows.

Theorem 3.1.1 (Theorem 3.8.1). *Let L be a primitive Hermitian lattice over $\mathscr{O}_F$ of signature $(1,n)$ with $n > 2$. Assume $(\heartsuit)$. Then, for a positive integer a, the line bundle $\mathcal{M}(a)$ is big if $\dim X_L = n$ or S is sufficiently large.*

It follows that the reflective obstructions can be resolved for $\mathscr{F}_L(\Gamma)$ with sufficiently large n or S. Next, we work on specific lattices. We call L is *unramified square-free* if $\det(L)$ is odd square-free and any prime divisor p of $\det(L)$ is unramified at F. We will prove that they satisfy $(\heartsuit)$ and more precise estimate holds (Lemma 3.5.2, Proposition 3.5.4, 3.5.5, Corollary 3.8.4, Subsection 3.8.4 and 3.8.5). Throughout this chapter, we denote by $F_0 \neq \mathbb{Q}(\sqrt{-3})$ an imaginary quadratic field, whose discriminant $-D$ is not a multiple of 4.

Corollary 3.1.2 (Corollary 3.8.2, Subsection 3.8.5). *(1) Up to scaling, assume that L is unramified square-free over $\mathscr{O}_{F_0}$. If $n > 138$, or $D > 30$ and n is even, then $\mathcal{M}(1)$ is big and hence, $\mathcal{M}(a)$ is big for any $a > 1$.*

(2) Moreover, for a fixed F_0, there are only finitely many unramified square-free lattices so that $\mathcal{M}(a)$ is not big with $n > 2$, up to scaling.

Remark 3.1.3 (Subsection 3.8.4). Note that a lower bound for n and S in Theorem 3.1.1 can be easily computed. This is essentially done by estimating certain functions f_F^{odd} and f_F^{even} below. For example, we will show that $\mathcal{M}(1)$ is big if $n > 582$ and then, $\mathcal{M}(a)$ is big for any $a > 1$. In the notation below, this is equivalent to $W(L,F,a) \leq W(L,F,1) < 0$ for $n > 582$.

3.1.2. Application I: Kodaira dimension. In this subsection, we assume $n \geq 13$ and $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3})$. These assumptions come from [**9**, Theorem 4], which asserts that $\mathscr{F}_L(\Gamma)$ has at worst canonical singularities and branch divisors of the map $D_L \to \mathscr{F}_L(\Gamma)$ do not exist at the boundary. Note that $\overline{X_L}$ contains no irregular cusps [**112**]. Under (B), we state an application to the birational type of X_L.

Theorem 3.1.4 (Corollary 3.8.2, Theorem 3.8.3). *Assume that* ($\heartsuit$) *holds and there exists a non-zero cusp form of weight lower than* $n+1$ *with respect to* $\mathrm{U}(L)$*. Then,* X_L *is of general type if* $\dim X_L = n$ *or* S *is sufficiently large.*

Remark 3.1.5. One way to construct a cusp form for $\mathrm{U}(1,n)$ is the theta lifting [**91**]. However, this produces only cusp forms of weight greater than n.

3.1.3. Application II: Reflectifve modular forms. Next, let us consider reflective modular forms. Let f be a modular form of some weight and character with respect to Γ on D_L. We say that f is *reflective* if the divisor of L is set-theoretically contained in the ramification divisors of $D_L \to \mathscr{F}_L(\Gamma)$. Reflective modular forms appear in many fields of mathematics; see [**52, 54, 61**]. Gritsenko-Nikulin [**61**, Conjecture 2.5.5] conjectured finiteness of quadratic lattices admitting a non-zero reflective modular form, and Ma [**107**, Corollary 1.9] proved it. Here, we consider an analogous problem for Hermitian lattices. We say that L is *reflective with slope* r for $r > 0$ if there exists a reflective modular form on D_L with its slope r; for the definition of the *slope* of a modular form, see [**107**, Subsection 1.3].

Conjecture 3.1.6 (Finiteness of Hermitian lattices admitting reflective modular forms). *For an* $r > 0$ *and a fixed* F*,*

$$\{\text{Hermitian reflective lattices with slope less than } r\}/\sim$$

is a finite set.

We can partially prove Conjecture 3.1.6 from a computation of the Hirzebruch-Mumford volumes.

Corollary 3.1.7 (Corollary 3.8.4). *For an* $r > 0$ *and a fixed* F_0*,*

$$\{\text{Unramified square-free reflective lattices with slope less than } r \mid n > 2\}/\sim$$

is a finite set.

3.1.4. Technical tools. To prove that $\mathcal{M}(a)$ is big, we will use the function $V(L,F)$ (see Definition 3.4.3), which represents the asymptotic growth of the dimension of the space of modular forms vanishing on the ramification divisors. This function depends only on L and F. To compute $V(L,F)$, we use Prasad's formula [**129**]. This approach is different from the one by Gritsenko-Hulek-Sankaran [**57**] and Ma [**107**] which use the calculation of local densities. We define

$$W(L,F,a) := V(L,F) - \frac{2a}{n} \cdot \begin{cases} \left(1+\dfrac{1}{a}\right)^{1-n} & (F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})), \\ 2\left(1+\dfrac{3}{a}\right)^{1-n} & (F = \mathbb{Q}(\sqrt{-1})), \\ 3\left(1+\dfrac{5}{a}\right)^{1-n} & (F \neq \mathbb{Q}(\sqrt{-3})), \end{cases}$$

for a positive integer $a > 0$. For the proof of Theorem 3.1.1, we use the following criterion. This is a unitary analog of [**107**, Proposition 4.3].

Proposition 3.1.8 (Proposition 3.4.4). *The line bundle $\mathcal{M}(a)$ is big if*

$$W(L, F, a) < 0.$$

This criterion reduces the proof of Theorem 3.1.1 to estimating $V(L, F)$ which occupies the bulk of this chapter. We define functions on m by

$$f_F^{odd}(m) := \frac{3 \cdot 2^5 \cdot (2\pi)^{2m+1}}{(2m)! \cdot L(2m+1)} \cdot \begin{cases} (1 + 2^{4m+1} + 2^{8m+2}) & (F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})), \\ 2(3 + 3 \cdot 2^{4m+1} + 2^{8m+2}) & (F = \mathbb{Q}(\sqrt{-1})), \\ 3(5 + 2 \cdot 3^{4m+1} + 2^{8m+2}) & (F = \mathbb{Q}(\sqrt{-3})), \end{cases}$$

$$f_F^{even}(m) := \frac{2^{2m+5/2} \cdot 3 \cdot (2\pi)^{2m}}{S \cdot (2m-1)! \cdot \zeta(2m)} \cdot \begin{cases} (1 + 2^{4m-1} + 2^{8m-2}) & (F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})), \\ (3 + 3 \cdot 2^{4m-1} + 2^{8m-2}) & (F = \mathbb{Q}(\sqrt{-1})), \\ (5 + 2 \cdot 3^{4m-1} + 2^{8m-2}) & (F = \mathbb{Q}(\sqrt{-3})). \end{cases}$$

Theorem 3.1.9 (Theorem 3.7.5, Theorem 3.7.6). *Let L be a primitive Hermitian lattice over $\mathscr{O}_F$ of signature $(1, 2m)$ (resp. $(1, 2m-1)$) with $m > 1$. Assume $(\heartsuit)$. Then, we obtain the following:*

$$V(L, F) \leq \frac{f_F^{odd}(m)}{S} \quad \left(\text{resp. } V(L, F) \leq \frac{f_F^{even}(m)}{S}\right).$$

Moreover, if L satisfies $P(M)$ (see Section 3.7) for some $M > 0$, we have

$$V(L, F) \leq \frac{f_F^{odd}(m)}{D(L)^{1/M} \cdot S} \quad \left(\text{resp. } V(L, F) \leq \frac{f_F^{even}(m)}{D(L)^{1/M} \cdot S}\right),$$

where $D(L)$ be the exponent of the discriminant group $L^\vee/L$.

Note that growth of $f_F^{odd}(m)$ and $f_F^{even}(m)$ with respect to m is $O(1/m!)$. This implies that for a fixed a, the inequality $W(L, F, a) < 0$ always holds for every pair (L, F) if n is sufficiently large, hence $\mathcal{M}(a)$ is big in that range of n.

Remark 3.1.10. We will discuss how large values of m, in Theorem 3.1.1, Corollary 3.1.2 and Theorem 3.1.9, we need to take to satisfy $W(L, F, a) < 0$ in Subsection 3.8.4 and 3.8.5.

Finally, we shall define the notion "principal" and prepare to discuss $(\heartsuit)$. Below, we use the special unitary group $G^1 := \mathrm{SU}(L)$, group scheme over $\mathbb{Z}$. To estimate $V(L, F)$, we need to compute the Hirzebruch-Mumford volume of $G^1(\mathbb{Z})$. Since G^1 is semi-simple and simply connected, we can use Prasad's formula [**129**, Theorem 3.7]. Prasad's theorem requires an arithmetic subgroup to be principal for some coherent parahoric family, so we consider when our arithmetic subgroups satisfy this condition. Below, v denotes a finite place. Let P_v be a parahoric subgroup of $G^1(\mathbb{Q}_v)$. We call $\{P_v\}_v$ *a coherent parahoric family* if $G^1(\mathbb{R}) \prod_v P_v \subset G^1(\mathbb{A})$ is an open compact subgroup. We call $G^1(\mathbb{Z})$ *principal* for a coherent parahoric family $\{P_v\}_v$ if $G^1(\mathbb{Z}) = G^1(\mathbb{Q}) \cap \prod_v P_v$ and the closure of the image of $G^1(\mathbb{Z})$ by the canonical embedding $\iota_v : G^1(\mathbb{Q}) \hookrightarrow G^1(\mathbb{Q}_v)$ is P_v. From the strong approximation theorem and the proof of [**132**, Proposition 1.6], the closure of $\iota(G^1(\mathbb{Z}))$ is $G^1(\mathbb{Z}_v)$. Moreover, it follows $G^1(\mathbb{Z}) = G^1(\mathbb{Q}) \cap \prod_v G^1(\mathbb{Z}_v)$. Hence, combining these observations, it follows that $G^1(\mathbb{Z})$ is principal with respect to $\{G^1(\mathbb{Z}_v)\}_v$ if $G^1(\mathbb{Z}_v)$ is parahoric for any v. Accordingly, we will compute the volume function $V(L,F)$ under ($\star$) on L.

$$(\star)\ \ \mathrm{SU}(L\otimes\mathbb{Z}_v) \text{ is a parahoric subgroup of } \mathrm{SU}(L\otimes\mathbb{Q}_v) \text{ for any } v\nmid\infty.$$

Condition ($\star$) on L implies that $\mathrm{SU}(L)$ is principal. Hence, ($\heartsuit$) can be rephrased as follows;

$$\ell\mathscr{O}_F\oplus(\ell^{\perp}\cap L) \text{ and } \ell^{\perp}\cap L \text{ satisfy } (\star) \text{ for any } [\ell]\in\mathcal{R}_L(F).$$

Remark 3.1.11.
(1) Hermitian lattice satisfying ($\star$) and Theorem 3.1.4 (2) exist; see Proposition 3.5.4 and 3.5.5.
(2) Condition ($\star$) holds for the special linear group [**138**, Example 3.2.4], i.e., $\mathrm{SL}_n(\mathbb{Z}_v)$ is parahoric for any v.
(3) From [**17**, Proposition 1.4 (iv)], if $G^1(\mathbb{Z})$ is maximal, then $G^1(\mathbb{Z}_v)$ is parahoric for any v. Note that the maximal arithmetic subgroups are classified in [**132**, Theorem 2.6].
(4) Hijikata [**66**, Introduction] stated that the maximal compact open subgroups of an algebraic group over p-adic fields can be obtained from the stabilizer of a maximal lattice. Bruhat [**20**, Section 5] proved it for unitary groups. On the other hand, Gan-Hanke-Yu [**49**, Introduction] stated that the stabilizer of any maximal Hermitian lattice in a unitary group over p-adic fields is a maximal parahoric subgroup except when the field extension is split.

Remark 3.1.12. We refer to the relationship between modular varieties of non-general type and reflective modular forms, and moduli representations of ball quotients.

(1) Gritsenko [**52, 54**] constructed reflective modular forms and showed that some orthogonal modular varieties have negative Kodaira dimension. The author and Odaka [**115**] formulated the notion "special reflective modular forms" and proved that some orthogonal or unitary modular varieties are Fano (e.g., the moduli space of Enriques surfaces). In these works, reflective modular forms played an important role. In this chapter, we deal with these modular forms in Subsection 3.8.3, and show a certain finiteness result (Corollary 3.8.4).
(2) Deligne-Mostow [**32**] realized some ball quotients as the periods of hypergeometric forms, and consequently, proved that they are related to moduli spaces of some weighted points in the projective line. On the other hand, Allcock-Carlson-Toledo [**4, 3**] showed that some ball quotients are moduli spaces of cubic surfaces or threefolds. In this context, Dolgachev-Kondō [**37**, Section 1] conjectured that all ball quotients arising from the Deligne-Mostow theory are related to the moduli spaces of K3 surfaces.

3.1.5. Outline of the proof of Theorem 3.1.1. First, we prove a criterion (Proposition 3.1.8) asserting when the line bundle $\mathcal{M}(a)$ is big. Since the branch divisors with higher branch indices may occur in our setting unlike orthogonal modular varieties, it needs to classify them in more detail than [**107**]. Based on the classification, Proposition 3.1.8 follows from the Hirzebruch-Mumford proportionality principle. Second, by using Prasad's formula [**129**, Theorem 3.7], we compute the Hirzebruch-Mumford volume of principal arithmetic subgroups. The application of Prasad's volume formula to the birational geometry seems to be new and is one of the differences from the previous studies on the geometry of modular varieties. Our work is based on the classification of the maximal reductive quotient of the reduction of the smooth integral models [**29, 30, 49**]. Combining this computation (Theorem 3.1.9) with the above criterion (Proposition 3.1.8), it follows that $\mathcal{M}(a)$ is big if n is sufficiently large. This implies Theorem 3.1.1. To obtain more explicit estimate, we will evaluate $f_F^{odd}(m)$ and $f_F^{even}(m)$ in Subsection 3.8.4 and 3.8.5.

3.1.6. Organization of this chapter. In Section 3.2, we describe the asymptotic behavior of the dimension of modular forms in terms of the Hirzebruch-Mumford volume. In Section 3.3, we clarify the description of ramification divisors in terms of Hermitian lattices. In Section 3.4, we show a criterion when the line bundle $\mathcal{M}(a)$ is big, by using the Hirzebruch-Mumford volume. In Section 3.5, we recall Prasad's formula. In Section 3.6, we compute the local factors appearing in the Hirzebruch-Mumford volume. In Section 3.7, we prove $V(L,F) \leq S^{-1}f_F^{odd}(m)$ or $S^{-1}f^{even}(m)$. This calculation shows that $\mathcal{M}(a)$ is big for sufficiently large n. In Section 3.8, we state the main results and estimate the value of the function $V(L,F)$ explicitly.

3.2. Dimension formula

In this section, we study the dimension formula of the space of modular forms. Gritsenko-Hulek-Sankaran [**57**] derived a formula for orthogonal modular forms from Hirzebruch's proportionality principle obtained by Mumford [**120**]. In this chapter, we assume that $3 < n+1$, which is the rank of a Hermitian lattice L.

Remark 3.2.1. Note that the definitions of "unimodular" considered in this chapter are Allcock's one [**2**], different from [**110, 112, 146**]; see also [**146**, Subsection 2.1]. We can also work on their ones, but for convenience, we restrict our definition.

Let D_L^c be the compact dual od D_L. In other words, D_L is the n-dimensional complex ball and D_L^c is the n-dimensional projective space. We recall the definition of modular forms from Section 2.6. We denote by $M_k(\Gamma,\chi)$ (resp. $S_k(\Gamma,\chi)$) the set consisting of modular (resp. cusp) forms of weight k with character χ and level Γ. Let $M_k(\Gamma) := M_k(\Gamma, \mathrm{id})$ and $S_k(\Gamma) := S_k(\Gamma, \mathrm{id})$.

For an arithmetic subgroup $\Gamma \subset \mathrm{U}(L \otimes_{\mathbb{Z}} \mathbb{Q})$, if Γ acts on D_L freely, the *Hirzebruch-Mumford volume* of Γ is defined by

$$\mathrm{vol}_{HM}(\Gamma) := \frac{e(D_L/\Gamma)}{e(D_L^c)} = \frac{e(D_L/\Gamma)}{n+1}.$$

If Γ does not act freely, we take a finite index normal subgroup $\Gamma' \lhd \Gamma$ which acts on D_L freely and define

$$\mathrm{vol}_{HM}(\Gamma) := \frac{\mathrm{vol}_{HM}(\Gamma')}{[\overline{\Gamma} : \Gamma']},$$

where $\overline{\Gamma}$ is Γ modulo center. Note that the Hirzebruch-Mumford volume does not depend on the choice of Γ'. Recall the following celebrated result.

Theorem 3.2.2 ([120, Corollary 3.5]). *Let $\mathscr{D}$ be a Hermitian symmetric domain, $\mathscr{D}^c$ be its compact dual, and Γ be a neat arithmetic group, acting on $\mathscr{D}$. We denote by $S_k^{geom}(\Gamma)$ the space of cusp forms on $\mathscr{D}$ of geometric weight k with respect to Γ. Then,*

$$\dim S_k^{geom}(\Gamma) = \mathrm{vol}_{HM}(\Gamma) h^0(\Gamma)(\omega_{\mathscr{D}^c}^{1-k}) + P_1(k),$$

for some polynomial $P_1(k)$ of degree at most $\dim(\mathscr{D}/\Gamma) - 1$ with respect to k.

We shall apply this result to unitary groups and obtain a formula for the asymptotic growth of the dimension of the space of cusp forms.

Proposition 3.2.3. *We assume that*

(1) If $-\mathrm{id} \in \Gamma$*, then* $\chi(-\mathrm{id}) = (-1)^k$*.*
(2) If $F = \mathbb{Q}(\sqrt{-1})$ *and* $\sqrt{-1}\,\mathrm{id} \in \Gamma$*, then* $\chi(\sqrt{-1}\,\mathrm{id}) = \sqrt{-1}^k$*.*
(3) If $F = \mathbb{Q}(\sqrt{-3})$ *and* $\omega\,\mathrm{id} \in \Gamma$*, then* $\chi(\omega\,\mathrm{id}) = \omega^k$*.*

Then,

$$\dim S_k(\Gamma, \chi) = \frac{1}{n!}\mathrm{vol}_{HM}(\Gamma)k^n + O(k^{n-1})$$

for sufficiently divisible k*.*

Proof. We follow the proof of [**57**, Proposition 1.2] or [**137**, Proposition 2.1]. By applying the Lefschetz fixed point theorem [**137**, Appendix to Section 2], we may assume that Γ is neat. Note that we use the assumption on χ here. For sufficiently divisible k, the asymptotic growth of the dimension of the space of cusp forms of weight k with character χ remains the same even when the character replaced with the trivial character because $\mathscr{L}$ and $\mathscr{L} \otimes \chi$ only differ by torsion, so we also assume that χ is trivial.

Note that $S_k(\Gamma) = H^0(\overline{\mathscr{F}_L(\Gamma)}, \mathscr{L}^{\otimes k}(-\Delta))$. We calculate the dimension of modular forms by using the Hrizebruch-Riemann-Roch theorem and Hirzebruch's proportionality principle (Theorem 3.2.2). First, since $\mathscr{L}$ is big and nef, by the Kawamata-Viehweg vanishing theorem, we obtain

$$\chi(\overline{\mathscr{F}_L(\Gamma)}, \mathscr{L}^{\otimes k}(-\Delta)) = h^0(\overline{\mathscr{F}_L(\Gamma)}, \mathscr{L}^{\otimes k}(-\Delta)) \tag{3.2.1}$$

for sufficiently divisible k. When we think of the above as a function of k, the Riemann-Roch polynomial is given by

$$\chi(\overline{\mathscr{F}_L(\Gamma)}, \mathscr{L}^{\otimes k}(-\Delta)) = \frac{c_1^n(\mathscr{L}^{\otimes k}(-\Delta))}{n!}k^n + O(k^{n-1}). \tag{3.2.2}$$

On the other hand, by Theorem 3.2.2,

$$\begin{aligned} h^0(\overline{\mathscr{F}_L(\Gamma)}, \mathscr{L}^{\otimes(n+1)k}(-\Delta)) &= h^0(\overline{\mathscr{F}_L(\Gamma)}, (\mathscr{L}^{\otimes k} \otimes \det{}^k)^{\otimes n+1}(-\Delta)) \\ &= \dim S_{(n+1)k}(\Gamma, \det{}^k) \\ &= \dim S_k^{geom}(\Gamma) \\ &= \mathrm{vol}_{HM}(\Gamma)h^0(\omega_{\mathbb{P}^n}^{1-k}) + O(k^{n-1}) \end{aligned}$$

for sufficiently divisible k. Note that the compact dual of D_L is $\mathbb{P}^n$, so by a standard calculation, for sufficiently divisible k, gives

$$\begin{aligned} \chi(\mathbb{P}^n, \omega_{\mathbb{P}^n}^{1-k}) &= h^0(\mathbb{P}^n, \omega_{\mathbb{P}^n}^{1-k}) \\ &= \frac{(n+1)^n}{n!}k^n + O(k^{n-1}) \end{aligned} \tag{3.2.3}$$

as a function of k. Hence, from (3.2.2) and (3.2.3), it follows

$$\frac{c_1^n(\mathscr{L}^{\otimes(n+1)k}(-\Delta))}{n!} = \frac{(n+1)^n}{n!}\mathrm{vol}_{HM}(\Gamma).$$

This implies

$$\frac{c_1^n(\mathscr{L}^{\otimes k}(-\Delta))}{n!} = \frac{1}{n!}\mathrm{vol}_{HM}(\Gamma).$$

Combining this with (3.2.1), we conclude that

$$\begin{aligned} \dim S_k(\Gamma) &= h^0(\overline{\mathscr{F}_L(\Gamma)}, \mathscr{L}^{\otimes k}(-\Delta)) \\ &= \frac{1}{n!}\mathrm{vol}_{HM}(\Gamma)k^n + O(k^{n-1}). \end{aligned}$$

□

Remark 3.2.4. (1) Gritsenko-Hulek-Sankaran [**57**, Proposition 1.2] derived a similar dimension formula for orthogonal groups.
(2) The asymptotic growth of the dimension of the space of modular forms is the same as that of cusp forms because only line bundles supported on the boundary contribute their difference; see [**57**].

3.3. Ramification divisors

We already know that the canonical divisor $K_{\overline{\mathscr{F}_L(\Gamma)}}$ is described as

$$\begin{cases} (n+1)\mathscr{L} - \dfrac{B_2}{2} - \Delta & (F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})), \\ (n+1)\mathscr{L} - \dfrac{B_2}{2} - \dfrac{3}{4}B_4 - \Delta & (F = \mathbb{Q}(\sqrt{-1})), \\ (n+1)\mathscr{L} - \dfrac{B_2}{2} - \dfrac{2}{3}B_3 - \dfrac{5}{6}B_6 - \Delta & (F = \mathbb{Q}(\sqrt{-3})), \end{cases}$$

in $\mathrm{Pic}(\overline{\mathscr{F}_L(\Gamma)}) \otimes_{\mathbb{Z}} \mathbb{Q}$ from [**9**]. In this section, we shall study the branch divisors B_i via Hermitian lattices. Geometrically, B_i is a quotient of Abelian varieties with complex multiplication by a finite group. Below, we shall mainly work on $\Gamma = \mathrm{U}(L)$.

Recall that the *reflection* $\sigma_{\ell,\xi}$ with respect to a primitive vector $\ell \in L$ with $\langle \ell, \ell \rangle < 0$ and $\xi \in \mathscr{O}_F^{\times} \backslash \{1\}$ is defined by

$$\sigma_{\ell,\xi} : V \to V, \quad v \to v - (1-\xi)\frac{\langle v, \ell \rangle}{\langle \ell, \ell \rangle}\ell.$$

By [**9**, Proposition 2], the ramification divisors are the union of fixed divisors of reflections:

$$B_2 = \bigcup_{\ell \in A_2} H(\ell),$$
$$B_3 = \bigcup_{\ell \in A_3} H(\ell) \quad (F = \mathbb{Q}(\sqrt{-3})),$$
$$B_4 = \bigcup_{\ell \in A_4} H(\ell) \quad (F = \mathbb{Q}(\sqrt{-1})),$$
$$B_6 = \bigcup_{\ell \in A_6} H(\ell) \quad (F = \mathbb{Q}(\sqrt{-3})),$$

where

$$A_2 = \{\ell \in L \mid \xi\, \mathrm{id} \cdot \sigma_{\ell,-1} \in \mathrm{U}(L) \text{ for some } \xi \in \mathscr{O}_F^{\times}\} \setminus (A_4 \coprod A_6),$$
$$A_3 = \{\ell \in L \mid \xi\, \mathrm{id} \cdot \sigma_{\ell,\omega^k} \in \mathrm{U}(L) \text{ for some } \xi \in \mathscr{O}_{\mathbb{Q}(\sqrt{-3})}^{\times} \text{ and } k \in \mathbb{Z}\backslash 3\mathbb{Z}\} \setminus A_6,$$
$$A_4 = \{\ell \in L \mid \xi\, \mathrm{id} \cdot \sigma_{\ell,\sqrt{-1}^k} \in \mathrm{U}(L) \text{ for some } \xi \in \mathscr{O}_{\mathbb{Q}(\sqrt{-1})}^{\times} \text{ and } k \in \mathbb{Z}\backslash 2\mathbb{Z}\},$$
$$A_6 = \{\ell \in L \mid \xi\, \mathrm{id} \cdot \sigma_{\ell,(-\omega)^k} \in \mathrm{U}(L) \text{ for some } \xi \in \mathscr{O}_{\mathbb{Q}(\sqrt{-3})}^{\times} \text{ and } k \in \mathbb{Z}\backslash (2\mathbb{Z} \cup 3\mathbb{Z})\}.$$

Here, $H(\ell)$ denotes a special divisor on D_L with respect to ℓ:

$$H(\ell) := \{v \in D_L \mid \langle v, \ell \rangle = 0\}.$$

We say that ℓ is *reflective* with index i if $\ell \in A_i$. We will investigate branch divisors that obstruct the automorphic line bundle with zeros on branch divisors from being big.

First, we classify them according to [**107**, Lemma 4.1]. For a primitive vector $l \in L$ with $\langle \ell, \ell \rangle < 0$, let $K_\ell := \ell^\perp \cap L$ be its orthogonal complement, $\mathrm{Div}(\ell)$ be the ideal generated by $\{\langle v, \ell \rangle \mid v \in L\}$, and

$$I_\ell := \langle \ell, \ell \rangle \cdot \mathrm{Div}(\ell)^{-1} \subset \mathscr{O}_F$$

be an $\mathscr{O}_F$-ideal. Then, we have

$$L/\mathscr{O}_F \ell \oplus K_\ell \cong \mathscr{O}_F / I_\ell.$$

Note that, unlike the case of orthogonal groups, $\mathrm{Div}(\ell)$ is not a principal ideal in general.

Lemma 3.3.1. *Let $F = \mathbb{Q}(\sqrt{-1})$. Then,*

(1) ℓ is reflective of index 2 if and only if $L \supset \mathscr{O}_{\mathbb{Q}(\sqrt{-1})}\ell \oplus K_\ell$ and $L/\mathscr{O}_{\mathbb{Q}(\sqrt{-1})}\ell \oplus K_\ell \cong \mathscr{O}_{\mathbb{Q}(\sqrt{-1})}/2\mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$ holds.

(2) ℓ is reflective of index 4 if and only if one of the following holds:

(a) $L = \mathscr{O}_{\mathbb{Q}(\sqrt{-1})}\ell \oplus K_\ell$.

(b) $L \supset \mathscr{O}_{\mathbb{Q}(\sqrt{-1})}\ell \oplus K_\ell$ and $L/\mathscr{O}_{\mathbb{Q}(\sqrt{-1})}\ell \oplus K_\ell \cong \mathscr{O}_{\mathbb{Q}(\sqrt{-1})}/(1+\sqrt{-1})\mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$.

Proof. (1) ℓ is reflective with index 2 if and only if

$$\frac{2\langle v, \ell \rangle}{\langle \ell, \ell \rangle} \in \mathscr{O}_F \text{ and } (1+\sqrt{-1})\frac{\langle v, \ell \rangle}{\langle \ell, \ell \rangle} \notin \mathscr{O}_F$$

for all $v \in L$, and this equals

$$2 \in I_\ell \text{ and } 1 + \sqrt{-1} \notin I_\ell.$$

This shows $I_\ell = 2\mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$. Thus the isomorphism $L/\mathscr{O}_{\mathbb{Q}(\sqrt{-1})}\ell \oplus K_\ell \cong \mathscr{O}_{\mathbb{Q}(\sqrt{-1})}/2\mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$ is proved. The sufficient condition can be proved in a similar way as proof of [**107**, Lemma 4.1].

(2) As in (1), it suffices to determine an ideal I_ℓ containing $1+\sqrt{-1}$. This holds if and only if $I_\ell = \mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$ or $(1+\sqrt{-1})\mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$. ☐

Lemma 3.3.2. *Let $F = \mathbb{Q}(\sqrt{-3})$. Then,*

(1) ℓ is reflective of index 2 if and only if $L \supset \mathscr{O}_F\ell \oplus K_\ell$ and $L/\mathscr{O}_F\ell \oplus K_\ell \cong \mathscr{O}_F/2\mathscr{O}_F$ holds.

(2) ℓ is reflective of index 3 if and only if $L \supset \mathscr{O}_{\mathbb{Q}(\sqrt{-3})}\ell \oplus K_\ell$ and $L/\mathscr{O}_{\mathbb{Q}(\sqrt{-3})}\ell \oplus K_\ell \cong \mathscr{O}_{\mathbb{Q}(\sqrt{-3})}/\sqrt{-3}\mathscr{O}_{\mathbb{Q}(\sqrt{-3})}$ holds.

(3) ℓ is reflective of index 6 if and only if $L = \mathscr{O}_{\mathbb{Q}(\sqrt{-3})}\ell \oplus K_\ell$ holds.

Proof. We follow the strategy in the proof of Lemma 3.3.1.

(1) It suffices to determine an ideal I_ℓ containing 2 and not containing $1+\omega = -\omega^2$. This holds if and only if $I_\ell = 2\mathscr{O}_{\mathbb{Q}(\sqrt{-3})}$.

(2) It suffices to determine an ideal I_ℓ containing $1-\omega = \sqrt{-3}\omega$ and not containing $-\omega^2$. This holds if and only if $I_\ell = \sqrt{-3}\mathscr{O}_{\mathbb{Q}(\sqrt{-3})}$.

(3) It suffices to determine an ideal I_ℓ containing $1+\omega = -\omega^2$. This holds if and only if $I_\ell = \mathscr{O}_{\mathbb{Q}(\sqrt{-3})}$. ☐

Lemma 3.3.3. *We assume that $F \neq \mathbb{Q}(\sqrt{-1})$ and the discriminant $-D$ of F is a multiple of 4. Then, ℓ is reflective of index 2 if and only if one of the following holds:*

(1) $L = \mathscr{O}_F\ell \oplus K_\ell$.

(2) $L \supset \mathscr{O}_F\ell \oplus K_\ell$ and $L/\mathscr{O}_F\ell \oplus K_\ell \cong \mathscr{O}_F/2\mathscr{O}_F$.

(3) $L \supset \mathscr{O}_F\ell \oplus K_\ell$ *and* $L/\mathscr{O}_F\ell \oplus K_\ell \cong \mathscr{O}_F/\mathfrak{p}\mathscr{O}_F$*, where* $\mathfrak{p}$ *is a prime ideal such that* $2 = \mathfrak{p}^2$.

Proof. This can be proved in a similar way as Lemma 3.3.1 or Lemma 3.3.2. □

Lemma 3.3.4. *We assume that the discriminant* $-D$ *of* F *satisfies* $-D \equiv 1 \bmod 8$. *Let* $\mathfrak{p}_1$ *and* $\mathfrak{p}_2$ *be prime ideals such that* $(2) = \mathfrak{p}_1\mathfrak{p}_2$. *Then,* ℓ *is reflective of index* 2 *if and only if one of the following holds:*

(1) $L = \mathscr{O}_F\ell \oplus K_\ell$.
(2) $L \supset \mathscr{O}_F\ell \oplus K_\ell$ *and* $L/\mathscr{O}_F\ell \oplus K_\ell \cong \mathscr{O}_F/2\mathscr{O}_F$.
(3) $L \supset \mathscr{O}_F\ell \oplus K_\ell$ *and* $L/\mathscr{O}_F\ell \oplus K_\ell \cong \mathscr{O}_F/\mathfrak{p}_1\mathscr{O}_F$.
(4) $L \supset \mathscr{O}_F\ell \oplus K_\ell$ *and* $L/\mathscr{O}_F\ell \oplus K_\ell \cong \mathscr{O}_F/\mathfrak{p}_2\mathscr{O}_F$.

Proof. This can be proved in a similar way as Lemma 3.3.1 or Lemma 3.3.2. □

Lemma 3.3.5. *We assume that* $F \neq \mathbb{Q}(\sqrt{-3})$ *and its discriminant* $-D$ *of* F *satisfies* $-D \equiv 5 \bmod 8$. *Then,* ℓ *is reflective of index* 2 *if and only if one of the following holds:*

(1) $L = \mathscr{O}_F\ell \oplus K_\ell$.
(2) $L \supset \mathscr{O}_F\ell \oplus K_\ell$ *and* $L/\mathscr{O}_F\ell \oplus K_\ell \cong \mathscr{O}_F/2\mathscr{O}_F$.

Proof. This can be proved in a similar way as Lemma 3.3.1 or Lemma 3.3.2. □

We denote by $\mathcal{R}_L(F,i)$ the set of $\mathrm{U}(L)$-equivalent classes of reflective vectors in L of index i and define the set

$$\mathcal{R}_L(F) := \coprod_i \mathcal{R}_L(F,i).$$

For convenience, we will write the imaginary quadratic field F, defining L, explicitly. Note that any element $[\ell] \in \mathcal{R}_L(F,i)$ corresponds to an irreducible component of the branch divisors with branch index i. Moreover, let

$$\begin{aligned} \mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_I &:= \{[\ell] \in \mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4) \mid L = \mathscr{O}_F\ell \oplus K_\ell\}, \\ \mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_{II} &:= \{[\ell] \in \mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4) \mid L/\mathscr{O}_F\ell \oplus K_\ell \cong \mathscr{O}_F/(1+\sqrt{-1})\mathscr{O}_F\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}_L(F,2)_I &:= \{[\ell] \in \mathcal{R}_L(F,2) \mid L = \mathscr{O}_F\ell \oplus K_\ell\}, \\ \mathcal{R}_L(F,2)_{II} &:= \{[\ell] \in \mathcal{R}_L(F,2) \mid L/\mathscr{O}_F\ell \oplus K_\ell \cong \mathscr{O}_F/2\mathscr{O}_F\}, \\ \mathcal{R}_L(F,2)_{III} &:= \begin{cases} \{[\ell] \in \mathcal{R}_L(F,2) \mid L/\mathscr{O}_F\ell \oplus K_\ell \cong \mathscr{O}_F/\mathfrak{p}\mathscr{O}_F\} & (D \neq 4 \text{ and } D \equiv 0 \bmod 4), \\ \emptyset & (\text{otherwise}), \end{cases} \\ \mathcal{R}_L(F,2)_{IV} &:= \begin{cases} \{[\ell] \in \mathcal{R}_L(F,2) \mid L/\mathscr{O}_F\ell \oplus K_\ell \cong \mathscr{O}_F/\mathfrak{p}_1\mathscr{O}_F\} & (-D \equiv 1 \bmod 8), \\ \emptyset & (\text{otherwise}), \end{cases} \\ \mathcal{R}_L(F,2)_V &:= \begin{cases} \{[\ell] \in \mathcal{R}_L(F,2) \mid L/\mathscr{O}_F\ell \oplus K_\ell \cong \mathscr{O}_F/\mathfrak{p}_2\mathscr{O}_F\} & (-D \equiv 1 \bmod 8), \\ \emptyset & (\text{otherwise}). \end{cases} \end{aligned}$$

From Lemma 3.3.1, 3.3.2, 3.3.3, 3.3.4 and 3.3.5, we have

$$\begin{aligned} &\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4) = \mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_I \coprod \mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_{II}, \\ &\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),2) = \mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),2)_{II}, \end{aligned}$$

$$\mathcal{R}_L(F,2)=\mathcal{R}_L(F,2)_I\coprod\mathcal{R}_L(F,2)_{II}\coprod\mathcal{R}_L(F,2)_{III}\coprod\mathcal{R}_L(F,2)_{IV}\coprod\mathcal{R}_L(F,2)_V,$$

for any imaginary quadratic field F. We call a reflective vector $[\ell]\in\mathcal{R}_L(F)$ *split type* if $L=\ell\mathscr{O}_F\oplus K_\ell$, according to [**107**]. This means that $[\ell]$ is contained in $R(F,2)_I$, $R(\mathbb{Q}(\sqrt{-1}),4)_I$ or $R(\mathbb{Q}(\sqrt{-3}),6)$. Otherwise, we call $[\ell]\in\mathcal{R}_L(F)$ *non-split type.*

Lemma 3.3.6. *Let $\Gamma_\ell\subset \mathrm{U}(K_\ell)$ be the stabilizer of a reflective vector $[\ell]\in\mathcal{R}_L(F)$.*

(1) For $[\ell]\in\mathcal{R}_L(F,2)_I,\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_I,\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),6)$, we have $\Gamma_\ell=\mathrm{U}(K_\ell)$.
(2) For $[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_{II}$, we have $[\mathrm{U}(K_\ell):\Gamma_\ell]<2^{r_{1+\sqrt{-1}}}$, where $r_{1+\sqrt{-1}}:=\ell((A_{K_\ell})_{1+\sqrt{-1}})$.
(3) For $[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),3)$, we have $[\mathrm{U}(K_\ell):\Gamma_\ell]<3^{r_{\sqrt{-3}}}$, where $r_{\sqrt{-3}}:=\ell((A_{K_\ell})_{\sqrt{-3}})$.
(4) For $[\ell]\in\mathcal{R}_L(F,2)_{II}$, we have $[\mathrm{U}(K_\ell):\Gamma_\ell]<4^{r_2}$, where $r_2:=\ell((A_{K_\ell})_2)$.
(5) For $[\ell]\in\mathcal{R}_L(F,2)_{III}$, we have $[\mathrm{U}(K):\Gamma_\ell]<2^{r_\mathfrak{p}}$, where $r_\mathfrak{p}:=\ell((A_{K_\ell})_\mathfrak{p})$.
(6) For $[\ell]\in\mathcal{R}_L(F,2)_{IV}$, we have $[\mathrm{U}(K):\Gamma_\ell]<2^{r_{\mathfrak{p}_1}}$, where $r_{\mathfrak{p}_1}:=\ell((A_{K_\ell})_{\mathfrak{p}_1})$.
(7) For $[\ell]\in\mathcal{R}_L(F,2)_V$, we have $[\mathrm{U}(K):\Gamma_\ell]<2^{r_{\mathfrak{p}_2}}$, where $r_{\mathfrak{p}_2}:=\ell((A_{K_\ell})_{\mathfrak{p}_2})$.

PROOF. This can be proved in the same way as [**107**, Lemma 4.2]. □

3.4. Reflective obstructions

We shall study when the line bundle $\mathcal{M}(a)$ is big in terms of the asymptotic growth of the dimension of the space of modular forms. The line bundle $\mathscr{L}$ is big, so the obstruction for $\mathcal{M}(a)$ being big is the branch divisors B_i. To estimate this obstruction, we use the unitary analog of the construction [**59**, Proposition 4.1].

For $F\neq\mathbb{Q}(\sqrt{-1}),\mathbb{Q}(\sqrt{-3})$, $\ell_1,\dots,\ell_r$ denotes a complete system of representatives of the set $\mathcal{R}_L(F,2)$. For $F=\mathbb{Q}(\sqrt{-1})$, let $\ell_{2,1},\dots,\ell_{2,s_2}$ (resp. $\ell_{4,1},\dots,\ell_{4,s_4}$) be a complete system of representatives of the set $\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),2)$ (resp. $\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)$). For $F=\mathbb{Q}(\sqrt{-3})$, let $\ell_{2,1},\dots,\ell_{2,t_2}$ (resp. $\ell_{3,1},\dots,\ell_{3,t_3}$, $\ell_{6,1},\dots,\ell_{6,t_6}$) be a complete system of representatives of the set $\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),2)$ (resp. $\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),3)$, $\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),6)$).

Lemma 3.4.1. *The following inequalities hold.*

(1) For $F\neq\mathbb{Q}(\sqrt{-1})\mathbb{Q}(\sqrt{-3})$, when k and ka are even, we have

$$h^0(k\cdot\mathcal{M}(a))\geq\dim M_{ka}(\mathrm{U}(L))-\sum_{i=1}^{r}\sum_{j=0}^{k/2-1}\dim M_{ka+2j}(\Gamma_i).$$

(2) For $F=\mathbb{Q}(\sqrt{-1})$, when k and ka are multiples of 4, we have

$$h^0(k\cdot\mathcal{M}(a))\geq\dim M_{ka}(\mathrm{U}(L))-\Big\{\sum_{i=1}^{s_2}\sum_{j_2=0}^{k/4-1}\dim M_{ka+4j_2}(\Gamma_i)+\sum_{i=1}^{s_4}\sum_{j_4=0}^{3k/4-1}\dim M_{ka+4j_4}(\Gamma_i)\Big\}.$$

(3) For $F=\mathbb{Q}(\sqrt{-3})$, when k and ka are multiples of 6, we have

$$h^0(k\cdot\mathcal{M}(a))\geq\dim M_{ka}(\mathrm{U}(L))-\Big\{\sum_{i=1}^{t_2}\sum_{j_2=0}^{k/6-1}\dim M_{ka+6j_2}(\Gamma_i)$$
$$+\sum_{i=1}^{t_3}\sum_{j_3=0}^{k/3-1}\dim M_{ka+6j_3}(\Gamma_i)+\sum_{i=1}^{t_6}\sum_{j_6=0}^{5k/6-1}\dim M_{ka+6j_6}(\Gamma_i)\Big\}.$$

Proof. (1) can be shown in a similar way as [**107**, Lemma 4.4]. For a non-negative j, there is the quasi-pullback:

$$\begin{aligned} H^0(ka\mathscr{L} - jB_2) &\to M_{ka+2j}(\Gamma_i) \\ F &\mapsto \frac{F}{\langle\ , \ell_i\rangle^{2j}}\Big|_{D_{K_i}}. \end{aligned}$$

From this, we derive the exact sequence,

$$0 \to H^0(ka\mathscr{L} - (j+1)B_2) \to H^0(ka\mathscr{L} - jB_2) \to \bigoplus_{i=1}^{r} M_{ka+2j}(\Gamma_i).$$

Iteration for $j = 0, \ldots, k/2 - 1$ yields the desired inequality.

(2) As in [**146**, Lemma 4.3 (1)], since $\sqrt{-1}\,\mathrm{id} \in \Gamma_i$, the vanishing order of F along D_{K_i} is a multiple of 4 and $M_t(\Gamma_i) = 0$ unless $4|t$. From this, we have the quasi-pullback maps:

$$\begin{aligned} H^0(ka\mathscr{L} - 2jB_2) &\to M_{ka+4j}(\Gamma_i) \\ F &\mapsto \frac{F}{\langle\ , \ell_i\rangle^{4j}}\Big|_{D_{K_i}}, \\ H^0(ka\mathscr{L} - jB_4) &\to M_{ka+4j}(\Gamma_i) \\ F &\mapsto \frac{F}{\langle\ , \ell_i\rangle^{4j}}\Big|_{D_{K_i}}. \end{aligned}$$

There exist exact sequences:

$$0 \to H^0(ka\mathscr{L} - 2(j_2+1)B_2) \to H^0(ka\mathscr{L} - 2j_2B_2) \to \bigoplus_{i=1}^{s_2} M_{ka+4j_2}(\Gamma_i), \tag{3.4.1}$$

$$0 \to H^0(ka\mathscr{L} - \frac{k}{2}B_2 - (j_4+1)B_4) \to H^0(ka\mathscr{L} - \frac{k}{2}B_2 - j_4B_4) \to \bigoplus_{i=1}^{s_4} M_{ka+4j_4}(\Gamma_i). \tag{3.4.2}$$

Iteration of (3.4.1) for $j_2 = 0, \ldots, k/4 - 1$ and (3.4.2) for $j_4 = 0, \ldots, 3k/4 - 1$ yields the desired inequality.

(3) As in [**146**, Lemma 4.3 (2)], since $-\omega\,\mathrm{id} \in \Gamma_i$, the vanishing order of F along D_{K_i} is a multiple of 6 and $M_t(\Gamma_i) = 0$ unless $6|t$. From this, we have the quasi-pullback maps:

$$\begin{aligned} H^0(ka\mathscr{L} - 3jB_2) &\to M_{ka+6j}(\Gamma_i) \\ F &\mapsto \frac{F}{\langle\ , \ell_i\rangle^{6j}}\Big|_{D_{K_i}}, \\ H^0(ka\mathscr{L} - 2jB_3) &\to M_{ka+6j}(\Gamma_i) \\ F &\mapsto \frac{F}{\langle\ , \ell_i\rangle^{6j}}\Big|_{D_{K_i}}, \\ H^0(ka\mathscr{L} - jB_6) &\to M_{ka+6j}(\Gamma_i) \\ F &\mapsto \frac{F}{\langle\ , \ell_i\rangle^{6j}}\Big|_{D_{K_i}}. \end{aligned}$$

There exist exact sequences:

$$0 \to H^0(ka\mathscr{L} - 3(j_2+1)B_2) \to H^0(ka\mathscr{L} - 3j_2B_2) \to \bigoplus_{i=1}^{t_2} M_{ka+6j_2}(\Gamma_i), \tag{3.4.3}$$

(3.4.4)

$$0 \to H^0(ka\mathscr{L} - \frac{k}{2}B_2 - 2(j_3+1)B_3) \to H^0(ka\mathscr{L} - \frac{k}{2}B_2 - j_3B_3) \to \bigoplus_{i=1}^{t_3} M_{ka+6j_3}(\Gamma_i),$$

$$\begin{aligned} 0 \to H^0(ka\mathscr{L} - \frac{k}{2}B_2 - \frac{2k}{3}B_3 - (j_6+1)B_6) &\to H^0(ka\mathscr{L} - \frac{2k}{3}B_3 - j_6B_6) \\ &\to \bigoplus_{i=1}^{t_6} M_{ka+6j_6}(\Gamma_i). \end{aligned} \tag{3.4.5}$$

Iteration of (3.4.3) for $j_2 = 0, \dots, k/6-1$, (3.4.4) for $j_3 = 0, \dots, k/3-1$ and (3.4.5) for $j_3 = 0, \dots, 5k/6-1$ yields the desired inequality. □

Remark 3.4.2. We cannot evaluate $h^0(\mathcal{M}(a) - \Delta)$ directly, because we don't know how to construct cusp forms vanishing on cusps with high order.

For $[\ell] \in \mathcal{R}_L(F)$, let

$$\mathrm{vol}_{HM}(L, K_\ell) := \frac{\mathrm{vol}_{HM}(\mathrm{U}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{U}(L))}.$$

Definition 3.4.3. For $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$, let

$$\begin{aligned} V(L,F) := \sum_{[\ell]\in R(F,2)_I} \mathrm{vol}_{HM}(L,K_\ell) + 2^n \sum_{[\ell]\in \mathcal{R}_L(F,2)_{III}, \mathcal{R}_L(F,2)_{IV}, \mathcal{R}_L(F,2)_V} \mathrm{vol}_{HM}(L,K_\ell) \\ + 4^n \sum_{[\ell]\in\mathcal{R}_L(F,2)_{II}} \mathrm{vol}_{HM}(L,K_\ell). \end{aligned}$$

For $F = \mathbb{Q}(\sqrt{-1})$, let

$$\begin{aligned} V(L,\mathbb{Q}(\sqrt{-1})) := 3 \sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_I} \mathrm{vol}_{HM}(L,K_\ell) + 3\cdot 2^n \sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_{II}} \mathrm{vol}_{HM}(L,K_\ell) \\ + 4^n \sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),2)_{II}} \mathrm{vol}_{HM}(L,K_\ell). \end{aligned}$$

For $F = \mathbb{Q}(\sqrt{-3})$, let

$$\begin{aligned} V(L,\mathbb{Q}(\sqrt{-3})) := 5 \sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),6)} \mathrm{vol}_{HM}(L,K_\ell) + 2\cdot 3^n \sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),3)} \mathrm{vol}_{HM}(L,K_\ell) \\ + 4^n \sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),2)} \mathrm{vol}_{HM}(L,K_\ell). \end{aligned}$$

Proposition 3.4.4. *Let a be a positive integer.*

(1) For $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$, $\mathcal{M}(a) = a\mathscr{L} - B_2/2$ is big if

$$V(L,F) < \left(1+\frac{1}{a}\right)^{1-n}\frac{2a}{n}. \tag{3.4.6}$$

(2) For $F = \mathbb{Q}(\sqrt{-1})$, $\mathcal{M}(a) = a\mathscr{L} - B_2/2 - 3B_4/4$ *is big if*

$$V(L, \mathbb{Q}(\sqrt{-1})) < \Big(1 + \frac{3}{a}\Big)^{1-n} \frac{4a}{n}. \tag{3.4.7}$$

(3) For $F = \mathbb{Q}(\sqrt{-3})$, $\mathcal{M}(a) = a\mathscr{L} - B_2/2 - 2B_3/3 - 5B_5/6$ *is big if*

$$V(L, \mathbb{Q}(\sqrt{-3})) < \Big(1 + \frac{5}{a}\Big)^{1-n} \frac{6a}{n}. \tag{3.4.8}$$

Proof. (1) We follow the strategy of [**107**, Proposition 4.3]. We calculate the right side of the inequality of Lemma 3.4.1 (1) in terms of Proposition 3.2.3.

First, we have

$$\dim M_{ka}(\mathrm{U}(L)) = \frac{1}{n!}\mathrm{vol}_{HM}(\mathrm{U}(L)) \cdot a^n \cdot k^n + O(k^{n-1}).$$

Second, we have

$$\begin{aligned}
&\sum_{i=1}^{r} \sum_{j=0}^{k/2-1} \dim M_{ka+2j}(\Gamma_i) \\
&= \sum_{i=1}^{r} \sum_{j=0}^{k/2-1} \Big\{ \frac{1}{(n-1)!}\mathrm{vol}_{HM}(\Gamma_i) \cdot (ka+2j)^{n-1} + O(k^{n-2}) \Big\} \\
&\leq \sum_{i=1}^{r} \frac{k}{2} \Big\{ \frac{1}{(n-1)!}\mathrm{vol}_{HM}(\Gamma_i) \cdot (a+1)^{n-1} \cdot k^{n-1} + O(k^{n-2}) \Big\} \\
&= \frac{(a+1)^{n-1}}{2 \cdot (n-1)!} \cdot \Big(\sum_{i=1}^{r} \mathrm{vol}_{HM}(\Gamma_i)\Big) \cdot k^n + O(k^{n-1}).
\end{aligned}$$

Combining the above, we get

$$\begin{aligned}
&h^0(k \cdot \mathcal{M}(a)) \\
&\geq \dim M_{ka}(\mathrm{U}(L)) - \sum_{i=1}^{r} \sum_{j=0}^{k/2-1} \dim M_{ka+2j}(\Gamma_i) \\
&\geq \frac{a^n}{n!}\mathrm{vol}_{HM}(\mathrm{U}(L)) \Big\{ 1 - \frac{n}{2a}\Big(1 + \frac{1}{a}\Big)^{n-1} \sum_{i=1}^{r} \frac{\mathrm{vol}_{HM}(\Gamma_i)}{\mathrm{vol}_{HM}(\mathrm{U}(L))} \Big\} k^n + O(k^{n-1}).
\end{aligned}$$

We need to estimate $\mathrm{vol}_{HM}(\Gamma_i)/\mathrm{vol}_{HM}(\mathrm{U}(L))$, in terms of $\mathrm{vol}_{HM}(L, K_\ell)$ from Lemma 3.3.6.

$$\begin{aligned}
\frac{\mathrm{vol}_{HM}(\Gamma_i)}{\mathrm{vol}_{HM}(\mathrm{U}(L))} &= [\mathrm{U}(K_\ell) : \Gamma_i]\mathrm{vol}_{HM}(L, K_\ell) \\
&\begin{cases} = \mathrm{vol}_{HM}(L, K_\ell) & ([\ell] \in \mathcal{R}_L(F,2)_I), \\ \leq 4^n \mathrm{vol}_{HM}(L, K_\ell) & ([\ell] \in \mathcal{R}_L(F,2)_{II}), \\ \leq 2^n \mathrm{vol}_{HM}(L, K_\ell) & ([\ell] \in \mathcal{R}_L(F,2)_{III} \coprod \mathcal{R}_L(F,2)_{IV} \coprod \mathcal{R}_L(F,2)_V). \end{cases}
\end{aligned}$$

Hence, since

$$\mathcal{R}_L(F) = \mathcal{R}_L(F,2) = \mathcal{R}_L(F,2)_I \coprod \mathcal{R}_L(F,2)_{II} \coprod \mathcal{R}_L(F,2)_{III} \coprod \mathcal{R}_L(F,2)_{IV} \coprod \mathcal{R}_L(F,2)_V,$$

the line bundle $\mathcal{M}(a) = a\mathscr{L} - B_2/2$ is big if

$$1 - \frac{n}{2a}\Big(1+\frac{1}{a}\Big)^{n-1}\Big\{\sum_{[\ell]\in\mathcal{R}_L(F,2)_I}\mathrm{vol}_{HM}(L,K_\ell)$$
$$+ 4^n\sum_{[\ell]\in\mathcal{R}_L(F,2)_{II}}\mathrm{vol}_{HM}(L,K_\ell) + 2^n\sum_{[\ell]\in\mathcal{R}_L(F,2)_{III}\coprod\mathcal{R}_L(F,2)_{IV}\coprod\mathcal{R}_L(F,2)_V}\mathrm{vol}_{HM}(L,K_\ell)\Big\} > 0$$

holds.

(2) Here, We calculate the right side of the inequality of Lemma 3.4.1 (2). As in the above calculation, we have

$$\sum_{i=1}^{s_2}\sum_{j_2=0}^{k/4-1}\dim M_{ka+4j_2}(\Gamma_i) + \sum_{i=1}^{s_4}\sum_{j_4=0}^{3k/4-1}\dim M_{ka+4j_4}(\Gamma_i)$$
$$\leq \frac{(a+3)^{n-1}}{4\cdot(n-1)!}\Big\{\sum_{i=1}^{s_2}\mathrm{vol}_{HM}(\Gamma_i) + 3\sum_{i=1}^{s_4}\mathrm{vol}_{HM}(\Gamma_i)\Big\}k^n + O(k^{n-1}).$$

Then,

$$h^0(k\cdot\mathcal{M}(a))$$
$$\geq \frac{a^n}{n!}\mathrm{vol}_{HM}(\mathrm{U}(L))\Big[1 - \frac{n}{4a}\Big(1+\frac{3}{a}\Big)^{n-1}\Big\{\sum_{i=1}^{s_2}\frac{\mathrm{vol}_{HM}(\Gamma_i)}{\mathrm{vol}_{HM}(\mathrm{U}(L))} + 3\sum_{i=1}^{s_4}\frac{\mathrm{vol}_{HM}(\Gamma_i)}{\mathrm{vol}_{HM}(\mathrm{U}(L))}\Big\}\Big]k^n + O(k^{n-1}).$$

Moreover, we need to estimate $\mathrm{vol}_{HM}(\Gamma_i)/\mathrm{vol}_{HM}(\mathrm{U}(L))$ in terms of $\mathrm{vol}_{HM}(L,K_\ell)$ from Lemma 3.3.6.

$$\frac{\mathrm{vol}_{HM}(\Gamma_i)}{\mathrm{vol}_{HM}(\mathrm{U}(L))} = [\mathrm{U}(K_\ell):\Gamma_i]\mathrm{vol}_{HM}(L,K_\ell)$$
$$\begin{cases} = \mathrm{vol}_{HM}(L,K_\ell) & ([\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_I),\\ \leq 2^n\mathrm{vol}_{HM}(L,K_\ell) & ([\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_{II}),\\ \leq 4^n\mathrm{vol}_{HM}(L,K_\ell) & ([\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),2)).\end{cases}$$

Hence, since

$$\mathcal{R}_L(\mathbb{Q}(\sqrt{-1})) = \mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),2)\coprod\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_I\coprod\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_{II},$$

the line bundle $\mathcal{M}(a) = a\mathscr{L} - B_2/2 - 3B_4/4$ is big if

$$1 - \frac{n}{4a}\Big(1+\frac{3}{a}\Big)^{n-1}\Big\{3\sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_I}\mathrm{vol}_{HM}(L,K_\ell)$$
$$+ 3\cdot 2^n\sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_{II}}\mathrm{vol}_{HM}(L,K_\ell) + 4^n\sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),2)}\mathrm{vol}_{HM}(L,K_\ell)\Big\} > 0$$

holds.

(3) Here, we calculate the right side of the inequality of Lemma 3.4.1 (3). As in the above calculation, we have

$$\sum_{i=1}^{t_2}\sum_{j_2=0}^{k/6-1}\dim M_{ka+6j_2}(\Gamma_i) + \sum_{i=1}^{t_3}\sum_{j_3=0}^{k/3-1}\dim M_{ka+6j_3}(\Gamma_i) + \sum_{i=1}^{t_6}\sum_{j_6=0}^{5k/6-1}\dim M_{ka+6j_6}(\Gamma_i)$$

$$\leq \frac{(a+5)^{n-1}}{6\cdot(n-1)!}\Big\{\sum_{i=1}^{t_2}\mathrm{vol}_{HM}(\Gamma_i)+2\sum_{i=1}^{t_3}\mathrm{vol}_{HM}(\Gamma_i)+5\sum_{i=1}^{t_6}\mathrm{vol}_{HM}(\Gamma_i)\Big\}k^n+O(k^{n-1}).$$

Then,

$$\begin{aligned}&h^0(k\cdot\mathcal{M}(a))\\ &\geq \frac{a^n}{n!}\mathrm{vol}_{HM}(\mathrm{U}(L))\Big[1-\frac{n}{6a}\Big(1+\frac{5}{a}\Big)^{n-1}\Big\{\sum_{i=1}^{t_2}\frac{\mathrm{vol}_{HM}(\Gamma_i)}{\mathrm{vol}_{HM}(\mathrm{U}(L))}+2\sum_{i=1}^{t_3}\frac{\mathrm{vol}_{HM}(\Gamma_i)}{\mathrm{vol}_{HM}(\mathrm{U}(L))}\\ &+5\sum_{i=1}^{t_6}\frac{\mathrm{vol}_{HM}(\Gamma_i)}{\mathrm{vol}_{HM}(\mathrm{U}(L))}\Big\}\Big]k^n+O(k^{n-1}).\end{aligned}$$

We need to estimate $\mathrm{vol}_{HM}(\Gamma_i)/\mathrm{vol}_{HM}(\mathrm{U}(L))$ in terms of $\mathrm{vol}_{HM}(L,K_\ell)$ from Lemma 3.3.6.

$$\begin{aligned}\frac{\mathrm{vol}_{HM}(\Gamma_i)}{\mathrm{vol}_{HM}(\mathrm{U}(L))}&=[\mathrm{U}(K_\ell):\Gamma_i]\mathrm{vol}_{HM}(L,K_\ell)\\ &\begin{cases}=\mathrm{vol}_{HM}(L,K_\ell) & ([\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),6)),\\ \leq 3^n\mathrm{vol}_{HM}(L,K_\ell) & ([\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),3)),\\ \leq 4^n\mathrm{vol}_{HM}(L,K_\ell) & ([\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),2)).\end{cases}\end{aligned}$$

Hence, since

$$\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}))=\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),2)\coprod\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),3)\coprod\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),6),$$

the line bundle $\mathcal{M}(a)=a\mathscr{L}-B_2/2-2B_3/3-5B_6/6$ is big if

$$\begin{aligned}&1-\frac{n}{6a}(1+\frac{5}{a})^{n-1}\Big\{5\sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),6)}\mathrm{vol}_{HM}(L,K_\ell)\\ &+2\cdot 3^n\sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),3)}\mathrm{vol}_{HM}(L,K_\ell)+4^n\sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),2)}\mathrm{vol}_{HM}(L,K_\ell)\Big\}>0\end{aligned}$$

holds. □

Next, we estimate the cardinality of the sets of split vectors. Let $\mathcal{R}_{\mathrm{split}}$ be the subset of $\mathcal{R}_L(F)$ consisting of the elements $[\ell]\in\mathcal{R}_L(F)$ satisfies $L=\ell\mathscr{O}_F\oplus K_\ell$. We divide up $\mathcal{R}_{\mathrm{split}}$ as

$$\mathcal{R}_{\mathrm{split}}=\coprod_{w|D(L)}\mathcal{R}_{\mathrm{split}}(w).$$

As in [**107**], $\mathcal{R}_{\mathrm{split}}(w)$ is canonically identified with the set of isometry classes of Hermitian lattices K such that $K\oplus\langle -w\rangle\cong L$. By the cancellation theorem [**144**, Theorem 10], if

$$\langle -w\rangle\oplus K\cong\langle -w\rangle\oplus K',$$

it follows $K\cong K'$ because K is indefinite of rank greater than or equals 3. Hence, the following holds.

Proposition 3.4.5.

$$|\mathcal{R}_{\mathrm{split}}(w)|\leq 1.$$

3.5. Prasad's formula

We will apply Prasad's formula to compute $V(L, F)$. The purposes in this section are followings;

(Subsection 3.5.1) to introduce Prasad′s formula,

(Subsection 3.5.2) to show that unramified square-free lattices satisfy ($\heartsuit$).

3.5.1. Preparation. Below, let v be a finite place. Let F_v be the completion of F at v, $\mathscr{O}_{F_v}$ be a maximal compact subring and $\mathfrak{p}_v$ be a maximal ideal. Let $\mathfrak{f}_v := \mathscr{O}_{F_v}/\mathfrak{p}_v$ and $q_v := |\mathfrak{f}_v|$. If v ramifies, let π be a uniformizer of F_v. Otherwise, let π be a uniformizer of $\mathbb{Q}_v$. Prasad [**129**, Theorem 3.7] proved the S-arithmetic volume formula of arithmetic subgroups. We shall apply it to our special unitary groups.

Now, let us assume that the arithmetic subgroup $\mathrm{SU}(L)$ is principal with respect to the coherent parahoric family $\{\mathrm{SU}(L\otimes\mathbb{Z}_v)\}_v$ in the sense of [**129**]. By the strong approximation theorem, it holds that

$$\mathrm{SU}(L) = \mathrm{SU}(L\otimes\mathbb{Q}) \cap \prod_{v\nmid\infty} \mathrm{SU}(L\otimes\mathbb{Z}_v).$$

Also, from the proof of [**132**, Proposition 2.6], the closure of the image of $\mathrm{SU}(L)$ in $\mathrm{SU}(L\otimes\mathbb{Q}_v)$ is $\mathrm{SU}(L\otimes\mathbb{Z}_v)$, so our assumption means that $\mathrm{SU}(L\otimes\mathbb{Z}_v)$ is a parahoric subgroup for all v.

By Prasad's formula, we obtain, for a Hermitian lattice L satisfying ($\star$),

$$\mathrm{vol}_{HM}(\mathrm{SU}(L)) = \begin{cases} D^{\frac{n(n+3)}{4}} \displaystyle\prod_{i=1}^{n} \frac{i!}{(2\pi)^{i+1}} \zeta(2)L(3)\zeta(4)\dots L(n+1) \prod_{v\nmid\infty}\lambda_v^L & (2\mid n), \\ D^{\frac{(n-1)(n+2)}{4}} \displaystyle\prod_{i=1}^{n} \frac{i!}{(2\pi)^{i+1}} \zeta(2)L(3)\zeta(4)\dots \zeta(n+1) \prod_{v\nmid\infty}\lambda_v^L & (2\nmid n). \end{cases}$$

Here, the local factor λ_v^L is defined as follows. By assumption, $\mathrm{SU}(L\otimes\mathbb{Z}_v)$ is a parahoric subgroup, so there exists the smooth integral model $\underline{H}$ in the sense of Bruhat-Tits [**140**] up to an isomorphism. Hence, there exists a reduction map $\underline{H}(\mathscr{O}_{F_v}) \to \underline{H}(\mathfrak{f}_v)$. Let M_v^L be the maximal reductive quotient $\mathcal{H}(\mathfrak{f}_v)$.

From [**130**, Subsection 2.4], if v is inert in F, then

$$\lambda_v^L = q_v^{(\dim M_v^L - n)/2} \cdot |M_v^L|^{-1} \cdot \prod_{i=2}^{n+1} (q_v^i - (-1)^i).$$

If v splits in F, then

$$\lambda_v^L = q_v^{(\dim M_v^L - n)/2} \cdot |M_v^L|^{-1} \cdot \prod_{i=2}^{n+1} (q_v^i - 1).$$

If v ramifies in F, then

$$\lambda_v^L = q_v^{(\dim M_v^L - [(n+1)/2])/2} \cdot |M_v^L|^{-1} \cdot \prod_{i=1}^{[n+1/2]} (q_v^{2i} - 1).$$

3.5.2. Local Jordan decomposition. For local Hermitian lattices, there exists the Jordan decomposition; see [**49**, Corollary 4.3] or [**73**, Section 4]:

$$L \otimes_{\mathbb{Z}} \mathbb{Z}_v = \bigoplus_{j=1}^{k_v} L_{v,j}(\pi^j),$$

where $L_{v,j}$ is a unimodular lattice over $\mathscr{O}_{F_v} = \mathscr{O}_F \otimes \mathbb{Z}_v$ and k_v is an integer. The local Jordan decomposition is unique up to its type in the sense of [**29**, Remark 2.3]. Let $n_{v,j} := \mathrm{rk}(L_{v,j})$, so $\sum_{j=1}^{k_v} n_{v,j} = n+1$ for all finite places v. Let

$$\langle \ell, \ell \rangle = D_\ell = \prod_{v \nmid \infty} v^{\nu_v},$$

$$K_\ell \otimes_{\mathbb{Z}} \mathbb{Z}_v = \bigoplus_{j=1}^{k_v} K_{\ell,v,j}(\pi^j) \quad (K_{\ell,v,j} : \text{unimodular}).$$

In this notation, it follows

$$K_{\ell,v,j} = L_{v,j} \quad (j \neq \nu_v),$$
$$\mathrm{rk}(K_{\ell,v,\nu_v}) = n_{v,\nu_v} - 1.$$

Remark 3.5.1. For a semisimple simply connected algebraic group over $\mathbb{Q}_v$, the stabilizer of a point in the affine Bruhat-Tits building is parahoric [**21**, Proposition 4.6.2], [**140**, Subsection 3.5.2]. Hence, if a Hermitian lattice $L \otimes \mathbb{Z}_v$ over $\mathbb{Z}_v$ defines a point in the affine Bruhat-Tits building, then $\mathrm{SU}(L \otimes \mathbb{Z}_v)$ is a parahoric subgroup of $\mathrm{SU}(L \otimes \mathbb{Q}_v)$. We can interpret a point in the affine Bruhat-Tits building as a lattice chain [**21**, Théorème 2.12], [**100**, Subsection 1.6] for unitary groups if $v \neq 2$ or $F_2/\mathbb{Q}_2$ is unramified; see [**21**, Subsection 2.2] or [**100**, Definition 1.5]. Note that the structure of the reduced building of a unitary group is the same as that of a special unitary group; see [**100**, Subsection 1.6].

Let us consider when a Hermitian lattice forms a lattice chain. We call a Hermitian lattice L over $\mathscr{O}_{F_v}$ *primitive* if there does not exist a Hermitian lattice L' of the same rank as L over $\mathscr{O}_{F_v}$ and a positive integer i satisfying $L = L'(\pi^i)$. A_L also denotes the discriminant group. Below, up to scaling, we will mainly consider primitive Hermitian lattices.

Lemma 3.5.2. *Let K be a quadratic extension of $\mathbb{Q}_p$, or be $\mathbb{Q}_p \times \mathbb{Q}_p$. Assume that K is not a ramified quadratic extension of $\mathbb{Q}_2$. Let M be a primitive Hermitian lattice over $\mathscr{O}_K$. If M satisfies*

$$A_M \cong (\mathscr{O}_K/\pi\mathscr{O}_K)^k$$

for some non-negative integer k, then $\mathrm{SU}(M)$ is a parahoric subgroup of $\mathrm{SU}(M \otimes \mathbb{Q}_p)$. Here, as before, π is a uniformizer of K if K is a ramified extension, and $\pi = p$ if not.

Proof. We denote by

$$M = \bigoplus_{j=0}^{t} M_j(\pi^j) \quad (m_j := \mathrm{rank}(M_j))$$

a Jordan decomposition of M for some integer t. First, we assume that K is unramified over $\mathbb{Q}_p$ or equals $\mathbb{Q}_p \times \mathbb{Q}_p$. Then, from [**73**, Section 7] or [**49**, Proposition 4.2, Section 9], it follows

$$M_j(\pi^j) \cong \langle \delta_{j,1}\pi^j \rangle \oplus \cdots \oplus \langle \delta_{j,m_j}\pi^j \rangle,$$

for some units $\delta_{j,i} \in \mathscr{O}_K$. In this situation, if M satisfies

$$M \subset \frac{1}{\pi} M^\sharp \subset \frac{1}{\pi} M,$$

then it defines a self-dual lattice chain; see [**136**, Subsection 2.1]. Here

$$M^\sharp := \{v \in M \otimes \mathbb{Q}_p \mid \langle v, w \rangle \in \pi \mathscr{O}_K \text{ for any } w \in M\}.$$

This implies $0 \leq j \leq 1$, that is, $M_j = 0$ for $j > 1$. Therefore, if the Jordan decomposition of M has the form

$$M = \bigoplus_{j=0}^{1} M_j(\pi^j), \tag{3.5.1}$$

then it defines a point in the affine Bruhat-Tits building. Since the stabilizer of this lattice chain in $\mathrm{SU}(M \otimes \mathbb{Q}_p)$ is $\mathrm{SU}(M)$, from Remark 3.5.1, this finishes the proof for the unramified or split cases.

Second, let us consider the case that K is a ramified extension of $\mathbb{Q}_p$ with $p \neq 2$. For odd j, from [**73**, Proposition 8.1 (b)] and invoking the same discussion as above, the condition

$$M_j(\pi^j) \subset \frac{1}{\pi} \{M_j(\pi^j)\}^\sharp \subset \frac{1}{\pi} M_j(\pi^j)$$

implies $M_j = 0$ for odd $j > 1$. Now, let j be even. Then, from [**73**, Proposition 8.1 (a)], it follows

$$M_j(\pi^j) \cong \langle \delta_{j,1} \pi^{j/2} \rangle \oplus \cdots \oplus \langle \delta_{j,m_j-1} \pi^{j/2} \rangle \oplus \langle \delta_{j,m_j} \pi^{(j+1)/2} \rangle,$$

for some units $\delta_{j,i} \in \mathscr{O}_K$. Then, the condition

$$M_j(\pi^j) \subset \frac{1}{\pi} \{M_j(\pi^j)\}^\sharp \subset \frac{1}{\pi} M_j(\pi^j)$$

implies $M_j = 0$ for even $j > 1$ through the effect of the last term. Combining these computation completes the proof for the ramified case. □

Remark 3.5.3. We can prove the above when K is a ramified extension over $\mathbb{Q}_2$ in a similar way as in [**73**, Section 9, 10, 11] or [**29**, Theorem 2.10]. However, in this case, points in the building constitute a subset of the set of self-dual lattice chains [**100**, Subsection 1.6], so more detailed calculation seems to be needed. For our purpose, it suffices to assume that $v = 2$ is unramified at F in the following examples because of the consideration of reflective vectors. Hence, we will restrict Lemma 3.5.2 to this case, for simplicity.

Below, for a reflective vector $\ell \in L$, we use the same notation for the local Jordan decomposition of $L' \otimes \mathbb{Z}_v$ of a Hermitian lattice $L' := \ell \mathscr{O}_F \oplus K_\ell$ over $\mathscr{O}_F$ as above. First, we shall explain that unimodular lattices satisfy (♡).

Proposition 3.5.4 (Unimodular). *A unimodular Hermitian lattice L of signature $(1, n)$ over $\mathscr{O}_{F_0}$ satisfy (♡).*

Proof. For a reflective vector $[\ell] \in \mathcal{R}_L(F_0)$, let $L' := \ell \mathscr{O}_{F_0} \oplus K_\ell$, where $K_\ell := \ell^\perp \cap L$. Then,

$$L/L' \cong \begin{cases} 1 & ([\ell] \in \mathcal{R}_L(F_0, 2)_I), \\ \mathscr{O}_F / 2\mathscr{O}_{F_0} & ([\ell] \in \mathcal{R}_L(F_0, 2)_{II}), \\ \mathscr{O}_F / \mathfrak{p}_i \mathscr{O}_{F_0} & ([\ell] \in \mathcal{R}_L(F_0, 2)_{IV} \coprod \mathcal{R}_L(F, 2)_V), \end{cases}$$

from the definition of reflective vectors and

$$\mathcal{R}_L(F_0,2) = \mathcal{R}_L(F_0,2)_I \coprod \mathcal{R}_L(F_0,2)_{II} \coprod \mathcal{R}_L(F_0,2)_{IV} \coprod \mathcal{R}_L(F_0,2)_V,$$

under the assumption on F_0.

If $[\ell] \in \mathcal{R}_L(F_0,2)_I$, then K_ℓ is also unimodular and local Jordan decompositions of L and K_ℓ have the trivial forms

$$L \otimes \mathbb{Z}_v = L_{v,0},$$
$$K_\ell \otimes \mathbb{Z}_v = K_{\ell,v,0}.$$

Now, consider the case of non-split vectors. Let $\ell \in L$ be a non-split vector, i.e., $[\ell] \in \mathcal{R}_L(F_0,2)_{II} \coprod \mathcal{R}_L(F_0,2)_{IV} \coprod \mathcal{R}_L(F_0,2)_V$. We refer to the proof of [**146**, Lemma 2.2]. Since L is unimodular, $\sigma_{\ell,-1} \in \mathrm{U}(L) = \widetilde{\mathrm{U}}(L)$. Hence,

$$\frac{2\langle v, \ell\rangle}{\langle \ell, \ell\rangle} \in \mathscr{O}_{F_0}$$

for any $v \in L = L^\vee$. Since ℓ is primitive, it follows $\langle \ell, \ell\rangle/2 \notin \mathscr{O}_{F_0} \setminus \mathscr{O}_{F_0}^\times$. Hence if $[\ell] \in \mathcal{R}_L(F_0,2)_{II}$, then we have $\langle \ell, \ell\rangle = -2$. This means that, since $I_\ell = (2)$, the discriminant groups of $L' = \ell\mathscr{O}_{F_0} \oplus K_\ell$ and K_ℓ are

$$A_{L'} \cong (\mathscr{O}_{F_0}/2\mathscr{O}_{F_0})^2, \ A_{K_\ell} \cong \mathscr{O}_{F_0}/2\mathscr{O}_{F_0}.$$

This concludes that the Jordan decompositions of $L' \otimes \mathbb{Z}_v$ and $K_\ell \otimes \mathbb{Z}_v$ are

$$L' \otimes \mathbb{Z}_v = \begin{cases} \bigoplus_{j=0}^{1} L'_{2,j}(\pi^j) & (v=2), \\ L'_{v,0} & (\text{otherwise}), \end{cases}$$

$$K_\ell \otimes \mathbb{Z}_v = \begin{cases} \bigoplus_{j=0}^{1} K_{\ell,2,j}(\pi^j) & (v=2), \\ K_{\ell,v,0} & (\text{otherwise}), \end{cases}$$

where

$$\mathrm{rk}(L'_{2,0}) = n-1, \ \mathrm{rk}(L'_{2,1}) = 2,$$
$$\mathrm{rk}(K_{\ell,2,0}) = n-1, \ \mathrm{rk}(K_{\ell,2,1}) = 1.$$

For $[\ell] \in \mathcal{R}_L(F_0,2)_{IV}$, from the same discussion as above, we have $\langle \ell, \ell\rangle = -2$. This means that, since $I_\ell = \mathfrak{p}_1$,

$$A_{L'} \cong \mathscr{O}_{F_0}/2\mathscr{O}_{F_0},$$

and K_ℓ is unimodular. This concludes that the Jordan decompositions of L' and K_ℓ are the same as above except $v = 2$. For $v = 2$, the local factors Jordan decompositions are

$$L' \otimes \mathbb{Z}_2 = \bigoplus_{j=0}^{1} L'_{2,j}(\pi^j),$$
$$K_\ell \otimes \mathbb{Z}_2 = K_{\ell,2,0},$$

where

$$\mathrm{rk}(L'_{v,0}) = n, \ \mathrm{rk}(L'_{v,1}) = 1.$$

In all cases, for any v, the local Jordan decompositions of $L' = \ell\mathscr{O}_F \oplus K_\ell$ and K_ℓ have the form (3.5.1). Hence, by Lemma 3.5.2, $\mathrm{SU}(L' \otimes \mathbb{Z}_v)$ and $\mathrm{SU}((\ell^\perp \cap L) \otimes \mathbb{Z}_v)$ are parahoric for any v. This implies ($\star$) for L' and K_ℓ, and from the discussion in Subsection 3.1.4, we conclude that L satisfies ($\heartsuit$). $\square$

Second, by generalizing the above proof, we prove that unramified square-free lattices satisfy ($\heartsuit$).

Proposition 3.5.5 (Unramified square-free). *A primitive unramified square-free lattice L over $\mathscr{O}_{F_0}$ of signature $(1,n)$ satisfy ($\heartsuit$).*

Proof. Let $\det(L) = p_1 \dots p_k$ be odd square-free. Here, any prime divisor p_i is unramified at F_0. For a split reflective vector $[\ell] \in \mathcal{R}_L(F_0)_I$, we denote by

$$\langle \ell, \ell \rangle = \prod_{v \nmid \infty} v^{\nu_v} = \prod_{i=1}^{k'} p_i,$$

for some order and $k' \leq k$. Then the local Jordan decomposition of $L \otimes \mathbb{Z}_v$ is

$$L \otimes \mathbb{Z}_v = \begin{cases} \bigoplus_{j=0}^{1} L_{p_i,j}(\pi^j) & (v = p_i \text{ for } i = 1, \cdots, k), \\ L_{v,0} & (\text{otherwise}), \end{cases}$$

where

$$\mathrm{rk}(L_{p_i,0}) = n,\ \mathrm{rk}(L_{p_i,1}) = 1,$$

for $i = 1, \dots, k$ .We also have

$$K_\ell \otimes \mathbb{Z}_v = \begin{cases} \bigoplus_{j=0}^{1} K_{\ell,p_i,j}(\pi^j) & (v = p_i \text{ for } i = k'+1, \cdots, k), \\ K_{\ell,v,0} & (\text{otherwise}), \end{cases}$$

where

$$\mathrm{rk}(K_{\ell,p_i,0}) = n-1,\ \mathrm{rk}(K_{\ell,p_i,1}) = 1,$$

for $i = k'+1, \cdots, k$, Now, We choose an element $e \in L$ so that

$$A_L \cong \mathscr{O}_{F_0}/p_1 \dots p_k \mathscr{O}_{F_0} = \langle \frac{1}{p_1 \dots p_k} e \rangle$$

holds as $\mathscr{O}_{F_0}$-modules. If $[\ell] \in \mathcal{R}_L(F_0, 2)_{II}$, first, we shall consider the case of $\sigma_{\ell,-1} \in \widetilde{\mathrm{U}}(L)$. This occurs if and only if $\langle e, \ell \rangle = 0$. In this situation, by the same discussion as Proposition 3.5.4, we have $\langle \ell, \ell \rangle = -2$, and

$$A_{L'} \cong (\mathscr{O}_{F_0}/2\mathscr{O}_{F_0})^2 \times \mathscr{O}_{F_0}/p_1 \dots p_k \mathscr{O}_{F_0},\ A_{K_\ell} \cong \mathscr{O}_{F_0}/2p_1 \dots p_k \mathscr{O}_{F_0}.$$

This concludes that the Jordan decompositions of $L' \otimes \mathbb{Z}_v$ and $K_\ell \otimes \mathbb{Z}_v$ are

$$L' \otimes \mathbb{Z}_v = \begin{cases} \bigoplus_{j=0}^{1} L'_{v,j}(\pi^j) & (v = 2, p_1, \cdots, p_k), \\ L'_{v,0} & (\text{otherwise}), \end{cases}$$

$$K_\ell \otimes \mathbb{Z}_v = \begin{cases} \bigoplus_{j=0}^{1} K_{\ell,v,j}(\pi^j) & (v = 2, p_1, \cdots, p_k), \\ K_{\ell,v,0} & (\text{otherwise}), \end{cases}$$

where for $v = p_1, \cdots, p_k$,

$$\begin{aligned} \mathrm{rk}(L'_{2,0}) &= n-1,\ \mathrm{rk}(L'_{2,1}) = 2, \\ \mathrm{rk}(L'_{v,0}) &= n,\ \mathrm{rk}(L'_{v,1}) = 1 \quad (v = p_1, \dots, p_k), \\ \mathrm{rk}(K_{\ell,v,0}) &= n-1,\ \mathrm{rk}(K_{\ell,v,1}) = 1 \quad (v = 2, p_1, \dots, p_k). \end{aligned}$$

Second, we consider the case of $\sigma_{\ell,-1} \notin \widetilde{\mathrm{U}}(L)$, i.e., $\langle e, \ell \rangle \neq 0$. From the definition of e, an integer $p_1 \dots p_k$ divides $\langle e, \ell \rangle$. Also since ℓ is primitive, it follows $\langle e, \ell \rangle = p_1 \dots p_k$ by replacing e with $-e$, if necessary. On the other hand, since $2e \in L' = \ell \mathscr{O}_{F_0} \oplus K_\ell$, we have

$$2e = a\ell + bk_\ell$$

for some $a \neq 0, b \in \mathscr{O}_{F_0}$ and $k_\ell \in K_\ell$. Taking an inner product of both sides with ℓ, we have

$$2\langle e, \ell \rangle = 2p_1 \dots p_k = a \langle \ell, \ell \rangle.$$

Now, the definition of $\mathcal{R}_L(F_0, 2)_{II}$ implies that 2 divides $\langle \ell, \ell \rangle$, so we have $\langle e, \ell \rangle = 2p_1 \dots p_{k'}$ for some integer $k' < k$, by changing the order of $p_1, \cdots p_k$, if necessary. Then, this implies

$$\begin{aligned} &A_{L'} \cong \mathscr{O}_{F_0}/2p_1 \dots p_{k'} \times \mathscr{O}_{F_0}/2p_{k'+1} \dots p_k \mathscr{O}_{F_0} \cong (\mathscr{O}_{F_0}/2\mathscr{O}_{F_0})^2 \times \mathscr{O}_{F_0}/p_1 \dots p_k \mathscr{O}_{F_0}, \\ &A_{K_\ell} \cong \mathscr{O}_{F_0}/2p_{k`+1} \dots p_k \mathscr{O}_{F_0}. \end{aligned}$$

Hence, the Jordan decompositions of $L' \otimes \mathbb{Z}_v$ and $K_\ell \otimes \mathbb{Z}_v$ are

$$L' \otimes \mathbb{Z}_v = \begin{cases} \bigoplus_{j=0}^{1} L'_{v,j}(\pi^j) & (v = 2, p_1, \cdots, p_k), \\ L'_{v,0} & (\text{otherwise}), \end{cases}$$

$$K_\ell \otimes \mathbb{Z}_v = \begin{cases} \bigoplus_{j=0}^{1} K_{\ell,v,j}(\pi^j) & (v = 2, p_{k'+1}, \cdots, p_k), \\ K_{\ell,v,0} & (\text{otherwise}), \end{cases}$$

where

$$\begin{aligned} \mathrm{rk}(L'_{2,0}) &= n-1,\ \mathrm{rk}(L'_{2,1}) = 2, \\ \mathrm{rk}(K_{\ell,v,0}) &= n-1,\ \mathrm{rk}(K_{\ell,v,1}) = 1 \quad (v = 2, p_{k'+1}, \cdots, p_k), \\ \mathrm{rk}(L'_{v,0}) &= n,\ \mathrm{rk}(L'_{v,1}) = 1 \quad (v \neq 2, p_{k'}, \cdots, p_k). \end{aligned}$$

For $[\ell] \in \mathcal{R}_L(F_0, 2)_{IV} \coprod \mathcal{R}_L(F_0, 2)_V$, we can also calculate the local Jordan decompositions in the same way, and get

$$A_{L'} \cong \mathscr{O}_{F_0}/2p_1 \dots p_k \mathscr{O}_{F_0},\ A_{K_\ell} \cong \mathscr{O}_{F_0}/p_1 \dots p_k \mathscr{O}_{F_0},$$

or

$$A_{L'} \cong \mathscr{O}_{F_0}/2p_1 \dots p_k \mathscr{O}_{F_0},\ A_{K_\ell} \cong \mathscr{O}_{F_0}/p_{k'+1} \dots p_k \mathscr{O}_{F_0},$$

for some integer k'.

In all cases, for any v, the local Jordan decompositions have the form (3.5.1). Hence, by Lemma 3.5.2, $\mathrm{SU}(L' \otimes \mathbb{Z}_v)$ and $\mathrm{SU}((\ell^\perp \cap L) \otimes \mathbb{Z}_v)$ are parahoric for any v. As before, it follows that L satisfies ($\heartsuit$). $\square$

3.6. Computation of local factors

Tits [**140**, Example 3.11] calculated the maximal reductive quotients in the case of special unitary groups of odd dimension. For unramified v, Gan-Yu [**49**] determined the structure of the maximal reductive quotient. For ramified $v \neq 2$, they determined the structure of the maximal reductive quotient. For ramified $v = 2$, Cho [**29, 30**] determined the structure of the maximal reductive quotient for ramified dyadic extension. On the other hand, Gan-Hanke-Yu [**49**] classified the maximal reductive quotient in the case of maximal lattices. As [**107**], up to scaling, we will mainly treat a primitive L. In the following, we will omit the notion of $\mathfrak{f}_v$-valued points and define $M_v^{K_\ell}$ for K_ℓ as M_v^L.

3.6.1. Unramified case. Gan-Yu clarified the structure of the maximal reductive quotient for unramified v.

3.6.1.1. **Inert case**. By [**49**, Proposition 6.2.3], according to local Jordan decompositions, the maximal reductive quotients of the mod $\mathfrak{p}$ reductions of the smooth integral models of $\mathrm{U}(L \otimes \mathbb{Z}_v)$ and $\mathrm{U}(K_\ell \otimes \mathbb{Z}_v)$ are

$$\mathrm{U}(n_{v,0}) \times \cdots \times \mathrm{U}(n_{v.\nu_v}) \times \cdots \times \mathrm{U}(n_{v,k_v})$$

and

$$\mathrm{U}(n_{v,0}) \times \cdots \times \mathrm{U}(n_{v,\nu_v} - 1) \times \cdots \times \mathrm{U}(n_{v,k_v}).$$

As in [**49**, Introduction], this also holds for $v = 2$. Hence, we have

$$M_v^L = \mathrm{Ker}(\det : \mathrm{U}(n_{v,0}) \times \cdots \times \mathrm{U}(n_{v.\nu_v}) \times \cdots \times \mathrm{U}(n_{v,k_v}) \to \mathfrak{f}_v^1),$$
$$M_v^{K_\ell} = \mathrm{Ker}(\det : \mathrm{U}(n_{v,0}) \times \cdots \times \mathrm{U}(n_{v.\nu_v} - 1) \times \cdots \times \mathrm{U}(n_{v,k_v}) \to \mathfrak{f}_v^1),$$

where $\mathfrak{f}_v^1$ denotes the set consisting of the elements of $\mathfrak{f}_v$ whose norm is 1. Note that these maps are surjective. This implies

$$\begin{aligned}\frac{|M_v^L|}{|M_v^{K_\ell}|} &= \frac{|\mathrm{U}(n_{v,0})| \times \cdots \times |\mathrm{U}(n_{v.\nu_v})| \times \cdots \times |\mathrm{U}(n_{v,k_v})|}{|\mathrm{U}(n_{v,0})| \times \cdots \times |\mathrm{U}(n_{v.\nu_v} - 1)| \times \cdots \times |\mathrm{U}(n_{v,k_v})|} \\ &= q_v^{n_{v,\nu_v}-1}(q_v^{n_{v,\nu_v}} - (-1)^{n_{v,\nu_v}})\end{aligned}$$

and

$$\dim M_v^L - \dim M_v^{K_\ell} = n_{v.\nu_v}^2 - (n_{v.\nu_v} - 1)^2 = 2n_{v.\nu_v} - 1.$$

Then,

$$\begin{aligned}\frac{\lambda_v^{K_\ell}}{\lambda_v^L} &= \left\{q_v^{(\dim M_v^{K_\ell} - n+1)/2} \cdot |M_v^{K_\ell}|^{-1} \cdot \prod_{i=2}^{n}(q_v^i - (-1)^i)\right\} \\ &\quad \cdot \left\{q_v^{(\dim M_v^L - n)/2} \cdot |M_v^L|^{-1} \cdot \prod_{i=2}^{n+1}(q_v^i - (-1)^i)\right\}^{-1} \\ &= \frac{q_v^{n_{v,\nu_v}} - (-1)^{n_{v,\nu_v}}}{q_v^{n+1} - (-1)^{n+1}}. \qquad (3.6.1)\end{aligned}$$

3.6.1.2. **Split case**. As Subsubsection 3.6.1.1, by [**49**, Proposition 6.2.3], the maximal reductive quotients of the mod $\mathfrak{p}$ reductions of the smooth integral models of $\mathrm{U}(L \otimes \mathbb{Z}_v)$ and $\mathrm{U}(K_\ell \otimes \mathbb{Z}_v)$ are

$$\mathrm{GL}(n_{v,0}) \times \cdots \times \mathrm{GL}(n_{v.\nu_v}) \times \cdots \times \mathrm{GL}(n_{v,k_v})$$

and

$$\mathrm{GL}(n_{v,0}) \times \cdots \times \mathrm{GL}(n_{v,\nu_v} - 1) \times \cdots \times \mathrm{GL}(n_{v,k_v}).$$

As in [**49**, Introduction], this also holds for $v = 2$. Hence, we have surjective maps

$$M_v^L = \mathrm{Ker}(\det : \mathrm{GL}(n_{v,0}) \times \cdots \times \mathrm{GL}(n_{v.\nu_v}) \times \cdots \times \mathrm{GL}(n_{v,k_v}) \to \mathfrak{f}_v^1),$$
$$M_v^{K_\ell} = \mathrm{Ker}(\det : \mathrm{GL}(n_{v,0}) \times \cdots \times \mathrm{GL}(n_{v.\nu_v} - 1) \times \cdots \times \mathrm{GL}(n_{v,k_v}) \to \mathfrak{f}_v^1).$$

This implies

$$\begin{aligned}\frac{|M_v^L|}{|M_v^{K_\ell}|} &= \frac{|\mathrm{GL}(n_{v,0})| \times \cdots \times |\mathrm{GL}(n_{v.\nu_v})| \times \cdots \times |\mathrm{GL}(n_{v,k_v})|}{|\mathrm{GL}(n_{v,0})| \times \cdots \times |\mathrm{GL}(n_{v.\nu_v} - 1)| \times \cdots \times |\mathrm{GL}(n_{v,k_v})|} \\ &= q_v^{n_{v,\nu_v}-1}(q_v^{n_{v,\nu_v}} - 1)\end{aligned}$$

and

$$\dim M_v^L - \dim M_v^{K_\ell} = n_{v,\nu_v}^2 - (n_{v,\nu_v} - 1)^2 = 2n_{v,\nu_v} - 1.$$

Then,

$$\begin{aligned}\frac{\lambda_v^{K_\ell}}{\lambda_v^L} &= \Big\{ q_v^{(\dim M_v^{K_\ell} - n+1)/2} \cdot |M_v^{K_\ell}|^{-1} \cdot \prod_{i=2}^{n} (q_v^i - 1) \Big\} \Big\{ q_v^{(\dim M_v^L - n)/2} \cdot |M_v^L|^{-1} \cdot \prod_{i=2}^{n+1} (q_v^i - 1) \Big\}^{-1} \\ &= \frac{q_v^{n_{v,\nu_v}} - 1}{q_v^{n+1} - 1}.\end{aligned} \tag{3.6.2}$$

3.6.2. Ramified case: $v \neq 2$. Fix a ramified prime $v \neq 2$. Recall the classification of the maximal reductive quotient of the reduction of the integral model by Gan-Yu [**49**]. For a positive integer x, let

$$\{x\} := \begin{cases} x & (x : \text{even}), \\ x - 1 & (x : \text{odd}). \end{cases}$$

Let

$$H(n_{v,i}) := \begin{cases} \mathrm{O}(n_{v,i}) \text{ or } {}^2\mathrm{O}(n_{v,i}) & (i : \text{even}), \\ \mathrm{Sp}(\{n_{v,i}\}) & (i : \text{odd}). \end{cases}$$

Here, ${}^2\mathrm{O}(i)$ denotes the quasi-split but nonsplit special orthogonal group if i is even. Note that $\mathrm{O}(i) = {}^2\mathrm{O}(i)$ is split if i is odd.

Accordingly, we obtain the following description of the maximal reductive quotients of the mod $\mathfrak{p}$ reduction of the smooth integral models of $\mathrm{U}(L \otimes \mathbb{Z}_v)$ and $\mathrm{U}(K_\ell \otimes \mathbb{Z}_v)$ from [**49**, Proposition 6.3.9];

$$H(n_{v,0}) \times \cdots \times H(n_{v,\nu_v}) \times \cdots \times H(n_{v,k_v})$$

and

$$H(n_{v,0}) \times \cdots \times H(n_{v,\nu_v} - 1) \times \cdots \times H(n_{v,k_v}).$$

If $(\nu_v, n_{v,\nu_v}) = (\text{even}, \text{even})$, then

$$M_v^L = \mathrm{Ker}(\det : H(n_{v,0}) \times \cdots \times \mathrm{Sp}(n_{v,\nu_v}) \times \cdots \times H(n_{v,k_v}) \to \mathfrak{f}_v^1),$$
$$M_v^{K_\ell} = \mathrm{Ker}(\det : H(n_{v,0}) \times \cdots \times \mathrm{Sp}(n_{v,\nu_v} - 2) \times \cdots \times H(n_{v,k_v}) \to \mathfrak{f}_v^1).$$

This implies

$$\frac{|M_v^L|}{|M_v^{K_\ell}|} \leq \frac{|\mathrm{Sp}(n_{v,\nu_v})|}{|\mathrm{Sp}(n_{v,\nu_v} - 2)|}$$

$$= q_v^{n_{v,\nu_v}-1}(q_v^{n_{v,\nu_v}}-1)$$

and

$$\dim M_v^L - \dim M_v^{K_\ell} = \frac{n_{v,\nu_v}(n_{v,\nu_v}+1)}{2} - \frac{(n_{v,\nu_v}-1)(n_{v,\nu_v}-2)}{2} = 2n_{v,\nu_v}-1.$$

Hence, if $n+1=2m+1$, then

$$\frac{\lambda_v^{K_\ell}}{\lambda_v^L} = \Big\{q_v^{(\dim M_v^{K_\ell}-m)/2}\cdot|M_v^{K_\ell}|^{-1}\cdot\prod_{i=1}^{m}(q_v^{2i}-1)\Big\}\Big\{q_v^{(\dim M_v^{L}-m)/2}\cdot|M_v^{L}|^{-1}\cdot\prod_{i=1}^{m}(q_v^{2i}-1)\Big\}^{-1}$$

(3.6.3)

$$\leq q_v^{-1/2}(q_v^{n_{v,nu_v}}-1).$$

If $n+1=2m$, then

$$\frac{\lambda_v^{K_\ell}}{\lambda_v^L} = \Big\{q_v^{(\dim M_v^{K_\ell}-m+1)/2}\cdot|M_v^{K_\ell}|^{-1}\cdot\prod_{i=1}^{m-1}(q_v^{2i}-1)\Big\}\Big\{q_v^{(\dim M_v^{L}-m)/2}\cdot|M_v^{L}|^{-1}\cdot\prod_{i=1}^{m}(q_v^{2i}-1)\Big\}^{-1}$$

(3.6.4)

$$\leq \frac{q_v^{n_{v,nu_v}}-1}{q_v^{n+1}-1}.$$

If $(\nu_p, n_{p,\nu_p}) = (\mathrm{even},\mathrm{odd})$, then

$$M_v^L = \mathrm{Ker}(\det: H(n_{v,0})\times\cdots\times\mathrm{Sp}(n_{v,\nu_v}-1)\times\cdots\times H(n_{v,k_v})\to\mathfrak{f}_v^1),$$
$$M_v^{K_\ell} = \mathrm{Ker}(\det: H(n_{v,0})\times\cdots\times\mathrm{Sp}(n_{v,\nu_v}-1)\times\cdots\times H(n_{v,k_v})\to\mathfrak{f}_v^1).$$

Hence, we have $M_v^L = M_v^{K_\ell}$, so if $n+1=2m+1$, then

$$\frac{\lambda_v^{K_\ell}}{\lambda_v^L} = \Big\{q_v^{(\dim M_v^{K_\ell}-m)/2}\cdot|M_v^{K_\ell}|^{-1}\cdot\prod_{i=1}^{m}(q_v^{2i}-1)\Big\}\Big\{q_v^{(\dim M_v^{L}-m)/2}\cdot|M_v^{L}|^{-1}\cdot\prod_{i=1}^{m}(q_v^{2i}-1)\Big\}^{-1}$$

(3.6.5)

$$= 1.$$

If $n+1=2m$, then

$$\frac{\lambda_v^{K_\ell}}{\lambda_v^L} = \Big\{q_v^{(\dim M_v^{K_\ell}-m+1)/2}\cdot|M_v^{K_\ell}|^{-1}\cdot\prod_{i=1}^{m-1}(q_v^{2i}-1)\Big\}\Big\{q_v^{(\dim M_v^{L}-m)/2}\cdot|M_v^{L}|^{-1}\cdot\prod_{i=1}^{m}(q_v^{2i}-1)\Big\}^{-1}$$

(3.6.6)

$$= \frac{q_v^{1/2}}{q_v^{n+1}-1}.$$

If $(\nu_p, n_{p,\nu_p}) = (\mathrm{odd},\mathrm{even})$, then

$$M_v^L = \mathrm{Ker}(\det: H(n_{v,0})\times\cdots\times^{(2)}\mathrm{O}(n_{v,\nu_v})\times\cdots\times H(n_{v,k_v})\to\mathfrak{f}_v^1),$$
$$M_v^{K_\ell} = \mathrm{Ker}(\det: H(n_{v,0})\times\cdots\times\mathrm{O}(n_{v,\nu_v}-1)\times\cdots\times H(n_{v,k_v})\to\mathfrak{f}_v^1).$$

Here, ${}^{(2)}\mathrm{O}(n_{v,\nu_v})$ denotes $\mathrm{O}(n_{v,\nu_v})$ or ${}^2\mathrm{O}(n_{v,\nu_v})$, so

$$\frac{|M_v^L|}{|M_v^{K_\ell}|} \leq \frac{|{}^{(2)}\mathrm{O}(n_{v,\nu_v})|}{|\mathrm{O}(n_{v,\nu_v}-1)|}$$
$$\leq q_v^{n_{v,\nu_v}/2-1}(q_v^{n_{v,\nu_v}/2}+1)$$

and

$$\dim M_v^L - \dim M_v^{K_\ell} = \frac{n_{v,\nu_v}(n_{v,\nu_v}-1)}{2} - \frac{(n_{v,\nu_v}-1)(n_{v,\nu_v}-2)}{2} = n_{v,\nu_v} - 1.$$

Hence, if $n+1 = 2m+1$, then

$$\begin{aligned} \frac{\lambda_v^{K_\ell}}{\lambda_v^L} &= \Big\{ q_v^{(\dim M_v^{K_\ell} - m)/2} \cdot |M_v^{K_\ell}|^{-1} \prod_{i=1}^{m} (q_v^{2i} - 1) \Big\} \Big\{ q_v^{(\dim M_v^L - m)/2} \cdot |M_v^L|^{-1} \cdot \prod_{i=1}^{m} (q_v^{2i} - 1) \Big\}^{-1} \\ &\le q_v^{-1/2} (q_v^{n_{v,\nu_v/2}} + 1). \end{aligned} \tag{3.6.7}$$

If $n+1 = 2m$, then

$$\begin{aligned} \frac{\lambda_v^{K_\ell}}{\lambda_v^L} &= \Big\{ q_v^{(\dim M_v^{K_\ell} - m+1)/2} \cdot |M_v^{K_\ell}|^{-1} \cdot \prod_{i=1}^{m-1} (q_v^{2i} - 1) \Big\} \Big\{ q_v^{(\dim M_v^L - m)/2} \cdot |M_v^L|^{-1} \cdot \prod_{i=1}^{m} (q_v^{2i} - 1) \Big\}^{-1} \\ &\le \frac{q_v^{n_{v,\nu_v}/2} + 1}{q_v^{n+1} - 1}. \end{aligned} \tag{3.6.8}$$

If $(\nu_p, n_{p,\nu_p}) = (\text{odd}, \text{odd})$, then

$$\begin{aligned} M_v^L &= \operatorname{Ker}(\det : H(n_{v,0}) \times \cdots \times \mathrm{O}(n_{v,\nu_v}) \times \cdots \times H(n_{v,k_v}) \to \mathfrak{f}_v^1), \\ M_v^{K_\ell} &= \operatorname{Ker}(\det : H(n_{v,0}) \times \cdots \times^{(2)} \mathrm{O}(n_{v,\nu_v} - 1) \times \cdots \times H(n_{v,k_v}) \to \mathfrak{f}_v^1). \end{aligned}$$

This implies

$$\begin{aligned} \frac{|M_v^L|}{|M_v^{K_\ell}|} &\le \frac{|\mathrm{O}(n_{v,\nu_v})|}{|{}^{(2)}\mathrm{O}(n_{v,\nu_v} - 1)|} \\ &\le q_v^{(n_{v,\nu_v} - 1)/2} (q_v^{(n_{v,\nu_v} - 1)/2} + 1). \end{aligned}$$

and

$$\dim M_v^L - \dim M_v^{K_\ell} = \frac{n_{v,\nu_v}(n_{v,\nu_v}-1)}{2} - \frac{(n_{v,\nu_v}-1)(n_{v,\nu_v}-2)}{2} = n_{v,\nu_v} - 1.$$

Hence, if $n+1 = 2m+1$, then

$$\begin{aligned} \frac{\lambda_v^{K_\ell}}{\lambda_v^L} &= \Big\{ q_v^{(\dim M_v^{K_\ell} - m)/2} \cdot |M_v^{K_\ell}|^{-1} \cdot \prod_{i=1}^{m} (q_v^{2i} - 1) \Big\} \Big\{ q_v^{(\dim M_v^L - m)/2} \cdot |M_v^L|^{-1} \cdot \prod_{i=1}^{m} (q_v^{2i} - 1) \Big\}^{-1} \\ &\le q_v^{(n_{v,\nu_v} - 1)/2} + 1. \end{aligned} \tag{3.6.9}$$

If $n+1 = 2m$, then

$$\begin{aligned} \frac{\lambda_v^{K_\ell}}{\lambda_v^L} &= \Big\{ q_v^{(\dim M_v^{K_\ell} - m+1)/2} \cdot |M_v^{K_\ell}|^{-1} \cdot \prod_{i=1}^{m-1} (q_v^{2i} - 1) \Big\} \Big\{ q_v^{(\dim M_v^L - m)/2} \cdot |M_v^L|^{-1} \cdot \prod_{i=1}^{m} (q_v^{2i} - 1) \Big\}^{-1} \\ &\le q_v^{1/2} \cdot \frac{q_v^{(n_{v,\nu_v} - 1)/2} + 1}{q_v^{n+1} - 1}. \end{aligned} \tag{3.6.10}$$

3.6.3. Ramified case: $v = 2$**.** Cho [**29, 30**] classified the maximal reductive quotient of the mod $\mathfrak{p}$ reduction of the integral models for a ramified quadratic extension $F_2/\mathbb{Q}_2$. He divided the problem into *Case I* and *Case II*, according to the structure of the lower ramification groups of the Galois group $\mathrm{Gal}(F_2/\mathbb{Q}_2)$; see [**29**, Introduction]. We also use his division.

3.6.3.1. **Case I**. Let

$$H_1^L(n_{2,i}) := \begin{cases} \mathrm{Sp}(\{n_{2,i}\}) & (i : \text{even and } L_{2,i} : \text{type } II), \\ \mathrm{Sp}(\{n_{2,i}-1\}) & (i : \text{even and } L_{2,i} : \text{type } I^o), \\ \mathrm{Sp}(\{n_{2,i}-2\}) & (i : \text{even and } L_{2,i} : \text{type } I^e), \\ {}^{(2)}\mathrm{O}(n_{2,i}) & (i : \text{odd and } L_{2,i} : \text{free}), \\ {}^{(2)}\,\mathrm{SO}(n_{2,i}+1) & (i : \text{odd and } L_{2,i} : \text{bounded}). \end{cases}$$

We define $H_1^{K_\ell}(n_{2,i}) := H_1^L(n_{2,i})$ if $i \neq \nu_2$ and

$$H_1^{K_\ell}(n_{2,\nu_2}-1) := \begin{cases} \mathrm{Sp}(\{n_{2,\nu_2}-1\}) & (\nu_2 : \text{even and } K_{\ell,2,\nu_2} : \text{type } II), \\ \mathrm{Sp}(\{n_{2,\nu_2}-2\}) & (\nu_2 : \text{even and } K_{\ell,2,\nu_2} : \text{type } I^o), \\ \mathrm{Sp}(\{n_{2,\nu_2}-3\}) & (\nu_2 : \text{even and } K_{\ell,2,\nu_2} : \text{type } I^e), \\ {}^{(2)}\mathrm{O}(n_{2,\nu_2}-1) & (\nu_2 : \text{odd and } K_{\ell,2,\nu_2} : \text{free}), \\ {}^{(2)}\,\mathrm{SO}(n_{2,\nu_2}) & (\nu_2 : \text{odd and } K_{\ell,2,\nu_2} : \text{bounded}). \end{cases}$$

See [**29**, Definition 2.1, Remark 2.6] for the definitions of the types of lattices. We will not use these definitions here, except that the type I^o (resp. I^e) means the rank is odd (resp. even) and evaluate the volume independently of the types of lattices. Moreover, while Cho [**29**, Remark 4.7] distinguishes between cases that even-dimensional orthogonal groups are split or non-split, we will not use this description. By [**29**, Theorem 4.12], we can determine the structure of the maximal reductive quotient of the mod $\mathfrak{p}$ reduction of the smooth integral model of $\mathrm{SU}(L \otimes \mathbb{Z}_2)$ and $\mathrm{SU}(K_\ell \otimes \mathbb{Z}_2)$.

$$\begin{aligned} M_2^L &= \mathrm{Ker}(\det : H_1^L(n_{2,0}) \times \cdots \times H_1^L(n_{2,\nu_2}) \times \cdots \times H_1^L(n_{2,k_2}) \times (\mathbb{Z}/2\mathbb{Z})^{\beta_L} \to \mathfrak{f}_v^1), \\ M_2^{K_\ell} &= \mathrm{Ker}(\det : H_1^{K_\ell}(n_{2,0}) \times \cdots \times H_1^{K_\ell}(n_{2,\nu_2}-1) \times \cdots \times H_1^{K_\ell}(n_{2,k_2}) \times (\mathbb{Z}/2\mathbb{Z})^{\beta_{K_\ell}} \to \mathfrak{f}_v^1). \end{aligned}$$

If $(\nu_2, n_{2,\nu_2}) = (\text{even}, \text{even})$, then $H_1^L(n_{2,\nu_2}) = \mathrm{Sp}(n_{2,\nu_2})$ or $\mathrm{Sp}(n_{2,\nu_2}-2)$, and $H_1^{K_\ell}(n_{2,\nu_2} - 1) = \mathrm{O}(n_{2,\nu_2}-1) \cong \mathrm{Sp}(n_{2,\nu_2}-2)$, according to the type of L_{2,ν_2}. The integers β_L and β_{K_ℓ} are defined in [**29**, Lemma 4.6] and satisfy $\beta_L, \beta_{K_\ell} \leq n+1$ and $\beta_L \leq \beta_{K_\ell} + 2$. Since

$$\begin{aligned} \frac{|\,\mathrm{Sp}(n_{2,\nu_2})|}{2^{\dim \mathrm{Sp}(n_{2,\nu_2})/2}} &\geq \frac{|\,\mathrm{Sp}(n_{2,\nu_2}-2)|}{2^{\dim \mathrm{Sp}(n_{2,\nu_2}-2)/2}} \quad (n_{2,\nu_2} > 2), \\ \frac{|\,\mathrm{Sp}(2)|}{2^{\dim \mathrm{Sp}(2)/2}} &= 3 \cdot 2^{-1/2}, \end{aligned}$$

we can bound the ratio of local factors independently of the type of a lattice:

$$\begin{aligned} \frac{|M_2^L|}{2^{\dim M_2^L/2}} \cdot \frac{2^{\dim M_2^{K_\ell}/2}}{|M_2^{K_\ell}|} &\leq \frac{|\,\mathrm{Sp}(n_{2,\nu_2})|}{2^{\dim \mathrm{Sp}(n_{2,\nu_2})/2}} \cdot \frac{2^{\dim \mathrm{Sp}(n_{2,\nu_2}-2)/2}}{|\,\mathrm{Sp}(n_{2,\nu_2}-2)|} \cdot 2^{(\beta_L - \beta_{K_\ell})/2} \\ &\leq 2^{\frac{1}{2}}(2^{n_{2,\nu_2}}-1) \quad (\text{This also holds for } n_{2,\nu_2} = 2). \end{aligned}$$

Hence, if $n+1=2m+1$, then

$$\frac{\lambda_2^{K_\ell}}{\lambda_2^L} = \Big\{2^{(\dim M_2^{K_\ell}-m)/2}\cdot\frac{\prod_{i=1}^{m}(2^{2i}-1)}{|M_2^{K_\ell}|}\Big\}\Big\{2^{(\dim M_2^L-m)/2}\cdot\frac{\prod_{i=1}^{m}(2^{2i}-1)}{|M_2^L|}\Big\}^{-1}$$
$$\le 2^{1/2}(2^{n_{2,\nu_2}}-1). \tag{3.6.11}$$

If $n+1=2m$, then

$$\frac{\lambda_2^{K_\ell}}{\lambda_2^L} = \Big\{2^{(\dim M_2^{K_\ell}-m+1)/2}\cdot\frac{\prod_{i=1}^{m-1}(2^{2i}-1)}{|M_2^{K_\ell}|}\Big\}\Big\{2^{(\dim M_2^L-m)/2}\cdot\frac{\prod_{i=1}^{m}(2^{2i}-1)}{|M_2^L|}\Big\}^{-1}$$
$$\le 2\cdot\frac{2^{n_{2,\nu_2}}-1}{2^{n+1}-1}. \tag{3.6.12}$$

If $(\nu_2, n_{2,\nu_2}) = (\text{even}, \text{odd})$, then $H_1^L(n_{2,\nu_2}) = \mathrm{O}(n_{2,\nu_2}) \cong \mathrm{Sp}(n_{2,\nu_2}-1)$, and $H_1^{K_\ell}(n_{2,\nu_2}-1) = \mathrm{Sp}(n_{2,\nu_2}-1)$ or $\mathrm{Sp}(n_{2,\nu_2}-3)$, according to the type of $K_{\ell,2,\nu_2}$. Thus, we can bound the ratio of local factors independently of the type of a lattice:

$$\frac{|M_2^L|}{2^{\dim M_2^L/2}}\cdot\frac{2^{\dim M_2^{K_\ell}/2}}{|M_2^{K_\ell}|} \le \frac{|\,\mathrm{Sp}(n_{2,\nu_2}-1)|}{2^{\dim \mathrm{Sp}(n_{2,\nu_2}-1)/2}}\cdot\frac{2^{\dim \mathrm{Sp}(n_{2,\nu_2}-3)/2}}{|\,\mathrm{Sp}(n_{2,\nu_2}-3)|}\cdot 2^{(\beta_L-\beta_{K_\ell})/2}$$
$$\le 2^{\frac12}(2^{n_{2,\nu_2}}-1).$$

Hence, if $n+1=2m+1$, then

$$\frac{\lambda_2^{K_\ell}}{\lambda_2^L} = \Big\{2^{(\dim M_2^{K_\ell}-m)/2}\cdot\frac{\prod_{i=1}^{m}(2^{2i}-1)}{|M_2^{K_\ell}|}\Big\}\Big\{2^{(\dim M_2^L-m)/2}\cdot\frac{\prod_{i=1}^{m}(2^{2i}-1)}{|M_2^L|}\Big\}^{-1}$$
$$\le 2^{1/2}(2^{n_{2,\nu_2}}-1). \tag{3.6.13}$$

If $n+1=2m$, then

$$\frac{\lambda_2^{K_\ell}}{\lambda_2^L} = \Big\{2^{(\dim M_2^{K_\ell}-m+1)/2}\cdot\frac{\prod_{i=1}^{m-1}(2^{2i}-1)}{|M_2^{K_\ell}|}\Big\}\Big\{2^{(\dim M_2^L-m)/2}\cdot\frac{\prod_{i=1}^{m}(2^{2i}-1)}{|M_2^L|}\Big\}^{-1}$$
$$\le 2\cdot\frac{2^{n_{2,\nu_2}}-1}{2^{n+1}-1}. \tag{3.6.14}$$

If $(\nu_2, n_{2,\nu_2}) = (\text{odd}, \text{even})$, then $H_1^L(n_{2,\nu_2}) = {}^{(2)}\,\mathrm{O}(n_{2,\nu_2})$ or $\mathrm{SO}(n_{2,\nu_2}+1)$, and $H_1^{K_\ell}(n_{2,\nu_2}-1) = \mathrm{O}(n_{2,\nu_2}-1)$ or ${}^{(2)}\,\mathrm{SO}(n_{2,\nu_2})$, according to the type of L_{2,ν_2} and $K_{\ell,2,\nu_2}$. Since

$$\frac{|\,\mathrm{SO}(n_{2,\nu_2}+1)|}{2^{\frac{\dim \mathrm{SO}(n_{2,\nu_2}+1)}{2}}} \ge \frac{|{}^2\mathrm{O}(n_{2,\nu_2})|}{2^{\dim^{(2)}\mathrm{O}(n_{2,\nu_2})/2}} \ge \frac{|\mathrm{O}(n_{2,\nu_2})|}{2^{\dim \mathrm{O}(n_{2,\nu_2})/2}} \ge 1 \quad (n_{2,\nu_2}>2),$$
$$3\cdot 2^{1/2} = \frac{|{}^2\mathrm{O}(2)|}{2^{\dim^2\mathrm{O}(2)/2}} \ge \frac{|\,\mathrm{SO}(3)|}{2^{\dim \mathrm{SO}(3)/2}} \ge \frac{|\mathrm{O}(2)|}{2^{\dim \mathrm{O}(2)/2}},$$
$$\frac{|{}^{(2)}\,\mathrm{SO}(n_{2,\nu_2})|}{2^{\dim^{(2)}\mathrm{SO}(n_{2,\nu_2})/2}} \ge \frac{|\mathrm{O}(n_{2,\nu_2}-1)|}{2^{\dim \mathrm{O}(n_{2,\nu_2}-1)/2}},$$

we can bound the ratio of local factors, independently of the type of a lattice:

$$\begin{aligned}\frac{|M_2^L|}{2^{\dim M_2^L/2}}\cdot\frac{2^{\dim M_2^{K_\ell}/2}}{|M_2^{K_\ell}|} &\leq 3\cdot 2^{1/2}\cdot\frac{|\,\mathrm{SO}(n_{2,\nu_2}+1)|}{2^{\dim \mathrm{SO}(n_{2,\nu_2}+1)/2}}\cdot\frac{2^{\dim \mathrm{O}(n_{2,\nu_2}-1)/2}}{|\mathrm{O}(n_{2,\nu_2}-1)|}\cdot 2^{(\beta_L-\beta_{K_\ell})/2}\\ &\leq 3\cdot 2(2^{n_{2,\nu_2}}-1).\end{aligned}$$

Hence, if $n+1=2m+1$, then

$$\begin{aligned}\frac{\lambda_2^{K_\ell}}{\lambda_2^L} &= \Big\{2^{(\dim M_2^{K_\ell}-m)/2}\cdot|M_2^{K_\ell}|^{-1}\cdot\prod_{i=1}^m(2^{2i}-1)\Big\}\Big\{2^{(\dim M_2^L-m)/2}\cdot|M_2^L|^{-1}\cdot\prod_{i=1}^m(2^{2i}-1)\Big\}^{-1}\\ &\leq 3\cdot 2(2^{n_{2,\nu_2}}-1).\end{aligned}\tag{3.6.15}$$

If $n+1=2m$, then

$$\begin{aligned}\frac{\lambda_2^{K_\ell}}{\lambda_2^L} &= \Big\{2^{(\dim M_2^{K_\ell}-m+1)/2}\cdot\frac{\prod_{i=1}^{m-1}(2^{2i}-1)}{|M_2^{K_\ell}|}\Big\}\Big\{2^{(\dim M_2^L-m)/2}\cdot\frac{\prod_{i=1}^{m}(2^{2i}-1)}{|M_2^L|}\Big\}^{-1}\\ &\leq 3\cdot 2^{3/2}\cdot\frac{2^{n_{2,\nu_2}}-1}{2^{n+1}-1}.\end{aligned}\tag{3.6.16}$$

If $(\nu_2,n_{2,\nu_2})=(\mathrm{odd},\mathrm{odd})$, then $H_1^L(n_{2,\nu_2})=\mathrm{O}(n_{2,\nu_2})$ or ${}^{(2)}\,\mathrm{SO}(n_{2,\nu_2}+1)$, and $H_1^{K_\ell}(n_{2,\nu_2}-1)={}^{(2)}\,\mathrm{O}(n_{2,\nu_2}-1)$ or $\mathrm{SO}(n_{2,\nu_2})$, according to the type of L_{2,ν_2} and $K_{\ell,2,\nu_2}$. we can bound the ratio of local factors, independently of the type of a lattice:

$$\begin{aligned}\frac{|M_2^L|}{2^{\dim M_2^L/2}}\cdot\frac{2^{\dim M_2^{K_\ell}/2}}{|M_2^{K_\ell}|} &\leq \frac{|{}^2\,\mathrm{SO}(n_{2,\nu_2}+1)|}{2^{\dim {}^2\,\mathrm{SO}(n_{2,\nu_2}+1)/2}}\cdot\frac{2^{\dim \mathrm{O}(n_{2,\nu_2}-1)/2}}{|\mathrm{O}(n_{2,\nu_2}-1)|}\cdot 2^{(\beta_L-\beta_{K_\ell})/2}\\ &\leq 2^{1/2}(2^{(n_{2,\nu_2}+1)/2}+1)(2^{(n_{2,\nu_2}-1)/2}+1).\end{aligned}$$

Hence, if $n+1=2m+1$, then

$$\begin{aligned}\frac{\lambda_2^{K_\ell}}{\lambda_2^L} &= \Big\{2^{(\dim M_2^{K_\ell}-m)/2}\cdot|M_2^{K_\ell}|^{-1}\cdot\prod_{i=1}^m(2^{2i}-1)\Big\}\Big\{2^{(\dim M_2^L-m)/2}\cdot|M_2^L|^{-1}\cdot\prod_{i=1}^m(2^{2i}-1)\Big\}^{-1}\\ &\leq 2^{1/2}(2^{(n_{2,\nu_2}+1)/2}+1)(2^{(n_{2,\nu_2}-1)/2}+1).\end{aligned}\tag{3.6.17}$$

If $n+1=2m$, then

$$\begin{aligned}\frac{\lambda_2^{K_\ell}}{\lambda_2^L} &= \Big\{2^{(\dim M_2^{K_\ell}-m+1)/2}\cdot|M_2^{K_\ell}|^{-1}\cdot\prod_{i=1}^{m-1}(2^{2i}-1)\Big\}\Big\{2^{(\dim M_2^L-m)/2}\cdot|M_2^L|^{-1}\cdot\prod_{i=1}^m(2^{2i}-1)\Big\}^{-1}\\ &\leq 2\cdot\frac{(2^{(n_{2,\nu_2}+1)/2}+1)(2^{(n_{2,\nu_2}-1)/2}+1)}{2^{n+1}-1}.\end{aligned}\tag{3.6.18}$$

3.6.3.2. **Case II**. Let

$$H_2^L(n_{2,i}) := \begin{cases} {}^{(2)}\mathrm{O}(n_{2,i}) & (i : \text{even and } L_{2,i} : \text{type } II, \text{ free}), \\ {}^{(2)}\,\mathrm{SO}(n_{2,i}+1) & (i : \text{even and } L_{2,i} : \text{type } II, \text{ bounded}), \\ {}^{(2)}\,\mathrm{SO}(n_{2,i}) & (i : \text{even and } L_{2,i} : \text{type } I^o), \\ {}^{(2)}\,\mathrm{SO}(n_{2,i}-1) & (i : \text{even and } L_{2,i} : \text{type } I^e), \\ \mathrm{Sp}(\{n_{2,i}\}) & (i : \text{odd and } L_{2,i} : \text{type } II, \text{ or type } I \text{ and bounded}), \\ \mathrm{Sp}(\{n_{2,i}-2\}) & (i : \text{odd and } L_{2,i} : \text{type } I, \text{ free}). \end{cases}$$

We define $H_2^{K_\ell}(n_{2,i}) := H_2^L(n_{2,i})$ if $i \neq \nu_2$ and

$$H_2^{K_\ell}(n_{2,\nu_2}-1) := \begin{cases} {}^{(2)}\mathrm{O}(n_{2,\nu_2}-1) & (\nu_2 : \text{even and } K_{\ell,2,\nu_2} : \text{type } II, \text{ free}), \\ {}^{(2)}\,\mathrm{SO}(n_{2,\nu_2}) & (\nu_2 : \text{even and } K_{\ell,2,\nu_2} : \text{type } II, \text{ bounded}), \\ {}^{(2)}\,\mathrm{SO}(n_{2,\nu_2}-1) & (\nu_2 : \text{even and } K_{\ell,2,\nu_2} : \text{type } I^o), \\ {}^{(2)}\,\mathrm{SO}(n_{2,\nu_2}-2) & (\nu_2 : \text{even and } K_{\ell,2,\nu_2} : \text{type } I^e), \\ \mathrm{Sp}(\{n_{2,\nu_2}-1\}) & (\nu_2 : \text{odd and } K_{\ell,2,\nu_2} : \text{type } II, \text{ or type } I \text{ and bounded}), \\ \mathrm{Sp}(\{n_{2,\nu_2}-3\}) & (\nu_2 : \text{odd and } K_{\ell,2,\nu_2} : \text{type } I, \text{ free}). \end{cases}$$

Although Cho [**30**, Remark 4.6] distinguishes cases in which the even-dimensional orthogonal groups are split or non-split we will not use this description. From [**30**, Theorem 4.11], we can determine the structure of the maximal reductive quotient of the mod $_p$ reduction of the smooth integral model of $\mathrm{SU}(L \otimes \mathbb{Z}_2)$ and $\mathrm{SU}(K_\ell \otimes \mathbb{Z}_2)$.

$$M_2^L = \mathrm{Ker}(\det : H_2^L(n_{2,0}) \times \cdots \times H_2^L(n_{2,\nu_2}) \times \cdots \times H_2^L(n_{2,k_2}) \times (\mathbb{Z}/2\mathbb{Z})^{\beta'_L} \to \mathfrak{f}_v^1),$$
$$M_2^{K_\ell} = \mathrm{Ker}(\det : H_2^{K_\ell}(n_{2,0}) \times \cdots \times H_2^{K_\ell}(n_{2,\nu_2}-1) \times \cdots \times H_2^{K_\ell}(n_{2,k_2}) \times (\mathbb{Z}/2\mathbb{Z})^{\beta'_{K_\ell}} \to \mathfrak{f}_v^1).$$

Here, β'_L and β'_{K_ℓ} are integers defined in [**30**, Lemma 4.5] and satisfying $\beta'_L, \beta'_{K_\ell} \leq n+1$ and $\beta'_L \leq \beta'_{K_\ell} + 4$.

Moreover, for later, we remark that

$$1 \leq \frac{|\,\mathrm{SO}(n_{2,\nu_2}-1)|}{2^{\mathrm{SO}(n_{2,\nu_2}-1)/2}} \leq \frac{|{}^{(2)}\mathrm{O}(n_{2,\nu_2})|}{2^{\dim^{(2)}\mathrm{O}(n_{2,\nu_2})/2}} \leq \frac{|\,\mathrm{SO}(n_{2,\nu_2}+1)|}{2^{\dim \mathrm{SO}(n_{2,\nu_2}+1)/2}} \leq \frac{|{}^{(2)}\,\mathrm{SO}(n_{2,\nu_2})|}{2^{(2)\dim \mathrm{SO}(n_{2,\nu_2})/2}} \ (n_{2,\nu_2} \neq 2 : \text{even}),$$

$$2^{-1/2} = \frac{|\,\mathrm{SO}(2)|}{2^{\dim \mathrm{SO}(2)/2}} \leq 1 = \frac{|\,\mathrm{SO}(1)|}{2^{\dim \mathrm{SO}(1)/2}} \leq \frac{|\mathrm{O}(2)|}{2^{\dim \mathrm{O}(2)/2}}$$
$$\leq \frac{|\,\mathrm{SO}(3)|}{2^{\dim \mathrm{SO}(3)/2}} = \frac{|^2\,\mathrm{SO}(2)|}{2^{\dim^2 \mathrm{SO}(2)/2}} \leq \frac{|^2\mathrm{O}(2)|}{2^{\dim^2 \mathrm{O}(2)/2}} = 2^{1/2} \cdot 3,$$

$$\frac{|{}^{(2)}\,\mathrm{SO}(n_{2,\nu_2}-1)|}{2^{\dim^{(2)}\mathrm{SO}(n_{2,\nu_2}-1)/2}} \leq \frac{|\mathrm{O}(n_{2,\nu_2})|}{2^{\dim \mathrm{O}(n_{2,\nu_2})/2}} = \frac{|\,\mathrm{SO}(n_{2,\nu_2})|}{2^{\dim \mathrm{SO}(n_{2,\nu_2})/2}} \leq \frac{|{}^{(2)}\,\mathrm{SO}(n_{2,\nu_2}+1)|}{2^{\dim^{(2)}\mathrm{SO}(n_{2,\nu_2}+1)/2}} \quad (n_{2,\nu_2} \neq 1 : \text{odd}),$$

$$2^{-1/2} = \frac{|\,\mathrm{SO}(2)|}{2^{\dim \mathrm{SO}(2)/2}} \leq 1 = \frac{|\mathrm{O}(1)|}{2^{\dim \mathrm{O}(1)/2}} = \frac{|\,\mathrm{SO}(1)|}{2^{\dim \mathrm{SO}(1)/2}} \leq \frac{|^2\,\mathrm{SO}(2)|}{2^{\dim^2 \mathrm{SO}(2)/2}} = 3 \cdot 2^{-1/2}.$$

If $(\nu_2, n_{2,\nu_2}) = (\text{even}, \text{even})$, then $H_2^L(n_{2,\nu_2}) =^{(2)} \mathrm{O}(n_{2,\nu_2})$, $\mathrm{SO}(n_{2,\nu_2}+1)$ or $\mathrm{SO}(n_{2,\nu_2}-1)$, and $H_2^{K_\ell}(n_{2,\nu_2}-1) = \mathrm{O}(n_{2,\nu_2}-1)$, ${}^{(2)}\,\mathrm{SO}(n_{2,\nu_2})$ or $\mathrm{SO}(n_{2,\nu_2}-1)$, according to the type of L_{2,ν_2} and $K_{\ell,2,\nu_2}$. Thus, we can bound the ratio of local factors independently of the type of a lattice:

$$\frac{|M_2^L|}{2^{\dim M_2^L/2}} \cdot \frac{2^{\dim M_2^{K_\ell}/2}}{|M_2^{K_\ell}|} \leq \frac{|\,\mathrm{SO}(n_{2,\nu_2}+1)|}{2^{\dim \mathrm{SO}(n_{2,\nu_2}+1)/2}} \cdot 2^{1/2} \cdot 3 \frac{2^{\dim \mathrm{SO}(n_{2,\nu_2}-1)/2}}{|\,\mathrm{SO}(n_{2,\nu_2}-1)|} \cdot 2^{(\beta'_L - \beta'_{K_\ell})/2}$$
$$\leq 2^2 \cdot 3(2^{n_{2,\nu_2}} - 1) \quad (\text{This also holds for } n_{2,\nu_2} = 2).$$

Hence, if $n+1=2m+1$, then

$$\frac{\lambda_2^{K_\ell}}{\lambda_2^L} = \Big\{2^{(\dim M_2^{K_\ell}-m)/2}\cdot|M_2^{K_\ell}|^{-1}\cdot\prod_{i=1}^{m}(2^{2i}-1)\Big\}\Big\{2^{(\dim M_2^L-m)/2}\cdot|M_2^L|^{-1}\cdot\prod_{i=1}^{m}(2^{2i}-1)\Big\}^{-1}$$

(3.6.19)

$$\leq 2^2\cdot 3(2^{n_{2,\nu_2}}-1).$$

If $n+1=2m$, then

$$\frac{\lambda_2^{K_\ell}}{\lambda_2^L} = \Big\{2^{(\dim M_2^{K_\ell}-m+1)/2}\cdot|M_2^{K_\ell}|^{-1}\cdot\prod_{i=1}^{m-1}(2^{2i}-1)\Big\}\Big\{2^{(\dim M_2^L-m)/2}\cdot|M_2^L|^{-1}\cdot\prod_{i=1}^{m}(2^{2i}-1)\Big\}^{-1}$$

(3.6.20)

$$\leq 2^{5/2}\cdot 3\cdot\frac{2^{n_{2,\nu_2}}-1}{2^{n+1}-1}.$$

If $(\nu_2, n_{2,\nu_2}) = (\text{even}, \text{odd})$, then $H_2^L(n_{2,\nu_2}) = \mathrm{O}(n_{2,\nu_2})$, ${}^{(2)}\mathrm{SO}(n_{2,\nu_2}+1)$ or $\mathrm{SO}(n_{2,\nu_2})$, and $H_2^{K_\ell}(n_{2,\nu_2}-1) ={}^{(2)}\mathrm{O}(n_{2,\nu_2}-1)$, $\mathrm{SO}(n_{2,\nu_2})$ or $\mathrm{SO}(n_{2,\nu_2}-2)$, according to the type of L_{2,ν_2} and $K_{\ell,2,\nu_2}$. Thus, we can bound the ratio of local factors independently of the type of a lattice:

$$\frac{|M_2^L|}{2^{\dim M_2^L/2}}\cdot\frac{2^{\dim M_2^{K_\ell}/2}}{|M_2^{K_\ell}|} \leq \frac{|{}^2\,\mathrm{SO}(n_{2,\nu_2}+1)|}{2^{\dim^2\mathrm{SO}(n_{2,\nu_2}+1)/2}}\cdot 2^{\frac{1}{2}}\cdot\frac{2^{\dim\mathrm{SO}(n_{2,\nu_2}-1)/2}}{|\mathrm{SO}(n_{2,\nu_2}-1)|}\cdot 2^{(\beta'_L-\beta'_{K_\ell})/2}$$

$$\leq 2^{3/2}\cdot(2^{(n_{2,\nu_2}+1)/2}+1)(2^{(n_{2,\nu_2}-1)/2}+1)\quad\text{(This also holds for } n_{2,\nu_2}=1\text{).}$$

Hence, if $n+1=2m+1$, then

$$\frac{\lambda_2^{K_\ell}}{\lambda_2^L} = \Big\{2^{(\dim M_2^{K_\ell}-m)/2}\cdot|M_2^{K_\ell}|^{-1}\cdot\prod_{i=1}^{m}(2^{2i}-1)\Big\}\Big\{2^{(\dim M_2^L-m)/2}\cdot|M_2^L|^{-1}\cdot\prod_{i=1}^{m}(2^{2i}-1)\Big\}^{-1}$$

(3.6.21)

$$\leq 2^{3/2}\cdot(2^{(n_{2,\nu_2}+1)/2}+1)(2^{(n_{2,\nu_2}-1)/2}+1).$$

If $n+1=2m$, then

$$\frac{\lambda_2^{K_\ell}}{\lambda_2^L} = \Big\{2^{(\dim M_2^{K_\ell}-m+1)/2}\cdot|M_2^{K_\ell}|^{-1}\cdot\prod_{i=1}^{m-1}(2^{2i}-1)\Big\}\Big\{2^{(\dim M_2^L-m)/2}\cdot|M_2^L|^{-1}\cdot\prod_{i=1}^{m}(2^{2i}-1)\Big\}^{-1}$$

(3.6.22)

$$\leq 2^2\cdot 3\cdot\frac{(2^{(n_{2,\nu_2}+1)/2}+1)(2^{(n_{2,\nu_2}-1)/2}+1)}{2^{n+1}-1}.$$

If $(\nu_2, n_{2,\nu_2}) = (\text{odd}, \text{even})$, then $H_2^L(n_{2,\nu_2}) = \mathrm{Sp}(n_{2,\nu_2})$ or $\mathrm{Sp}(n_{2,\nu_2}-2)$, and $H_2^{K_\ell}(n_{2,\nu_2}-1) = \mathrm{Sp}(n_{2,\nu_2}-2)$ or $\mathrm{Sp}(n_{2,\nu_2}-4)$, according to the type of L_{2,ν_2} and $K_{\ell,2,\nu_2}$. Thus, we can bound the ratio of local factors independently of the type of a lattice:

$$\frac{|M_2^L|}{2^{\dim M_2^L/2}}\cdot\frac{2^{\dim M_2^{K_\ell}/2}}{|M_2^{K_\ell}|} \leq \frac{|\mathrm{Sp}(n_{2,\nu_2})|}{2^{\dim\mathrm{Sp}(n_{2,\nu_2})/2}}\cdot\frac{2^{\dim\mathrm{Sp}(n_{2,\nu_2}-4)/2}}{|\mathrm{Sp}(n_{2,\nu_2}-4)|}\cdot 2^{(\beta'_L-\beta'_{K_\ell})/2}$$

$$\leq 2^{3/2}\cdot(2^{n_{2,\nu_2}}-1)(2^{n_{2,\nu_2}-2}-1)\quad\text{(This also holds for } n_{2,\nu_2}=2,4\text{).}$$

Hence, if $n+1=2m+1$, then

$$\frac{\lambda_2^{K_\ell}}{\lambda_2^L} = \Big\{2^{(\dim M_2^{K_\ell}-m)/2}\cdot|M_2^{K_\ell}|^{-1}\cdot\prod_{i=1}^{m}(2^{2i}-1)\Big\}\Big\{2^{(\dim M_2^L-m)/2}\cdot|M_2^L|^{-1}\cdot\prod_{i=1}^{m}(2^{2i}-1)\Big\}^{-1}$$

(3.6.23)
$$\leq 2^{3/2} \cdot (2^{n_{2,\nu_2}} - 1)(2^{n_{2,\nu_2}-2} - 1).$$

If $n+1 = 2m$, then

$$\frac{\lambda_2^{K_\ell}}{\lambda_2^L} = \Big\{ 2^{(\dim M_2^{K_\ell} - m+1)/2} \cdot |M_2^{K_\ell}|^{-1} \cdot \prod_{i=1}^{m-1} (2^{2i} - 1) \Big\} \Big\{ 2^{(\dim M_2^L - m)/2} \cdot |M_2^L|^{-1} \cdot \prod_{i=1}^{m} (2^{2i} - 1) \Big\}^{-1}$$

(3.6.24)
$$\leq 2^2 \cdot \frac{(2^{n_{2,\nu_2}} - 1)(2^{n_{2,\nu_2}-2} - 1)}{2^{n+1} - 1}.$$

If $(\nu_2, n_{2,\nu_2}) = (\text{odd}, \text{odd})$, then $H_2^L(n_{2,\nu_2}) = \mathrm{Sp}(n_{2,\nu_2} - 1)$ or $\mathrm{Sp}(n_{2,\nu_2} - 3)$, and $H_2^{K_\ell}(n_{2,\nu_2} - 1) = \mathrm{Sp}(n_{2,\nu_2} - 1)$ or $\mathrm{Sp}(n_{2,\nu_2} - 3)$, according to the type of L_{2,ν_2} and $K_{\ell,2,\nu_2}$. Thus, we can bound the ratio of local factors independently of the type of a lattice:

$$\begin{aligned} \frac{|M_2^L|}{2^{\dim M_2^L/2}} \cdot \frac{2^{\dim M_2^{K_\ell}/2}}{|M_2^{K_\ell}|} &\leq \frac{|\,\mathrm{Sp}(n_{2,\nu_2} - 1)|}{2^{\dim \mathrm{Sp}(n_{2,\nu_2}-1)/2}} \cdot \frac{2^{\dim \mathrm{Sp}(n_{2,\nu_2}-3)/2}}{|\,\mathrm{Sp}(n_{2,\nu_2} - 3)|} \cdot 2^{(\beta'_L - \beta'_{K_\ell})/2} \\ &\leq 2^{3/2} \cdot (2^{n_{2,\nu_2}} - 1) \quad \text{(This also holds for } n_{2,\nu_2} = 1, 3\text{)}. \end{aligned}$$

Hence, if $n+1 = 2m+1$, then

$$\frac{\lambda_2^{K_\ell}}{\lambda_2^L} = \Big\{ 2^{(\dim M_2^{K_\ell} - m)/2} \cdot |M_2^{K_\ell}|^{-1} \cdot \prod_{i=1}^{m} (2^{2i} - 1) \Big\} \Big\{ 2^{(\dim M_2^L - m)/2} \cdot |M_2^L|^{-1} \cdot \prod_{i=1}^{m} (2^{2i} - 1) \Big\}^{-1}$$

(3.6.25)
$$\leq 2^{3/2} \cdot (2^{n_{2,\nu_2}} - 1).$$

If $n+1 = 2m$, then

$$\frac{\lambda_2^{K_\ell}}{\lambda_2^L} = \Big\{ 2^{(\dim M_2^{K_\ell} - m+1)/2} \cdot |M_2^{K_\ell}|^{-1} \cdot \prod_{i=1}^{m-1} (2^{2i} - 1) \Big\} \Big\{ 2^{(\dim M_2^L - m)/2} \cdot |M_2^L|^{-1} \cdot \prod_{i=1}^{m} (2^{2i} - 1) \Big\}^{-1}$$

(3.6.26)
$$\leq 2^2 \cdot \frac{2^{n_{2,\nu_2}} - 1}{2^{n+1} - 1}.$$

3.7. Volume estimation

In this section, we will prove

$$V(L, F) \leq \frac{f_F^{odd}(m)}{S} \text{ or } \frac{f_F^{even}(m)}{S}$$

according to whether $n+1$ is odd or even. Consequently, this implies that $V(L, F)$ converges to 0 faster than the exponential function with respect to m.

Let $M > 0$ be a fixed positive integer. We say that L *satisfies* $P(M)$ if any prime divisor p_i of $D(L)$ is unramified and the inequality $2(n + 1 - n_{p_i,\nu_{p_i}}) \geq a_i/M$ holds for any p_i and any $[\ell] \in \mathcal{R}_{\text{split}}$, where a_i is defined by the exponent $D(L) = \prod p_i^{a_i}$.

3.7.1. Non-split vectors. Here, we need to prepare some tools to treat the "non-split case" as in [**107**] for unitary groups. For more details, see [**107**, Subsection 6.2].

Let $[\ell] \in \mathcal{R}_L(F,i)$ be a non-split vector so that it defines the proper sublattice $L' := \ell\mathscr{O}_F \oplus K_\ell \subsetneq L$. From Lemma 3.3.1, 3.3.2, 3.3.3, 3.3.4 and 3.3.5 , $[\ell] \in \mathcal{R}_L(F) \setminus \mathcal{R}_{\mathrm{split}}$ means

$$[\ell] \in \begin{cases} \mathcal{R}_L(F) \setminus \mathcal{R}_L(F,2)_I & (F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})), \\ \mathcal{R}_L(\mathbb{Q}(\sqrt{-1})) \setminus \mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_I & (F = \mathbb{Q}(\sqrt{-1})), \\ \mathcal{R}_L(\mathbb{Q}(\sqrt{-3})) \setminus \mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),6) & (F = \mathbb{Q}(\sqrt{-3})). \end{cases}$$

We call these vectors *non-split type* in accordance with [**107**]. Let

$$\Gamma_{L'} := \mathrm{U}(L) \cap \mathrm{U}(L')$$

in $\mathrm{U}(L \otimes_{\mathbb{Z}} \mathbb{Q})$.

On the basis of the definition of $R(F,2)_{II}$, let

$$\begin{aligned} T_L(F,2)_{II} &:= \{L' : \text{sublattice of } L \mid L' = \mathscr{O}_F\ell \oplus K_\ell \text{ for some } [\ell] \in \mathcal{R}_L(F,2)_{II}\}, \\ \mathcal{T}_L(F,2)_{II} &:= T_L(F,2)_{II}/\mathrm{U}(L). \end{aligned}$$

For $L' \in T_L(F,2)$, define

$$\begin{aligned} R[L'](F,2)_{II} &:= \{\ell' \in L' : \text{primitive in } L' \mid L' = \mathscr{O}_F\ell' \oplus (\ell'^{\perp} \cap L')\}, \\ \mathcal{R}[L'](F,2)_{II} &:= R[L'](F,2)_{II}/\mathrm{U}(L'). \end{aligned}$$

In accordance with $\mathcal{R}_L(F,2)_{III}, \mathcal{R}_L(F,2)_{IV}, \mathcal{R}_L(F,2)_V, \mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_{II}, \mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),3)$, for $\diamond \in \{2,3,6\}$ and $* \in \{II, III, IV, V\}$, define $T_L(F,\diamond)_*, \mathcal{T}_L(F,\diamond)_*, R[L'](F,\diamond)_*$ and $\mathcal{R}[L'](F,\diamond)_*$ as above. Note that

$$\mathcal{R}[L'](F,\diamond)_* = \begin{cases} \mathcal{R}_{L'}(F,2)_I & (F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})), \\ \mathcal{R}_{L'}(\mathbb{Q}(\sqrt{-1}),4) & (F = \mathbb{Q}(\sqrt{-1})), \\ \mathcal{R}_{L'}(\mathbb{Q}(\sqrt{-3}),6) & (F = \mathbb{Q}(\sqrt{-3})). \end{cases}$$

Lemma 3.7.1 ([**107**, Lemma 6.5]**).** *Fix $\diamond \in \{2,3,6\}$ and $* \in \{II, III, IV, V\}$. Then for a possible pair $(\diamond, *)$ that makes sense with $\mathcal{R}_L(F,\diamond)_*$, we obtain*

$$\sum_{[\ell] \in \mathcal{R}_L(F,\diamond)_*} \mathrm{vol}_{HM}(L, K_\ell) \leq \sum_{[L'] \in \mathcal{T}_L(F,\diamond)_*} [\mathrm{U}(L) : \Gamma_{L'}]\Big(\sum_{[\ell] \in \mathcal{R}[L'](F,\diamond)_*} \mathrm{vol}_{HM}(L', K'_\ell)\Big).$$

Proof. This can be proved in a similar way as [**107**, Lemma 6.5]. We can embed $\mathcal{R}_L(F,\diamond)_*$ into the formal disjoint union

$$\coprod_{[L'] \in \mathcal{T}_L(F,\diamond)_*} R[L']/\Gamma_{L'}.$$

Then, we have

$$\begin{aligned} \sum_{[\ell] \in \mathcal{R}_L(F,\diamond)_*} \mathrm{vol}_{HM}(L, K_\ell) &= \sum_{[\ell] \in \mathcal{R}_L(F,\diamond)_*} \frac{[\mathrm{U}(L) : \Gamma_{L'}]}{[\mathrm{U}(L') : \Gamma_{L'}]} \mathrm{vol}_{HM}(L', K_\ell) \\ &\leq \sum_{[L'] \in \mathcal{T}_L(F,\diamond)_*} \frac{[\mathrm{U}(L) : \Gamma_{L'}]}{[\mathrm{U}(L') : \Gamma_{L'}]}\Big(\sum_{[\ell] \in R[L'](F,\diamond)_*} \mathrm{vol}_{HM}(L', K'_\ell)\Big). \end{aligned}$$

Since the number of elements of fibers of the projection $R[L'](F,\diamond)_* \to \mathcal{R}[L'](F,\diamond)_*$ is at most $[\mathrm{U}(L'):\Gamma_{L'}]$, we find that

$$\sum_{[\ell]\in R[L'](F,\diamond)_*} \mathrm{vol}_{HM}(L',K'_\ell) \leq [\mathrm{U}(L'):\Gamma_{L'}]\cdot \sum_{[\ell]\in \mathcal{R}[L'](F,\diamond)_*} \mathrm{vol}_{HM}(L',K'_\ell).$$

□

Now, $[\mathrm{U}(L'):\Gamma_{L'}]$ equals the cardinality of the $\mathrm{U}(L)$-orbit of L' in $T_L(F,\diamond)_*$, so

$$\begin{aligned} &\sum_{[L']\in \mathcal{T}_L(F,\diamond)_*} [\mathrm{U}(L):\Gamma_{L'}] \\ &= |T_L(F,\diamond)_*| \\ &< \begin{cases} 2^{n+1} & ((F,\diamond,*)=(\mathrm{any},2,III),(\mathrm{any},2,IV),(\mathrm{any},2,V),(\mathbb{Q}(\sqrt{-1}),4,II)), \\ 3^{n+1} & ((F,\diamond,*)=(\mathbb{Q}(\sqrt{-3}),3,\emptyset)), \\ 4^{n+1} & ((F,\diamond,*)=(\mathrm{any},2,II)). \end{cases} \end{aligned} \tag{3.7.1}$$

Below, we bound the value

$$\sum_{[\ell]\in \mathcal{R}[L'](F,\diamond)_*} \mathrm{vol}_{HM}(L',K'_\ell)$$

independently of L', K'_ℓ and L. Note that $\mathcal{R}[L'](F,\diamond)_*$ is the set consisting of split reflective vectors of L'.

Let $\mathrm{SU}(L)$ be the subgroup of $\mathrm{U}(L)$ consisting of elements whose determinant is 1. An easy calculation allows us to prove the following propositions.

Proposition 3.7.2. *Let $F\neq \mathbb{Q}(\sqrt{-1}),\mathbb{Q}(\sqrt{-3})$. If n is even, then*

$$\mathrm{vol}_{HM}(\mathrm{U}(L)) = \mathrm{vol}_{HM}(\mathrm{SU}(L)).$$

If n is odd, then

$$\mathrm{vol}_{HM}(\mathrm{SU}(L)) \leq \mathrm{vol}_{HM}(\mathrm{U}(L)) \leq 2\cdot \mathrm{vol}_{HM}(\mathrm{SU}(L)).$$

Proposition 3.7.3. *Let $F=\mathbb{Q}(\sqrt{-1})$. If n is even, then*

$$\mathrm{vol}_{HM}(\mathrm{U}(L)) = \mathrm{vol}_{HM}(\mathrm{SU}(L)).$$

Otherwise,

$$\mathrm{vol}_{HM}(\mathrm{SU}(L)) \leq \mathrm{vol}_{HM}(\mathrm{U}(L)) \leq \begin{cases} 2\cdot \mathrm{vol}_{HM}(\mathrm{SU}(L)) & (n\equiv 1 \bmod 4), \\ 4\cdot \mathrm{vol}_{HM}(\mathrm{SU}(L)) & (n\equiv 3 \bmod 4). \end{cases}$$

Proposition 3.7.4. *Let $F=\mathbb{Q}(\sqrt{-3})$. If $n\equiv 0,4 \bmod 6$, then*

$$\mathrm{vol}_{HM}(\mathrm{U}(L)) = \mathrm{vol}_{HM}(\mathrm{SU}(L)).$$

Otherwise,

$$\mathrm{vol}_{HM}(\mathrm{SU}(L)) \leq \mathrm{vol}_{HM}(\mathrm{U}(L)) \leq \begin{cases} 2\cdot \mathrm{vol}_{HM}(\mathrm{SU}(L)) & (n\equiv 1,3 \bmod 6), \\ 3\cdot \mathrm{vol}_{HM}(\mathrm{SU}(L)) & (n\equiv 2 \bmod 6), \\ 6\cdot \mathrm{vol}_{HM}(\mathrm{SU}(L)) & (n\equiv 5 \bmod 6). \end{cases}$$

3.7.2. Odd-dimensional case $\mathrm{SU}(1,2m)$**.** Here, we consider the case of odd-dimensional unitary groups; i.e., we assume that L is primitive of signature $(1,2m)$ with $m>1$. Let

$$\epsilon_{v,j}(1) := \frac{q_v^j - (-1)^j}{q_v^{2m+1} - (-1)^{2m+1}} \leq 1,$$
$$\epsilon_{v,j}(2) := \frac{q_v^j - 1}{q_v^{2m+1} - 1} \leq 1,$$

and

$$\epsilon_v(1) := \sum_{j, L_{v,j} \neq 0} \epsilon_{v,j}(1) \leq 1,$$
$$\epsilon_v(2) := \sum_{j, L_{v,j} \neq 0} \epsilon_{v,j}(2) \leq 1.$$

Note that since L is primitive, if p does not divide $\det(L)$, then $n_{p,\nu_p} < 2m+1$. For $m>1$, from (3.6.1), (3.6.2), (3.6.3), (3.6.5), (3.6.7), (3.6.9), (3.6.11), (3.6.13), (3.6.15), (3.6.17), (3.6.19), (3.6.21), (3.6.23) and (3.6.25), we have

$$\begin{aligned}
&\sum_{[\ell]\in\mathcal{R}_{\mathrm{split}}} \frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))} \\
&\leq \frac{(2\pi)^{2m+1}}{D^{2m+1/2}\cdot(2m)!\cdot L(2m+1)} \\
&\cdot \sum_{[\ell]\in\mathcal{R}_{\mathrm{split}}} \Big\{ \prod_{v:\mathrm{inert}} \epsilon_{v,n_{v,\nu_v}}(1) \prod_{v:\mathrm{split}} \epsilon_{v,n_{v,\nu_v}}(2)\cdot 2 \prod_{v\neq 2:\mathrm{ram}} q_v^{n_{v,\nu_v}-1/2} \cdot \prod_{v=2:\mathrm{ram}} 2^2\cdot 3\cdot 2^{2n_2} \Big\} \\
&\leq \frac{3\cdot 2^4\cdot(2\pi)^{2m+1}}{S\cdot(2m)!\cdot L(2m+1)} \sum_{[\ell]\in\mathcal{R}_{\mathrm{split}}} \Big\{ \prod_{v:\mathrm{inert}} \epsilon_{v,n_{v,\nu_v}}(1) \prod_{v:\mathrm{split}} \epsilon_{v,n_{v,\nu_v}}(2) \Big\} \\
&\leq \frac{3\cdot 2^4\cdot(2\pi)^{2m+1}}{S\cdot(2m)!\cdot L(2m+1)} \sum_J \Big\{ \prod_{v|D(L):\mathrm{inert}} \epsilon_{v,j(v)}(1) \prod_{v|D(L):\mathrm{split}} \epsilon_{v,j(v)}(2) \Big\} \quad \text{(Proposition 3.4.5)} \\
&= \frac{3\cdot 2^4\cdot(2\pi)^{2m+1}}{S\cdot(2m)!\cdot L(2m+1)} \prod_{v|D(L):\mathrm{inert}} \epsilon_v(1) \prod_{v|D(L):\mathrm{split}} \epsilon_v(2) \\
&\leq \frac{3\cdot 2^4\cdot(2\pi)^{2m+1}}{S\cdot(2m)!\cdot L(2m+1)}.
\end{aligned} \tag{3.7.2}$$

Here, $J = (j(v))_{v|D(L)}$ runs through multi-indices such that $L_{v,j(v)} \neq 0$ for every v; see [**107**, Definition 5.7].

Besides, if L satisfies $P(M)$, then we have

$$\begin{aligned}
&\sum_{[\ell]\in\mathcal{R}_{\mathrm{split}}} \frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))} \\
&\leq \frac{3\cdot 2^4\cdot(2\pi)^{2m+1}}{S\cdot(2m)!\cdot L(2m+1)} \prod_{v|D(L):\mathrm{inert}} \epsilon_v(1) \prod_{v|D(L):\mathrm{split}} \epsilon_v(2)
\end{aligned}$$

$$\leq \frac{3 \cdot 2^4 \cdot (2\pi)^{2m+1}}{S \cdot (2m)! \cdot L(2m+1) \cdot D(L)^{1/M}}. \tag{3.7.3}$$

We apply these estimates to $V(L, F)$ in Proposition 3.4.4.

3.7.2.1. $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ **case**. Let $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$. From (3.7.2), we have

$$\begin{aligned}
&V(L,F)\\
&:= \sum_{[\ell]\in R(F,2)_I} \mathrm{vol}_{HM}(L,K_\ell) + 2^{2m} \sum_{[\ell]\in\mathcal{R}_L(F,2)_{III},\mathcal{R}_L(F,2)_{IV},\mathcal{R}_L(F,2)_V} \mathrm{vol}_{HM}(L,K_\ell)\\
&+ 4^{2m} \sum_{[\ell]\in\mathcal{R}_L(F,2)_{II}} \mathrm{vol}_{HM}(L,K_\ell)\\
&\leq 2\cdot \sum_{[\ell]\in R(F,2)_I} \frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))} + 2\cdot 2^{2m} \sum_{[\ell]\in\mathcal{R}_L(F,2)_{III},\mathcal{R}_L(F,2)_{IV},\mathcal{R}_L(F,2)_V} \frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))}\\
&+ 2\cdot 4^{2m} \sum_{[\ell]\in\mathcal{R}_L(F,2)_{II}} \frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))} \quad \text{(Proposition 3.7.2)}\\
&\leq 2(1 + 2^{2m}\cdot 2^{2m+1} + 4^{2m}\cdot 4^{2m+1}) \cdot \frac{3 \cdot 2^4 \cdot (2\pi)^{2m+1}}{S \cdot (2m)! \cdot L(2m+1)} \quad (3.7.2)\\
&= (1 + 2^{4m+1} + 2^{8m+2}) \cdot \frac{3 \cdot 2^5 \cdot (2\pi)^{2m+1}}{S \cdot (2m)! \cdot L(2m+1)}.
\end{aligned}$$

Moreover, if L satisfies $P(M)$, we have

$$V(L,F) \leq (1 + 2^{4m+1} + 2^{8m+2}) \cdot \frac{3 \cdot 2^5 \cdot (2\pi)^{2m+1}}{S \cdot (2m)! \cdot L(2m+1) \cdot D(L)^{1/M}} \tag{3.7.4}$$

by (3.7.3).

3.7.2.2. $F = \mathbb{Q}(\sqrt{-1})$ **case**. Let $F = \mathbb{Q}(\sqrt{-1})$. From (3.7.2), we have

$$\begin{aligned}
&V(L,\mathbb{Q}(\sqrt{-1}))\\
&:= 3 \sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_I} \mathrm{vol}_{HM}(L,K_\ell) + 3\cdot 2^{2m} \sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_{II}} \mathrm{vol}_{HM}(L,K_\ell)\\
&+ 4^{2m} \sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),2)_{II}} \mathrm{vol}_{HM}(L,K_\ell)\\
&\leq 4\cdot 3 \sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_I} \frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))} + 4\cdot 3\cdot 2^{2m} \sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_{II}} \frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))}\\
&+ 4\cdot 4^{2m} \sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),2)_{II}} \frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))} \quad \text{(Proposition 3.7.3)}\\
&\leq 4(3 + 3\cdot 2^{2m}\cdot 2^{2m+1} + 4^{2m}\cdot 4^{2m+1}) \cdot \frac{3 \cdot 2^4 \cdot (2\pi)^{2m+1}}{S \cdot (2m)! \cdot L(2m+1)} \quad (3.7.2)\\
&= (3 + 3\cdot 2^{4m+1} + 2^{8m+2}) \cdot \frac{3 \cdot 2^6 \cdot (2\pi)^{2m+1}}{S \cdot (2m)! \cdot L(2m+1)}.
\end{aligned}$$

Moreover, if L satisfies $P(M)$, we have

$$V(L,F) \le (3+3\cdot 2^{4m+1}+2^{8m+2})\cdot \frac{3\cdot 2^6\cdot(2\pi)^{2m+1}}{S\cdot(2m)!\cdot L(2m+1)\cdot D(L)^{1/M}} \tag{3.7.5}$$

by (3.7.3).

3.7.2.3. $F=\mathbb{Q}(\sqrt{-3})$ **case**. Let $F=\mathbb{Q}(\sqrt{-3})$. From (3.7.2), we have

$$\begin{aligned}
&V(L,\mathbb{Q}(\sqrt{-3}))\\
&:= 5\sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),6)} \mathrm{vol}_{HM}(L,K_\ell) + 2\cdot 3^{2m}\sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),3)} \mathrm{vol}_{HM}(L,K_\ell)\\
&+ 4^{2m}\sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),2)} \mathrm{vol}_{HM}(L,K_\ell)\\
&\le 6\cdot 5\sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),6)} \frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))} + 6\cdot 2\cdot 3^{2m}\sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),3)} \frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))}\\
&+ 6\cdot 4^{2m}\sum_{[\ell]\in\mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),2)} \frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))} \quad \text{(Proposition 3.7.4)}\\
&\le 6(5+2\cdot 3^{2m}\cdot 3^{2m+1}+4^{2m}\cdot 4^{2m+1})\cdot\frac{3\cdot 2^4\cdot(2\pi)^{2m+1}}{S\cdot(2m)!\cdot L(2m+1)} \quad (3.7.2)\\
&= (5+2\cdot 3^{4m+1}+2^{8m+2})\cdot\frac{3^2\cdot 2^5\cdot(2\pi)^{2m+1}}{S\cdot(2m)!\cdot L(2m+1)}.
\end{aligned}$$

Moreover, if L satisfies $P(M)$, we have

$$V(L,F) \le (5+2\cdot 3^{4m+1}+2^{8m+2})\cdot \frac{3^2\cdot 2^5\cdot(2\pi)^{2m+1}}{S\cdot(2m)!\cdot L(2m+1)\cdot D(L)^{1/M}} \tag{3.7.6}$$

by (3.7.3).

3.7.2.4. **Summary: odd-dimensional case**. Upon collecting the above statements, we can assert as follows.

Theorem 3.7.5. *Let L be primitive of signature $(1,2m)$ with $m>1$. Assume $(\heartsuit)$. Then, if m or S is sufficiently large, the line bundle $\mathcal{M}(a)$ is big. More precisely,*

$$V(L,F)\le\frac{f_F^{odd}(m)}{S}.$$

Moreover, if L satisfies $P(M)$ for some $M>0$, we have

$$V(L,F)\le\frac{f_F^{odd}(m)}{D(L)^{1/M}\cdot S}.$$

3.7.3. Even-dimensional case $\mathrm{SU}(1,2m-1)$**.** Let

$$\epsilon_{v,j} := \frac{q_v^j-1}{q_v^{2m+1}-1}\le 1,$$

and

$$\epsilon_v := \sum_{j,L_{v,j}\neq 0}\epsilon_{v,j}\le 1.$$

Now, let L be primitive of signature $(1,2m)$ with $m>1$. Note that since L is primitive, if p does not divide $\det(L)$, then $n_{p,\nu_p}<2m$. For $m>1$, from (3.6.1), (3.6.2), (3.6.4), (3.6.6), (3.6.8), (3.6.10), (3.6.12), (3.6.14), (3.6.16), (3.6.18), (3.6.20), (3.6.22), (3.6.24) and (3.6.26), we have

$$\begin{aligned}
&\sum_{[\ell]\in\mathcal{R}_{\mathrm{split}}}\frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))}\\
&\leq \frac{(2\pi)^{2m}}{(2m-1)!\cdot\zeta(2m)}\sum_{[\ell]\in\mathcal{R}_{\mathrm{split}}}\Big\{\prod_{v:\mathrm{unram}}\epsilon_{v,\nu_v}\prod_{v\neq 2:\mathrm{ram}}\frac{q_v^{n_{v,\nu_v}}-1}{q_v^{2m}-1}\prod_{v=2:\mathrm{ram}}\frac{2^{5/2}\cdot 3\cdot 2^{2n_2}}{2^{2m}-1}\Big\}\\
&\leq \frac{2^{2m+5/2}\cdot 3\cdot(2\pi)^{2m}}{S\cdot(2m-1)!\cdot\zeta(2m)}\sum_{[\ell]\in\mathcal{R}_{\mathrm{split}}}\prod_{v|D(L):\mathrm{unram}}\epsilon_{v,\nu_v}\\
&\leq \frac{2^{2m+5/2}\cdot 3\cdot(2\pi)^{2m}}{S\cdot(2m-1)!\cdot\zeta(2m)}\sum_{J}\prod_{v|D(L):\mathrm{unram}}\epsilon_{v,j(v)}\quad\text{(Proposition 3.4.5)}\\
&= \frac{2^{2m+5/2}\cdot 3\cdot(2\pi)^{2m}}{S\cdot(2m-1)!\cdot\zeta(2m)}\prod_{v|D(L):\mathrm{unram}}\epsilon_{v}\\
&\leq \frac{2^{2m+5/2}\cdot 3\cdot(2\pi)^{2m}}{S\cdot(2m-1)!\cdot\zeta(2m)}. \qquad (3.7.7)
\end{aligned}$$

More strongly, if L satisfies $P(M)$, we have

$$\begin{aligned}
&\sum_{[\ell]\in\mathcal{R}_{\mathrm{split}}}\frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))}\\
&\leq \frac{2^{2m+5/2}\cdot 3\cdot(2\pi)^{2m}}{S\cdot(2m-1)!\cdot\zeta(2m)}\prod_{v|D(L):\mathrm{unram}}\epsilon_{v}\\
&\leq \frac{2^{2m+5/2}\cdot 3\cdot(2\pi)^{2m}}{S\cdot(2m-1)!\cdot\zeta(2m)\cdot D(L)^{1/M}}. \qquad (3.7.8)
\end{aligned}$$

Below, we apply these estimates to $V(L,F)$ in Proposition 3.4.4.

3.7.3.1. $F\neq\mathbb{Q}(\sqrt{-1}),\mathbb{Q}(\sqrt{-3})$ **case**. Let $F\neq\mathbb{Q}(\sqrt{-1}),\mathbb{Q}(\sqrt{-3})$. From (3.7.7), we have

$$\begin{aligned}
&V(L,F)\\
&:=\sum_{[\ell]\in R(F,2)_I}\mathrm{vol}_{HM}(L,K_\ell)+2^{2m-1}\sum_{[\ell]\in\mathcal{R}_L(F,2)_{III},\mathcal{R}_L(F,2)_{IV},\mathcal{R}_L(F,2)_V}\mathrm{vol}_{HM}(L,K_\ell)\\
&+4^{2m-1}\sum_{[\ell]\in\mathcal{R}_L(F,2)_{II}}\mathrm{vol}_{HM}(L,K_\ell)\\
&\leq\sum_{[\ell]\in R(F,2)_I}\frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))}+2^{2m-1}\sum_{[\ell]\in\mathcal{R}_L(F,2)_{III},\mathcal{R}_L(F,2)_{IV},\mathcal{R}_L(F,2)_V}\frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))}\\
&+4^{2m-1}\sum_{[\ell]\in\mathcal{R}_L(F,2)_{II}}\frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))}\quad\text{(Proposition 3.7.2)}
\end{aligned}$$

$$\leq (1 + 2^{2m-1} \cdot 2^{2m} + 4^{2m-1} \cdot 4^{2m}) \cdot \frac{2^{2m+5/2} \cdot 3 \cdot (2\pi)^{2m}}{S \cdot (2m-1)! \cdot \zeta(2m)} \quad (3.7.7)$$
$$= (1 + 2^{4m-1} + 2^{8m-2}) \cdot \frac{2^{2m+5/2} \cdot 3 \cdot (2\pi)^{2m}}{S \cdot (2m-1)! \cdot \zeta(2m)}.$$

Moreover, if L satisfies $P(M)$, we have

$$V(L, F) \leq (1 + 2^{4m-1} + 2^{8m-2}) \cdot \frac{2^{2m+5/2} \cdot 3 \cdot (2\pi)^{2m}}{S \cdot (2m-1)! \cdot \zeta(2m) \cdot D(L)^{1/M}} \tag{3.7.9}$$

by (3.7.8).

3.7.3.2. $F = \mathbb{Q}(\sqrt{-1})$ **case**. Let $F = \mathbb{Q}(\sqrt{-1})$. From (3.7.7), we have

$$\begin{aligned}
&V(L, \mathbb{Q}(\sqrt{-1})) \\
&:= 3 \sum_{[\ell] \in \mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_I} \mathrm{vol}_{HM}(L, K_\ell) + 3 \cdot 2^{2m-1} \sum_{[\ell] \in \mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_{II}} \mathrm{vol}_{HM}(L, K_\ell) \\
&+ 4^{2m-1} \sum_{[\ell] \in \mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),2)_{II}} \mathrm{vol}_{HM}(L, K_\ell) \\
&\leq 3 \sum_{[\ell] \in \mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_I} \frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))} + 3 \cdot 2^{2m-1} \sum_{[\ell] \in \mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),4)_{II}} \frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))} \\
&+ 4^{2m-1} \sum_{[\ell] \in \mathcal{R}_L(\mathbb{Q}(\sqrt{-1}),2)_{II}} \frac{\mathrm{vol}_{HM}(\mathrm{SU}(K_\ell))}{\mathrm{vol}_{HM}(\mathrm{SU}(L))} \quad \text{(Proposition 3.7.3)} \\
&\leq (3 + 3 \cdot 2^{2m-1} \cdot 2^{2m} + 4^{2m-1} \cdot 4^{2m}) \cdot \frac{2^{2m+5/2} \cdot 3 \cdot (2\pi)^{2m}}{S \cdot (2m-1)! \cdot \zeta(2m)} \quad (3.7.7) \\
&= (3 + 3 \cdot 2^{4m-1} + 2^{8m-2}) \cdot \frac{2^{2m+5/2} \cdot 3 \cdot (2\pi)^{2m}}{S \cdot (2m-1)! \cdot \zeta(2m)}.
\end{aligned}$$

Moreover, if L satisfies $P(M)$, we have

$$V(L, F) \leq (3 + 3 \cdot 2^{4m-1} + 2^{8m-2}) \cdot \frac{2^{2m+5/2} \cdot 3 \cdot (2\pi)^{2m}}{S \cdot (2m-1)! \cdot \zeta(2m) \cdot D(L)^{1/M}} \tag{3.7.10}$$

by (3.7.8).

3.7.3.3. $F = \mathbb{Q}(\sqrt{-3})$ **case**. Let $F = \mathbb{Q}(\sqrt{-3})$. From (3.7.7), we have

$$\begin{aligned}
&V(L, \mathbb{Q}(\sqrt{-3})) \\
&:= 5 \sum_{[\ell] \in \mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),6)} \mathrm{vol}_{HM}(L, K_\ell) + 2 \cdot 3^{2m-1} \sum_{[\ell] \in \mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),3)} \mathrm{vol}_{HM}(L, K_\ell) \\
&+ 4^{2m-1} \sum_{[\ell] \in \mathcal{R}_L(\mathbb{Q}(\sqrt{-3}),2)} \mathrm{vol}_{HM}(L, K_\ell) \quad \text{(Proposition 3.7.4)} \\
&\leq (5 + 2 \cdot 3^{2m-1} \cdot 3^{2m} + 4^{2m-1} \cdot 4^{2m}) \cdot \frac{2^{2m+5/2} \cdot 3 \cdot (2\pi)^{2m}}{S \cdot (2m-1)! \cdot \zeta(2m)} \quad (3.7.7) \\
&= (5 + 2 \cdot 3^{4m-1} + 2^{8m-2}) \cdot \frac{2^{2m+5/2} \cdot 3 \cdot (2\pi)^{2m}}{S \cdot (2m-1)! \cdot \zeta(2m)}.
\end{aligned}$$

Moreover, if L satisfies $P(M)$, we have

$$V(L,F) \le (5 + 2\cdot 3^{4m-1} + 2^{8m-2}) \cdot \frac{2^{2m+5/2}\cdot 3 \cdot (2\pi)^{2m}}{S\cdot (2m-1)!\cdot \zeta(2m)\cdot D(L)^{1/M}} \tag{3.7.11}$$

by (3.7.8).

3.7.3.4. **Summary: even-dimensional case**. Upon collecting the above statements, we obtain the following.

Theorem 3.7.6. *Let L be primitive of signature $(1, 2m-1)$ with $m > 1$. Assume $(\heartsuit)$. Then, if m or S is sufficiently large, the line bundle $\mathcal{M}(a)$ is big. More precisely,*

$$V(L,F) \le \frac{f_F^{even}(m)}{S}.$$

Moreover, if L satisfies $P(M)$ for some $M > 0$, we have

$$V(L,F) \le \frac{f_F^{even}(m)}{D(L)^{1/M}\cdot S}.$$

3.8. Conclusion

3.8.1. Main results. We shall restate our main results in this chapter. This gives a solution to the problem (A) in Section 3.1.

Theorem 3.8.1. *Let L be a primitive Hermitian lattice over $\mathscr{O}_F$ of signature $(1, 2m)$ (resp. $(1, 2m-1)$) with $m > 1$. Assume $(\heartsuit)$. Then, for a positive integer a, if m or S is sufficiently large, the line bundle $\mathcal{M}(a)$ is big. More precisely,*

$$V(L,F) \le \frac{f_F^{odd}(m)}{S} \quad \Big(\text{resp. } V(L,F) \le \frac{f_F^{even}(m)}{S}\Big).$$

Moreover, if L satisfies $P(M)$ (see Section 3.7) for some $M > 0$, we have

$$V(L,F) \le \frac{f_F^{odd}(m)}{D(L)^{1/M}\cdot S} \quad \Big(\text{resp. } V(L,F) \le \frac{f_F^{even}(m)}{D(L)^{1/M}\cdot S}\Big).$$

Proof. Combine Proposition 3.4.4 with Theorem 3.7.5 and Theorem 3.7.6. □

For unrmaified square-free lattices, we obtain more strict estimate because one can see that $\lambda_v^{K_\ell}/\lambda_v^L \le 1$ for $v|D$ and such lattices satisfy $P(1)$. Hence, we have the following:

$$V(L,F) \le (1 + 2^{4m+1} + 2^{8m+2}) \cdot \begin{cases} \dfrac{2\cdot (2\pi)^{2m+1}}{D^{2m+1/2}\cdot (2m)!\cdot L(2m+1)\cdot D(L)^{1/M}} & (n = 2m), \\ \dfrac{(2\pi)^{2m}}{(2m-1)!\cdot \zeta(2m)\cdot D(L)^{1/M}} & (n = 2m-1). \end{cases} \tag{3.8.1}$$

Corollary 3.8.2 (Unramified square-free case). *Up to scaling, assume that L is unramified square-free over $\mathscr{O}_{F_0}$. Then, for a positive integer a, if n is sufficiently large, or D is sufficiently large and n is even, then the line bundle $\mathcal{M}(a)$ is big.*

Proof. Since we obtain stronger estimate (3.8.1), to prove that $\mathcal{M}(a)$ is big, it suffices to show that L and K_ℓ satisfy $(\star)$ for any $[\ell] \in \mathcal{R}_L(F, 2)$ under the assumption on L and F. This was shown in Proposition 3.5.4 and 3.5.5. □

3.8.2. Application I: Unitary modular varieties of general type.

Theorem 3.8.3. *Let L be primitive, $n \geq 13$ and $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3})$. Assume that ($\heartsuit$) holds and there exists a non-zero cusp form of weight lower than $n+1$ with respect to $\mathrm{U}(L)$. Then, X_L is of general type if $\dim X_L = n$ or S is sufficiently large.*

Proof. The canonical divisor $K_{\overline{X_L}}$ is big by combining Proposition 3.4.4, 3.7.5 and 3.7.6 with the existence of a cusp form. Here, we use the result of Behrens [**9**, Theorem 4] which asserts there are no branch divisors at boundary $\overline{X_L} \backslash X_L$ and the author [**112**] which asserts that there are no irregular cusps for $\mathrm{U}(L)$. Then, every pluricanonical form on $\overline{X_L}$ extends to its desingularization since it has at worst canonical singularities [**9**, Theorem 4]. This means that X_L is of general type. □

3.8.3. Application II: Finiteness of Hermitian lattices admitting reflective modular forms. One might expect that there exist only finitely many Hermitian lattices of signature $(1, n)$ admitting reflective modular forms. We can prove this consideration for unramified square-free lattices from (3.7.4), (3.7.5), (3.7.6), (3.7.9), (3.7.10) and (3.7.11).

Corollary 3.8.4 (Finitness of Hermitian lattices admitting reflective modular forms). *Up to scaling, the set of reflective lattices with slope less than r, satisfying $P(M)$ and ($\heartsuit$), is finite for fixed $M, r > 0$. In particular, the set*

$$\{\text{Unramified square-free reflective lattices with slope less than } r \mid n > 2\}/\sim$$

is finite for a fixed F_0.

Proof. We will only consider the odd-dimensional case of $F \neq \mathbb{Q}(\sqrt{-3})$ because the other cases can be proved in the same way. Let L be a Hermitian reflective lattice of signature $(1, n)$ with $n > 2$, satisfying $P(M)$. We may assume that L is primitive. From (3.7.3) and the fact that there are only finitely many Hermitian lattices with bounded discriminant, it follows that the set of Hermitian lattices satisfying $P(M)$ is finite, up to scaling; see also [**107**, Proof of Theorem 1.5]. If L is unramified square-free, then the primitivity implies that L satisfies $P(1)$. Therefore, we also obtain finiteness of unramified square-free reflective lattices. □

3.8.4. Explicit estimation: General case. In the rest of the chapter, we estimate $V(L, F)$ and $W(L, F, 1)$ explicitly. We investigate how large values of m we need to take in Theorem 3.8.3. First, we consider odd-dimensional cases so that assume that L has signature $(1, 2m)$ with $m > 1$. Then, from Theorem 3.7.5, $W(L, F, 1) < 0$ if

$$m > \begin{cases} 277 & (F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})), \\ 550 & (F = \mathbb{Q}(\sqrt{-1})), \\ 823 & (F = \mathbb{Q}(\sqrt{-3})). \end{cases}$$

Second, when L has signature $(1, 2m-1)$ with $m > 1$, from Theorem 3.7.5, $W(L, F, 1) < 0$ if

$$m > \begin{cases} 390 & (F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})), \\ 776 & (F = \mathbb{Q}(\sqrt{-1})), \\ 1163 & (F = \mathbb{Q}(\sqrt{-3})). \end{cases}$$

3.8.5. Explicit estimation: Unramified square-free case. We assume that L is unramified square-free over $\mathscr{O}_{F_0}$. From (3.8.1), we have $W(L, F, 1) < 0$ if $n > 138$ where $n = \dim X_L$ as usual. On the other hand, if $D > 30$, then for any even $n \geq 4$, it follows $W(L, F, 1) < 0$.

CHAPTER 4

Fano modular varieties with mostly branched cusps

4.1. Introduction

We prove that the Baily-Borel compactification of certain modular varieties are Fano varieties or with ample canonical divisor by means of special modular forms (see Theorem 4.2.1). Their unbranched open subsets are always quasi-affine, and in Fano modular varieties case, we observe that most of the cusps are covered by the closure of branch divisors. In Section 4.3, we give various concrete examples, which include the moduli of (log) Enriques surfaces, those corresponding to $II_{2,26}$, and those associated to various Hermitian lattices which we construct.

The study of birational types of modular varieties is a semi-classical topic; Tai [**137**], Freitag [**40**] and Mumford [**121**] (resp. Kondō [**82, 84**], Gritsenko-Hulek-Sankaran [**58**] and Ma [**107**]) showed some Siegel (resp. orthogonal) modular varieties are of general type. Recently, the first author studied a similar problem for unitary modular varieties [**114**]. On the other hand, in order to prove that modular varieties have negative Kodaira dimension, one of the powerful tools for it is the use of certain reflective modular forms [**52, 106, 55, 54, 110**].

For this recurring theme, our main idea in this chapter is to focus on the Baily-Borel compactification, study it through modern birational geometry adapted to singular varieties and give applications. In this chapter, we define "special" reflective modular forms, motivated by the work of Gritsenko-Hulek [**55**], and show a criterion for proving the Baily-Borel compactification of modular varieties are Fano varieties. Then, we discuss examples in Section 4.3, including new ones, to which we apply our criterion. For instance, it follows that the Baily-Borel compactification of the moduli spaces of unpolarized (log) Enriques surfaces are Fano varieties; see Example 4.3.13, 4.3.17. We also give some applications to the understanding of cusps and rationality problems. More precisely, for these Fano-like modular varieties, all but one compact cusps are shown to be contained in the closure of branch divisors. In the same setup, we also show that if there are no such compact cusps, two general points are connected by a rational curve i.e., rationally connected by [**153**]. See Corollary 4.2.6 for details. The former uses [**5, 43**], and in particular it logically relies on a vanishing theorem proven in *loc.cit.* We do not know of another proof which does not use a vanishing theorem (Problem 4.2.12). See Corollaries 4.2.6, 4.2.8, 4.2.10 for the details and more assertions proved. For instance, the moduli space of (unpolarized) Enriques surface is shown to be rationally connected, which is a weaker version of a famous result of Kondō [**83**].

4.2. Main results and proofs

In this section, we prove general theorems which are mentioned in the introduction. In the later Section 4.3, we apply them to various concrete examples.

4.2.1. Main general results and proofs. Now, we shall prove the first main theorem in this chapter. For the notation, see Subsection 1.5.1.

Theorem 4.2.1 (Birational properties). *We follow the notation as above. If there is a reflective modular form which satisfies Assumption 1.5.1* (1) *with some* $s(X) \in \mathbb{Q}_{>0}$, *then the Baily-Borel compactification* $\overline{X}^{\mathrm{BB}}$ *of* $X = \Gamma\backslash D$ *only has log canonical singularities and* X^o *is quasi-affine. In addition,*

(1) if $s(X) > 1$*, then* $\overline{X}^{\mathrm{BB}}$ *is a Fano variety i.e.,* $-K_{\overline{X}^{\mathrm{BB}}}$ *is ample (*$\mathbb{Q}$*-Cartier),*
(2) if $s(X) = 1$*, then* $\overline{X}^{\mathrm{BB}}$ *is a Calabi-Yau variety i.e.,* $K_{\overline{X}^{\mathrm{BB}}} \sim_{\mathbb{Q}} 0$*, or*
(3) if $s(X) < 1$*, then* $K_{\overline{X}^{\mathrm{BB}}}$ *is ample.*

Terminology. In this chapter, we often say a normal variety is a *log canonical model* (resp. *canonical model*) in the sense that it only has log canonical singularities (resp. canonical singularities) and the canonical class is ample. Hence, in the case (3) above, $\overline{X}^{\mathrm{BB}}$ is a log canonical model. For the basics of birational geometry, we refer to e.g., [**80**].

Proof. Note that the codimension of the boundary of the Baily-Borel compactification $\partial\overline{X}^{\mathrm{BB}} := \overline{X}^{\mathrm{BB}} \setminus X$ is at least 2, following from our assumption that $\mathbb{G}$ is not isogenous to SL(2). Indeed, for such G, any maximal real parabolic subgroup P has unipotent radical of dimension at least 2 so that Levi part of P has real codimension at least 3. The existence of the special reflective modular form implies

$$\sum_i \frac{d_i - 1}{d_i} B_i \sim_{\mathbb{Q}} s(X)\mathscr{L}. \tag{4.2.1}$$

If we regard the holomorphic section satisfying Assumption 1.5.1 (1) as a section of the ample line bundle $\mathscr{L}^{\otimes s(X)N}$, it follows that the complement of the vanishing locus is affine but that is nothing but $\overline{X}^{\mathrm{BB}} \setminus \cup_i \overline{B_i}^{\mathrm{BB}}$ which includes X^o. This proof reflects the idea of [**13**].

From (4.2.1) and the definition of $\mathscr{L}$ it follows that

$$-K_{\overline{X}^{\mathrm{BB}}} \sim_{\mathbb{Q}} (s(X) - 1)\mathscr{L} \tag{4.2.2}$$

in $\mathrm{Pic}(\overline{X}^{\mathrm{BB}}) \otimes \mathbb{Q}$. Hence, $-K_{\overline{X}^{\mathrm{BB}}}$ is ample $\mathbb{Q}$-Cartier if $s(X) > 1$. Similarly, $K_{\overline{X}^{\mathrm{BB}}}$ is ample $\mathbb{Q}$-Cartier (resp. $K_{\overline{X}^{\mathrm{BB}}} = 0$) if $s(X) < 1$ (resp. if $s(X) = 1$). On the other hand, from [**120**, 3.4, 4.2 (also see 1.3)], $\overline{X}^{\mathrm{BB}}$ is obtained as a projective spectrum of a certain log canonical ring, hence the pair $(\overline{X}^{\mathrm{BB}}, \sum_i \frac{d_i-1}{d_i}\overline{B}_i^{\mathrm{BB}})$ has only log canonical singularity (as a pair) and $K_{\overline{X}^{\mathrm{BB}}} + \sum_i \frac{d_i-1}{d_i}\overline{B}_i^{\mathrm{BB}}$ is ample (see also [**1**, 3.4, 3.5]). Thus $\sum_i \frac{d_i-1}{d_i}\overline{B}_i^{\mathrm{BB}}$ is also $\mathbb{Q}$-Cartier so that X itself is also log canonical.

On the other hand, recall that the construction of the Baily-Borel compactification [**7**] is a projective spectrum of the graded ring of automorphic forms and $\mathscr{L}$ is the c multiple tensors of its tautological line bundle $\mathcal{O}(1)$ in the construction. Hence, it is ample so that our latter statements of the above theorem all follow from (4.2.2). This fact is more clarified in [**120**, Section 3, Section 4]. We complete the proof. □

Remark 4.2.2. The above results are analogous to the Fanoness results in [**39**], (resp. [**70**, Section 2] also [**101**, Section 4]) in the context of moduli of (semi)stable bundles over curves (resp. surfaces). For the case over surfaces, the determinant line bundle which descends to the Donaldson-Uhlenbeck compactification is used in the place of the automorphic line bundle $\mathscr{L}$.

Remark 4.2.3. Case (3) is a variant of the so-called "low weight cusp form trick" (cf. e.g., [**58**]). See also [**52**], [**52**, Section 5.5] and references therein.

We review the following well-known fact for convenience.

Lemma 4.2.4 (cf. [**40**, Hilfsatz 2.1]**,** [**58**, Section 6.1]**).** *In the orthogonal case $G = \mathrm{O}^+(2,n)$ (resp. in the unitary case $G = \mathrm{U}(1,n)$), the canonical weight c in the sense of Section 1.5.1 is n (resp. $n+1$).*

Proof. Recall that the compact dual D^c of D in the orthogonal case $G = \mathrm{O}^+(2,n)$ is the n-dimensional quadratic hypersurface (resp. $D^c = \mathbb{P}^n$ in the unitary case $G = \mathrm{U}(1,n)$), its canonical divisor is $K_{D^c} = \mathcal{O}_{\mathbb{P}^{n+1}}(-n)|_{Q^n}$ (resp. $K_{D^c} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$) so that the canonical weight c is n (resp. $n+1$). $\square$

We introduce the following notion.

Definition 4.2.5. We call a cusp F of $\overline{X}^{\mathrm{BB}}$ *naked* if it is not contained in $\mathrm{Supp}(\overline{B_i}^{\mathrm{BB}}) \cap \partial\overline{X}^{\mathrm{BB}}$ for any i. Further, we call it *minimal naked* if it is minimal with respect to the closure relation among naked cusps, i.e., $\overline{F} \setminus F$ is contained in $(\cup_i \mathrm{Supp}(\overline{B_i}^{\mathrm{BB}})) \cap \partial\overline{X}^{\mathrm{BB}}$. Also, we call $\partial\overline{X}^{\mathrm{BB}} \setminus \bigcup_i \overline{B_i}^{\mathrm{BB}}$ *the naked locus.*

Below, we observe a certain weakening of connected-ness of cusps closure in the case of $s(X) > 1$, i.e., Fano case. This follows from [**5**, 4.4, 6.6 (ii)], [**44**, Section 3], [**43**, 8.1], [**46**, 1.2] as the proof below, which is essentially just a review to make our logic more self-contained. Compare with our examples of the modular varieties given in the next section.

Corollary 4.2.6 (Boundary structure for Fano modular varieties). *Let us assume the same assumption of Theorem 4.2.1 and further that $s(X) > 1$. Then, the naked locus*

$$\partial\overline{X}^{\mathrm{BB}} \setminus \bigcup_i \overline{B_i}^{\mathrm{BB}}$$

is connected and its closure is nothing but the non-log-terminal locus of $\overline{X}^{\mathrm{BB}}$. More strongly, there is at most one minimal naked cusp with respect to the closure relation.

Furthermore, if we suppose such a minimal naked cusp F exists, there is an effective $\mathbb{Q}$-divisor D_F such that $(\overline{F}, D_F)$ has only klt singularities and is a log Fano pair, i.e., $-K_F - D_F$ is ample and $\mathbb{Q}$-Cartier. For instance, if F is a modular curve, it is rational i.e., $\overline{F} \simeq \mathbb{P}^1$ (with "Hauptmodul").

Proof. Firstly, we prepare the following general lemma (compare with e.g., [**1**, Section 3]).

Lemma 4.2.7 (Log canonical centers). *(1) Under the notation of Section 1.5.1 for general modular varieties, without the above assumptions in Corollary 4.2.6, the log canonical centers of $(\overline{X}^{\mathrm{BB}}, \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{BB}})$ are nothing but cusps of the Baily-Borel compactification $\overline{X}^{\mathrm{BB}}$.*

(2) Under the above assumptions in Corollary 4.2.6, the log canonical centers of $\overline{X}^{\mathrm{BB}}$ are nothing but cusps of the Baily-Borel compactification $\overline{X}^{\mathrm{BB}}$ which are not contained in $\cup_i \mathrm{Supp}(\overline{B_i}^{\mathrm{BB}})$.

PROOF OF LEMMA 4.2.7. As in [**6**, Chapter III, Section 7], we replace the (implicit dividing) discrete group Γ in Section 1.5.1 by its neat subgroup (cf. [**6**]) of finite index. In that way, we replace X (and $\overline{X}^{\mathrm{BB}}$) by its finite cover so that the first desired claim (1) for the log canonical centers of $(\overline{X}^{\mathrm{BB}}, \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{BB}})$ is reduced to the case when there is no B_i.

Then, there is a log resolution of $(\overline{X}^{\mathrm{BB}}, \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{BB}})$ as a toroidal compactificaftion [**6**, chapter III], see especially *loc.cit* 6.2. By its construction in *op.cit* of toroidal nature (see again e.g., [**1**, Section 3]), all the exceptional prime divisors have the discrepancy -1 and hence the claim (1) for the log canonical centers of $(\overline{X}^{\mathrm{BB}}, \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{BB}})$ follows.

For the proof of latter claim (2), note that the existence of special reflective modular form implies $\sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{BB}}$ is a $\mathbb{Q}$-Cartier divisor by (4.2.1) of the proof of Theorem 4.2.1. Hence, the note that log canonical centers of $\overline{X}^{\mathrm{BB}}$ form a subset of the lc centers of (1) which are not contained in the support of the effective $\mathbb{Q}$-Cartier divisor $\sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{BB}}$. Hence, the claim of Lemma 4.2.7 (2). □

Now we start the proof of Corollary 4.2.6. We take the union of the minimal naked cusps of $\overline{X}^{\mathrm{BB}}$ as W and put the reduced scheme structure on it. We denote the corresponding coherent ideal sheaf of $\mathcal{O}_{\overline{X}^{\mathrm{BB}}}$ as I_W.

From a vanishing theorem of [**5**, 4.4],[**43**, 8.1], whose absolute non-log version is enough for our particular purpose here, we have $H^1(\overline{X}^{\mathrm{BB}}, I_W) = 0$. Now, $H^0(\overline{X}^{\mathrm{BB}}, I_W) = 0$ also holds since it is a linear subspace of $H^0(\overline{X}^{\mathrm{BB}}, \mathcal{O})$ which is identified with $\mathbb{C}$ because of the properness of $\overline{X}^{\mathrm{BB}}$, combined with the fact that $W \neq \emptyset$. Hence, combined with standard cohomology exact sequence arguments, $H^0(\mathcal{O}_W) \simeq \mathbb{C}$ follows. Hence, it implies the connectivity of W, so that there is at most 1 minimal naked cusp F.

For such F, the existence of D_F on the closure $\overline{F}$ follows from applying the log canonical subadjunction [**46**, Subsection 1.2] to $F \subset (\overline{X}^{\mathrm{BB}}, 0)$. □

We make a caution that the above Corollary 4.2.6 does not claim the naked cusp always has log terminal singularity. Nevertheless, in the $\mathbb{Q}$-rank 1 case, we have the following.

Corollary 4.2.8 ($\mathbb{Q}$-rank 1 case). *Under the same assumptions of Theorem 4.2.1 with >1, if further $\mathbb{Q}$-rank of $\mathbb{G}$ is 1 (e.g., when $G \simeq \mathrm{U}(1,n)$ for some n so that G/K is an n-dimensional complex unit ball), only either one of the followings hold.*

(1) *There is exactly one naked cusp F of $\overline{X}^{\mathrm{BB}}$ which is an isolated non-log-terminal locus but at worst log canonical. Furthermore, there is an effective $\mathbb{Q}$-divisor D_F such that (F, D_F) is a klt log Fano pair hence in particular, the modular branch divisor in F is nonzero effective.*
(2) *No naked cusp exists and X is rationally connected, i.e., two general points are connected by a rational curve and has at worst log terminal singularities. Furthermore, $X \setminus \mathrm{Supp} \cup_i B_i$ is affine (not only quasi-affine).*

PROOF. Note that the condition that $\mathbb{Q}$-rank of $\mathbb{G}$ is 1 implies that the boundary strata of the Baily-Borel compactification of X are all compact and do not have closure relations. Thus, among the above statements, the only assertion which does not follow trivially from Corollary 4.2.6 is the rationally connected assertion for the latter case (2). We confirm it as follows: the non-existence of naked cusp means $\overline{X}^{\mathrm{BB}} \setminus X$ is included in $\cup_i \mathrm{Supp}(\overline{B_i}^{\mathrm{BB}})$ which implies the log terminality of X. Hence, it is rationally connected by a theorem of Zhang [**153**]. Finally, $X \setminus \operatorname{Supp} \cup_i B_i$ is affine by the proof of Theorem 4.2.1 and the assumption that there are no naked cusps. □

Here is a version of the converse direction of Theorem 4.2.1.

Theorem 4.2.9 (Abstract existence of special modular forms). *We follow the notation of Theorem 4.2.1. If $\overline{X}^{\mathrm{BB}}$ satisfies either*

- $K_{\overline{X}^{\mathrm{BB}}} \equiv 0$ *or*
- *either $K_{\overline{X}^{\mathrm{BB}}}$ or $-K_{\overline{X}^{\mathrm{BB}}}$ is ample with Picard number* 1*,*

then there are special reflective modular forms satisfying Assumption 1.5.1 (1) *for some $s(X) \in \mathbb{Q}_{>0}$ and sufficiently divisible $N \in \mathbb{Z}_{>0}$. Furthermore, if it is of a certain orthogonal type, i.e., $\mathbb{G}$ is isogenous to $\mathrm{SO}^+(\Lambda)$ for $\Lambda = U \oplus U(l) \oplus N$ with some negative definite lattice N and $l \in \mathbb{Z}_{>0}$, the modular forms are necessarily Borcherds lift of some nearly holomorphic elliptic $Mp_2(\mathbb{Z})$-modular forms of a specific principal part of the Fourier expansion in the sense of* [**14**], [**22**, Section 1.3, Section 3.4].

Proof. Given the proof of Theorem 4.2.1, we can almost trace back the arguments as follows. In either cases, the automorphic line bundle $\mathscr{L}$ is proportional to $K_{\overline{X}^{\mathrm{BB}}}$ in $\operatorname{Pic}(\overline{X}^{\mathrm{BB}})$, hence so is it to $\sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{BB}}$. Therefore, $\mathcal{O}(N(s(X)\mathscr{L} - \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{BB}}))$ is trivial for some $s(X), N$. The last assertion follows from [**22**, 5.12], [**24**, 1.2]. □

4.2.2. Modular varieties with big anti-canonical classes. Recall that Gritsenko-Hulek [**55**] (resp. Maeda [**110**]) discuss the classes of reflective orthogonal modular forms (resp. unitary modular forms) satisfying Assumption 1.5.1 (2) with $s(X) > 1$ and proved uniruledness of X and constructs some examples.

This subsection proves the following a slight refinement of their results, which applies to the examples constructed in *loc.cit.*

Theorem 4.2.10 (cf. [55, 2.1], [110, 4.1]). *We follow the notation of Section 1.5.1, and discuss modular varieties $X = \Gamma\backslash D$ for a priori general G. If there is a reflective modular form Φ which satisfies Assumption 1.5.1* (2) *with some $s(X) \in \mathbb{Q}_{>1}$, , we define $V_\Phi := \cup_F \overline{F} \subset \partial\overline{X}^{\mathrm{BB}}$ where F runs through all cusps along which Φ does not vanish (as a function, or a section of $\mathscr{L}^{\otimes s(X)N}$). Then, the following holds.*

1. *The Baily-Borel compactification $\overline{X}^{\mathrm{BB}}$ of $X = \Gamma\backslash D$ only has log canonical singularities, X^o is quasi-affine and $-K_{\overline{X}^{\mathrm{BB}}}$ is big.*
2. *For any two closed points $x, y \in \overline{X}^{\mathrm{BB}}$, there are union of rational curves C such that $C \cup V_\Phi$ is connected (i.e., rationally chain connected modulo V_Φ cf.* [**63**, 1.1]*). In particular, X is uniruled. If $G = \mathrm{U}(1,n)$ for some n, then $\overline{X}^{\mathrm{BB}}$ is even rationally chain connected.*
3. *If we consider the set of cusps outside V_Φ, there is at most* 1 *minimal element (cusp) with respect to the closure relation.*

Proof. We first consider (i) of the above theorem. From the existence of Φ, it follows in the same way that

$$-K_{\overline{X}^{\mathrm{BB}}} \sim_{\mathbb{Q}} (s(X)-1)\mathscr{L} + \sum_i \left(\frac{d_i - 1}{d_i} - c_i\right)\overline{B_i}^{\mathrm{BB}},$$

hence it is big. The proofs of the other assertions in (i) are the same as those of Theorem 4.2.1. For (ii), note that the non-klt locus of $(\overline{X}^{\mathrm{BB}}, \sum_i(\frac{d_i-1}{d_i} - c_i)\overline{B_i}^{\mathrm{BB}})$ is the union of log canonical centers of $(\overline{X}^{\mathrm{BB}}, \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{BB}})$ which are not inside Supp(div(Φ)). Hence, the assertion (ii) directly follows from [**63**, 1.2] for $(\overline{X}^{\mathrm{BB}}, \sum_i \frac{d_i-1}{d_i}\overline{B_i}^{\mathrm{BB}})$. The assertion for the unitary case holds since the cusps are all 0-dimensional (cf. e.g., [**9**, Section 4]). Indeed, it follows since the Levi part of real parabolic subgroup of G corresponding to the cusps are $\mathrm{U}(0, n-1)$, which is trivial. For (iii), the same arguments as Corollary 4.2.6, similarly applying [**5**, 4.4, 6.6(ii)] or [**43**, 8.1] to the log canonical Fano pair $(\overline{X}^{\mathrm{BB}}, \sum_i(\frac{d_i-1}{d_i} - c_i)\overline{B_i}^{\mathrm{SBB}})$, give a proof. □

Remark 4.2.11. We can also show a variant of Corollary 4.2.6, Theorem 4.2.10 (iii) under general *meromorphic* modular forms if we replace the use of [**5**, 6.6(ii)] by [**5**, 4.4] or [**45**, 6.1.2]. However, because the obtained statement is rather complicated and no interesting applications have been found (yet at least), we omit it in this chapter.

We conclude this section by posing a natural problem.

Problem 4.2.12. In specific situations, e.g., when $\mathbb{G} = \mathrm{SO}^+(\Lambda \otimes \mathbb{Q})$ for a quadratic lattice Λ, or in the unitary modular case corresponding to a Hermitian lattice as later subsection 4.3.4, the assertions of Corollaries 4.2.6, 4.2.8, Theorem 4.2.10 (iii) can be phrased in a purely lattice theoretic manner. Is there a more lattice theoretic or number theoretic proof without the use of a vanishing theorem in algebraic geometry?

4.3. Examples of Fano and K-ample cases

We provide examples of which Theorems 4.2.1, Corollary 4.2.6, Corollary 4.2.8, Theorem 4.2.9 in Section 4.2.1 apply. In the examples, the compactified modular varieties are either Fano varieties or with ample canonical classes. There are also some examples with $s(X) = 1$, for instance [**41**] (cf. also earlier [**8**] with a weaker statement) but we do not focus such cases in this chapter.

4.3.1. Siegel modular cases. We start by discussing the Baily-Borel compactifications of some semi-classical modular varieties, which we show to fit our picture. The examples in this subsection and the next Subsection 4.3.2 do not use explicit modular forms but they are Fano varieties so that the converse theorem 4.2.9 applies to imply the (abstract) existence of special reflective modular forms.

The examples with explicit special reflective modular forms, to which we can apply Theorem 4.2.1 will be discussed from the next Section 4.3.3. Here are two examples of Siegel modular varieties whose Baily-Borel compactifications are Fano varieties.

Example 4.3.1 ([72]). The Baily-Borel compactification of the moduli of principally polarized abelian surfaces $\overline{A_2}^{\mathrm{BB}}$ is known to be a weighted projective hypersurface in $\mathbb{P}(4, 6, 10, 12, 35)$ of degree 70 with the coarse moduli isomorphic to $\mathbb{P}(2, 3, 5, 6)$ by relating to the invariants of genus 2 curves, hence binary sextics. Note that the adjunction does not work due to non-well-formedness, as indeed one has non-trivial isotropy (μ_2) along a divisor in the moduli stack. The reduction of the natural Faltings-Chai model over $\mathbb{F}_p$ are also determined (cf. [**71, 142**]).

Example 4.3.2 (cf. [141, 5.2] (also [72])). The Baily-Borel compcatification of the moduli of principally polarized abelian surfaces with level 2 structure $\overline{\Gamma(2)\backslash\mathfrak{H}}^{\rm BB}$ is known to be a quartic 3-fold

$$\sum_{i=0}^{5} x_i = (\sum_{i=0}^{5} x_i^2)^2 - 4(\sum_{i=0}^{5} x_i^4) = 0, \tag{4.3.1}$$

with non-isolated singularities along 15 lines. Since this is a hypersurface, it is clearly Gorenstein and has ample anticanonical class. It also follows from [**120**, Section 3, 4] (cf. also [**1**, 3.5]) again that it is at least log canonical.

4.3.2. Orthogonal modular cases, Part I. Below, we consider the cases where $\mathbb{G} = \mathrm{SO}^+(\Lambda \otimes \mathbb{Q})$ for a quadratic lattice $(\Lambda, (\ ,\ ))$ of signature $(2, n)$ with $n \in \mathbb{Z}_{>0}$. We realize the Hermitian symmetric domain $X = G/K$ as $G/K \simeq \mathscr{D}_\Lambda$ which is defined as one of (the isomorphic two) connected components of

$$\{v \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid (v, v) = 0,\ (v, \overline{v}) > 0\}.$$

We keep this notation throughout in the discussion of orthogonal modular varieties. Our first two examples in this Part I are understood via moduli-theoretic methods and GIT as follows.

Example 4.3.3 (Hilbert). The GIT compactification of the moduli of cubic surfaces ([**128**, Subsection 4.2]) is known to be isomorphic to the Baily-Borel compactification of the stable locus which admits uniformization of complex ball (cf. [**4**]). Hilbert's invariant calculation in his thesis tells this is $\mathbb{P}(1, 2, 3, 4, 5)$, hence the only cusp is not naked because of the log terminality. Obviously, it is also a ($\mathbb{Q}$-)Fano variety. This is also one of the simplest examples of the K-moduli variety of Fano varieties ([**128**, Subsection 4.2]).

Given [**107**], it is reasonable to ask the following problem in general.

Problem 4.3.4. Classify the lattices Λ of signature $(2, n)$ such that the Baily-Borel compactification $\Gamma\backslash\mathcal{D}_\Lambda$ are Fano varieties, especially when $\Gamma = \mathrm{O}^+(\Lambda)$ or $\widetilde{\mathrm{O}}^+(\Lambda)$.

From what follows, our arithmetic subgroup satisfies Γ is either $\mathrm{O}^+(\Lambda)$ or the stable orthogonal group $\widetilde{\mathrm{O}}^+(\Lambda)$.

Example 4.3.5 (Moduli of elliptic K3 surfaces). We consider the moduli M_W of Weierstrass elliptic K3 surfaces, which is an open subset of $\mathrm{O}^+(\Lambda)\backslash\mathcal{D}_\Lambda$ for $\Lambda := U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}$. We consider its Baily-Borel compactification ([**127**, Theorem 7.9]), which we denote $\overline{M_W}^{\rm BB}$ here. Recall from *loc.cit* Subsection 7.1 that there are exactly two 1-cusps intersecting at the only 0-cusp. Two 1-cusps are $M_W^{\rm nn}$ with canonical Gorenstein singularity and $M_W^{\rm seg}$ with toroidal singularity (including the 0-cusp $\overline{M_W^{\rm nn}} \cap \overline{M_W^{\rm seg}}$) hence $\overline{M_W}^{\rm BB}$ also only has log terminal singularity ([**126**, Part I, Section 2]). The notation of our superscripts "nn" and "seg" follow that of [**127**, Chapter 7] where some collapsing of hyperKähler metrics to *segment* i.e., $[0, 1]$ is partially observed along $M_W^{\rm seg}$, and also that *non-normal* degenerations are parametrized by $M_W^{\rm nn}$.

We recall that $\overline{M_W}^{\rm BB}$ coincides with a certain GIT quotient of a weighted projective space ([**127**, Theorem 7.9]). Using the fact as well as some analysis of singularities along the 1-cusps in [**126**, Part I], we prove the following.

Theorem 4.3.6. $\overline{M_W}^{\mathrm{BB}}$ *is a* 18*-dimensional log terminal rational Fano variety of Picard rank* 1*, although not isomorphic to any weighted projective space. Its two* 1*-cusps* M_W^{seg} *and* M_W^{nn} *are both non-naked.*

Proof. The description of $\overline{M_W}^{\mathrm{BB}}$ as a GIT quotient [**127**, Theorem 7.9] allows us to apply [**19**, Corollary 3] to confirm there is an effective $\mathbb{Q}$-divisor D on $\overline{M_W}^{\mathrm{BB}}$ such that $-K_{\overline{M_W}^{\mathrm{BB}}} - D$ is ample. Therefore, $-K_{\overline{M_W}^{\mathrm{BB}}}$ is big. On the other hand, $\overline{M_W}^{\mathrm{BB}}$ has Picard rank 1 because of the same GIT quotient description. Hence, the bigness of $-K_{\overline{M_W}^{\mathrm{BB}}}$ implies it is actually even ample i.e., $\overline{M_W}^{\mathrm{BB}}$ is a Fano variety.

The fact that both 1-cusps are non-naked are follows from Corollary 4.2.6, because $\overline{M_W}^{\mathrm{BB}}$ is log terminal as proven in [**126**, Part I, Section 2]. (The log terminality also follows from [**19**, Theorem1] combined again with the fact that $\overline{M_W}^{\mathrm{BB}}$ has Picard rank 1.) As for the rationality of M_W, [**99**] proved it, based on more classical rationality result of the moduli space of hyperelliptic curves (of genus 5).

The only remained thing to prove in the above theorem is that $\overline{M_W}^{\mathrm{BB}}$ is not a weighted projective space. From the analysis of singularity type along 1-cusp M_W^{nn} in [**126**, Part I, Theorem 2.2], it easily follows that the local fundamental group along the transversal slice is $(\mathbb{Z}/2\mathbb{Z})^4$ hence not cyclic. In particular, $\overline{M_W}^{\mathrm{BB}}$ can not be a weighted projective space. We complete the proof of Theorem 4.3.6. □

As a corollary, we also observe the following.

Corollary 4.3.7. *On the orthogonal modular variety* $\overline{M_W}^{\mathrm{BB}}$*, there are special reflective modular forms which satisfy Assumption 1.5.2* (1) *(of Section 1.5.2) for some* $s(X) > 1$ *and sufficiently divisible* $N \in \mathbb{Z}_{>0}$*.*

Proof. By the above theorem 4.3.6, we can apply Theorem 4.2.9 to complete the proof. □

4.3.3. Orthogonal modular cases, Part II. From here, we use the Borcherds products to show that various Baily-Borel compactifications of orthogonal modular varieties are Fano varieties or log canonical models.

Notation. Let

$$\mathcal{H}(\ell) := \{v \in \mathscr{D}_\Lambda \mid (v, \ell) = 0\}$$

be the special divisor with respect to $\ell \in \Lambda$ with $(\ell, \ell) < 0$. For any primitive element $r \in \Lambda$ satisfying $(r, r) < 0$, we define the *reflection* $\sigma_r \in \mathrm{O}^+(\Lambda)(\mathbb{Q})$ with respect to r as follows:

$$\sigma_r(\ell) := \ell - \frac{2(\ell, r)}{(r, r)} r.$$

Then, the union of ramification divisors of $\pi_\Gamma \colon \mathscr{D}_\Lambda \to \Gamma\backslash\mathscr{D}_\Lambda$ is

$$\bigcup_{\substack{r\in\Lambda/\pm:\text{primitive}\\ \sigma_r\in\Gamma \text{ or } -\sigma_r\in\Gamma}} \mathcal{H}(r)$$

by [**58**] for $\Gamma \subset \mathrm{O}^+(\Lambda)$ and $n > 2$. They also showed that the ramification degrees are 2. We sometimes denote π_Γ as π. We also define

$$\mathcal{H}_{-2} := \bigcup_{\ell\in\Lambda,\ \ell^2=-2} \mathcal{H}(\ell)$$

$$\mathcal{H}_{-4} := \bigcup_{\ell\in\Lambda,\ \ell^2=-4} \mathcal{H}(\ell)$$

$$\mathcal{H}_{-4,special\text{-}even} := \bigcup_{\ell\in\Lambda:special\text{-}even,\ \ell^2=-4} \mathcal{H}(\ell).$$

Here we say a vector $r \in \Lambda$ is special-even (also called even type e.g., in [**86**]) if $(\ell.r)$ is even for any $\ell \in \Lambda$, i.e., $\mathrm{div}(r)$ is even integer, so that the corresponding reflection lies in Γ. We define $\mathrm{div}(r)$ is the positive generator of the ideal

$$\{(\ell, r) \mid \ell \in \Lambda\}.$$

Remark 4.3.8. Below, for orthogonal cases, if f is a modular form corresponding to a section satisfying Assumption 1.5.2 (i), we can compute $s(X) = \frac{k}{2mn}$. Here, k is the weight of f and m is the multiplicity of $\mathrm{div} f$, and $n = \dim X$.

Example 4.3.9. Let $II_{2,26} = U\oplus U\oplus E_8(-1)\oplus E_8(-1)\oplus E_8(-1)$ be an even unimodular lattice of signature $(2, 26)$. We consider the case $\Gamma = \mathrm{O}^+(\Lambda)$. There is the modular form Φ_{12} of weight 12 on $\mathscr{D}_{II_{2,26}}$ by Borcherds [**12**] with

$$\mathrm{div}\Phi_{12} = \mathcal{H}_{-2}. \tag{4.3.2}$$

On the other hand, the ramification divisors of the map $\pi\colon II_{2,26} \to X := \mathrm{O}^+(II_{2,26})\backslash\mathscr{D}_{II_{2,26}}$ are $\mathcal{H}_{-2}$ by the even unimodularity of Λ and [**58**].

Now $\Phi_{12}^{2\times 26}$ satisfies Assumption 1.2 (i) with $s(X) = \frac{3}{13}$ and by Theorem 4.2.1 (iii) so that the Baily-Borel compactification $\overline{X}^{\mathrm{BB}}$ of the 26-dimensional orthogonal modular variety $X = \mathrm{O}^+(II_{2,26})\backslash\mathcal{D}_{II_{2,26}}$ is a log canonical model i.e., with ample canonical divisor $K_{\overline{X}^{\mathrm{BB}}}$ and at worst log canonical singularities. Let us specify and study the non-log-terminal locus or the log canonical center.

First, recall that there are exactly 24 1-cusps, which correspond to Niemeier lattices and all intersect at a common closed point (cf. e.g., [**53**, 1.1]). In particular, there is a 1-cusp which is the compactification of the modular curve $SL(2,\mathbb{Z})\backslash\mathbb{H}$ corresponding to the Leech lattice. We denote the particular 1-cusp as C_{Leech}.

For the Harish-Chandra-Borel embedding

$$\mathcal{D}_{II_{2,26}} \subset \mathcal{D}^c_{II_{2,26}} \subset \mathbb{P}(II_{2,26}\otimes\mathbb{C}),$$

$\mathcal{O}_{\mathbb{P}(II_{2,26}\otimes\mathbb{C})}(1)$ restricts to $\mathcal{O}_{\mathbb{P}^1}(1)|_{\mathbb{H}}$ for any 1-cusp $\mathbb{H} \subset \mathbb{P}^1$. For instance, by [**12**, Section 10], [**53**, 1.2], Φ_{12} restricts to the Ramanujan cusp form $\Delta_{12}(q) := q\prod_{n\geq 1}(1-q^n)^{24}$ of weight 12 on C_{Leech}. Since the only modular branch divisor is $\mathcal{H}_{-2}$, together with (4.3.2) and Lemma 4.2.7, it implies that the only log canonical center is the C_{Leech}. Recall that through the well-known isomorphism $SL(2,\mathbb{Z})\backslash\mathbb{H} \simeq \mathbb{A}^1(\mathbb{C}) \subset \mathbb{P}^1(\mathbb{C})$, the elliptic modular forms of weight $12k$ can be regarded with a section of $\mathcal{O}_{\mathbb{P}^1}(k)$, at the level of coarse moduli. In other words, $\mathcal{O}_{\mathbb{P}^1}(12k)|_{\mathbb{H}}$ descends to a line bundle $\mathcal{O}_{\mathbb{P}^1}(k)$ on $\mathbb{P}^1 \simeq SL(2,\mathbb{Z})\backslash\overline{\mathbb{H}}$ where $\overline{\mathbb{H}}$ denotes the rational closure of $\mathbb{H}$.

In particular, $(2s(X)L.C_{\mathrm{Leech}}) = 1$, where L follows the notation of Section 1.5.1. Equivalently $(K_{\overline{X}^{\mathrm{BB}}}.C_{\mathrm{Leech}}) = \frac{5}{3}$, $(\overline{B}.C_{\mathrm{Leech}}) = 1$ as $s(X) = \frac{3}{13}$. We summarize our conclusion in this case neatly as $II_{2,26}$ attracts special attention.

Corollary 4.3.10 (**$II_{2,26}$ case**)**.** *The Baily-Borel compactification $\overline{X}^{\mathrm{BB}}$ of the 26-dimensional orthogonal modular variety $X = \mathrm{O}^+(II_{2,26})\backslash\mathcal{D}_{II_{2,26}}$ is a log canonical model* *i.e., with ample canonical divisor $K_{\overline{X}^{\mathrm{BB}}}$ and at worst log canonical singularities. Further, the non-log-terminal locus is the single $C_{\mathrm{Leech}} \simeq \mathbb{P}^1$ in the boundary ∂X^{BB} which compactifies 1-cusp $SL(2,\mathbb{Z})\backslash\mathbb{H}$ and is characterized by that the corresponding isotropic plane $p \subset II_{2,26} \otimes \mathbb{R}$ satisfies that $(p^{\perp} \cap II_{2,26})/(p \cap II_{2,26})$ is the Leech lattice i.e., contains no roots. Its degree is $(K_{\overline{X}^{\mathrm{BB}}}.C_{\mathrm{Leech}}) = \frac{5}{3}$. (resp. $(\overline{B}.C_{\mathrm{Leech}}) = 1$).*

Later in Example 4.3.32, we also construct a 13-dimensional unitary modular subvariety which also compactifies with ample canonical class as the Baily-Borel compactification.

Example 4.3.11. Let $\Lambda := U \oplus U \oplus E_8(-1)$ be an even unimodular lattice of signature $(2,10)$. We again consider the case $\Gamma = \mathrm{O}^+(\Lambda)$. Borcherds constructed a reflective modular form on $\mathscr{D}_\Lambda$.

Theorem 4.3.12 ([12, 10.1, 16.1]). *There is a reflective modular form Φ_{252} of weight 252 on $\mathscr{D}_\Lambda$ such that*

$$\mathrm{div}\Phi_{252} = \mathcal{H}_{-2}.$$

Here, by the map $\pi\colon \mathscr{D}_\Lambda \to X := \mathrm{O}^+(\Lambda)\backslash\mathscr{D}_\Lambda$, the divisors $\mathcal{H}_{-2}$ maps to the unique branch divisors (cf. [58, Section 2]). Hence Φ_{252}^{10t} satisfies Assumption 1.5.2 (i) with $s(X) = \frac{63}{5}$ for some $t \in \mathbb{Z}$, and by Theorem 4.2.1 (i), the compactified modular variety $\overline{X}^{\mathrm{BB}}$ is a Fano variety. Actually, [64, 1.1], [34, 4.1] (also attributed to H.Shiga and [104]) shows it is the weighted projective space $\mathbb{P}(2,5,6,8,9,11,12,14,15,18,21)$.

Example 4.3.13 (Moduli of Enriques surfaces). The well-studied moduli space M_{Enr} of (unpolarized) Enriques surfaces (cf. e.g., [**124, 135, 13, 83**]) also fit into our setting. Let $\Lambda_{Enr} := U \oplus U(2) \oplus E_8(-2)$ be an even lattice of signature $(2,10)$. Then the modular variety

$$M_{Enr} := \mathrm{O}^+(\Lambda_{Enr})\backslash\mathscr{D}_{\Lambda_{Enr}}$$

is a 10-dimensional quasi-projective variety. Now we review the ramification divisors of the natural map $\pi : \mathscr{D}_{\Lambda_{\mathrm{Enr}}} \to M_{\mathrm{Enr}}$ and moduli discription. From [58] and [56], the ramification divisors are

$$\mathcal{H}_{-2} \cup \mathcal{H}_{-4,special\text{-}even}.$$

On the other hand, let

$$\widetilde{M_{Enr}} := \widetilde{\mathrm{O}}^+(\Lambda_{Enr})\backslash\mathscr{D}_{L_{Enr}}$$

be a finite cover of M_{Enr}. Then the following are known.

Proposition 4.3.14. *(1) $M_{Enr}\backslash\pi(\mathcal{H}_{-2})$ is the so-called moduli space of Enriques surfaces (cf. e.g., [**124**]). Moreover this is rational (Kondo [**83**]).*

*(2) $\widetilde{M_{Enr}}\backslash\pi(\mathcal{H}_{-2})$, which is a finite cover of M_{Enr}, is the moduli space of Enriques surfaces with a certain level-2 structure. Moreover $\widetilde{M_{Enr}}$ and $\widetilde{M_{Enr}}\backslash\pi(\mathcal{H}_{-2})$ are of general type (Gritsenko-Hulek cf. [**56**]).*

(3) $M_{Enr}\backslash(\pi(\mathcal{H}_{-2}) \cup \pi(\mathcal{H}_{-4,special\text{-}even}))$ is the moduli space of non-nodal Enriques surfaces.

Going back to our situation, we need special reflective modular forms satisfying Assumption 1.5.2 (i). Our input here is the following.

Lemma 4.3.15 ([13, 86]). *There exist two reflective modular forms Φ_4 and Φ_{124} on $\mathscr{D}_{L_{Enr}}$ of weights 4, 124 respectively such that;*

$$\mathrm{div}\Phi_4 = \mathcal{H}_{-2},$$
$$\mathrm{div}\Phi_{124} = \mathcal{H}_{-4,special\text{-}even}.$$

We put $F_{128} := \Phi_4\Phi_{124}$. Then this is a weight 128 modular form on $\mathscr{D}_{L_{Enr}}$ and $\mathrm{div}(F_{128})$ is exactly the ramification divisors of the map $\pi : \mathscr{D}_{L_{Enr}} \to M_{Enr}$ with coefficients 1. Now F_{128}^2 has a trivial character and satisfies Assumption 1.5.2 (i) with $s(X) = \frac{32}{5}$ and by Theorem 4.2.1 (i), $\overline{M_{Enr}}^{\mathrm{BB}}$ is a log canonical Fano variety.

Actually, it is even log terminal without naked cusps as we confirm in the folowing. By [**135**, 3.3, 4.5], there are only two 0-cusps which correspond to an isotropic vector e in the first summand U and an isotropic vector e' the second summand $U(2)$ of Λ_{Enr}. They belong to the same 1-cusp which corresponds to isotropic plane $\mathbb{Q}e \oplus \mathbb{Q}e'$. That 1-cusp is contained in the closure of $\mathcal{H}_{-4,special\text{-}even}$ since e and e' are orthogonal to the (norm-doubled) root of $E_8(-2)$, the third summand of L_{Enr}. By *loc.cit*, the only other 1-cusp corresponds to another isotropic plane

$$p = \mathbb{Q}e' \oplus \mathbb{Q}(2e + 2f + \alpha)$$

where e, f is the standard basis of the first summand U and α is norm -8 integral vector in the third summand $E_8(-2)$. Since p is obviously orthogonal to the -2 vector $e - f \in U$, the corresponding 1-cusp is also contained in the closure of the Coble locus $\mathcal{H}_{-2}$. Hence there are no naked cusps so that we conclude the following.

Corollary 4.3.16. *The Baily-Borel compactification $\overline{M_{Enr}}^{\mathrm{BB}}$ of the moduli of Enriques surfaces M_{Enr} is a log terminal Fano variety.*

Example 4.3.17 (Moduli of log Enriques surfaces). For each $1 \leq k \leq 10$ $(k \neq 2)$, let $\Lambda_{\mathrm{logEnr},k} := U(2) \oplus A_1 \oplus A_1(-1)^{\oplus 9-k}$ be an even lattice of signature $(2, 10-k)$. Then the associated modular variety $\mathrm{O}^+(\Lambda_{logEnr,k})\backslash\mathscr{D}_{L_{logEnr}}$ is a (partial compactification of) the moduli space of log Enriques surface with k $\frac{1}{4}(1,1)$ singularities. For the definition of log Enriques surfaces with $\frac{1}{4}(1,1)$ singularities, see [**31**, Definition 2.1, 2.6]. Yoshikawa [**149**] and Ma [**106**] constructed reflective modular forms on $\mathscr{D}_{L_{logEnr}}$ for $k \leq 7$ which we use.

Theorem 4.3.18 ([149, Theorem 4.2(i)]). *There is a reflective modular form Ψ_4 of weight $4+k$ on $\mathscr{D}_{\Lambda_{logEnr,k}}$ with*

$$\mathrm{div}\Psi_{4+k} = \mathcal{H}_{-2}.$$

Theorem 4.3.19 ([106, Appendix by Yoshikawa; A.4, proof of A.5]). *There is a reflective modular form $\Psi_{124,k}$ of weight $-k^2 - 9k + 124$ on $\mathscr{D}_{\Lambda_{logEnr,k}}$ with*

$$\mathrm{div}\Psi_{124,k} = \mathcal{H}_{-4}.$$

Now, the ramification divisors of the map $\mathscr{D}_{L_{logEnr,k}} \to \mathrm{O}^+(L_{logEnr,k})\backslash\mathscr{D}_{L_{logEnr,k}}$ is the union of special divisors with respect to (-2)-vectors and (-4)-vectors by the same discussion. As $(\Psi_{4+k}\Psi_{124,k})^{t(10-k)}$ with $t \in \mathbb{Z}_{>0}$ satisfies Assumption 1.5.2 (i) with $s(X) = \frac{-k^2-8k+128}{2(10-k)}$ for $k \leq 7$, by Theorem 1.3 (i), we conclude the following.

Corollary 4.3.20. *For the above (partially compactified) moduli spaces of log Enriques surface with k $\frac{1}{4}(1,1)$ singularities with $1 \leq k \leq 7$ $(k \neq 2)$ $X = \mathrm{O}^+(\Lambda_{logEnr,k})\backslash\mathscr{D}_{L_{logEnr}}$, the Baily-Borel compactifications $\overline{X}^{\mathrm{BB}}$ are Fano varieties.*

Actually, they are also unirational, by [**106**].

Example 4.3.21 (Simple lattices case). Let Λ be a quadratic lattice over $\mathbb{Z}$ of signature $(2, n)$. We recall from [**25**] that Λ is called *simple* if the space of cusp forms of weight $1+\frac{n}{2}$ associated with a finite quadratic form $\Lambda^\vee/\Lambda$ is zero. Then the special divisors on $\mathscr{D}_\Lambda$ are all given by the divisors of Borcherds lift, so that we can apply Theorem 4.2.1.

In fact, Wang-Williams [**145**] showed that for every simple lattice Λ of signature $(2, n)$ with $3 \leq n \leq 10$, the graded algebra of modular forms for certain subgroups of the orthogonal group is freely generated. From this, we have the associated modular varieties are weighted projective spaces, in particular, log terminal $\mathbb{Q}$-Fano.

From Theorem 4.2.1, all Borcherds product satisfying Assumption 1.5.2 (i) should have $s(X) > 1$. Also from Corollary 4.2.6, the boundary of the Baily-Borel compactification is in the closure of the branch divisors. See the tables of examples in [**145**].

We remark that before [**145**], [**25**] showed there are only finitely many isometry classes of even simple lattices Λ of signature $(2, n)$.

4.3.4. Preparation for unitary case - Hermitian lattice. Here, we recall some material on Hermitian lattices treated in [**68**] to prepare for constructing some examples of unitary modular varieties from the next subsection. There, we similarly apply Theorem 4.2.1 to certain restriction of Borcherds products to explore their birational properties.

Here is the setup. For a Hermitian lattice Λ, we define $\Lambda(a) := (\Lambda, a\langle\ ,\ \rangle)$ for $a \in \delta\mathscr{O}_F$. Analogously to quadratic forms, we also have the following proposition.

Proposition 4.3.22. *There exists a unimodular Hermitian lattice M and an element $b \in \mathscr{O}_F$ such that $\Lambda = M(b)$ if and only if the ideal $\{\langle v, w\rangle \in \delta\mathscr{O}_F \mid w \in \Lambda\}$ with respect to $v \in \Lambda$ is equal $b\delta\mathscr{O}_F$ for every primitive element $v \in \Lambda$.*

Let D_Λ be the Hermitian symmetric domain (complex ball) with respect to $U(\Lambda)(\mathbb{R})$, equivalently,

$$D_\Lambda := \{v \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid \langle v, v\rangle > 0\}$$

and $H(v)$ be the special divisor with respect to $v \in \Lambda$. For any element $r \in \Lambda$ satisfying $\langle r, r\rangle < 0$ and $\xi \in \mathscr{O}_F^\times \backslash \{1\}$, we define the *quasi-reflection* $\sigma_{r,\xi} \in \mathrm{U}(\Lambda)(\mathbb{Q})$ with respect to r, ξ as follows:

$$\sigma_{r,\xi}(\ell) := \ell - (1-\xi)\frac{\langle \ell, r\rangle}{\langle r, r\rangle} r.$$

Note that for $\xi = -1$, we have the usual reflection. See also [**4**]. We also remark that, for example, for $F = \mathbb{Q}(\sqrt{-1})$, we get $\sigma^2_{r,\sqrt{-1}} = \sigma_{r,-1}$ and for $F = \mathbb{Q}(\sqrt{-3})$, we get $\sigma^2_{r,\omega} = \sigma_{r,\overline{\omega}}$ for any $r \in \Lambda$ where ω is a primitive third root of unity.

The union of ramification divisors of $\pi_\Gamma \colon D_\Lambda \to \Gamma\backslash D_\Lambda$ is

$$\bigcup_r H(r)$$

by [**9**, Corollary 3] for $\Gamma \subset \mathrm{U}(\Lambda)$ and $n > 1$. Here, the union runs thorough primitive elements $r \in \Lambda/\mathscr{O}_F^\times$ with $\langle r, r\rangle < 0$ such that $\eta\sigma_{r,\xi} \in \Gamma$ for some $\eta \in \mathscr{O}_F^\times$ and $\xi \in \mathscr{O}_F^\times \backslash \{1\}$. We consider the natural embedding of the type I domain to the type IV domain

$$\iota \colon D_\Lambda \hookrightarrow \mathscr{D}_{\Lambda_Q}$$

where $(\Lambda_Q, (\ ,\ ))$ is the quadratic lattice associated with $(\Lambda, \langle\ ,\ \rangle)$, i.e., $\Lambda_Q := \Lambda$ as a $\mathbb{Z}$-module and $(\ ,\ ) := \mathrm{Tr}_{F/\mathbb{Q}}\langle\ ,\ \rangle$. For the analysis of ramification divisors on D_Λ, we first prepare the following lemma.

Lemma 4.3.23. *For $F = \mathbb{Q}(\sqrt{d})$, assume $d \equiv 2, 3 \bmod 4$ or $d = -3$. Then*

$$\iota\Big(\bigcup_{\substack{r\in\Lambda/\mathscr{O}_F^\times:\text{primitive}\\ \eta\sigma_{r,\xi}\in U(\Lambda)\ \text{for}\ \exists\eta\in\mathscr{O}_F^\times,\ \exists\xi\in\mathscr{O}_F^\times\setminus\{1\}}} H(r)\Big) \subset \bigcup_{\substack{r\in\Lambda_Q/\pm:\text{primitive}\\ \sigma_r\in O^+(\Lambda_Q)\ \text{or}\ -\sigma_r\in O^+(\Lambda_Q)}} \mathcal{H}(r) \cap \iota(D_\Lambda).$$

Proof. For $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$, it suffices to show that if

$$\frac{2\langle \ell, r\rangle}{\langle r, r\rangle} \in \mathscr{O}_F,$$

then

$$\alpha := \frac{2(\ell, r)}{(r, r)} = \frac{2\,\mathrm{Tr}_{F/Q}\langle \ell, r\rangle}{\mathrm{Tr}_{F/\mathbb{Q}}\langle r, r\rangle} \in \mathbb{Z}.$$

Since $\langle r, r\rangle \in \mathbb{Q}$, we have

$$\alpha = \Re\frac{2\langle \ell, r\rangle}{\langle r, r\rangle}.$$

Hence for $d \equiv 2, 3 \bmod 4$ with $d \neq -1$, this concludes lemma.

For $F = \mathbb{Q}(\sqrt{-1})$, it needs to show that if

$$(1-\sqrt{-1})\frac{\langle \ell, r\rangle}{\langle r, r\rangle} \in \mathscr{O}_F \text{ or } (1+\sqrt{-1})\frac{\langle \ell, r\rangle}{\langle r, r\rangle} \in \mathscr{O}_F,$$

then $\alpha \in \mathbb{Z}$. In the following, let a, b be rational integers. First, we assume

$$(1-\sqrt{-1})\frac{\langle \ell, r\rangle}{\langle r, r\rangle} = a + \sqrt{-1}b \in \mathscr{O}_F.$$

Then $\alpha = a - b \in \mathbb{Z}$. Second, we assume

$$(1+\sqrt{-1})\frac{\langle \ell, r\rangle}{\langle r, r\rangle} = a + \sqrt{-1}b \in \mathscr{O}_F.$$

Then $\alpha = a + b \in \mathbb{Z}$. This concludes lemma for $F = \mathbb{Q}(\sqrt{-1})$.

For $F = \mathbb{Q}(\sqrt{-3})$, assume that one of the following holds.

$$(1 \pm \omega)\frac{\langle \ell, r\rangle}{\langle r, r\rangle} \in \mathscr{O}_F, \tag{4.3.3}$$

$$(1 \pm \omega^2)\frac{\langle \ell, r\rangle}{\langle r, r\rangle} \in \mathscr{O}_F, \tag{4.3.4}$$

$$2\frac{\langle \ell, r\rangle}{\langle r, r\rangle} \in \mathscr{O}_F. \tag{4.3.5}$$

Through some simple computation, when (4.3.3) or (4.3.4) hold, then we have $\alpha \in \mathbb{Z}$. Finally, we assume (4.3.5). Let

$$\alpha = \alpha_1 = \frac{2(\ell, r)}{(r, r)} = a - \frac{b}{2},$$

$$\alpha_2 = \frac{2(\ell, \omega r)}{(\omega r, \omega r)} = -\frac{a}{2} + b,$$

$$\alpha_3 = \frac{2(\ell, \omega^2 r)}{(\omega^2 r, \omega^2 r)} = -\frac{a+b}{2}.$$

Hence, the assumption $a + \omega b \in \mathscr{O}_F$ implies one of α_i for $i = 1, 2, 3$ is an element of $\mathbb{Z}$. On the other hand, we have $H(r) = H(\omega r) = H(\omega^2 r)$ and $\iota(H(r)) \subset \mathcal{H}(r)$, thus this concludes lemma for $F = \mathbb{Q}(\sqrt{-3})$. ☐

For the computation of multiplicities of unitary modular forms later, we need the following converse to [**68**, Remark after 6.1].

Lemma 4.3.24. *Let $r \in \Lambda$ be a primitive element with $\langle r, r\rangle < 0$.*

(1) The special divisor $H(r)$ is contained in exactly $\frac{\#\mathscr{O}_F^\times}{2}$ special divisors of the form $\mathcal{H}(r') \subset \mathcal{D}_{\Lambda_Q}$ for some primitive $r' \in \Lambda_Q$.
(2) The restriction of the special divisor $\mathcal{H}_r|_{D_\Lambda}$ is $H(r)$ with multiplicity 1 i.e., reduced.

Proof. We fix $\sqrt{d} \in \mathbb{C}$ and the corresponding embedding $F \hookrightarrow \mathbb{C}$. First, we prove (1). Note $\mathcal{H}(r)|_{D_\Lambda} = \mathcal{H}(r')|_{D_\Lambda}$ if and only if $\mathbb{C}r' = \mathbb{C}r$ for $r, r' \in \Lambda$. This implies $r = ar'$ for some $a \in \mathbb{C}^\times$. Since r is primitive, we have $a \in \mathscr{O}_F^\times$. On the other hand, as $\mathcal{H}(r')$ only depends on $\mathbb{R}r'$ so that $\mathcal{H}(r') = \mathcal{H}(-r')$, the number we concern is $\frac{\#\mathscr{O}_F^*}{2}$.

The proof of (2) is as follows. Since $\langle r, r\rangle < 0$, $\mathcal{H}(r)$ is again an orthogonal symmetric domain which is an (analytic) open subset of a quadric hypersurface, say $Q^{n-1} \subset Q^n \subset \mathbb{P}^{n+1}$. Thus the restriction of the Cartier divisor $r = 0$ to Q^n is reduced and $\mathcal{H}(r)$ is its open subset. $H(r)$ is also an open subset of the restriction of $r = 0$ to the linear subspace, which is also clearly reduced. Hence the assertion follows. ☐

4.3.5. Unramifiedness of unitary modular varieties.

Theorem 4.3.25. *Let $F = \mathbb{Q}(\sqrt{d})$ $(d \neq -1)$ be an imaginary quadratic field and Λ be a Hermitian unimodular lattice over $\mathscr{O}_F$ of signature $(1, n)$ for $n > 1$. We assume $d \equiv 2, 3 \bmod 4$. Then for any arithmetic subgroup $\Gamma \subset \mathrm{U}(\Lambda)$, the canonical map $\pi_\Gamma \colon D_\Lambda \to \Gamma\backslash D_\Lambda$ does not ramify in codimension 1, so that $\overline{X}^{\mathrm{BB}}$ is a log canonical model.*

Proof. It suffices to show the claim for $\Gamma = \mathrm{U}(\Lambda)$. The ramification divisors are defined by $\sigma_{r,\xi}$ for some primitive $r \in \Lambda$ and $\xi \in \mathscr{O}_F^\times \backslash \{1\}$ and by Lemma 4.3.23, they are included in the set

$$\bigcup_{r\in\Lambda, b\in\mathbb{Z}, \xi\in\mathscr{O}_F^\times\backslash\{1\}} \quad \bigcup_{\substack{r\in\Lambda/\mathscr{O}_F^\times \\ \langle r,r\rangle = -\frac{b}{2},\ \sigma_{r,\xi}\in U(\Lambda)}} H(r).$$

Now

$$\sigma_{r,\xi}(\ell) = \ell - (1-\xi)\frac{\langle \ell, r\rangle}{\langle r, r\rangle} r.$$

We assume that $r \in \Lambda$ is a reflective element, that is, $\sigma_{r,\xi} \in U(\Lambda)$ for some $\xi \in \mathscr{O}_F^\times \backslash \{1\}$. Then

$$(1-\xi)\frac{\langle \ell, r\rangle}{\langle r, r\rangle} = -\frac{2(1-\xi)\langle \ell, r\rangle}{b}.$$

Since r is primitive and Λ is unimodular, by Proposition 4.3.22, there exists an $\ell \in \Lambda$ such that $\langle \ell, r\rangle = \frac{1}{2\sqrt{d}}$, so we have

$$(1-\xi)\frac{\langle \ell, r\rangle}{\langle r, r\rangle} = -\frac{1-\xi}{b\sqrt{d}} \notin \mathscr{O}_F$$

for $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$. This implies $\sigma_{r,\xi} \notin \mathrm{U}(\Lambda)$ and this is contradiction. The last assertion then follows from [**120**] (or as a special case of our Theorem 4.2.1 (iii)). □

Note that we can also deduce this result from [**146**, Lemma 2.2].

Corollary 4.3.26. *Let $F = \mathbb{Q}(\sqrt{d})$ $(d \neq -1)$ be an imaginary quadratic field and $(\Lambda, \langle\ ,\ \rangle) = M(b)$ be a Hermitian lattice over $\mathscr{O}_F$ of signature $(1, n)$ for $n > 1$ where M is a unimodular Hermitian lattice and $b \in \mathscr{O}_F$. We assume $d \equiv 2, 3 \bmod 4$, and $\frac{b}{\sqrt{d}} \notin \mathscr{O}_F$. Then for any arithmetic subgroup $\Gamma \subset \mathrm{U}(\Lambda)$, the canonical map $\pi_\Gamma\colon D_\Lambda \to \Gamma\backslash D_\Lambda$ does not ramify in codimension 1.*

4.3.6. Unitary modular cases, Part I - Fano cases. Below, for the definition of Hermitian lattices; see Appendix 4.A.

Remark 4.3.27. We can estimate the value $s(X)$ as orthogonal modular varieties and use it to determine the birational types of ball quotients. Note that the ramification degrees arising from unitary cases may differ from orthogonal ones [**9**], so we have to pay attention to the computation of a; compare with Remark 4.3.8.

For $F = \mathbb{Q}(\sqrt{-1})$, let B_2 (resp. B_4) be a union of ramification divisor with ramification degree 2 (resp. 4). If a modular form f of weight k vanishes on B_2 (resp. B_4) with order $2m$ (resp. $3m$) for some $m \in \mathbb{Z}_{>0}$, then f satisfies Assumption 1.5.1 (i) and $s(X) = \frac{k}{4mn}$.

Example 4.3.28. For $F = \mathbb{Q}(\sqrt{-1})$, let $\Lambda := \Lambda_{U\oplus U} \oplus \Lambda_{E_8(-1)}$ be an even unimodular Hermitian lattice over $\mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$ of signature $(1, 5)$ whose associated quadratic lattice is $\Lambda_Q = U \oplus U \oplus E_8(-1)$.

The only ramification divisors of the map $D_\Lambda \to X := \mathrm{U}(\Lambda)\backslash D_\Lambda$ are

$$\bigcup_{\substack{r\in\Lambda/\mathscr{O}_F^\times:\text{primitive}\\ \langle r,r\rangle=-1}} H(r)$$

with ramification degree 2. For more details, see Example 4.3.32.

By Example 4.3.11, $f := \Phi_{252}|_{D_\Lambda}$ is a weight 252 modular form with

$$\mathrm{div} f = 2 \sum_{\substack{r\in L/\mathscr{O}_{\mathbb{Q}(\sqrt{-1})}^\times:\text{primitive}\\ \langle r,r\rangle=-1}} H(r)$$

whose coefficient comes from Lemma 4.3.24. Therefore applying Theorem 4.2.1 (i) for f^{12} with $s(X) = \frac{21}{2}$, we have the following.

Corollary 4.3.29. *The Baily-Borel compactification $\overline{X}^{\mathrm{BB}}$ of the modular variety $X := U(\Lambda)\backslash D_\Lambda$ is a Fano variety, where $\Lambda := \Lambda_{U\oplus U} \oplus \Lambda_{E_8(-1)}$ for $F = \mathbb{Q}(\sqrt{-1})$.*

Example 4.3.30. For $F = \mathbb{Q}(\sqrt{-1})$, let $\Lambda := \Lambda_{U\oplus U(2)} \oplus \Lambda_{E_8(-1)}(2)$ be an even Hermitian lattice over $\mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$ of signature $(1, 5)$ whose associated quadratic lattice is $\Lambda_Q = U \oplus U(2) \oplus E_8(-2)$. The ramification divisors on $\mathscr{D}_{\Lambda_Q}$ with respect to $\mathrm{O}^+(\Lambda_Q)$ is the union of special divisors with respect to (-2)-vectors and special-even (-4)-vectors, so the ramification divisors on D_Λ with respect to $U(\Lambda)$ are included in the union of special divisors with respect to (-1)-vectors and special-even (-2)-vectors since $\langle v, v\rangle$ is real for all $v \in \Lambda$. Here we say a vector $r \in \Lambda$ is special-even if $\Re\langle r, v\rangle \in \mathbb{Z}$ for any $v \in \Lambda$. The only ramification divisors of π are

$$\bigcup_{\substack{r\in L/\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{primitive} \\ \langle r,r\rangle=-1,\ \sigma_{r,-1}\in U(\Lambda)}} H(r) \quad \cup \quad \bigcup_{\substack{r\in L/\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{special-even, primitive} \\ \langle r,r\rangle=-2,\ \sigma_{r,-1}\in U(\Lambda)}} H(r)$$

with ramification degree $d_i = 2$ and

$$\bigcup_{\substack{r\in \Lambda/\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{primitive} \\ \langle r,r\rangle=-1,\ \sigma_{r,\sqrt{-1}}\in U(\Lambda)}} H(r) \quad \cup \quad \bigcup_{\substack{r\in \Lambda/\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{special-even, primitive} \\ \langle r,r\rangle=-2,\ \sigma_{r,\sqrt{-1}}\in U(\Lambda)}} H(r).$$

with ramification degree $d_i = 4$. For any primitive element $r \in \Lambda$ with $\langle r, r\rangle = -1$, we have

$$\sigma_{r,-1}(\ell) = \ell + 2\langle \ell, r\rangle r.$$

By the description of Hermitian lattices $\Lambda_{U\oplus U(2)}$ and $\Lambda_{E_{8(-1)}(2)}$,

$$2\langle \ell, r\rangle \in \mathscr{O}_{\mathbb{Q}(\sqrt{-1})}.$$

Hence $\sigma_{r,-1} \in U(\Lambda)$ for any (-1)-primitive element $r \in \Lambda$. For any special-even primitive element $r \in \Lambda$ with $\langle r, r\rangle = -2$, we have

$$\sigma_{r,-1}(\ell) = \ell + \langle \ell, r\rangle r.$$

By the definition of $\Lambda_{U\oplus U(2)}$, if $\Re\langle \ell, r\rangle \in \mathbb{Z}$, then $\Im\langle \ell, r\rangle \in \mathbb{Z}$ for any $\ell \in \Lambda$. Also by the definition of $\Lambda_{E_8}(-2)$, we have $\langle \ell, r\rangle \in \mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$ for any $\ell \in \Lambda$. Hence $\sigma_{r,-1} \in \mathrm{U}(\Lambda)$ for any special-even (-2)-primitive vector $r \in \Lambda$. Therefore the map $D_\Lambda \to X := \mathrm{U}(\Lambda)\backslash D_\Lambda$ ramifies along

$$\bigcup_{\substack{r\in L/\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{primitive} \\ \langle r,r\rangle=-1}} H(r) \quad \cup \quad \bigcup_{\substack{r\in L/\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{special-even, primitive} \\ \langle r,r\rangle=-2}} H(r).$$

For (-1)-primitive vector $r \in \Lambda$,

$$\sigma_{r,\sqrt{-1}}(\ell) = \ell + (1-\sqrt{-1})\langle \ell, r\rangle r.$$

If $r \in \Lambda_{E_{8(-1)}}(2)$, then by the description of the Hermitian matrix defining $\Lambda_{E_{8(-2)}}$, we have $\langle \ell, r\rangle \in \mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$, so $(1-\sqrt{-1})\langle \ell, r\rangle \in \mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$. If $r \in \Lambda_{U\oplus U(2)}$, then the ideal

$$\{\langle \ell, r\rangle \mid \ell \in \Lambda_{U\oplus U(2)}\}$$

is generated by $\frac{1+\sqrt{-1}}{2}$ since $\det(L_{U\oplus U(2)}) = \frac{1}{2}$, so $(1-\sqrt{-1})\langle \ell, r\rangle \in \mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$. From a similar discussion as above, we have $\sigma_{r,\sqrt{-1}} \in U(\Lambda)$ for any (-1)-primitive vector $r \in \Lambda$.

For special-even (-2)-primitive vector $r \in \Lambda$,

$$\sigma_{r,\sqrt{-1}}(\ell) = \ell + \frac{(1-\sqrt{-1})}{2}\langle \ell, r\rangle r.$$

If $r \in \Lambda_{E_{8(-1)}}(2)$, then there exists an $\ell \in \Lambda_{E_{8(-1)}}(2)$ such that $\langle \ell, r\rangle = 1$, so we have $\frac{(1-\sqrt{-1})\langle \ell, r\rangle}{2} = \frac{1-\sqrt{-1}}{2} \notin \mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$. If $r \in \Lambda_{U\oplus U(2)}$, then there exists an $\ell \in \Lambda_{U\oplus U(2)}$ such that $\langle \ell, r\rangle = \frac{1+\sqrt{-1}}{2}$, so we have $\frac{(1-\sqrt{-1})\langle \ell, r\rangle}{2} = \frac{1}{2} \notin \mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$. Thus, we have $\sigma_{r,\sqrt{-1}} \notin \mathrm{U}(\Lambda)$ for any special-even (-2)-primitive vector $r \in \Lambda$.

Therefore, the ramification in codimension 1 only occurs along

$$\bigcup_{\substack{r\in\Lambda/\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{primitive}\\ \langle r,r\rangle=-1}} H(r)$$

with ramification degree 2, and along

$$\bigcup_{\substack{r\in\Lambda/\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{special-even, primitive}\\ \langle r,r\rangle=-2}} H(r)$$

with ramification degree 4.

This example implies Theorem 4.3.25 does not hold for non-unimodular lattices and $F=\mathbb{Q}(\sqrt{-1})$. By Example 4.3.13, we have modular forms $\Phi_4|_{D_\Lambda}$ and $\Phi_{124}|_{D_\Lambda}$ such that

$$\mathrm{div}\Phi_4|_{D_\Lambda} = 2\sum_{\substack{r\in\Lambda/\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{primitive}\\ \langle r,r\rangle=-1}} H(r)$$

$$\mathrm{div}\Phi_{124}|_{D_\Lambda} = 2\sum_{\substack{r\in\Lambda/\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{special-even, primitive}\\ \langle r,r\rangle=-2}} H(r)$$

whose coefficient again comes from Lemma 4.3.24.

Hence, applying Theorem 4.2.1 (i) to $(\Phi_4|_{D_\Lambda}^2\Phi_{124}|_{D_\Lambda}^3)^{12}$ with $s(X)=62$, we have the following.

Corollary 4.3.31. *The Baily-Borel compactification $\overline{X}^{\mathrm{BB}}$ of the modular variety $X := \mathrm{U}(\Lambda)\backslash D_\Lambda$ is a Fano variety, where $\Lambda := \Lambda_{U\oplus U(2)}\oplus\Lambda_{E_8(-1)}(2)$ for $F=\mathbb{Q}(\sqrt{-1})$.*

4.3.7. Unitary modular cases, Part II - with ample canonical class.

Example 4.3.32. For $F=\mathbb{Q}(\sqrt{-1})$, let $\Lambda := \Lambda_{U\oplus U}\oplus\Lambda_{E_8(-1)}\oplus\Lambda_{E_8(-1)}\oplus\Lambda_{E_8(-1)}$ be an even unimodular Hermitian lattice of signature $(1,13)$ whose associated quadratic lattice is $\Lambda_Q = II_{2,26} = U\oplus U\oplus E_8(-1)\oplus E_8(-1)\oplus E_8(-1)$. The ramification divisors on $\mathscr{D}_{\Lambda_Q}$ with respect to $\mathrm{O}^+(\Lambda_Q)$ is the union of special divisors with respect to (-2)-vectors, so the ramification divisors on D_Λ with respect to $U(\Lambda)$ are included in the union of special divisors with respect to (-1)-vectors as $\langle v,v\rangle$ is real for all $v\in\Lambda$. There exist possibly double ramification divisors i.e., those with $d_i=2$, and quadruple ramification divisors i.e., those with $d_i=4$, of the natural morphism $\pi\colon D_\Lambda\to X := \mathrm{U}(\Lambda)\backslash D_\Lambda$. It ramifies in codimension 1 along

$$\bigcup_{\substack{r\in\Lambda/\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{primitive}\\ \langle r,r\rangle=-1,\ \sigma_{r,-1}\in U(\Lambda)}} H(r)$$

with ramification degree 2, and

$$\bigcup_{\substack{r\in\Lambda/\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{primitive}\\ \langle r,r\rangle=-1,\ \sigma_{r,\sqrt{-1}}\in U(\Lambda)}} H(r)$$

with ramification degree 4.

For any primitive element $r\in\Lambda$ with $\langle r,r\rangle=-1$, we have

$$\sigma_{r,\sqrt{-1}}(\ell) = \ell + (1-\sqrt{-1})\langle \ell, r\rangle r,$$

but by Proposition 4.3.22 and unimodularity of Λ, $\langle \ell, r\rangle = \frac{1}{2\sqrt{-1}}$ for some $\ell \in \Lambda$. Hence $\sigma_{r,-1} \notin \mathrm{U}(\Lambda)$ for any (-1)-primitive element $r \in \Lambda$, that is, there is no quadruple ramification divisors.

For any primitive element $r \in \Lambda$ with $\langle r, r\rangle = -1$, we have

$$\sigma_{r,-1}(\ell) = \ell + 2\langle \ell, r\rangle r.$$

Here,

$$\langle \ell, r\rangle \in \delta\mathscr{O}_F = \frac{1}{2\sqrt{-1}}\mathscr{O}_{\mathbb{Q}(\sqrt{-1})},$$

so $2\langle \ell, r\rangle \in \mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$. Thus, $\sigma_{r,-1} \in \mathrm{U}(\Lambda)$ for any (-1)-primitive element $r \in \Lambda$, that is, there are only double ramification divisors along

$$\bigcup_{\substack{r\in\Lambda/\mathscr{O}^{\times}_{\mathbb{Q}(\sqrt{-1})}:\text{primitive} \\ \langle r,r\rangle=-1}} H(r)$$

with ramification degree 2. By Example 4.3.9, $f := \Phi_{12}|_{D_\Lambda}$ is a weight 12 modular form whose divisors are equal to double ramification divisors;

$$\mathrm{div} f = 2\sum_{\substack{r\in\Lambda/\mathscr{O}^{\times}_{F}:\text{primitive} \\ \langle r,r\rangle=-1}} H(r)$$

whose coefficient again comes from Lemma 4.3.24. Therefore applying Theorem 4.2.1 (iii) to f^{28} with $s(X) = \frac{3}{14}$, we have the following the following.

Corollary 4.3.33. *The Baily-Borel compactification $\overline{X}^{\mathrm{BB}}$ of the modular variety $X := \mathrm{U}(\Lambda)\backslash D_\Lambda$ is a log canonical model, where $\Lambda := \Lambda_{U\oplus U} \oplus \Lambda_{E_8(-1)} \oplus \Lambda_{E_8(-1)} \oplus \Lambda_{E_8(-1)}$ for $F = \mathbb{Q}(\sqrt{-1})$. Recall from Terminology after Theorem 4.2.1 that a log canonical model in this chapter means it has only log canonical singularities and ample canonical class.*

Example 4.3.34. For $F = \mathbb{Q}(\sqrt{-2})$, let $\Lambda := \Lambda'_{U\oplus U(2)} \oplus \Lambda'_{E_8(-1)}(2)$ be an even Hermitian lattice over $\mathscr{O}_{\mathbb{Q}(\sqrt{-2})}$ of signature $(1,5)$. The union of ramification divisors of the map $\pi : D_\Lambda \to X := \mathrm{U}(\Lambda)\backslash D_\Lambda$ are the union of special divisors with respect to (-1)-vectors only, unlike $F = \mathbb{Q}(\sqrt{-1})$ case. Of course, these divisors ramify with ramification degree 2, so we can also show $\overline{X}^{\mathrm{BB}}$ is a log canonical model. (Applying Theorem 4.2.1 (iii) to f^{12} with $s(X) = \frac{1}{6}$, where $f := \Phi_4|_{D_\Lambda}$.) This example implies Theorem 4.3.25 does not hold for non-unimodular lattices and there exist Hermitian lattices, whose quadratic lattices are the same, admitting modular varieties with various birational types according to imaginary quadratic fields.

Corollary 4.3.35. *The Baily-Borel compactification $\overline{X}^{\mathrm{BB}}$ of the modular variety $X := U(\Lambda)\backslash D_\Lambda$ is a log canonical model, where $\Lambda := \Lambda'_{U\oplus U(2)} \oplus \Lambda'_{E_8(-1)}(2)$ for $F = \mathbb{Q}(\sqrt{-2})$.*

Remark 4.3.36. For $F = \mathbb{Q}(\sqrt{-2})$, let $\Lambda := \Lambda'_{U\oplus U} \oplus \Lambda'_{E_8(-1)} \oplus \Lambda'_{E_8(-1)} \oplus \Lambda'_{E_8(-1)}$ be an even unimodular Hermitian lattice over $\mathscr{O}_{\mathbb{Q}(\sqrt{-2})}$ of signature $(1,13)$, whose associated quadratic lattice Λ_Q is $U \oplus U \oplus E_8(-1) \oplus E_8(-1) \oplus E_8(-1)$.

Now, we know that for any arithmetic subgroup $\Gamma \subset \mathrm{U}(\Lambda)$, the map $\pi : D_\Lambda \to \Gamma\backslash D_\Lambda$ does not ramify in codimension 1. This is exactly an example of Theorem 4.3.25. Thus the Baily-Borel compactification $\overline{\Gamma\backslash D_\Lambda}^{\mathrm{BB}}$ is a log canonical model.

Remark 4.3.37. For any imaginary quadratic field with class number 1, we can construct $\Lambda_{U\oplus U}$ and Λ_{E_8}; see [**110**, Appendix A]. As in Theorem 4.3.25, we can show that the corresponding map does not ramify in codimension 1 for any arithmetic subgroup so that the Baily-Borel compactification is log canonical model again.

Remark 4.3.38. By the same reason as Remark 4.3.36, for $F \neq \mathbb{Q}(\sqrt{-1})$, the map $\pi : D_\Lambda \to \Gamma\backslash D_\Lambda$ does not ramify in codimension 1, where $\Lambda := \Lambda_{U\oplus U} \oplus \Lambda_{E_8(-1)}$ and $\Gamma \subset \mathrm{U}(\Lambda)$ is any arithmetic subgroup. This is also an example of Theorem 4.3.25 and $\overline{\Gamma\backslash D_\Lambda}^{\mathrm{BB}}$ is a log canonical model.

4.3.8. More examples. For $F = \mathbb{Q}(\sqrt{-1})$, let $\Lambda_{-1} := \Lambda_{U\oplus U} \oplus \Lambda_{E_8(-1)}(2)$. Then, the map $\pi : D_{\Lambda_{-1}} \to U(\Lambda_{-1})\backslash D_{\Lambda_{-1}}$ ramifies at the union of special divisors with respect to (-1)-vectors and (-2)-special-even vectors. By [**150**, Theorem 8.1], there exists a reflective modular form Ψ_{12} of weight 12 on $\mathscr{D}_{(\Lambda_{-1})_Q}$ such that

$$\mathrm{div}\Psi_{12}|_{D_\Lambda} = 2 \sum_{\substack{r\in\Lambda_{-1}/\mathscr{O}^\times_{\mathbb{Q}(\sqrt{-1})}:\text{primitive} \\ \langle r,r\rangle=-1}} H(r)$$

whose coefficient again comes from Lemma 4.3.24. Thus, $\iota^\star\Psi_{12} = \Psi_{12}|_{D_{\Lambda_{-1}}}$ is a reflective modular form on $D_{\Lambda_{-1}}$, but this does not satisfy Assumption 1.5.2 (ii) because the ramification divisors properly include the divisors of $\Psi_{12}|_{D_{\Lambda_{-1}}}$, i.e.,

$$\mathrm{Supp}(\mathrm{div}\Psi_{12}|_{D_\Lambda}) \subsetneq \bigcup_{\substack{r\in L/\mathscr{O}^\times_{\mathbb{Q}(\sqrt{-1})}:\text{primitive} \\ \langle r,r\rangle=-1}} H(r) \quad \cup \bigcup_{\substack{r\in L/\mathscr{O}^\times_{\mathbb{Q}(\sqrt{-1})}:\text{special-even, primitive} \\ \langle r,r\rangle=-2}} H(r),$$

where the right-hand side is the ramification divisor. Hence, we can not show the Fanoness of $\overline{(\mathrm{U}(\Lambda_{-1})\backslash D_{\Lambda_{-1}})}^{\mathrm{BB}}$ in this way (but we can show the uniruledness or more strongly, rationally-chain-connectedness of $\mathrm{U}(\Lambda_{-1})\backslash D_{\Lambda_{-1}}$ by [**110**, Theorem 5.1]).

On the other hand, for $F = \mathbb{Q}(\sqrt{-2})$, let Λ_{-2} be the Herimtian lattice over $\mathscr{O}_{\mathbb{Q}(\sqrt{-2})}$ of signature $(1, 5)$ whose associated quadratic lattice is $U \oplus U \oplus E_8(-2)$. Then the map $\pi : D_{\Lambda_{-2}} \to U(\Lambda_{-2})\backslash D_{\Lambda_{-2}}$ has no ramification divisors, so we can not even show the uniruledness.

4.A. Definition of matrices

The following matrices are taken from [**110**, Appendix A].

4.A.1. $\mathbb{Q}(\sqrt{-1})$ cases. Let $\Lambda_{U\oplus U}$ be an even unimodular Hermitian lattice of signature $(1, 1)$ over $\mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$ defined by the matrix

$$\frac{1}{2\sqrt{-1}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

whose associated quadratic lattice $(\Lambda_{U\oplus U})_Q$ is $U \oplus U$.

Let $\Lambda_{U\oplus U(2)}$ be an even Hermitian lattice of signature $(1, 1)$ over $\mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$ defined by the matrix

$$\frac{1}{2}\begin{pmatrix} 0 & 1+\sqrt{-1} \\ 1-\sqrt{-1} & 0 \end{pmatrix}$$

whose associated quadratic lattice $(\Lambda_{U\oplus U(2)})_Q$ is $U \oplus U(2)$.

Let $\Lambda_{E_8(-1)}$ be an even unimodular Hermitian lattice of signature $(0,4)$ over $\mathscr{O}_{\mathbb{Q}(\sqrt{-1})}$ defined by the matrix

$$-\frac{1}{2}\begin{pmatrix} 2 & -\sqrt{-1} & -\sqrt{-1} & 1 \\ \sqrt{-1} & 2 & 1 & \sqrt{-1} \\ \sqrt{-1} & 1 & 2 & 1 \\ 1 & -\sqrt{-1} & 1 & 2 \end{pmatrix}$$

whose associated quadratic lattice $(\Lambda_{E_8(-1)})_Q$ is $E_8(-1)$. This matrix is called Iyanaga's matrix.

4.A.2. $\mathbb{Q}(\sqrt{-2})$ **cases.** Let $\Lambda'_{U\oplus U}$ be an even unimodular Hermitian lattice of signature $(1,1)$ over $\mathscr{O}_{\mathbb{Q}(\sqrt{-2})}$ defined by the matrix

$$\frac{1}{2\sqrt{-2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

whose associated quadratic lattice $(\Lambda'_{U\oplus U})_Q$ is $U \oplus U$.

Let $\Lambda'_{U\oplus U(2)}$ be a Hermitian lattice of signature $(1,1)$ over $\mathscr{O}_{\mathbb{Q}(\sqrt{-2})}$ defined by the matrix

$$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

whose associated quadratic lattice $(\Lambda'_{U\oplus U(2)})_Q$ is $U \oplus U(2)$.

Let $\Lambda'_{E_8(-1)}$ be an even unimodular Hermitian lattice of signature $(0,4)$ over $\mathscr{O}_{\mathbb{Q}(\sqrt{-2})}$ defined by the matrix

$$-\frac{1}{2}\begin{pmatrix} 2 & 0 & \sqrt{-2}+1 & \frac{1}{2}\sqrt{-2} \\ 0 & 2 & \frac{1}{2}\sqrt{-2} & 1-\sqrt{-2} \\ 1-\sqrt{-2} & -\frac{1}{2}\sqrt{-2} & 2 & 0 \\ -\frac{1}{2}\sqrt{-2} & \sqrt{-2}+1 & 0 & 2 \end{pmatrix}$$

whose associated quadratic lattice $(\Lambda'_{E_8(-1)})_Q$ is $E_8(-1)$.

CHAPTER 5

Revisiting the moduli space of 8 points on $\mathbb{P}^1$

5.1. Introduction

It was shown by Casalaina-Martin-Grushevsky-Hulek-Laza [**28**] that the Kirwan blow-up and the toroidal compactification of the moduli space of (non-marked) smooth cubic surfaces are not isomorphic. In this chapter, we prove analogous results for the moduli space of unordered 8 points on $\mathbb{P}^1$, denoted by $\mathcal{M}^{\mathrm{GIT}}$. The proof we give here is inspired by that of [**28**], but requires further ideas. As we shall discuss in Section 5.6, the behavior observed here is shared by other ball quotients as well, thus pointing towards a much more general, and yet not fully understood, phenomenon.

The case of 8 points on $\mathbb{P}^1$ is of special interest for more than one reason. One is that it has more than one modular interpretation. Besides being a moduli space of points, it is, by work of Kondō [**88**], also closely related to moduli of K3 surfaces and to automorphic forms. A further reason is that it is a so-called *ancestral* Deligne-Mostow variety in the sense of the discussion by Gallardo-Kerr-Scheffler [**48**]. This means that any Deligne-Mostow variety over the Gaussian integers with arithmetic monodromy group, and which has cusps, can be embedded into this ball quotient. The other ancestral case is that of 12 points on $\mathbb{P}^1$, which plays the same role for the Eisenstein integers. In this chapter, we shall concentrate on the Gaussian case and only briefly discuss the Eisenstein case, which will be treated in forthcoming work.

5.1.1. Main results. First, we prove that the Deligne-Mostow isomorphism does not lift between the Kirwan blow-up and the toroidal compactification.

Theorem 5.1.1 (Theorem 5.3.15). *Neither the Deligne-Mostow isomorphism* $\phi : \mathcal{M}^{\mathrm{GIT}} \to \overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}$ *nor its inverse* ϕ^{-1} *lift to a morphism between the Kirwan blow-up* $\mathcal{M}^{\mathrm{K}}$ *and the unique toroidal compactification* $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$.

Second, we compute the cohomology groups of the varieties appearing in this chapter.

Theorem 5.1.2 (Theorem 5.5.1, 5.5.2, 5.5.6, 5.5.8). *All the odd degree cohomology of the following projective varieties vanishes. In even degrees, their Betti numbers are given by:*

j	0	2	4	6	8	10
$\dim H^j(\mathcal{M}^{\mathrm{K}})$	1	2	3	3	2	1
$\dim IH^j(\overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}})$	1	1	2	2	1	1
$\dim H^j(\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}})$	1	2	3	3	2	1
$\dim H^j(\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}})$	1	43	99	99	43	1
$\dim IH^j(\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}})$	1	8	29	29	8	1
$\dim IH^j(\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}})$	1	43	99	99	43	1

thus, all the Betti numbers of $\mathcal{M}^{\mathrm{K}}$ *and* $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ *are the same.*

These results leave the possibility that the varieties $\mathcal{M}^{\mathrm{K}}$ and $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ is isomorphic, but finally we show that these are actually not isomorphic as abstract varieties.

Theorem 5.1.3 (Theorem 5.4.6). *The Kirwan blow-up* $\mathcal{M}^{\mathrm{K}}$ *and the toroidal compactification* $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ *are not* K*-equivalent and hence, in particular, not isomorphic as abstract varieties.*

5.1.2. Outline of the proof of Theorem 5.1.1. The strategy of the proof of Theorem 5.1.1 is as follows. As in [**28**] the argument is divided into two steps. We first prove that the discriminant divisor and the boundary divisor intersect non-transversally in the Kirwan blow-up. This is done in terms of local computation by using the Luna slice. Secondly, we show that the corresponding divisors intersect generically transversally in the toroidal compactification of the 5-dimensional ball quotient. Here is a major difference to [**28**]. This is because we cannot use Naruki's compactification. Instead, we work on a sequence of blow-ups of the Baily-Borel compactification of the 5-dimensional ball quotient. This was studied in detail in [**48, 75**] and can be described in terms of moduli spaces of weighted pointed stable curves [**75**]. The discriminant divisor and boundary divisor exist as normal crossing divisors in these spaces, thus we can use this to prove the generic transversality of the divisors in the toroidal compactification.

5.1.3. Organization of this chapter. In Section 5.2, we describe the relationship between GIT quotients and ball quotients. In Section 5.3, we prove Theorem 5.1.1 through local computations. In Section 5.4, we compute the top self-intersection number of canonical bundles and deduce Theorem 5.1.3. In Section 5.5, we compute the cohomology by using the Kirwan method. In Section 5.6, we will briefly discuss other Deligne-Mostow varieties.

5.2. GIT and ball quotients

Below, we consider the moduli spaces of *ordered* and *unordered* 8 points on $\mathbb{P}^1$. Throughout this chapter, the phrase "8 points on $\mathbb{P}^1$" will always mean "unordered 8 points on $\mathbb{P}^1$" for simplicity. Let

$$\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}} := (\mathbb{P}^1)^8/\!/\operatorname{SL}_2(\mathbb{C}), \quad \mathcal{M}^{\mathrm{GIT}} := \mathbb{P}^8/\!/\operatorname{SL}_2(\mathbb{C}).$$

Here, the GIT quotients are taken with respect to the symmetric linearisation $\mathscr{O}(1, \cdots, 1)$ and $\mathscr{O}(1)$. We also note, see [**75**, Theorem 1.1], that

$$\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}}/\mathfrak{S}_8 \cong \mathcal{M}^{\mathrm{GIT}}.$$

We denote by $\varphi_1 : \mathcal{M}^{\mathrm{K}}_{\mathrm{ord}} \to \mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}}$ and $f : \mathcal{M}^{\mathrm{K}} \to \mathcal{M}^{\mathrm{GIT}}$ the Kirwan blow-ups [**76**].

As in [**88**], we consider the free $\mathbb{Z}[\sqrt{-1}]$-module of rank 2 equipped with the Hermitian form defined by the following matrix

$$\begin{pmatrix} 0 & 1+\sqrt{-1} \\ 1-\sqrt{-1} & 0 \end{pmatrix}, \quad \begin{pmatrix} -2 & 1-\sqrt{-1} \\ 1+\sqrt{-1} & -2 \end{pmatrix}.$$

The underlying integral lattices are isomorphic to $U \oplus U(2)$ and $D_4(-1)$, where U denotes the hyperbolic plane, $U(2)$ is the hyperbolic plane where the form has been multiplied by 2 and $D_4(-1)$ is the negative D_4-lattice. By abuse of notation, we will also denote the Hermitian lattices by these symbols.

Here, let $L := U \oplus U(2) \oplus D_4(-1)^{\oplus 2}$ be the Hermitian lattice of signature $(1,5)$ over $\mathbb{Z}[\sqrt{-1}]$, defined by above Hermitian forms. Let $\mathrm{U}(L)$ be the unitary group scheme over $\mathbb{Z}$ and $\Gamma := \mathrm{U}(L)(\mathbb{Z})$. Now, there is the Hermitian symmetric domain $\mathbb{B}^5$ associated with the reductive group $\mathrm{U}(L)(\mathbb{R}) \cong \mathrm{U}(1,5)$ defined by

$$\mathbb{B}^5 := \{v \in L \otimes_{\mathbb{Z}[\sqrt{-1}]} \mathbb{C} \mid \langle v, v \rangle > 0\}/\mathbb{C}^\times$$

which is isomorphic to the 5-dimensional complex ball. Let $L^\vee$ be the *dual lattice* of L, which contains L as a finite $\mathbb{Z}[\sqrt{-1}]$-module, and $A_L := L^\vee/L$ be the *discriminant group*, isomorphic to $\left(\mathbb{Z}[\sqrt{-1}]/(1+\sqrt{-1})\mathbb{Z}[\sqrt{-1}]\right)^6$ in this situation. Now, let us introduce an important arithmetic subgroup $\Gamma_{\mathrm{ord}} \subset \Gamma$, which is called the *discriminant kernel*:

$$\Gamma_{\mathrm{ord}} := \{g \in \Gamma \mid g(v) \equiv v \bmod L \ (\forall v \in A_L)\}.$$

This data gives us the notion of the *ball quotients* $\mathbb{B}^5/\Gamma_{\mathrm{ord}}$ and $\mathbb{B}^5/\Gamma$ which are quasi-projective varieties over $\mathbb{C}$. We denote by $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}$ and $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}$ (resp. $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}}$ and $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$) the Baily-Borel compactifications (resp. toroidal compactifications) of the corresponding ball quotients. Note that the toroidal compactifications of ball quotients are canonical as there is no choice of a fan involved. Further, let

$$H := \bigcup_{\langle \ell, \ell \rangle = -2} H(\ell)$$

be the discriminant divisor where

$$H(\ell) = \{v \in \mathbb{B}^5 \mid \langle v, \ell \rangle = 0\}$$

is the special divisor with respect to a root $\ell \in L$, see [**88**, Subsection 3.4].

Next, we describe the stable, semi-stable and polystable loci on $\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}}$ and $\mathcal{M}^{\mathrm{GIT}}$. This goes back to very classical results of GIT, in fact Mumford's seminal work, see [**122**, Chapter 4, Section 2]. In our cases, this is spelled out as follows. In the ordered case, 8 points define a stable (resp. semi-stable) GIT-point if and only if no 4 points (resp. 5 points) coincide, see also [**88**, Subsection 4.4] or [**38**, Example 2, p31]. Polystable points (that is, strictly semi-stable points whose orbit is closed) correspond to the points $(4,4)$, which means that we have two different points, each with multiplicity 4; for the notation, see [**88**, Subsection 4.4]. In the unordered case, stable, semi-stable and polystable points are described in the same way as above, see also [**119**, Subsection 7.2 (c)].

A crucial result of Kondō, [**88**, Theorem 4.6], says that there are $\mathfrak{S}_8$-equivariant isomorphisms

$$\phi_{\mathrm{ord}} : \mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}} \xrightarrow{\sim} \overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}$$
$$\phi : \mathcal{M}^{\mathrm{GIT}} \xrightarrow{\sim} \overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}},$$

where the second isomorphism goes back to [**32**].

These isomorphisms also allow us to describe the subloci of 8-tuples consisting of different points, the discriminant locus of stable, but not distinct, 8-tuples and the properly polystable loci. For this, let $(\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}})^o \subset \mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}}$ (resp. $(\mathcal{M}^{\mathrm{GIT}})^o \subset \mathcal{M}^{\mathrm{GIT}}$) be the moduli space of distinct ordered 8 points on $\mathbb{P}^1$ (resp. the moduli space of distinct 8 points on $\mathbb{P}^1$). By [**88**, Theorem 3.3], the morphisms ϕ_{ord} and ϕ restrict to isomorphisms:

$$\phi_{\mathrm{ord}}|_{(\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}})^o} : (\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}})^o \xrightarrow{\sim} (\mathbb{B}^5 \setminus H)/\Gamma_{\mathrm{ord}}$$
$$\phi|_{(\mathcal{M}^{\mathrm{GIT}})^o} : (\mathcal{M}^{\mathrm{GIT}})^o \xrightarrow{\sim} (\mathbb{B}^5 \setminus H)/\Gamma.$$

Also the isomorphisms ϕ_{ord} and ϕ identify the discriminant locus of stable, but not distinct 8 points on $\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}}$ and $\mathcal{M}^{\mathrm{GIT}}$ with H/Γ_{ord} and H/Γ respectively. It turns out that the discriminant divisor H/Γ_{ord} has 28 irreducible components, whereas H/Γ is irreducible. See also [**88**, Subsection 4.2], asserting that A_L contains 64 vectors: 1 zero vector, 35 isotropic vectors and 28 non-isotropic vectors.

Finally, the properly polystable points are identified with the cusps of the Borel compactification, namely $\overline{(\mathbb{B}^5/\Gamma_{\mathrm{ord}})}^{\mathrm{BB}} \setminus (\mathbb{B}^5/\Gamma_{\mathrm{ord}})$ and $\overline{(\mathbb{B}^5/\Gamma)}^{\mathrm{BB}} \setminus (\mathbb{B}^5/\Gamma))$ respectively. There are 35 cusps on $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}$ (also corresponding to the 35 isotropic vectors in A_L), but $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}$ has a unique cusp. This directly follows from [**88**, Subsection 4.2, Proposition 4.4], but we will see this in detail when we study the blow-up sequences.

The moduli spaces under consideration are also closely related to moduli spaces of stable curves. We do not repeat all details of the general theory here, but recall some notions as they are relevant for our purposes. Let $\overline{\mathcal{M}}_{0,8(\frac{1}{4}+\epsilon)}$ be the smooth projective variety which is the coarse moduli space representing the moduli problem of weighted pointed stable curves of type $(0, 8(\frac{1}{4}+\epsilon))$ with $0 < \epsilon < 1$ in the sense of [**75**, Theorem 2.1] or [**75**, Definition 2.1, Theorem 2.2], see also [**48**, Lemma 2.3, Remark 2.4, Remark 2.11, Example 2.12]. This is also realized as the KSBA compactification [**48**, Subsection 3.2]. $\overline{\mathcal{M}}_{0,8}$ is defined in the same way, but in this case, this is exactly the GIT quotient of $\mathbb{P}^1[8]$, the Fulton-Macpherson compactification of the configuration space of 8 points on $\mathbb{P}^1$ [**47**], by SL_2; see also [**123**, p55]. More generally, this is interpreted as the wonderful compactification [**102**, p536, Subsection 4.2] (or the Deligne-Mumford compactification [**48**, Remark 2.9]).

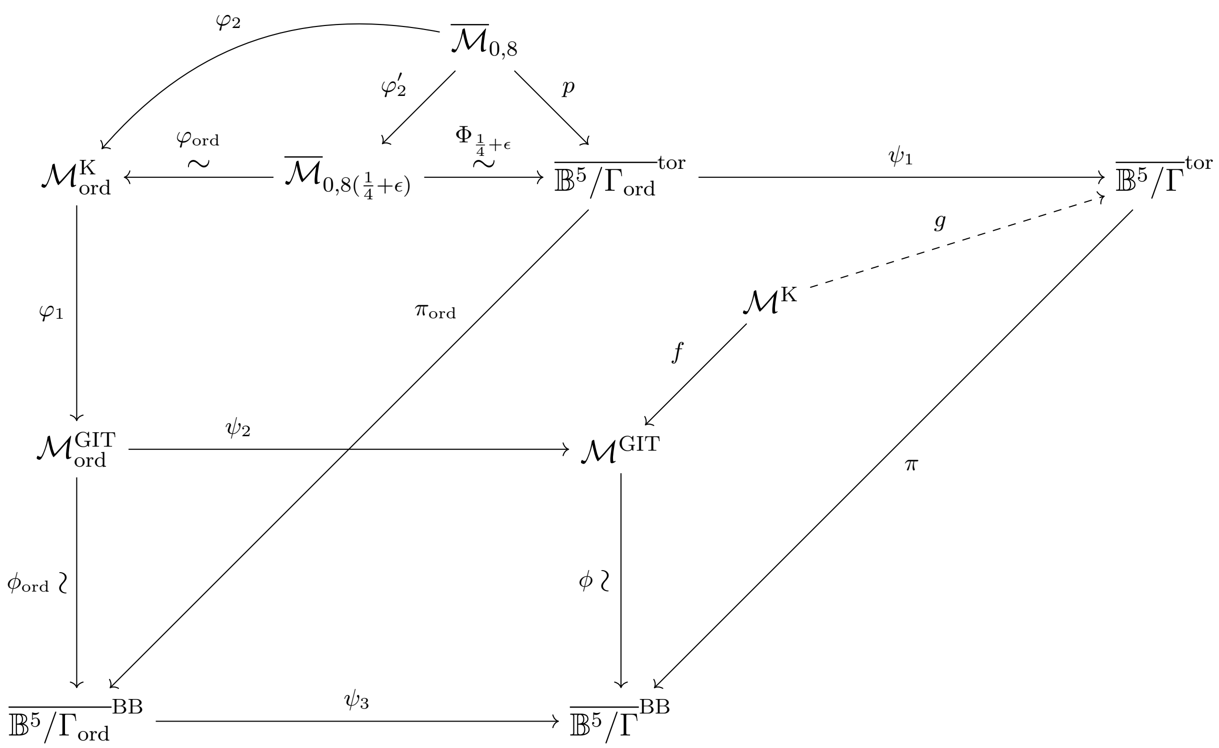

FIGURE 5.2.1. Relationship between several compactifications

We describe the relation of these spaces in Figure 5.2.1.

(1) ψ_i is a morphism by the above discussion about stable conditions for $i = 1, 2, 3$.
(2) ϕ is an isomorphism [**32**].
(3) ϕ_{ord} is an $\mathfrak{S}_8$-equivalent isomorphism [**88**, Theorem 4.6].
(4) φ_{ord} is an isomorphism [**75**, Theorem 1.1].
(5) $\Phi_{\frac{1}{4}+\epsilon}$ is an isomorphism [**48**, Theorem 1.1].
(6) p is a morphism [**48**, Proposition 2.13].
(7) The blow-up sequences φ_1, φ_2 are considered in [**75**, Theorem 4.1 (i), (iii)]. About the contraction of divisors of these maps, see [**75**, Proposition 4.5] or [**75**, p1121]. We study these maps in detail in Subsection 5.3.2.
(8) $\overline{\mathcal{M}}_{0,8}$ is a normal crossing compactification of $(\mathbb{B}^5 \setminus H)/\Gamma_{\mathrm{ord}}$, see [**65**, p345] or [**48**, Proposition 2.13].
(9) $\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}} \cong \overline{\mathcal{M}}_{0,8(\frac{1}{4}+\epsilon)}$ is nonsingular [**75**, Section 4].

We conclude this section with a remark about the toroidal boundary, which is defined by $T_{\mathrm{ord}} := \overline{(\mathbb{B}^5/\Gamma_{\mathrm{ord}})}^{\mathrm{tor}} \backslash (\mathbb{B}^5/\Gamma_{\mathrm{ord}})$ and $T := \overline{(\mathbb{B}^5/\Gamma)}^{\mathrm{tor}} \backslash (\mathbb{B}^5/\Gamma)$ respectively. The divisor T_{ord} has 35 irreducible components (mapping to the 35 cusps in the Baily-Borel compactification). We write them as $T_{\mathrm{ord},i}$ for $i = 1, \ldots 35$. Note that $T_{\mathrm{ord},i} \cong \mathbb{P}^2 \times \mathbb{P}^2$ by [**118**, Remark 6] or [**48**, Example 2.12]. The boundary divisor T is irreducible (and maps to the unique cusp in the Baily-Borel compactification); see also [**88**, Proposition 4.7]. We study T_{ord} and T in detail in Lemma 5.3.11.

5.3. (Non-) Extendability of the Deligne-Mostow isomorphism

5.3.1. Non-transversality in the Kirwan blow-up. In this subsection, we show that the discriminant divisor and the boundary divisor do not intersect transversally in $\mathcal{M}^{\mathrm{K}}$. To prove this statement, we will need a detailed analysis of stabilizer groups. For an algebraic group G we will denote the connected component of the identity by G°. The following two lemmas are modeled on [**28**, Lemma 2.3] and [**28**, Lemma 2.4]. Below, we denote by x_0, x_1 the homogeneous coordinate of $\mathbb{P}^1$. In this terminology, the polystable point $c_{4,4}$ corresponds to $x_0^4 x_1^4$.

Lemma 5.3.1. *The following equalities hold:*

$$R := \mathrm{Stab}(c_{4,4}) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \right\} \bigcup \left\{ \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \right\} \cong \mathbb{C}^\times \rtimes \mathfrak{S}_2$$
$$R^\circ := \mathrm{Stab}(c_{4,4})^\circ \cong \mathbb{C}^\times.$$

Now, let us prepare for the local computations. The Luna slice theorem gives us a tool to study them as handled in the case of the moduli space of cubic threefolds [**27**, Subsection 4.3.1] or cubic surfaces [**28**, Lemma 3.4]; see also [**152**, Subsection 7.1].

Lemma 5.3.2. *A Luna slice for $c_{4,4}$, normal to the orbit $\mathrm{SL}_2(\mathbb{C}) \cdot \{c_{4,4}\} \subset \mathbb{P}^8$, is isomorphic to $\mathbb{C}^6$, spanned by the 6 monomials*

$$x_0^8, \quad x_1^8, \quad x_0^7 x_1, \quad x_0 x_1^7, \quad x_0^6 x_1^2, \quad x_0^2 x_1^6$$

in the tangent space $H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(8))$. Projectively,

$$\mathbb{P}^6 = \{\alpha_0 x_0^8 + \alpha_1 x_1^8 + \beta_0 x_0^7 x_1 + \beta_1 x_0 x_1^7 + \gamma_0 x_0^6 x_1^2 + \gamma_1 x_0^2 x_1^6 + k x_0^4 x_1^4\}$$
$$\subset \mathbb{P} H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(8)) = \mathbb{P}^8.$$

Proof. This can be proven in the same way as [**27**, Subsection 4.3.1]. We note that the (affine) tangent space of the orbit is given by the entries of the matrix

$$\begin{pmatrix} x_0^4x_1^4 & x_0^3x_1^5 \\ x_0^5x_1^3 & x_0^4x_1^4 \end{pmatrix}.$$

□

Let

$$\mathrm{diag}(\lambda,\lambda^{-1}) := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},\ \mathrm{antidiag}(\lambda,-\lambda^{-1}) := \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}.$$

Then, the action of an element of $\mathrm{Stab}(c_{4,4})$ is given by

$$\mathrm{diag}(\lambda,\lambda^{-1})\cdot(\alpha_0,\alpha_1,\beta_0,\beta_1,\gamma_0,\gamma_1) = (\lambda^8\alpha_0,\lambda^{-8}\alpha_1,\lambda^6\beta_0,\lambda^{-6}\beta_1,\lambda^4\gamma_0,\lambda^{-4}\gamma_1) \tag{5.3.1}$$

$$\mathrm{antidiag}(\lambda,-\lambda^{-1})\cdot(\alpha_0,\alpha_1,\beta_0,\beta_1,\gamma_0,\gamma_1) = (\lambda^{-8}\alpha_1,\lambda^8\alpha_0,-\lambda^{-6}\beta_1,-\lambda^6\beta_0,\lambda^{-4}\gamma_1,\lambda^4\gamma_0). \tag{5.3.2}$$

We write the coordinates of the Kirwan blow-up $Bl_0\mathbb{C}^6 \subset \mathbb{C}^6 \times \mathbb{P}^5$ of the Luna slice as $(\alpha_0,\alpha_1,\beta_0,\beta_1,\gamma_0,\gamma_1)$ and $[S_0 : S_1 : T_0 : T_1 : U_0 : U_1]$.

Lemma 5.3.3. *The unstable locus of the action of the stabilizer* $\mathrm{GL}(c_{4,4})$ *of* $c_{4,4}$ *in* $\mathrm{GL}_2(\mathbb{C})$ *is the codimension three locus*

$$\{S_0 = T_0 = U_0 = 0\} \cup \{S_1 = T_1 = U_1 = 0\} \subset \mathbb{P}^5.$$

Proof. From (5.3.1), the action of $R^\circ \cong \mathbb{C}^\times$ is given by

$$\mathrm{diag}(\lambda,\lambda^{-1})\cdot(S_0,S_1,T_0,T_1,U_0,U_1) = (\lambda^8S_0,\lambda^{-8}S_1,\lambda^6T_0,\lambda^{-6}T_1,\lambda^4U_0,\lambda^{-4}U_1).$$

Thus, the representation of $\mathbb{C}^\times$ on $\mathbb{C}^6$ decomposes into 6 characters. By the same discussion as in the proof of [**28**, Lemma 3.6], the points in the unstable locus are characterized by the property that the convex hull spanned by the weights appearing in the above representation does not contain the origin. This condition holds if and only if $\{S_0 = T_0 = U_0 = 0\}$ or $\{S_1 = T_1 = U_1 = 0\}$. □

We denote by $\mathscr{D}_{\mathrm{ord}}$ (resp. $\mathscr{D}$) the discriminant divisor, corresponding to the closure of H/Γ_{ord} (resp. H/Γ), through the isomorphism $\phi_{\mathrm{ord}} : \mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}} \to \overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}$ (resp. $\phi : \mathcal{M}^{\mathrm{GIT}} \to \overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}$). Let $\widetilde{\mathscr{D}}$ be the strict transform of the discriminant divisor $\mathscr{D}$ in the blow-up $\mathcal{M}^{\mathrm{K}} \to \mathcal{M}^{\mathrm{GIT}}$. Besides, let Δ_{ord} (resp. Δ) be the union of boundary divisors of $\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}}$ (resp. $\mathcal{M}^{\mathrm{K}}$).

Theorem 5.3.4. *The strict transform* $\widetilde{\mathscr{D}}$ *and the boundary divisor* Δ *do not meet generically transversally in* $\mathcal{M}^{\mathrm{K}}$.

Proof. We work on the local computation via the Luna slice described in Lemma 5.3.2. Before taking the GIT quotient, we have the blow-up

$$Bl_0\mathbb{C}_6 \to \mathbb{C}_6,$$

where the coordinates of the affine space (the Luna slice) $\mathbb{C}^6$ are $\alpha_0,\alpha_1,\beta_0,\beta_1,\gamma_0,\gamma_1$ (this is the first step of the Kirwan blow-up). In this Luna slice, $\widetilde{\mathscr{D}}$ locally near the origin, is given by

$$\begin{aligned} &\mathrm{disc}(x^4+\alpha_0x^2+\beta_0x+\gamma_0)\cdot\mathrm{disc}(y^4+\alpha_1y^2+\beta_1y+\gamma_1) \\ &= (256\gamma_0^3 - 128\alpha_0^2\gamma_0^2 + 144\alpha_0\beta_0^2\gamma_0 - 27\beta_0^4 + 16\alpha_0^4\gamma_0 - 4\alpha_0^3\beta_0^2) \end{aligned}$$

$$\cdot (256\gamma_1^3 - 128\alpha_1^2\gamma_1^2 + 144\alpha_1\beta_1^2\gamma_1 - 27\beta_1^4 + 16\alpha_1^4\gamma_1 - 4\alpha_1^3\beta_1^2)$$
$$= 0.$$

The reason for this is that we consider the polystable point given by x^4y^4 and that the versal deformation of the quadruple point $x^4 = 0$ is given by $x^4 + \alpha_0x^2 + \beta_0x + \gamma_0 = 0$. We write this as $V := V_1 \cup V_2$ with $\mathfrak{S}_2$ permuting the two components. We consider the affine loci

$$\mathcal{P} := (S_0 \neq 0), \quad \mathcal{Q} := (T_0 \neq 0), \quad \mathcal{R} := (U_0 \neq 0).$$

First, on $\mathcal{P}$, the proper transform of V is

$$\alpha_0^6(256u_0^3 - 128\alpha_0u_0^2 + 144\alpha_0t_0^2u_0 - 27\alpha_0t_0^4 + 16\alpha_0^2u_0 - 4\alpha_0^2t_0^2)$$
$$\cdot (256u_1^3 - 128\alpha_0s_1^2u_1^2 + 144\alpha_0s_1t_1^2u_1 - 27\alpha_0t_1^4 + 16\alpha_0^2s_1^4u_1 - 4\alpha_0^2s_1^3t_1^2)$$
$$= 0,$$

where

$$s_1 := \frac{S_1}{S_0}, \quad t_i := \frac{T_i}{S_0}, \quad u_i := \frac{U_i}{S_0}$$

and the coordinates of $\mathcal{P}$ are $(\alpha_0, s_1, t_0, t_1, u_0, u_1)$. Hence, the strict transform of V is given by

$$(256u_0^3 - 128\alpha_0u_0^2 + 144\alpha_0t_0^2u_0 - 27\alpha_0t_0^4 + 16\alpha_0^2u_0 - 4\alpha_0^2t_0^2)$$
$$\cdot (256u_1^3 - 128\alpha_0s_1^2u_1^2 + 144\alpha_0s_1t_1^2u_1 - 27\alpha_0t_1^4 + 16\alpha_0^2s_1^4u_1 - 4\alpha_0^2s_1^3t_1^2)$$
$$= 0,$$

since the exceptional divisor of the blow-up is $(\alpha_0 = 0)$. The Luna slice for the action $\mathbb{T} \subset R$ is given by $(s_1 = 1)$ in $\mathcal{P}$ because for any point $(\alpha_0, s_1, t_0, t_1, u_0, u_1) \in \mathcal{P}$ with $s_1 \neq 0$, there exists a complex number λ such that $\lambda^{-16} = s_1$. Thus, the intersection of V with this Luna slice is given by

$$\{256u_0^3 - \alpha_0(128u_0^2 + 144t_0^2u_0 - 27t_0^4 + 16\alpha_0u_0 - 4\alpha_0t_0^2)\}$$
$$\cdot \{256u_1^3 - \alpha_0(128u_1^2 + 144t_1^2u_1 - 27t_1^4 + 16\alpha_0u_1 - 4\alpha_0t_1^2)\}$$
$$= 0.$$

This shows that the first (resp. second) factor intersect the exceptional divisor $(\alpha_0 = 0)$ non-transversally along $(u_0 = 0)$ (resp. $(u_1 = 0)$).

Next, on $\mathcal{Q}$, the proper transform of V is

$$\beta_0^6(256u_0^3 - 128\beta_0s_0^2u_0^2 + 144\beta_0s_0u_0 - 27\beta_0 + 16\beta_0^2s_0u_0 - 4\beta_0^2s_0)$$
$$\cdot (256u_1^3 - 128\beta_0s_1^2u_1^2 + 144\beta_0s_1t_1^2u_1 - 27\beta_0t_1^4 + 16\beta_0^2s_1^4u_1 - 4\beta_0^2\alpha_1^3u_1^2)$$
$$= 0,$$

where

$$s_i := \frac{S_i}{T_0}, \quad t_1 := \frac{T_1}{T_0}, \quad u_i := \frac{U_i}{T_0}$$

and the coordinates of $\mathcal{P}$ is $(s_0, s_1, \beta_0, t_1, u_0, u_1)$. Hence, the strict transform of V is given by

$$(256u_0^3 - 128\beta_0s_0^2u_0^2 + 144\beta_0s_0u_0 - 27\beta_0 + 16\beta_0^2s_0u_0 - 4\beta_0^2s_0)$$
$$\cdot (256u_1^3 - 128\beta_0s_1^2u_1^2 + 144\beta_0s_1t_1^2u_1 - 27\beta_0t_1^4 + 16\beta_0^2s_1^4u_1 - 4\beta_0^2\alpha_1^3u_1^2)$$
$$= 0,$$

since the exceptional divisor of the blow-up is $(\beta_0 = 0)$. The Luna slice for the action $\mathbb{T} \subset R$ is given by $(t_1 = 1)$ in $\mathcal{P}$ because for any point $(s_0, s_1, \beta_0, t_1, u_0, u_1) \in \mathcal{Q}$ with $t_1 \neq 0$, there exists a complex number λ such that $\lambda^{-12} = t_1$. Thus, the intersection of V with this Luna slice is given by

$$\begin{aligned}
&\{256u_0^3 - \beta_0(128s_0^2u_0^2 + 144s_0u_0 - 27 + 16\beta_0s_0u_0 - 4\beta_0s_0)\}\\
&\cdot \{256u_1^3 - \beta_0(128s_1^2u_1^2 + 144s_1u_1 - 27 + 16\beta_0s_1^4u_1 - 4\beta_0\alpha_1^3u_1^2)\}\\
&= 0.
\end{aligned}$$

This shows that the first (resp. second) factor intersect the exceptional divisor $(\beta_0 = 0)$ non-transversally along $(u_0 = 0)$ (resp. $(u_1 = 0)$).

Finally, on $\mathcal{R}$, the proper transform of V is

$$\begin{aligned}
&\gamma_0^6(256 - 128\gamma_0s_0^2 + 144\gamma_0s_0t_0^2 - 27\gamma_0t_0^4 + 16\gamma_0^2s_0^4 - 4\gamma_0^2s_0^3t_0^2)\\
&\cdot (256u_1^3 - 128\gamma_0s_1^2 + 144\gamma_0s_1t_1^2 - 27\gamma_0t_1^4 + 16\gamma_0^2s_1^4 - 4\gamma_0^2s_1^3t_1^2)\\
&= 0,
\end{aligned}$$

where

$$s_i := \frac{S_i}{U_0}, \quad t_i := \frac{T_i}{U_0}, \quad u_1 := \frac{U_1}{U_0}$$

and the coordinates of $\mathcal{R}$ are $(s_0, s_1, t_0, t_1, \gamma_0, u_1)$. Hence the strict transform of V is given by

$$\begin{aligned}
&(256 - 128\gamma_0s_0^2 + 144\gamma_0s_0t_0^2 - 27\gamma_0t_0^4 + 16\gamma_0^2s_0^4 - 4\gamma_0^2s_0^3t_0^2)\\
&\cdot (256u_1^3 - 128\gamma_0s_1^2 + 144\gamma_0s_1t_1^2 - 27\gamma_0t_1^4 + 16\gamma_0^2s_1^4 - 4\gamma_0^2s_1^3t_1^2)\\
&= 0,
\end{aligned}$$

since the exceptional divisor of the blow-up is $(\gamma_0 = 0)$. The Luna slice for the action $\mathbb{T} \subset R$ is given by $(g_1 = 1)$ in $\mathcal{R}$ because for any point $(s_0, s_1, t_0, t_1, \gamma_0, u_1) \in \mathcal{R}$ with $u_1 \neq 0$, there exists a complex number λ such that $\lambda^{-8} = \gamma_1$. Thus, the intersection of V with this Luna slice is given by

$$\begin{aligned}
&(256 - 128\gamma_0s_0^2 + 144\gamma_0s_0t_0^2 - 27\gamma_0t_0^4 + 16\gamma_0^2s_0^4 - 4\gamma_0^2s_0^3t_0^2)\\
&\cdot \{256u_1^3 - \gamma_0(128s_1^2 + 144s_1t_1^2 - 27t_1^4 + 16\gamma_0s_1^4 - 4\gamma_0s_1^3t_1^2)\}.
\end{aligned}$$

This shows that the first factor has an empty intersection with the exceptional divisor $(\gamma_0 = 0)$, whereas the second factor intersects the exceptional divisor non-transversally along $(u_1 = 0)$.

Next, we consider the action of the finite quotient $\mathfrak{S}_2 \cong R/R^\circ$. We only consider $\mathcal{P}$ (the other cases being the same). If $\mathrm{diag}(\lambda, \lambda^{-1})$ fixes a general point in $\mathcal{P} \cap (s_1 = 1)$, by the condition on t_0, we have $\lambda^2 = 1$. This implies that $\mathrm{diag}(\lambda, \lambda^{-1})$ is trivial as an element of $\mathrm{PGL}_2(\mathbb{C})$.

Thus, finally, let us consider the case of the form $\mathrm{antidiag}(\lambda, -\lambda^{-1})$. For a general point $p = (t_0, t_1, u_0, u_1) \in \mathcal{P} \cap (s_1 = 1)$, we have

$$\mathrm{antidiag}(\lambda, -\lambda^{-1}) \cdot (t_0, t_1, u_0, u_1) = (-\lambda^{-2}t_1, -\lambda^{14}t_0, \lambda^4u_1, \lambda^{12}u_0)$$

by (5.3.2).

For a point p to be invariant under the above action, one finds the conditions $t_0 = -\lambda^{-2}t_1$ and $t_1 = -\lambda^{-14}t_0$. This implies that $t_0^8 = t_1^8$ which is clearly not the case for a general point p. $\square$

Remark 5.3.5. The situation in the ordered case is different. Indeed, a similar calculation, again using a Luna slice argument, shows that the discriminant divisors and the boundary divisors meet transversally everywhere on $\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}}$.

Remark 5.3.6. In Theorem 5.5.2, we shall see that $\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}}$ and $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}}$ have the same cohomology. Note that this proof does not require a priori knowledge that the two spaces are isomorphic. Using the information of their Betti numbers,, we can give a short independent proof that $\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}} \cong \overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}}$ which is independent of [**48**]. This argument follows a similar argument given by Casalaina-Martin for cubic surfaces. By the Borel extension theorem [**16**, Theorem A], the map $\mathcal{M}_{\mathrm{ord}} \to \mathbb{B}^5/\Gamma_{\mathrm{ord}}$ extends to a morphism $\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}} \to \overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}}$. Since both spaces have the same Betti numbers, this must be an isomorphism or a small contraction. But the latter is impossible since $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}}$ is $\mathbb{Q}$-factorial (and in fact smooth).

In the rest of this subsection, we work on the stabilizers of points in the exceptional divisor in $\mathcal{M}^{\mathrm{K}}$. Let $\mathscr{E} \subset \mathcal{M}^{\mathrm{K}}$ be the exceptional divisor of the Kirwan blow-up. The following proposition plays a critical role in the proof of Theorem 5.4.6.

Proposition 5.3.7. *For any point in $x \in \mathscr{E}$, the order of its stabilizer $S_x := \mathrm{Stab}_R(x)$ is* not *divisible by 5.*

Proof. Since the order of the finite part of R is not divisible by 5, it is enough to concentrate on the connected component R°, which is isomorphic to $\mathbb{C}^\times$. For simplicity, we will also use S_x to denote the stabilizer of x in R°. By the $\mathfrak{S}_2$ symmetry, it suffices to show the claim for the affine open sets $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$.

First, let us consider the points $(\alpha_0, s_1, t_0, t_1, u_0, u_1) \in \mathcal{P}$. In this locus, the exceptional divisor corresponds to $(\alpha_0 = 0)$, and the action of $\mathrm{diag}(\lambda, \lambda^{-1})$ is given by

$$\mathrm{diag}(\lambda, \lambda^{-1}) \cdot (0, s_1, t_0, t_1, u_0, u_1) = (0, \lambda^{-16} s_1, \lambda^{-2} t_0, \lambda^{-14} t_1, \lambda^{-4} u_0, \lambda^{-12} u_1).$$

Since the Kirwan blow-up is completed after one step, it is enough to consider the stable points after blowing up the orbit $\mathrm{SL}_2(\mathbb{C}) \cdot \{c_{4,4}\}$. It follows from Lemma 5.3.3 that both $\{t_0 \neq 0 \text{ or } u_0 \neq 0\}$ and $\{s_1 \neq 0 \text{ or } t_1 \neq 0 \text{ or } u_1 \neq 0\}$. If $t_0 \neq 0$, then $S_x \cong \mathbb{Z}/2\mathbb{Z}$. If $u_0 \neq 0$, then

$$S_x \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} & (s_1 \neq 0 \text{ or } u_1 \neq 0) \\ \mathbb{Z}/2\mathbb{Z} & (t_1 \neq 0). \end{cases}$$

The other cases are similar, but we nevertheless state them for completeness, starting with the points $(s_0, s_1, 0, t_1, u_0, u_1) \in \mathcal{P} \cap (\beta_0 = 0)$. The action of $\mathrm{diag}(\lambda, \lambda^{-1})$ is given by

$$\mathrm{diag}(\lambda, \lambda^{-1}) \cdot (s_0, s_1, 0, t_1, u_0, u_1) = (\lambda^2 s_0, \lambda^{-14} s_1, 0, \lambda^{-12} t_1, \lambda^{-2} u_0, \lambda^{-10} u_1).$$

Again by Lemma 5.3.3, we can assume that $\{s_0 \neq 0 \text{ or } u_0 \neq 0\}$ and $\{s_1 \neq 0 \text{ or } t_1 \neq 0 \text{ or } u_1 \neq 0\}$. In all cases, we obtain $S_x \cong \mathbb{Z}/2\mathbb{Z}$. Finally, let $(s_0, s_1, t_0, t_1, 0, u_1) \in \mathcal{P} \cap (\gamma_0 = 0)$. The action of $\mathrm{diag}(\lambda, \lambda^{-1})$ is given by

$$\mathrm{diag}(\lambda, \lambda^{-1}) \cdot (s_0, s_1, t_0, t_1, 0, u_1) = (\lambda^4 s_0, \lambda^{-4} s_1, \lambda^2 t_0, \lambda^{-10} t_1, 0, \lambda^{-8} u_1).$$

As above, we study the case holding both of $\{s_0 \neq 0 \text{ or } t_0 \neq 0\}$ and $\{s_1 \neq 0 \text{ or } t_1 \neq 0 \text{ or } u_1 \neq 0\}$. If $t_0 \neq 0$, then $S_x \cong \mathbb{Z}/2\mathbb{Z}$. If $s_0 \neq 0$, then

$$S_x \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} & (s_1 \neq 0 \text{ or } u_1 \neq 0) \\ \mathbb{Z}/2\mathbb{Z} & (t_1 \neq 0). \end{cases}$$

This calculation completes the proof.

□

5.3.2. Transversality in the toroidal compactification. In this subsection, we prove that the discriminant divisors and the boundary divisors intersect generically transversally in $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}}$. We will see that this also implies the transversality at a generic point in $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$. Throughout this subsection, let $\mathbb{N}_8 := \{1, 2, \cdots, 8\}$ and $I \subset \mathbb{N}_8$. As before, $(\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}})^o$ denotes the set of 8-tuples where all points are different; see Section 5.2. Below, we shall recall the construction of the blow-up sequence $\overline{\mathcal{M}}_{0,8} \to \mathcal{M}^{\mathrm{K}}_{\mathrm{ord}} \to \mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}}$. By the explicit description of the blow-ups or the interpretation as the configuration space, the locus $(\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}})^o$ does not meet the centers of each blow-up step. Thus, we consider $(\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}})^o$ to be also an open subset of $\overline{\mathcal{M}}_{0,8}$ and $\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}}$ via birational maps.

First, we work on $\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}}$. The boundary divisor $\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}} \setminus (\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}})^o$ is

$$D_2^{(0)} := \bigcup_{|I|=2} D_2^{(0)}(I) = \mathscr{D}_{\mathrm{ord}} \subset \mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}}$$

by [**75**, p1134] $(m = 4, k = 0)$. Here, $D_2^{(0)}(I)$ is defined by

$$D_2^{(0)}(I) := \overline{\{(x_1, \cdots, x_8) \in (\mathbb{P}^1)^8 \mid x_i = x_j \text{ if } i, j \in I\}} /\!/ \mathrm{SL}_2(\mathbb{C}).$$

The number of such I is 28. As in Section 5.2, the morphism $\varphi_1 : \mathcal{M}^{\mathrm{K}}_{\mathrm{ord}} \to \mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}}$ is the Kirwan blow-up whose center is the locus of polystable orbits, consisting of 35 orbits (which in turn correspond to the 35 cusps, see below). We interpret φ_1 in terms of configuration spaces as follows. Let

$$\Sigma_4^{(0)}(I, I^{\perp}) := \overline{\{(x_1, \cdots, x_8) \in (\mathbb{P}^1)^8 \mid x_i = x_j \text{ if and only if } \{i, j\} \subset I \text{ or } I^{\perp}\}} /\!/ \mathrm{SL}_2(\mathbb{C})$$

for $|I| = |I^{\perp}| = 4$ and $I \sqcup I^{\perp} = \mathbb{N}_8$. We also denote by $\Sigma_4^{(0)}$ their union running through such I and $I^{\perp}$. Note that there are 35 pairs $(I, I^{\perp})$ satisfying $|I| = |I^{\perp}| = 4$ and $I \sqcup I^{\perp} = \mathbb{N}_8$. In this terminology, the center of φ_1 is described by

$$\Sigma_4^{(0)} = \{c_{\mathrm{ord},i}\}_{i=1}^{35}$$

where $\{c_{\mathrm{ord},i}\}_{i=1}^{35}$ are the polystable points of $\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}}$, corresponding to 35 Baily-Borel cusps.

Next, we consider $\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}} (\cong \overline{\mathcal{M}}_{0,8(\frac{1}{4}+\epsilon)})$. Let

$$D_4^{(1)}(I) := \varphi_1^{-1}\left(\Sigma_4^{(0)}(I, I^{\perp})\right)$$

for $|I| = 4$. Then, the exceptional divisor of φ_1 is

$$D_4^{(1)} := \bigcup_{|I|=4} D_4^{(1)}(I) = \varphi_1^{-1}\left(\Sigma_4^{(0)}\right) = \Delta_{\mathrm{ord}}.$$

Note that each irreducible component of Δ_{ord} is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$ by [**75**, Proposition 4.5], [**118**, Remark 6] or [**48**, Example 2.12]. Besides, let

$$D_2^{(1)}(I) := \overline{\varphi_1^{-1}\left(D_2^{(0)}(I) \setminus \Sigma_4^{(0)}\right)}$$

be the strict transform of $D_2^{(0)}(I)$ for $|I| = 2$, and $D_2^{(1)}$ be their union. Then $D_2^{(1)}$ is the strict transform of $\mathscr{D}_{\mathrm{ord}}$, i.e.,

$$D_2^{(1)} = \widetilde{\mathscr{D}_{\mathrm{ord}}}$$

and has 28 irreducible components. In this setting, the boundary divisor $\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}} \setminus (\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}})^o$ is

$$D^{(1)}_2 \bigcup D^{(1)}_4 = \widetilde{\mathscr{D}_{\mathrm{ord}}} \bigcup \Delta_{\mathrm{ord}}$$

by [**75**, p1134] ($m = 4, k = 1$).

Next, we describe the center of the blow-up $\varphi_2 := \varphi'_2 \circ \varphi_{\mathrm{ord}} : \overline{\mathcal{M}}_{0,8} \to \mathcal{M}^{\mathrm{K}}_{\mathrm{ord}}$, where is a codimension 2 locus. Let

$$\Sigma^{(0)}_3(I) := \overline{\{(x_1, \cdots, x_8) \in (\mathbb{P}^1)^8 \mid x_i = x_j \text{ if and only if } i, j \in I\}} // \mathrm{SL}_2(\mathbb{C})$$
$$\Sigma^{(1)}_3(I) := \overline{\varphi_1^{-1}\left(\Sigma^{(0)}_3(I) \setminus \Sigma^{(0)}_4\right)}$$

for $|I| = 3$ and

$$\Sigma^{(1)}_3 := \bigcup_{|I|=3} \Sigma^{(1)}_3(I).$$

Then, the center of the blow up $\varphi_2 : \overline{\mathcal{M}_{0,8}} \to \mathcal{M}^{\mathrm{K}}_{\mathrm{ord}}$ is $\Sigma^{(1)}_3$.

Finally, we study $\overline{\mathcal{M}}_{0,8}$. For $|I| = 3$, let

$$D^{(2)}_3(I) := \varphi_2^{-1}\left(\Sigma^{(1)}_3(I)\right)$$

be an irreducible component of the exceptional divisor of φ_2. Then, the variety

$$D^{(2)}_3 := \bigcup_{|I|=3} D^{(2)}_3(I) = \varphi_2^{-1}\left(\Sigma^{(1)}_3\right)$$

is exactly the exceptional divisor of the blow-up φ_2. For $|I| = 2, 4$, we denote by $D^{(2)}_{|I|}(I)$ the strict transform of $D^{(1)}_{|I|}$ and define

$$D^{(2)}_2 := \bigcup_{|I|=2} D^{(2)}_2(I), \quad D^{(2)}_4 := \bigcup_{|I|=4} D^{(2)}_4(I).$$

Now, the boundary divisor $\overline{\mathcal{M}}_{0,8} \setminus (\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}})^o$ is

$$D^{(2)}_2 \bigcup D^{(2)}_3 \bigcup D^{(2)}_4$$

by [**75**, p1134] ($m = 4, k = 2$).

The boundaries which are contractied through the map φ_2 can also be calculated as follows. By [**75**, Theorem 4.1], there exists the reduction map

$$\varphi'_2 : \overline{\mathcal{M}}_{0,8} \to \overline{\mathcal{M}}_{0,8(\frac{1}{4}+\epsilon)}.$$

The map φ'_2 is a divisorial contraction, more preisely:

Lemma 5.3.8 (cf. [**75**, Proposition 4.5]**).** *The morphism φ'_2 contracts the boundary divisors $D^{(2)}_3$.*

PROOF. By [**75**, p1121], the exceptional locus of φ'_2 is the union of $D^{(2)}_{|I|}(I)$ with $I = \{i_1, \ldots, i_r\}$ for $r > 2$ so that

$$r \times (\frac{1}{4} + \epsilon) \leq 1.$$

This implies $r = 3$. □

By construction, $D_2^{(2)} \cup D_3^{(2)} \cup D_4^{(2)}$ is normal crossing (since $\overline{M}_{0,8}$ is a normal crossing compactification of $(\mathbb{B}^5 \setminus H)/\Gamma_{\mathrm{ord}}$). We denote $\mathscr{H}_{\mathrm{ord}} := \overline{H/\Gamma_{\mathrm{ord}}}$ and $\mathscr{H} := \overline{H/\Gamma}$, where the closures are taken in the respective Baily-Borel compactifications. We further denote by $\widetilde{\mathscr{H}_{\mathrm{ord}}}$ the strict transform of $\mathscr{H}_{\mathrm{ord}}$ under $\pi_{\mathrm{ord}} : \overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}} \to \overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}$. Since the contraction divisor of φ_2 is only $D_3^{(2)}$, we now obtain the following:

Theorem 5.3.9. *The boundary $\widetilde{\mathscr{H}_{\mathrm{ord}}} \cup T_{\mathrm{ord}}$ is a normal crossing divisor. In particular, $\widetilde{\mathscr{H}_{\mathrm{ord}}}$ and T_{ord} intersect transversally everywhere in $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}}$.*

Again, by this formulation, we mean that $\widetilde{\mathscr{H}_{\mathrm{ord}}}$ and T_{ord} intersect transversally everywhere along any component of their intersection. As a consequence, we obtain the following corollary, where $\widetilde{\mathscr{H}}$ is the strict transform of $\mathscr{H}$ under $\pi : \overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}} \to \overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}$.

Corollary 5.3.10. *The divisor $\widetilde{\mathscr{H}} \cup T$ is a normal crossing divisor, up to finite quotients.*

Next, we discuss the generical transversality of the intersection of $\widetilde{\mathscr{H}}$ and T in $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$. Note that $\Gamma/\Gamma_{\mathrm{ord}} \cong \mathfrak{S}_8$ acts on $\{T_{\mathrm{ord},i}\}_{i=1}^{35}$ transitively and

$$1 \to \mathfrak{S}_4 \times \mathfrak{S}_4 \to \mathrm{Stab}_{\mathfrak{S}_8}(T_{\mathrm{ord},i}) \to \mathfrak{S}_2 \to 1.$$

Next, we study the description of the boundary and group actions via the Hermitian form. The claim of the following lemma is already known in terms of a moduli description by [**118**, Remark 6] or [**48**, Example 2.12], but we need the details in the proof of Theorem 5.3.14.

Lemma 5.3.11. *The following holds.*

(1) $T_{\mathrm{ord},i} \cong \mathbb{P}^2 \times \mathbb{P}^2$.
(2) $T \cong (\mathbb{P}^2/\mathfrak{S}_4 \times \mathbb{P}^2/\mathfrak{S}_4)/\mathfrak{S}_2$.

Proof. We orientate ourselves along the strategy of the proof of [**27**, Proposition 7.8]. First, we take an isotropic vector $h = (1,0,0,0,0,0) \in L$ and denote by F the corresponding cusp. As the unitary group acts transitively on the set of all cusps, this means no loss of generality. Also, taking $h^\vee = (0,1,0,0,0,0)$ as a further basis vector, we can replace our Hermitian form by

$$\begin{pmatrix} & & 1-\sqrt{-1} \\ & B & \\ 1+\sqrt{-1} & & \end{pmatrix}$$

where

$$B := \begin{pmatrix} -2 & 1+\sqrt{-1} & & \\ 1-\sqrt{-1} & -2 & & \\ & & -2 & 1+\sqrt{-1} \\ & & 1-\sqrt{-1} & -2 \end{pmatrix}.$$

Then,

$$N(F) := \mathrm{Stab}_\Gamma(F) = \left\{ g = \begin{pmatrix} u & v & w \\ & X & y \\ & & s \end{pmatrix} \;\middle|\; \begin{array}{l} s\overline{u} = 1,\ \overline{X^t}BX = B \\ \overline{X^t}By + (1-\sqrt{-1})\overline{v^t}s = 0 \\ \overline{y^t}By + (1+\sqrt{-1})\overline{s}w + (1-\sqrt{-1})s\overline{w} = 0 \end{array} \right\}.$$

Its unipotent radical is

$$W(F) = \left\{ g = \begin{pmatrix} 1 & v & w \\ & 1 & y \\ & & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} By + (1-\sqrt{-1})\overline{v^t} = 0 \\ \overline{y^t}By + (1+\sqrt{-1})w + (1-\sqrt{-1})\overline{w} = 0 \end{array} \right\}$$

and its center is

$$Z(F) = \left\{ g = \begin{pmatrix} 1 & 1 & \sqrt{-1}(1-\sqrt{-1})w \\ & I_4 & 1 \\ & & 1 \end{pmatrix} \;\middle|\; w \in \mathbb{Z} \right\}.$$

We take the partial quotient

$$\begin{array}{ccc} \mathbb{B}^5 & \hookrightarrow & \mathbb{C}^\times \times \mathbb{C}^4 \\ (z_0, z_1, z_2, z_3, z_4) & \mapsto & \left(t = \exp\left(2\pi z_0/(1-\sqrt{-1})\right), z_1, z_2, z_3, z_4\right). \end{array}$$

We shall here consider the quotient of $\mathbb{C}^4$ by $W(F)$. For an element $g \in W(F)$, its action on $\underline{z} := (z_1, z_2, z_3, z_4)$ is given by

$$g \cdot \underline{z}^t = \frac{1}{s}(X\underline{z} + y).$$

A straight forward computation shows that for given $y^t \in \mathbb{Z}[\sqrt{-1}]^4$, we can find suitable elements $w \in \mathbb{Z}[\sqrt{-1}]$ and $v \in \mathbb{Z}[\sqrt{-1}]^4$ such that $g = \begin{pmatrix} 1 & v & w \\ & 1 & y \\ & & 1 \end{pmatrix} \in W(F)$. This implies that

$$\mathbb{C}^4/W(F) \cong (E_{\sqrt{-1}})^4,$$

where $E_{\sqrt{-1}}$ is the CM-elliptic curve $\mathbb{C}/\left(\mathbb{Z} + \sqrt{-1}\mathbb{Z}\right)$. Now, we consider the effect of an element of the form

$$g = \begin{pmatrix} u & & \\ & 1 & \\ & & s \end{pmatrix} \in N(F).$$

Here, from the above action, $s \in \mathbb{Z}[\sqrt{-1}]^\times$ acts on $(E_{\sqrt{-1}})^4$ diagonally by multiplication with powers of $\sqrt{-1}$. However, this element is already in $\mathrm{U}(D_4^{\oplus 2})$, thus it follows that $T \cong (E_{\sqrt{-1}})^4/\mathrm{U}(D_4^{\oplus 2})$. Here, we note that $X = \mathrm{U}(D_4^{\oplus 2})$. By [**35**, Table 2], we have

$$\mathrm{U}(D_4^{\oplus 2}) \cong \left((\mathbb{Z}/2\mathbb{Z})^2 \times \mathfrak{S}_2) \rtimes \mathfrak{S}_4\right)^2 \rtimes \mathfrak{S}_2.$$

See also [**134**, Subsection 6.4]. Since, the action of this group, described in [**35**, Subsection 3.2, Table 2], gives

$$(E_{\sqrt{-1}})^2/\mathrm{U}(D_4) \cong (\mathbb{P}^1)^2/\left(\mathfrak{S}_2 \rtimes \mathfrak{S}_4\right) \cong \mathbb{P}^2/\mathfrak{S}_4,$$

where $\mathfrak{S}_4$ acts on $\mathbb{P}^2$ by the standard representation, we obtain

$$(E_{\sqrt{-1}})^4/\mathrm{U}(D_4^{\oplus 2}) \cong (\mathbb{P}^2/\mathfrak{S}_4)^2/\mathfrak{S}_2.$$

For the ordered case, a straightforward computation shows that $\widetilde{\mathrm{U}}(D_4) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathfrak{S}_2$, thus this gives

$$T_{\mathrm{ord},i} \cong \mathbb{P}^2 \times \mathbb{P}^2.$$

□

Remark 5.3.12. This description allows us to describe the geometry of the toroidal boundary T explicitly. By the above Lemma 5.3.11 we know that $T = (\mathbb{P}^2/\mathfrak{S}_4 \times \mathbb{P}^2/\mathfrak{S}_4)/\mathfrak{S}_2$ where $\mathfrak{S}_4$ acts on $\mathbb{P}^2$ by the standard 3-dimensional representation and $\mathfrak{S}_2$ exchanges the two factors. We claim that $\mathbb{P}^2/\mathfrak{S}_4 \cong \mathbb{P}(1,2,3)$ where $\mathbb{P}(1,2,3)$ denotes the weighted projective space with weights $(1,2,3)$. This follows since the invariants are freely generated by the restriction of the elementary symmetric polynomials of degree $2,3,4$ on $\mathbb{P}^3$ restricted to the hyperplane $\sum_{i=0}^{3} x_i = 0$. Hence $\mathbb{P}^2/\mathfrak{S}_4 \cong \mathbb{P}(2,3,4) \cong \mathbb{P}(1,2,3)$. In conclusion we find that $T \cong S^2(\mathbb{P}(1,2,3))$.

Before discussing the intersection of divisors on the toroidal compactifications, we recall the discriminant form, see [**88**, Subsection 2.2] (where the lattice is called N compared to our L):

$$q_L : A_L \to \mathbb{F}_2.$$

Associated with q_L, there is an associated bilinear form $b_L(\ ,\ )$ on A_L. Note that q_L is isomorphic to the direct sum of 3 copies of the hyperbolic plane u over $\mathbb{F}_2$ by [**88**, Subsection 2.2] or explicit computation in terms of the concrete form of L. We have to pay attention to the norm of a vector because our quadratic form exists over $\mathbb{F}_2$. In other words, the norm is measured by q_L, not $b_L(\ ,\ )$.

Lemma 5.3.13. *For a given isotropic vector h in the finite quadratic space $\mathbb{P}(A_L) \cong \mathbb{P}(\mathbb{F}_2^6)$, the orthogonal complement $h^\perp \cong \mathbb{P}(\mathbb{F}_2^5)$ contains 19 isotropic vectors and 12 non-isotropic vectors. In addition, the stabilizer of* $\operatorname{Stab}(h)$ *in $\mathfrak{S}_8$ acts on the set consisting of all 12 non-isotropic vectors transitively.*

Proof. Since the symmetric group $\mathfrak{S}_8$ acts on the set of isotropic vectors transitively, it suffices to choose one isotropic vector $h = (1,0,0,0,0,0) \in u^{\oplus 3}$. Then, the non-isotropic vectors are given by the

$$(0,0,1,1,0,0), (0,0,1,1,1,0), (0,0,1,1,0,1), (0,1,1,1,0,0), (0,1,1,1,1,0), (0,1,1,1,0,1)$$

and the vectors which arise from these by applying the switching to the last two components of $u^{\oplus 3}$. One can easily obtain a similar result for isotropic vectors. The latter half of the statement is clear because for any two non-isotropic vectors v_1 and v_2, orthogonal to h, we can define an element $g \in \operatorname{Stab}(h)$ permuting v_1 and v_2, and extend it by the identity to $\langle v_1, v_2, h\rangle^\perp \subset \mathbb{F}_2^6$. Here, we used the fact that there is no relation such as $h = v_1 + v_2$, i.e., that v_1, v_2 and h are independent. $\square$

The goal of this subsection is the following theorem.

Theorem 5.3.14. *The divisors $\widetilde{\mathscr{H}}$ and T meet generically transversally in $\overline{\mathbb{B}/\Gamma}^{\mathrm{tor}}$.*

Proof. First, we take an irreducible component $T_{\mathrm{ord},i}$ of T_{ord}, namely the divisor over the cusp corresponding to the isotropic vector $h = (1,1,0,0,0,0)$. Then, we choose the component of $\widetilde{\mathscr{H}_{\mathrm{ord}}} \cap T_{\mathrm{ord},i}$ given by taking the divisor orthogonal to the vector $\ell = (0,0,1,1,0,0)$. We can perform both choices without loss of generality due to Lemma 5.3.13, which tells us that the group $\mathfrak{S}_8$ acts transitively on the components of $\widetilde{\mathscr{H}_{\mathrm{ord}}} \cap T_{\mathrm{ord},i}$.

Thus, it suffices to consider the component $\mathcal{T}$ of $\widetilde{\mathscr{H}_{\mathrm{ord}}} \cap T_{\mathrm{ord},i}$ chosen above. Now, $\mathcal{T}$ is the fixed locus of the reflection with respect to ℓ. In addition, through the isomorphism $\Gamma/\Gamma_{\mathrm{ord}} \cong \mathfrak{S}_8 \cong \mathrm{O}(\mathbb{F}_2^6)$ by [**42**, Section 3] or [**117**, Proposition 3.2], the choice of ℓ implies that this reflection acts on $\mathbb{P}^2 \times \mathbb{P}^2$ by

$$\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2 \times \mathbb{P}^2$$

$$([a_1 : b_1 : c_1], [a_2 : b_2 : c_2]) \mapsto ([b_1 : a_1 : c_1], [a_2 : b_2 : c_2]).$$

Also, a straightforward computation shows that $\mathcal{T}$ is not fixed by any other reflection with respect to a non-isotropic vector set-theoretically. Hence, we consider a general point $p = (p_1, p_2) \in \mathcal{T} \subset \mathbb{P}^2 \times \mathbb{P}^2$, where general means the following: the point $p_1 = [1 : 1 : c] \in \mathbb{P}^2$ satisfies $\mathrm{Stab}_{\mathfrak{S}_4}(p_1) = \langle (1\ 2) \rangle$, where $(1\ 2)$ denotes the transposition in $\mathfrak{S}_4$ of the first two components, and p_2 is general in the sense that $p_1 \neq p_2$ and $\mathrm{Stab}_{\mathfrak{S}_4}(p_2) = 1$. Clearly, the set of these points is non-empty. Here, we have used the fact that $\mathfrak{S}_4$ acts on $\mathbb{P}^2$ by the standard representation; see the proof of Lemma 5.3.11 and [**35**, Subsection 3.2]. By construction, the stabilizer of p is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, generated by a non-trivial involution in the first factor of $\mathfrak{S}_4 \times \mathfrak{S}_4$.

Using the coordinates taken in the proof of Lemma 5.3.11, by Theorem 5.3.9, taking the quotients, we can choose the defining equation of T_{ord} (resp. $\widetilde{\mathscr{H}_{\mathrm{ord}}}$) as $(t = 0)$ (resp. $(z_1 = 0)$). Then, the non-trivial involution in $\mathrm{Stab}(p)$ acts on p as $(t, z_1, z_2, z_3, z_4) \mapsto (t, -z_1, z_2, z_3, z_4)$. Hence, we obtain the new coordinates (t, w_1, z_2, z_3, z_4) of $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$, where $w_1 = z_1^2$. Therefore, the divisors T and $\widetilde{\mathscr{H}}$, defined by $(t = 0)$ and $(w_1 = 0)$ respectively, meet transversally.

□

5.3.3. Proof of Theorem 5.1.1. We shall now restate one of the main results in this chapter. Its proof uses our computation of the Betti numbers of the Kirwan blow-up $\mathcal{M}^{\mathrm{K}}$ and the toroidal compactification $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ which we will perform in Section 5.5.

Theorem 5.3.15. *Neither the Deligne-Mostow isomorphism $\phi : \mathcal{M}^{\mathrm{GIT}} \to \overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}$ nor its inverse ϕ^{-1} lift to a morphism between the Kirwan blow-up $\mathcal{M}^{\mathrm{K}}$ and the unique toroidal compactification $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$.*

Proof. We shall prove this for ϕ, the argument for ϕ^{-1} being the same. By Theorem 5.3.4 and Theorem 5.3.14, the birational map $g : \mathcal{M}^{\mathrm{K}} \dashrightarrow \overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ cannot be an isomorphism. By Theorems 5.5.6 and 5.5.8 the Betti numbers $b_2(\mathcal{M}^{\mathrm{K}}) = b_2(\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}) = 2$ agree. Hence g cannot contract a divisor and must thus be a small contraction. This, however, contradicts the fact that both $\mathcal{M}^{\mathrm{K}}$ and $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ are $\mathbb{Q}$-factorial. (See also the proof of [**28**, Theorem 1.1]). □

Since the compactifications concerned are $\mathfrak{S}_8$-equivariant we obtain as a byproduct that $\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}}/\mathfrak{S}_8 \not\cong \mathcal{M}^{\mathrm{K}}$.

5.4. Canonical bundles

On the way we shall use a modular form constructed by Kondō, which will be essential for us. In this section, we focus on the canonical bundles, and as a result, we shall show Theorem 5.1.3.

5.4.1. Computation involving blow-ups. We first recall some basic facts about the birational geometry of the relevant moduli spaces and noticeably the maps φ_1 and φ_2. From [**75**, Lemma 5.3], in their $\mathbb{Q}$-Picard groups, we obtain

$$\begin{aligned} \varphi_1^*(D_2^{(0)}) &= D_2^{(1)} + 6D_4^{(1)} \\ &= \widetilde{\mathscr{D}_{\mathrm{ord}}} + 6\Delta_{\mathrm{ord}}. \end{aligned} \tag{5.4.1}$$

Of course, this implies

$$\pi_{\mathrm{ord}}^*(\mathscr{H}_{\mathrm{ord}}) = \widetilde{\mathscr{H}_{\mathrm{ord}}} + 6T_{\mathrm{ord}}. \tag{5.4.2}$$

Note that this can be obtained from Lemma 5.3.13. For the sake of completeness, though this will not be used in this thesis, we note that

$$\varphi_{2*}(D_2^{(2)}) = D_2^{(1)} = \widetilde{\mathscr{D}_{\mathrm{ord}}}, \quad \varphi_{2*}(D_3^{(2)}) = 0, \quad \varphi_{2*}(D_4^{(2)}) = D_4^{(1)} = T_{\mathrm{ord}},$$
$$\varphi_{1*}(D_2^{(1)}) = D_2^{(0)} = \mathscr{D}_{\mathrm{ord}}, \quad \varphi_{1*}(D_4^{(1)}) = 0,$$
$$\varphi_2^*(D_2^{(1)}) = D_2^{(2)} + 3D_3^{(2)}, \quad \varphi_2^*(D_4^{(1)}) = D_4^{(2)}.$$

All of these equalities hold in the relevant $\mathbb{Q}$-Picard groups.

Moreover, the canonical divisors are described as

$$\begin{aligned} K_{\overline{\mathcal{M}}_{0,8}} &= -\frac{2}{7}D_2^{(2)} + \frac{1}{7}D_3^{(2)} + \frac{2}{7}D_4^{(2)} \\ K_{\mathcal{M}_{\mathrm{ord}}^{\mathrm{K}}} &= -\frac{2}{7}D_2^{(1)} + \frac{2}{7}D_4^{(1)} \\ K_{\mathcal{M}_{\mathrm{ord}}^{\mathrm{GIT}}} &= -\frac{2}{7}D_2^{(0)} \end{aligned} \tag{5.4.3}$$

where the number 7 in the denominators comes from $n-1$ in [**75**, Proposition 5.4, Lemma 5.5]. It follows that

$$\begin{aligned} K_{\mathcal{M}_{\mathrm{ord}}^{\mathrm{K}}} &= \varphi_1^*(K_{\mathcal{M}_{\mathrm{ord}}^{\mathrm{GIT}}}) + 2D_4^{(1)} \\ K_{\overline{\mathcal{M}}_{0,8}} &= \varphi_2^*(K_{\mathcal{M}_{\mathrm{ord}}^{\mathrm{K}}}) + D_3^{(2)}. \end{aligned}$$

In addition, there is a specific modular form of weight 14 on $\mathbb{B}^5$ vanishing exactly on H [**88**, Theorem 6.2], and hence

$$14\mathscr{L}_{\mathrm{ord}} = \frac{1}{2}\mathscr{H}_{\mathrm{ord}} \tag{5.4.4}$$

in $\mathrm{Pic}(\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}) \otimes \mathbb{Q}$. Here $\mathscr{L}_{\mathrm{ord}}$ denotes the automorphic line bundle of weight 1. By (standard) abuse of notation, we use the same notation for this line bundle on both the Baily-Borel and toroidal compactifications. Thus,

$$\begin{aligned} K_{\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}} &= -\frac{2}{7}\mathscr{D}_{\mathrm{ord}} \\ &= -8\mathscr{L}_{\mathrm{ord}}. \end{aligned}$$

Now, we compute the canonical bundles of $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}} \cong \mathcal{M}_{\mathrm{ord}}^{\mathrm{K}}$ in two ways: the realization as a ball quotient and the blow-up sequence.

Remark 5.4.1. The finite map $\mathbb{B}^3 \to \mathbb{B}^3/\Gamma_{\mathrm{ord}}$ (resp. $\mathbb{B}^3/\Gamma_{\mathrm{ord}} \to \mathbb{B}^3/\Gamma$) branches along H/Γ_{ord} (resp. H/Γ) with branch index 2. We illustrate a sketch of the proof below. First, for $r \in L$ let

$$\sigma_{\ell,\zeta}(r) := r + (1-\zeta)\frac{\langle \ell, r\rangle}{2} \in L \otimes \mathbb{Q}(\sqrt{-1})$$

where $\ell \in L$ is (-2)-vector and $\zeta \in \{1, \sqrt{-1}\}$. Then, a straightforward calculation shows $\sigma_{r,-1} \in \Gamma_{\mathrm{ord}}$ and $\sigma_{r,\sqrt{-1}} \in \Gamma \setminus \Gamma_{\mathrm{ord}}$. This concludes the claim.

On the one hand, by Remark 5.4.1, the standard application of Hilzebruch's proportionality principle gives

$$\begin{aligned}
K_{\overline{\mathbb{B}^5/\Gamma_{\rm ord}}^{\rm tor}} &= 6\mathscr{L}_{\rm ord} - \frac{1}{2}\widetilde{\mathscr{H}_{\rm ord}} - T_{\rm ord} \\
&= 6\mathscr{L}_{\rm ord} - \frac{1}{2}\left\{\pi^*_{\rm ord}(\mathscr{H}_{\rm ord}) - 6T_{\rm ord}\right\} - T_{\rm ord} \quad \text{(by (5.4.2))} \\
&= -8\mathscr{L}_{\rm ord} + 2T_{\rm ord} \quad \text{(by (5.4.4))}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
K_{\mathcal{M}^{\rm K}_{\rm ord}} &= -\frac{2}{7}\widetilde{\mathscr{D}_{\rm ord}} + \frac{2}{7}\Delta_{\rm ord} \quad \text{(by (5.4.3))} \\
&= -\frac{2}{7}\left\{\varphi_1^*(\mathscr{D}_{\rm ord}) - 6\Delta_{\rm ord}\right\} + \frac{2}{7}\Delta_{\rm ord} \quad \text{(by (5.4.1))} \\
&= -\frac{2}{7}\varphi_1^*\phi^*_{\rm ord}(\mathscr{H}_{\rm ord}) + 2\Delta_{\rm ord} \\
&= -8\varphi_1^*\phi^*_{\rm ord}(\mathscr{L}_{\rm ord}) + 2\Delta_{\rm ord} \quad \text{(by (5.4.4))} \\
&= \tau^*(-8\mathscr{L}_{\rm ord} + 2T_{\rm ord}) \quad \text{(by Figure 5.2.1)},
\end{aligned}$$

for $\tau := \Phi_{\frac{1}{4}+\epsilon} \circ \phi_{\rm ord}$. Thus, this calculation recovers the fact $K_{\mathcal{M}^{\rm K}_{\rm ord}} = \tau^*(K_{\overline{\mathbb{B}^5/\Gamma_{\rm ord}}^{\rm tor}})$ under the isomorphism $\tau : \mathcal{M}^{\rm K}_{\rm ord} \cong \overline{\mathbb{B}^5/\Gamma_{\rm ord}}^{\rm tor}$.

Remark 5.4.2. The above modular form constructed by Kondō is a "special reflective modular form" in the sense of [**115**, Assumption 2.1]. Hence, both $\mathcal{M}^{\rm GIT}_{\rm ord}$ and $\mathcal{M}^{\rm GIT}$ are Fano varieties from the above computation or [**115**, Theorem 2.4].

Now, we need the description of normal bundles along the toroidal boundary.

Proposition 5.4.3. *The normal bundle of* $T_{{\rm ord},i}$ *in* $\overline{\mathbb{B}^5/\Gamma}^{\rm tor}$ *is given by*

$$\mathcal{N}_{T_{{\rm ord},i}/\overline{\mathbb{B}^5/\Gamma}^{\rm tor}} = \mathscr{O}(-1,-1).$$

Proof. First, we obtain

$$(K_{\overline{\mathbb{B}^5/\Gamma_{\rm ord}}^{\rm tor}} + T_{{\rm ord},i})|_{T_{{\rm ord},i}} = (-8\mathscr{L}_{\rm ord} + 2T_{\rm ord} + T_{{\rm ord},i})|_{T_{{\rm ord},i}}.$$

The left-hand side gives

$$\begin{aligned}
(K_{\overline{\mathbb{B}^5/\Gamma_{\rm ord}}^{\rm tor}} + T_{{\rm ord},i})|_{T_{{\rm ord},i}} &= K_{T_{{\rm ord},i}} \\
&= \mathscr{O}(-3,-3)
\end{aligned}$$

by the adjunction formula. On the other hand, the right-hand side is

$$\begin{aligned}
(-8\mathscr{L}_{\rm ord} + 2T_{\rm ord} + T_{{\rm ord},i})|_{T_{{\rm ord},i}} &= 3T_{{\rm ord},i}|_{T_{{\rm ord},i}} \\
&= 3\mathcal{N}_{T_{{\rm ord},i}/\overline{\mathbb{B}^5/\Gamma}^{\rm tor}}.
\end{aligned}$$

This completes the proof.

□

Remark 5.4.4. This is an analog of Naruki's result [**125**, Proposition 12.1] on the moduli spaces of cubic surfaces. He constructed a cross ratio variety and analyzed its singularity at the boundary. Later, Gallardo-Kerr-Schaffler [**48**, Theorem 1.4] showed that the toroidal compactification and Naruki's compactification are isomorphic and Casalaina-Martin-Grushevsky-Hulek-Laza [**28**, Theorem 1.2] used this to compute the top self-intersection number of the canonical bundles. In the case of the moduli spaces of 8 points, there also exists the cross ratio variety constructed by [**42**, Theorem 2.4], [**88**, Theorem 7.2] or [**117**, Theorem 1.1]. However, these coincide with the Baily-Borel compactification $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}$ of the ball quotient unlike the case of cubic surfaces. This is why we used the results on the moduli spaces of stable curves in our case.

Now, we study the behavior of the boundary divisors along $\psi_1 : \overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}} \to \overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$. We recall that the toroidal compactifications are constructed by taking a "partial compactification in the direction of each cusp" [**6**, Section III. 5]. Here, this is done by choosing a polyhedral decomposition of a cone in the center of the unipotent part of the stabilizer of a cusp (which is canonical in our case). Hence, this group, which is denoted by $U(F)$ in [**6**], describes the toroidal boundary.

Lemma 5.4.5. *The map $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}} \to \overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ does not branch along T.*

Proof. The quotient $\Gamma/\Gamma_{\mathrm{ord}} \cong \mathfrak{S}_8$ acts on the set $\{T_{\mathrm{ord},i}\}_{i=1}^{35}$ transitively. Hence, it suffices to take one component $T_{\mathrm{ord},i}$, corresponding to the following isotropic vector $h \in L$, and prove that the center, denoted as $Z(F)$ in Lemma 5.3.11, of the unipotent radical of $\mathrm{Stab}_{\Gamma}(h)$ and $\mathrm{Stab}_{\Gamma_{\mathrm{ord}}}(h)$ are equal. Now, we choose an isotropic vector $h := (1, 0, 0, 0, 0) \in U \oplus U(2) \oplus D_4(-1)^{\oplus 2}$. Then, the corresponding center of the unipotent part of $\mathrm{Stab}_{\Gamma}(h)$ is given by

$$\left\{ \left(\begin{array}{c|c|c} 1 & & \sqrt{-1}(1-\sqrt{-1})w \\ \hline & I_4 & \\ \hline & & 1 \end{array} \right) \middle| \; w \in \mathbb{Z} \right\}.$$

Then, one can check that each matrix of the above form acts on $A_L \cong \left(\mathbb{Z}/(1+\sqrt{-1})\mathbb{Z}\right)^6$ trivially. This proves the above claim. □

On the one hand, in a similar way as [**28**, Proposition 5.8], it follows that

$$K_{\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}} = \pi^* K_{\overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}} + 7T \tag{5.4.5}$$

by Lemma 5.4.5. On the other hand, we can calculate the canonical bundle of $\mathcal{M}^{\mathrm{K}}$ by [**28**, Lemma 6.4], where a general approach to calculating the canonical bundle of Kirwan blow-ups was developed:

$$K_{\mathcal{M}^{\mathrm{K}}} = f^* K_{\mathcal{M}^{\mathrm{GIT}}} + 5\mathscr{E}, \tag{5.4.6}$$

where $\mathscr{E}$ is the exceptional divisor of the blow-up $f : \mathcal{M}^{\mathrm{K}} \to \mathcal{M}^{\mathrm{GIT}}$. Here, we apply the method [**28**, Lemma 6.4] for our case $c = 6$ (Lemma 5.3.2) and $|G_X| = |G_F| = 2$ (Lemma 5.3.1) in their notation. Note that there is no divisorial locus having a strictly bigger stabilizer than G_X.

5.4.2. Proof of Theorem 5.1.3. We can now prove that these two compactifications are not K-equivalent.

Theorem 5.4.6. *The compactifications $\mathcal{M}^{\mathrm{K}}$ and $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ are not K-equivalent.*

Proof. It suffices to show that $K^5_{\mathcal{M}^{\mathrm{K}}} \neq K^5_{\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}}$. By (5.4.5) and (5.4.6), we need to show that

$$(5\mathscr{E})^5 \neq (7T)^5.$$

Now, $T^5_{\mathrm{ord},i} = 6$ by Proposition 5.4.3. Hence, we have $T^5_{\mathrm{ord}} = 210$ and

$$T^5 = \frac{210}{8!} = \frac{1}{192}.$$

Here, if $(5\mathscr{E})^5$ and $(7T)^5$ are equal, then the denominator of $\mathscr{E}^5$ must be divided by 5 from the above calculation. On the other hand, [**28**, Proposition 6.10] implies

$$\mathscr{E}^5 \in \frac{1}{e}\mathbb{Z},$$

where e is the least common multiple of the orders of S_x for any $x \in \mathscr{E}$. However, the quantity e is not divisible by 5 by Proposition 5.3.7. This contradicts to the above. □

5.5. Cohomology

In this section, we compute the cohomology of the varieties appearing in this chapter.

5.5.1. The cohomology of $\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}}$, $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}$, $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}}$ and $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}$. We first collect the results due to Kirwan-Lee-Weintraub [**78**] and Kirwan [**77**] who determined the Betti numbers of $\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}}$ and $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}$, and $\mathcal{M}^{\mathrm{GIT}} \cong \overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}$ respectively. We summarize this in

Theorem 5.5.1 ([**78**, Table III, Theorem 8.6], [**77**, Table, p.40])**.** *All the odd degree cohomology of $\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}}$, $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}$ and $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}$ vanishes. In even degrees, the Betti numbers are as follows:*

j	0	2	4	6	8	10
$\dim H^j(\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}})$	1	43	99	99	43	1
$\dim IH^j(\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}})$	1	8	29	29	8	1
$\dim IH^j(\mathcal{M}^{\mathrm{GIT}})$	1	1	2	2	1	1
$\dim IH^j(\overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}})$	1	1	2	2	1	1

By an application of an easy version of the decomposition theorem, we can also compute the cohomology of $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}}$ (without using that this space is isomorphic to $\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}}$).

Theorem 5.5.2. *All the odd degree cohomology of $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}}$ vanishes. In even degrees, the Betti numbers are as follows:*

j	0	2	4	6	8	10
$\dim H^j(\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}})$	1	43	99	99	43	1

Proof. We use the form of the decomposition theorem as given in [**62**, Lemma 9.1]. Here we have 35 cusps and the toroidal boundary at each cusp is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$. The even Betti numbers of this space are given by $(1, 2, 3, 2, 1)$ and the result then follows from the Betti numbers of $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}$ together with the fact that there are 35 cusps. □

5.5.2. The cohomology of $\mathcal{M}^{\mathrm{K}}$. Now, we compute the cohomology of $\mathcal{M}^{\mathrm{K}}$. This will be done using the Kirwan method [**79, 76, 77**], studying the cohomology of the Kirwan blow-ups. We mainly follow [**27**, Chapter 3, 4], in particular, the case of cubic threefolds with precisely $2A_5$-singularities. Let us consider $X = \mathbb{P}^8$, acted on by $G = \mathrm{SL}_2(\mathbb{C})$ with the usual linearization and let Z_R^{ss} be the fixed locus of the action of R on X^{ss}, which is the semi-stable locus. We denote by $\widetilde{X}^{\mathrm{ss}} := Bl_{G\cdot Z_R^{\mathrm{ss}}}(X)$ the blow-up whose center is the unique polystable orbit $G \cdot Z_R^{\mathrm{ss}}$. From [**77**, Section 3 Eq. 3.2] or [**27**, Subsection 4.12, (4.22)], the Poincare series of $\widetilde{X}^{\mathrm{ss}}$ is given by

$$P_t^G(\widetilde{X}^{\mathrm{ss}}) = P_t^G(X^{\mathrm{ss}}) + A_R(t),$$

where $A_R(t)$ is a correction term consisting of a "main term" and an "extra term" with respect to the unique stabilizer R; see [**27**, Section 4.1.2] for precise definitions.

This method reduces the computation of $H^k(\mathcal{M}^{\mathrm{K}})$ to the estimation of

(1) the semi-stable locus (Subsection 5.5.2.1) ,
(2) the main correction term (Subsection 5.5.2.2) and
(3) the extra correction term (Subsection 5.5.2.3).

5.5.2.1. **Equivariant cohomology of the semi-stable locus**. Here we proceed according to [**27**, Chapter 3]. We can compute the cohomology of the semi-stable locus by using the stratification introduced by Kirwan. We omit details, but will still need to introduce some notation in order to describe the outline. Let $\{S_\beta\}_{\beta\in\mathcal{B}}$ be the stratification defined in [**79**, Theorem 4.16] and $d(\beta)$ be the codimension of S_β in X^{ss}. Here, the index set $\mathcal{B}$ consists of the point which is closest to the origin of the convex hull spanned by some weights in the closure of a positive Weyl chamber in the Lie algebra of a maximal torus in SO(2); see [**27**, Chapter 3] or [**79**, Definition 3.13] for details.

Proposition 5.5.3.

$$P_t^G(X^{\mathrm{ss}}) \equiv 1 + t^2 + 2t^4 \bmod t^6.$$

Proof. We shall prove $2d(\beta) \geq 6$ for any $0 \neq \beta \in \mathcal{B}$. This implies

$$\begin{aligned} P_t^G(X^{\mathrm{ss}}) &\equiv P_t(X)P_t(B\,\mathrm{SL}_2(\mathbb{C})) \bmod t^6 \\ &\equiv (1-t^2)^{-1}(1-t^4)^{-1} \bmod t^6 \\ &\equiv 1 + t^2 + 2t^4 \bmod t^6. \end{aligned}$$

In the same way, as in the proof of [**27**, Proposition 3.5] we obtain

$$d(\beta) \geq 7 - r(\beta),$$

where $r(\beta)$ is the number of weights α satisfying $\beta\cdot\alpha \geq ||\beta||^2$. Now, we have

$$\mathcal{B} = \{(1,-1),(2,-2),(3,-3),(4,-4)\}.$$

For each $(a,-a) \in \mathcal{B}$, it easily follows

$$r(\beta) = 5 - a,$$

and this implies $d(\beta) \geq 3$.

□

5.5.2.2. **The main correction term**. The following is based on [**27**, Chapter 4].

Proposition 5.5.4. *The main correction term in* $A_R(t)$ *is given by*

$$(1-t^4)^{-1}(t^2+t^4) \equiv t^2+t^4 \bmod t^6.$$

Proof. In the same way as in [**27**, Proposition B.1 (4)], the normalizer of R is computed to be

$$N := N(R) \cong \mathbb{T} \rtimes \mathbb{Z}/2\mathbb{Z}.$$

Hence, it follows that

$$\begin{aligned} H^\bullet_N(Z^{\mathrm{ss}}_R) &= (H^\bullet_{\mathbb{T}}(Z^{\mathrm{ss}}_R))^{\mathbb{Z}/2\mathbb{Z}} \\ &= (H^\bullet(BR) \otimes H^\bullet_{\mathbb{T}/R}(Z^{\mathrm{ss}}_R))^{\mathbb{Z}/2\mathbb{Z}} \\ &= (H^\bullet(BR) \otimes H^\bullet(*))^{\mathbb{Z}/2\mathbb{Z}} \\ &= \mathbb{Q}[c^4] \end{aligned}$$

where $*$ denotes a set of 1 point and the degree of c is 1. The last equation follows from the discussion in the proof of [**27**, Proposition 4.4]. Hence,

$$P^N_t(Z^{\mathrm{ss}}_R) = (1-t^4)^{-1}.$$

Combining this with [**27**, (4.24)] completes the proof. □

5.5.2.3. **The extra correction term**. Let $\mathcal{N}$ be the normal bundle to the orbit $G \cdot Z^{\mathrm{ss}}_R$. Then, for a generic point $x \in Z^{\mathrm{ss}}_R$, we have a representation ρ of R on $\mathcal{N}_x$. Let $\mathcal{B}(\rho)$ be the set consisting of the closest point to 0 of the convex hull of a nonempty set of weights of the representation ρ. For $\beta' \in \mathcal{B}(\rho)$, let $n(\beta')$ be the number of weights less than β'.

Proposition 5.5.5. *The extra correction term vanishes modulo* t^6*, i.e., does not contribute to* $A_R(t)$*.*

Proof. In our case we have $Z^{\mathrm{ss}}_R = \{c_{4,4}\}$. Thus, to describe $\mathcal{N}_x$, we have to compute

$$\Big(T_{c_{4,4}}(\mathrm{SL}_2(\mathbb{C}) \cdot \{c_{4,4}\})\Big)^{\perp}.$$

This was calculated in Lemma 5.3.2. Moreover, $\mathrm{diag}(\lambda, \lambda^{-1})$ acts on $T_{c_{4,4}}\mathbb{C}^9 \cong \mathbb{C}^9$ by the weights

$$0, \pm 2, \pm 4, \pm 6, \pm 8.$$

It follows that $T_{c_{4,4}}(\mathrm{SL}_2(\mathbb{C}) \cdot \{c_{4,4}\})$ is generated by the weights $\{0, \pm 2\}$, and hence we obtain

$$\mathcal{B}(\rho) = \{\pm 4, \pm 6, \pm 8\}.$$

This shows that

$$\begin{aligned} d(|\beta'|) &= n(|\beta'|) \\ &= 1 + \frac{|\beta'|}{2} \\ &\geq 3 \end{aligned}$$

for $\beta' \in \mathcal{B}(\rho)$. This in turn implies that

$$\text{“extra correction term”} \equiv 0 \bmod t^6$$

by [**27**, (4.25)]. □

5.5.2.4. **Computation of the cohomology of $\mathcal{M}^{\mathrm{K}}$.** From Propositions 5.5.3, 5.5.4 and 5.5.5, it follows that

$$\begin{aligned} P_t(\mathcal{M}^{\mathrm{K}}) &= P_t^G(\widetilde{X}^{\mathrm{ss}}) \\ &\equiv (1+t^2+t^4) + (t^2+t^4) \bmod t^6 \\ &\equiv 1 + 2t^2 + 3t^4 \bmod t^6. \end{aligned}$$

Therefore, we obtain the following:

Theorem 5.5.6. *All the odd degree cohomology of $\mathcal{M}^{\mathrm{K}}$ vanishes. In even degrees, its Betti numbers are given as follows:*

j	0	2	4	6	8	10
$\dim H^j(\mathcal{M}^{\mathrm{K}})$	1	2	3	3	2	1

5.5.3. The cohomology of $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$. Now, we compute the cohomology of the toroidal compactification of the 5-dimensional ball quotient. Our main tool is the decomposition in the easy form stated in theorem [**62**, Lemma 9.1], see also [**27**, chapter 6]. This allows us to combine the cohomology of $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}$ and the toroidal boundary. To do this, we first study the cohomology of the toroidal boundary.

Proposition 5.5.7. *All the odd degree cohomology of the boundary T vanishes. In even degrees, its Betti numbers are given as follows:*

j	0	2	4	6	8
$\dim H^j(T)$	1	1	2	1	1

Proof. This amounts to the computation of the invariant cohomology of the action of the stabilizer of a toroidal boundary component as in the proof of [**27**, Proposition 7.13]. More precisely, we have to determine the cohomology ring

$$H^\bullet(\mathbb{P}^2\times\mathbb{P}^2)^{(\mathfrak{S}_4\times\mathfrak{S}_4)\rtimes\mathfrak{S}_2} = H^\bullet((\mathbb{P}^2/\mathfrak{S}_4)^2,\mathbb{Q})^{\mathfrak{S}_2} = H^\bullet((\mathbb{P}(1,2,3)^2,\mathbb{Q})^{\mathfrak{S}_2}.$$

Since $H^\bullet(\mathbb{P}^2/\mathfrak{S}_4) = H^\bullet((\mathbb{P}(1,2,3)) \cong \mathbb{Q}[x]/(x^3)$, this is equivalent to compute the $\mathfrak{S}_2$-invariant parts of the tensor product $\mathbb{Q}[x]/(x^3)\otimes\mathbb{Q}[y]/(y^3)$. Hence the invariant cohomology is given by

$$P_t(T) = 1 + t^2 + 2t^4 + t^6 + t^8.$$

□

We can now summarize the above computations in the

Theorem 5.5.8. *All the odd degree cohomology of $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ vanishes. In even degrees, the Betti numbers are given by the following table:*

j	0	2	4	6	8	10
$\dim H^j(\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}})$	1	2	3	3	2	1

In particular, all the Betti numbers of $\mathcal{M}^{\mathrm{K}}$ and $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ are the same.

Proof. This follows now from an application of the decomposition theorem as stated in [**62**, Lemma 9.1], applied to the last line in Theorem 5.5.1 and Proposition 5.5.7. □

5.6. Other cases of the Deligne-Mostow list

Here we very briefly discuss some further cases of the Deligne-Mostow list where a similar analysis can be made. More concretely, we consider N points on $\mathbb{P}^1$ for $5 \leq N \leq 12$ with symmetric weights; see [**32**] or [**139**, Appendix]. Note that the notions of stable and semi-stable coincide for odd N. Remarkably, the beahviour which was observed for the moduli spaces of cubic surfaces and 8 points on $\mathbb{P}^1$, can also be found in other cases, thus pointing towards a much more general phenomenon.

5.6.1. 5 points. The moduli space of 5 points on $\mathbb{P}^1$ is associated with K3 surfaces with an automorphism of order 5 [**87**]. In this case, the Deligne-Mostow isomorphism gives

$$\mathcal{M}_{\mathrm{ord}}^{\mathrm{GIT}} \cong \overline{\mathbb{B}^2/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}$$

for the discriminant kernel group Γ_{ord} [**87**, Subsection 6.3, (6.5)]. Here, the weight in the sense of Deligne-Mostow is

$$\left(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}\right).$$

This is the quintic del Pezzo surface [**85**, Proposition 6.2 (2)]. Now, $\mathbb{B}^2/\Gamma'$ is compact ([**87**, Subsection 6.5] or [**139**, Appendix]). Hence, we have

$$\mathcal{M}^{\mathrm{K}} = \mathcal{M}^{\mathrm{GIT}} \cong \overline{\mathbb{B}^2/\Gamma}^{\mathrm{BB}} = \overline{\mathbb{B}^2/\Gamma}^{\mathrm{tor}}$$

for the full modular unitary group Γ.

5.6.2. 7, 9, 10 or 11 points. The moduli space of 7 points on $\mathbb{P}^1$ was studied in [**36**]. In this chapter, we apply the theory of the moduli spaces of stable curves to analyze the geometry of our ball quotients. In order to apply the work by Hassett, Kiem-Moon and others, the weights appearing in the Deligne-Mostow theory, that is the linearization of a line bundle, must be linearised as $\mathscr{O}(1, \cdots, 1)$; see [**75**, Section 1]. Thus, in particular, the case of 7, 9, 10 and 11 points are out of scope in this thesis.

5.6.3. 6 points and 12 points. These are Eisenstein cases, which will be treated in upcoming work.

5.6.3.1. **6 points**. The moduli space of 6 points on $\mathbb{P}^1$ is closely related to the theory of the Igusa quartic and the Segre cubic [**89, 90, 116**]. It is known that the Segre cubic is realised as the Baily-Borel compactification of a 3-dimensional ball quotient. We recall the setting of [**89**]. Let $\Lambda := \mathbb{Z}[\omega]^{\oplus 4}$ be the Hermitian lattice over $\mathbb{Z}[\omega]$ of signature $(1, 3)$ equipped with the Hermitian matrix $\mathrm{diag}(1, -1, -1, -1)$, where ω is a primitive third root of unity. Let $\Gamma := \mathrm{U}(\Lambda)(\mathbb{Z})$ and

$$\Gamma_{\mathrm{ord}} := \{g \in \Gamma \mid g|_{\Lambda/\sqrt{-3}\Lambda} = \mathrm{id}\}.$$

The ball quotient $\overline{\mathbb{B}^3/\Gamma}^{\mathrm{BB}}$ (resp. $\overline{\mathbb{B}^3/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}$) is isomorphic to the moduli space of unordered (resp. ordered) 6 points on $\mathbb{P}^1$. Here, $\mathbb{B}^3$ is the 3-dimensional complex ball. The approach developed in the current thesis can be fully carried over to this case. In particular, the analogs of Theorems 5.1.1 and 5.1.3 hold unchanged.

5.6.3.2. **12 points**. The moduli space of unordered 12 points on $\mathbb{P}^1$ is known to be the moduli space of (non-hyperelliptic) curves of genus 4 [**85**]. In particular, this moduli space is the 9-dimensional ball quotient taken by the full unitary group for the Hermitian lattice with underlying integral lattice $U(3) \oplus U \oplus E_8(-1)^{\oplus 2}$. There is, however, an important difference here to the cases discussed previously: the arithmetic subgroup defining the moduli space of ordered 12 points on $\mathbb{P}^1$ is not known, see [**81**], although it is expected to be the discriminant kernel as in the case of 6 or 8 points.

In this case, there is the blow-up sequence

$$\overline{\mathcal{M}}_{0,12} \to \overline{\mathcal{M}}_{0,12(\frac{1}{4}+\epsilon)} \to \overline{\mathcal{M}}_{0,12(\frac{1}{5}+\epsilon)} \to \overline{\mathcal{M}}_{0,12(\frac{1}{6}+\epsilon)} \cong \mathcal{M}_{\mathrm{ord}}^{\mathrm{K}} \overset{\varphi_1}{\to} \mathcal{M}_{\mathrm{ord}}^{\mathrm{GIT}}.$$

Note that $\mathcal{M}_{\mathrm{ord}}^{\mathrm{GIT}}$ has 464 cusps. Combining the above observation with modular forms constructed by Kondō [**81**, Corollary 2.9] using Borcherds product, we strongly expect an analog of Theorems 5.1.1 and 5.1.3. This is further confirmed by an observation by Casalaina-Martin (private communication), who also expects that Theorem 5.1.1 should hold.

CHAPTER 6

Modularity of the generating series of special cycles on orthogonal Shimura varieties

6.1. Introduction

We study special cycles on a Shimura variety of orthogonal type over a totally real field of degree d associated with a quadratic form in $n+2$ variables whose signature is $(n,2)$ at e real places and $(n+2,0)$ at the remaining $d-e$ real places for $1 \leq e < d$. Recently, these cycles were constructed by Kudla and Rosu-Yott and they proved that the generating series of special cycles in the cohomology group is a Hilbert-Siegel modular form of half integral weight. We prove that, assuming the Beilinson-Bloch conjecture on the injectivity of the higher Abel-Jacobi map, the generating series of special cycles of codimension er in the Chow group is a Hilbert-Siegel modular form of genus r and weight $1+n/2$. Our result is a generalization of *Kudla's modularity conjecture*, solved by Yuan-Zhang-Zhang unconditionally when $e=1$.

In this chapter, we prove that, assuming the Beilinson-Bloch conjecture on the injectivity of the higher Abel-Jacobi map, the generating series of special cycles in the Chow groups of a Shimura variety of orthogonal type is a Hilbert-Siegel modular form of half integral weight. These cycles were constructed by Kudla [**95**] and Rosu-Yott [**131**].

Historically, Kudla and Millson studied the cohomology groups in [**96**]. Kudla conjectured the modularity of the generating series of special cycles in the Chow groups in [**94**] and he proved it for one-codimensional Chow cycles, using the results of Borcherds [**15**]. This conjecture is often called *Kudla's modularity conjecture*. In his thesis [**154**], Zhang proved it for higher codimensional Chow cycles on Shimura varieties of orthogonal type associated with a quadratic form of signature $(n,2)$ over $\mathbb{Q}$ by his modularity criterion. His criterion works only over $\mathbb{Q}$ because its proof depends on the results of Borcherds [**14**]. Yuan-Zhang-Zhang [**151**] extended Zhang's results [**154**] to totally real fields. Their proof is similar to Zhang's proof over $\mathbb{Q}$ in view of using induction on the codimension of Chow cycles and calculating element-wise modularity.

Recently, Kudla [**95**] and Rosu-Yott [**131**] generalized Kudla-Millson's work by changing the signature of the quadratic form. Rosu-Yott [**131**] studied special cycles in the cohomology groups only, so did not generalize Yuan-Zhang-Zhang's work. In this chapter, we shall generalize the results of Yuan-Zhang-Zhang [**151**] under the Beilinson-Bloch conjecture. In the same setting as [**95**], [**131**] and assuming the Beilinson-Bloch conjecture on the injectivity of the higher Abel-Jacobi map, we prove the modularity of the generating series of special cycles in the Chow groups. (For the precise statement, see Theorem 6.1.5 and Theorem 6.1.6.)

After the first version of the paper [**111**] was written, the author learned that Kudla independently obtained similar results in his recent preprint [**95**]. His results and proof are different from ours. More precisely, in [**95**], he assumed the Beilinson-Bloch conjecture for Chow cycles of codimension er, and proved the absolute convergence and the modularity of the generating series. In contrast, even if $r \geq 2$, we assume the Beilinson-Bloch conjecture for Chow cycles of codimension e only. However, we cannot prove the absolute convergence. Assuming the absolute convergence, we prove the modularity by induction on r by the methods of [**151**]. (For details, see Remark 1.7.2 (4).)

6.1.1. Beilinson-Bloch conjecture. In the 1980s, Beilinson and Bloch formulated a series of influential conjectures on algebraic cycles. We review the statement of a part of the Beilinson-Bloch conjecture which is needed in the main theorem of this chapter. Our main reference is [**10**]. More generally, the Beilinson-Bloch conjecture is formulated in the theory of mixed motives, but we do not need the full version and need only a part of it for smooth projective varieties over number fields. We recommend [**74**] to the readers who want to know the Beilinson-Bloch conjecture in the theory of mixed motives.

In this subsection, let k be a field of characteristic 0 embedded in $\mathbb{C}$. Let X be a smooth projective variety over k. Let

$$cl^m\colon \mathrm{CH}^m(X) \to H^{2m}(X, \mathbb{Q}) := H^{2m}(X(\mathbb{C}), \mathbb{Q})$$

be the cycle map. We put $\mathrm{CH}^m_{\mathrm{hom}}(X) := \mathrm{Ker}(cl^m)$.

The following is a generalization of the Birch and Swinnerton-Dyer conjecture.

Conjecture 6.1.1. *(Beilinson-Bloch conjecture* [**10**, Conjecture 5.0]*) Assume that k is a number field. Then the group $\mathrm{CH}^m_{\mathrm{hom}}(X)$ is finitely generated and the rank of $\mathrm{CH}^m_{\mathrm{hom}}(X)$ is equal to the order of zero of the Hasse-Weil L-function $L(H^{2m-1}_{\acute{e}t}(X \otimes_k \overline{k}, \mathbb{Q}_\ell), s)$ at $s = m$ for any prime ℓ.*

We recall another conjecture which is also considered as a part of the Beilinson-Bloch conjecture. By Hodge theory, we have the Hodge decomposition

$$H^r(X, \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q},$$

where $H^{p,q} := H^q(X, \Omega^p)$ and a Hodge filtration $\left\{F^i H^m\right\}_{i=0}^m$ on H^m is defined by

$$F^i H^m := \bigoplus_{p \geq i} H^{p,m-p}.$$

The *the m-th intermediate Jacobian of X* (or *the Griffiths Jacobian of X*) is defined by

$$J^{2m-1}(X) := H^{2m-1}(X, \mathbb{C})/(F^m H^{2m-1}(X, \mathbb{C}) \oplus H^{2m-1}(X, \mathbb{Z}(m))).$$

Then we have *the m-th higher Abel-Jacobi map*:

$$AJ^m\colon \mathrm{CH}^m_{\mathrm{hom}}(X)_{\mathbb{Q}} := \mathrm{CH}^m_{\mathrm{hom}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to J^{2m-1}(X)_{\mathbb{Q}} := J^{2m-1}(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Here we can state another conjecture which is a part of a version of the Beilinson-Bloch conjecture.

Conjecture 6.1.2. *(Beilinson-Bloch conjecture* [**10**, Lemma 5.6]*) The m-th higher Abel-Jacobi map AJ^m is injective.*

Conjecture 6.1.1 or Conjecture 6.1.2 suggests the following is true. Recall that $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$ embedded in $\mathbb{C}$.

Conjecture 6.1.3. *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$. If $H^{2m-1}(X, \mathbb{Q}) = 0$, then $\mathrm{CH}^m_{\mathrm{hom}}(X)_{\mathbb{Q}} = 0$. In particular, the cycle map tensored with $\mathbb{Q}$*

$$cl^m_{\mathbb{Q}}\colon \mathrm{CH}^m(X)_{\mathbb{Q}} := \mathrm{CH}^m(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^{2m}(X, \mathbb{Q})$$

is injective.

Remark 6.1.4. When $m = 1$ and X is a smooth projective curve over $\mathbb{C}$, the map AJ^1 is the usual Abel-Jacobi map, so we get an isomorphism between the Picard group and the Jacobian. See [**74**, Section 1.4]. From this, it is easy to see that Conjecture 6.1.2 and Conjecture 6.1.3 are true when $m = 1$.

6.1.2. Main results. Let notation be as in Section 1.2.2. If $n \geq 3$, our main results in this chapter are below.

Theorem 6.1.5. *Assume $n \geq 3$ and Conjecture 6.1.3 for the Shimura variety M_{K_f} for $m = e$. Let $r \geq 1$ be a positive integer.*

(1) If $\ell\colon \mathrm{CH}^{er}(M_{K_f})_{\mathbb{C}} \to \mathbb{C}$ is a linear map over $\mathbb{C}$ such that $\ell(Z_{\phi_f})(\tau)$ is absolutely convergent, then $\ell(Z_{\phi_f})(\tau)$ defines a Hilbert-Siegel modular form of genus r and weight $1 + n/2$.

(2) If $r = 1$, for any linear map $\ell\colon \mathrm{CH}^{e}(M_{K_f})_{\mathbb{C}} \to \mathbb{C}$, the formal power series $\ell(Z_{\phi_f})(\tau)$ is absolutely convergent and we get a Hilbert modular form of weight $1 + n/2$.

If $n \leq 2$, we need to embed M_{K_f} into a larger Shimura variety. Let W be a totally positive quadratic space of dimension ≥ 3 over E_0 and we put $\mathcal{G}' := \mathrm{Res}_{E_0/\mathbb{Q}} \mathrm{GSpin}(V \oplus W)$. We may assume there is an open compact subgroup $K'_f \subset \mathcal{G}'(\mathbb{A}_f)$ such that $K_f = K'_f \cap \mathcal{G}(\mathbb{A}_f)$. Let $M'_{K'_f}$ be the Shimura variety associated with $\mathcal{G}'$ and K'_f defined over $\overline{\mathbb{Q}}$. Then we have an embedding of Shimura varieties $M_{K_f} \hookrightarrow M'_{K'_f}$ defined over $\overline{\mathbb{Q}}$.

Theorem 6.1.6. *Assume $n \leq 2$ and Conjecture 6.1.3 for the larger Shimura variety $M'_{K'_f}$ for $m = e$. Let $r \geq 1$ be a positive integer.*

(1) If $\ell\colon \mathrm{CH}^{er}(M_{K_f})_{\mathbb{C}} \to \mathbb{C}$ is a linear map over $\mathbb{C}$ such that $\ell(Z_{\phi_f})(\tau)$ is absolutely convergent, then $\ell(Z_{\phi_f})(\tau)$ defines a Hilbert-Siegel modular form of genus r and weight $1 + n/2$.

(2) If $r = 1$, for any linear map $\ell\colon \mathrm{CH}^{e}(M_{K_f})_{\mathbb{C}} \to \mathbb{C}$, the formal power series $\ell(Z_{\phi_f})(\tau)$ is absolutely convergent and we get a Hilbert modular form of weight $1 + n/2$.

6.1.3. Outline of the proof of Theorem 6.1.5 and Theorem 6.1.6. We mostly follow the strategy of Yuan-Zhang-Zhang [**151**]. However, we have to treat higher codimensional cycles rather than 1 even in the case of $r = 1$ different from [**151**], so we need algebraic geometrical consideration, such as the Beilinson-Bloch conjecture.

First, we shall prove Theorem 6.1.5 (2). To prove Theorem 6.1.5 (2), we calculate the cohomology of the Shimura variety M_{K_f}. By the Matsushima formula, we conclude

$$H^{2e-1}(M_{K_f}, \mathbb{C}) = 0.$$

Since we are assuming Conjecture 6.1.3 holds for M_{K_f} and $m = e$, the cycle map tensored with $\mathbb{C}$

$$cl^e_{\mathbb{C}}\colon \mathrm{CH}^e(M_{K_f})_{\mathbb{C}} \to H^{2e}(M_{K_f}, \mathbb{C})$$

is injective. Hence every $\mathbb{C}$-linear map $\mathrm{CH}^e(M_{K_f})_{\mathbb{C}} \to \mathbb{C}$ is extended to a $\mathbb{C}$-linear map $H^{2e}(M_{K_f}, \mathbb{C}) \to \mathbb{C}$. We can deduce Theorem 6.1.5 (2) from the results of Kudla [**95**, Section 5.3] and Rosu-Yott [**131**, Theorem 1.1].

Then we shall prove Theorem 6.1.6 (2) by the intersection formula [**151**, Proposition 2.6] and the pull-back formula [**151**, Proposition 3.1].

Finally, we deduce Theorem 6.1.5 (1) from Theorem 6.1.5 (2) and deduce Theorem 6.1.6 (1) from Theorem 6.1.6 (2). When $r \geq 2$, we prove Theorem 6.1.5 (1) and Theorem 6.1.6 (1) by induction on r. We put $J := \begin{pmatrix} 0 & -1_r \\ 1_r & 0 \end{pmatrix} \in \mathrm{GL}_{2r}(E_0)$. The symplectic group

$$\mathrm{Sp}_{2r}(E_0) := \left\{ g \in \mathrm{GL}_{2r}(E_0) \;\middle|\; {}^t g J g = J \right\}$$

is generated by the Siegel parabolic subgroup $P(E_0)$ and an element $w_1 \in \mathrm{Sp}_{2r}(E_0)$. Here

$$P(F) := \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathrm{Sp}_{2r}(E_0) \right\}$$

and w_1 is the image of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ by the injection

$$\mathrm{SL}_2 \hookrightarrow \mathrm{Sp}_{2r}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1_{r-1} & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1_{r-1} \end{pmatrix}.$$

We consider a function $Z_{\phi_f}(g')$ on the metaplectic group $\mathrm{Mp}_{2r}(\mathbb{A}_{E_0})$ which is a lift of $Z_{\phi_f}(\tau)$. It suffices to prove that the function $Z_{\phi_f}(g')$ is invariant under the action of $P(E_0)$ and w_1. A direct calculation shows the invariance under the action of an element of $P(E_0)$. To prove the invariance under w_1, we use the Poisson summation formula to reduce to the case $r = 1$.

6.1.4. Organization of this chapter. In Section 6.2, we recall some facts about special cycles and Weil representations. In Section 6.3, we calculate the cohomology of a Shimura variety and prove Theorem 6.1.5 (2) and Theorem 6.1.6 (2). Finally, in Section 6.4, we complete a proof of Theorem 6.1.5 and Theorem 6.1.6.

6.2. Special cycles and Weil representations

In this section, we recall and extend some properties of special cycles in Chow groups. We also note about Weil representations since in the proof of our main results, we use the function on $\mathrm{Mp}_{2r}(\mathbb{A}_{E_0})$, the metaplectic double cover of $\mathrm{Sp}_{2r}(\mathbb{A}_{E_0})$, lifting $Z_{\phi_f}(\tau)$. For more details, see [**151**].

6.2.1. Special cycles. Let W be an E_0-vector subspace of

$$\widehat{V} := V \otimes_{\mathbb{Q}} \mathbb{A}_f.$$

We say W is *admissible* if the restriction of the inner product to W is E_0-valued and totally positive. We say an element $x = (x_1, \ldots, x_r) \in \widehat{V}^r$ is *admissible* if the E_0-subspace of $\widehat{V}$ spanned by $x_1, \ldots, x_r$ is admissible. The following lemma shows admissibility is useful for description of special cycles.

Lemma 6.2.1. *An E_0-subspace W of $\widehat{V}$ is admissible if and only if there exists an E_0-subspace W' of V and $g \in \mathcal{G}(\mathbb{A}_f)$ such that $W = gW'$.*

PROOF. See [**151**, Lemma 2.1]. □

By the above lemma, for an admissible subspace $W = g^{-1}W'$, we define $Z(W)_{K_f} := Z(W', g)_{K_f}$. In the same way, for an admissible element $x = g^{-1}x'$, we write $Z(x)_{K_f} := Z(x', g)_{K_f}$. Moreover, for $\tau \in (\mathscr{H}_r)^d$ and $g' \in \mathrm{Mp}_{2r}(\mathbb{A}_{E_0})$, we get the following descriptions:

$$Z_{\phi_f}(\tau) = \sum_{\substack{x \in K_f \backslash \widehat{V}^r \\ \text{admissible}}} \phi_f(x) Z(x)_{K_f} q^{T(x)}$$

$$Z_{\phi_f}(g') = \sum_{\substack{x \in K_f \backslash \widehat{V}^r \\ \text{admissible}}} \omega_f(g'_f)\phi_f(x) Z(x)_{K_f} W_{T(x)}(g'_\infty).$$

By [**151**, Proposition 2.2], the scheme-theoretic intersection of two cycles $Z(W_1)_{K_f}$ and $Z(W_2)_{K_f}$ is the union of $Z(W)$ indexed by admissible classes W in

$$K_f \backslash (K_f W_1 + K_f W_2).$$

We shall investigate the intersection of two cycles in the Chow group.

Proposition 6.2.2. *The intersection of two cycles $Z(W_1)_{K_f}$ and $Z(W_2)_{K_f}$ in the Chow group are proper if and only $k_1W_1 \cap k_2W_2 = 0$ for all admissible classes $k_1W_1 + k_2W_2$.*

Proof. We recall that $\dim Z(W_i)_{K_f} = e(n - \dim W_i)$. The intersection is proper if and only if the following inequality holds:

$$\begin{aligned} \dim(Z(W_1)_{K_f} \cap Z(W_2)_{K_f}) &\leq \dim Z(W_1)_{K_f} + \dim Z(W_2)_{K_f} - \dim M_{K_f} \\ &= e(n - (\dim W_1 + \dim W_2)) \end{aligned}$$

On the other hand,

$$Z(W_1)_{K_f} \cap Z(W_2)_{K_f} = \sum_{\substack{W \in K_f \backslash (K_f W_1 + K_f W_2) \\ \text{admissible}}} Z(W)_{K_f}$$

and

$$\dim Z(k_1W_1 + k_2W_2)_{K_f} = e(n - (\dim k_1W_1 + \dim k_2W_2) + \dim(k_1W_1 \cap k_2W_2)).$$

Therefore the above inequality holds if and only if $k_1W_1 \cap k_2W_2 = 0$ for all admissible classes $k_1W_1 + k_2W_2$.

□

Proposition 6.2.3. *The intersection of two cycles $Z(W_1)_{K_f}$ and $Z(W_2)_{K_f}$ in the Chow group is given by the sum of $Z(W)_{K_f}$ indexed by admissible classes W in*

$$K_f \backslash (K_f W_1 + K_f W_2).$$

Proof. In the same way as the proof of [**151**, Proposition 2.6], we have to check that if $\dim W_2 = 1, Z(W_1)_{K_f} \subset Z(W_2)_{K_f}$, then $Z(W_1)_{K_f} \cdot Z(W_2)_{K_f} = Z(W_1)_{K_f} \cdot \mathrm{c}_1(\mathscr{L})$. Let $\mathscr{N}$ be the restriction of the normal bundle $\mathscr{N}_{Z(W_2)_{K_f}}(M_{K_f})$ to $Z(W_2)_{K_f}$. Now,

$$Z(W_1)_{K_f} \cdot Z(W_2)_{K_f} = \mathrm{c_e}(\mathscr{N}) \cap Z(W_2)_{K_f}$$

and by the calculation of normal bundles in [**95**, Chapter 4], we have

$$\mathrm{c_e}(\mathscr{N}) \cap Z(W_2)_{K_f} = Z(W_1)_{K_f} \cdot \mathrm{c}_1(\mathscr{L}).$$

□

6.2.2. The pull-back formula. Here we study the behavior of special cycles under the pull-back map. Let $W \subset V$ be a totally positive E_0-vector subspace. There exists a natural morphism

$$i_W \colon M_{K_f,W} \to M_{K_f},$$

which is a closed embedding if K_f is sufficiently small. Therefore we get a pull-back map of Chow groups:

$$i_W^* \colon \mathrm{CH}^{er}(M_{K_f}) \to \mathrm{CH}^{er}(M_{K_f,W}).$$

For $g' = (g'_f, g'_\infty) \in \mathrm{Mp}_{2r}(\mathbb{A}_{E_0}) = \mathrm{Mp}_{2r}(\mathbb{A}_{E_0,f}) \times \mathrm{Mp}_{2r}(E_{0_\infty})$, we define

$$i_W^*(Z_{\phi_f})(g') := \sum_{\substack{x \in K_f \backslash \widehat{V}^r \\ \text{admissible}}} \omega_f(g'_f)\phi_f(x) i_W^*(Z(x)_{K_f}) W_{T(x)}(g'_\infty).$$

For a Bruhat-Schwartz function $\phi_{2,f} \in \mathbf{S}((\widehat{W^\perp})^r)$, the theta function is defined by

$$\theta_{\phi_{2,f}}(g') := \sum_{z \in W^r} \omega_{\mathbb{A}}(g')(\phi_{2,f} \otimes \varphi_+^d)(z).$$

Proposition 6.2.4. *For a K_f-invariant Bruhat-Schwartz function*

$$\phi_f = \phi_{1,f} \otimes \phi_{2,f} \in \mathbf{S}(\widehat{V}^r) = \mathbf{S}(\widehat{W}^r) \otimes_{\mathbb{C}} \mathbf{S}((\widehat{W^\perp})^r),$$

we have

$$i_W^*(Z_{\phi_f}(g')) = Z_{\phi_{1,f}}(g')\theta_{\phi_{2,f}}(g').$$

PROOF. Proposition 6.2.3 implies that the assertion is proved by the same way as [**151**, Proposition 3.1]. □

6.2.3. Weil representations. Let $\psi \colon E_0 \backslash \mathbb{A}_{E_0} \to \mathbb{C}^\times$ be the composite of the trace map $E_0 \backslash \mathbb{A}_F \to \mathbb{Q} \backslash \mathbb{A}$ and the usual additive character

$$\mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$$
$$(x_v)_v \mapsto \exp(2\pi\sqrt{-1}(x_\infty - \sum_{v<\infty} \overline{x_v})),$$

where $\overline{x_v}$ is the class of x_v in $\mathbb{Q}_p/\mathbb{Z}_p$.

Let W be a symplectic vector space of dimension $2r$ over E_0. We consider a reductive dual pair $(\mathrm{O}(V), \mathrm{Sp}(W))$ in $\mathrm{Sp}(V \otimes_{E_0} W)$. Then we get a Weil representation ω which is the action of $\mathrm{Mp}_{2r}(\mathbb{A}_{E_0}) \times \mathrm{O}(V(\mathbb{A}_{E_0}))$ on $\mathbf{S}(V(\mathbb{A}_{E_0})^r)$. Let ω_f and $\omega_{\mathbb{A}}$ the action of $\mathrm{Mp}_{2r}(\mathbb{A}_{E_0,f})$ on $\mathbf{S}(V(\mathbb{A}_{E_0,f})^r)$ and $\mathrm{Mp}_{2r}(\mathbb{A}_{E_0})$ on $\mathbf{S}(V(\mathbb{A}_{E_0})^r)$ respectively. Here we put $E_{0_\infty} := E_0 \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{i=1}^d \mathbb{R}$.

Now, we introduce the degenerate Whittaker function. We shall use the same notation as in [**92**]. Let $(V_+, (\ ,\ )_+)$ be a positive definite quadratic space of dimension $n+2$ over $\mathbb{R}$ and ω_+ be an action of $\mathrm{Mp}_{2r}(\mathbb{R})$ to $\mathbf{S}(V_+^r)$. Let $\varphi_+ \in \mathbf{S}(V_+^r)$ be the Gaussian defined by

$$\varphi_+(x) := \exp(-\pi((x_1,x_1)_+ + \cdots + (x_r,x_r)_+)) \quad (x = (x_1,\ldots,x_r) \in V_+^r).$$

Let

$$K_\infty := \left\{ \begin{pmatrix} p & q \\ -q & p \end{pmatrix} \in \mathrm{Sp}_{2r}(\mathbb{R}) \;\middle|\; (p+\sqrt{-1}q)\,{}^t(p-\sqrt{-1}q) = 1_r \right\}.$$

be the maximal compact subgroup in $\mathrm{Sp}_{2r}(\mathbb{R})$ and $\widetilde{K}_\infty$ be the inverse image of K_∞ in $\mathrm{Mp}_{2r}(\mathbb{R})$. Then the function φ_+ is an eigenvector with respect to the Weil representation ω_+:

$$\omega_+(k)\varphi_+ = \det(k)^{(n+2)/2}\varphi_+ \quad (k \in \widetilde{K}_\infty).$$

For a symmetric matrix $T \in \mathrm{Sym}_r(\mathbb{R})$ of size $r \times r$, we take an element $x \in V_+^r$ satisfying $\frac{1}{2}(x,x)_+ = T$. For $g_\infty \in \mathrm{Mp}_{2r}(\mathbb{R})$, we define the degenerate Whittaker function by

$$W_T(g_\infty) := \omega_+(g_\infty)\varphi_+(x).$$

For $T \in \mathrm{Sym}_r(E_0)$ and

$$g'_\infty = (g'_{\infty,1}, \ldots, g'_{\infty,d}) \in \mathrm{Mp}_{2r}(E_{0\infty}) \cong \prod_{i=1}^{d} \mathrm{Mp}_{2r}(\mathbb{R}),$$

we set

$$W_T(g'_\infty) := W_{T^{\sigma_1}}(g'_{\infty,1}) \cdots W_{T^{\sigma_d}}(g'_{\infty,d}).$$

For $g' = (g'_f, g'_\infty) \in \mathrm{Mp}_{2r}(\mathbb{A}_{E_0}) = \mathrm{Mp}_{2r}(\mathbb{A}_{E_0,f}) \times \mathrm{Mp}_{2r}(E_{0\infty})$, we put

$$Z_{\phi_f}(g') := \sum_{x \in G(\mathbb{Q})\backslash V^r} \sum_{g \in G_x(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K_f} \omega_f(g'_f)\phi_f(g^{-1}x) Z(x,g)_{K_f} W_{T(x)}(g'_\infty).$$

By the Fourier expansion, we consider $Z_{\phi_f}(g')$ as a formal power series with coefficients in $\mathrm{CH}^{er}(M_{K_f})_{\mathbb{C}}$. Therefore, the modularity of the generating series $Z_{\phi_f}(\tau)$ is equivalent to the left $\mathrm{Sp}_{2r}(E_0)$-invariance of the function $Z_{\phi_f}(g')$ on $\mathrm{Mp}_{2r}(\mathbb{A}_{E_0})$.

6.3. Proof for the case of $r = 1$

6.3.1. Cohomology of Shimura varieties of orthogonal type. In this subsection, we shall prove if $n \geq 3$, then

$$H^{2e-1}(M_{K_f}, \mathbb{C}) = 0.$$

Recall that we have

$$M_{K_f}(\mathbb{C}) \cong \coprod_{\Gamma} X_\Gamma,$$

where $X_\Gamma = \Gamma\backslash D$ and Γ is a cocompact congruence subgroup of

$$\mathrm{SO}_0(V \otimes_{\mathbb{Q}} \mathbb{R}) \cong \mathrm{SO}_0(n,2)^e \times \mathrm{SO}(n+2)^{d-e}.$$

Here $\mathrm{SO}_0(V \otimes_{\mathbb{Q}} \mathbb{R}), \mathrm{SO}_0(n,2)$ denote the identity components of $\mathrm{SO}(V \otimes_{\mathbb{Q}} \mathbb{R}), \mathrm{SO}(n,2)$, respectively. Therefore, it is enough to show $H^{2e-1}(X_\Gamma, \mathbb{C}) = 0$.

We put $\mathcal{G}' := (\mathrm{Res}_{E_0/\mathbb{Q}} \mathrm{SO}(V))(\mathbb{R})$ and $\mathfrak{g}' := (\mathrm{Lie}\,\mathcal{G}') \otimes_{\mathbb{R}} \mathbb{C}$. We put

$$\mathcal{G}'_i := \begin{cases} \mathrm{SO}_0(n,2) & (1 \leq i \leq e) \\ \mathrm{SO}(n+2) & (e+1 \leq i \leq d), \end{cases}$$

and $\mathfrak{g}'_i := \mathrm{Lie}(G'_i) \otimes_{\mathbb{R}} \mathbb{C}$. We also put

$$\mathcal{K}'_i := \begin{cases} \mathrm{SO}(n) \times \mathrm{SO}(2) & (1 \leq i \leq e) \\ \mathrm{SO}(n+2) & (e+1 \leq i \leq d) \end{cases}$$

and $\mathcal{K}' := \mathcal{K}'_1 \times \cdots \times \mathcal{K}'_d$. By the Matsushima formula, we can write the cohomology of X_Γ as follows:

$$H^{2e-1}(X_\Gamma, \mathbb{C}) \cong \bigoplus_{\pi \in \widehat{\mathcal{G}'(\mathbb{R})}} \mathrm{Int}_\Gamma(\pi) \otimes_{\mathbb{C}} H^{2e-1}(\mathfrak{g}', \mathcal{K}'; \pi).$$

Here $\widehat{\mathcal{G}'(\mathbb{R})}$ is the set of irreducible unitary representations of $\mathcal{G}'(\mathbb{R})$, $\mathrm{Int}_\Gamma(\pi)$ is the one appearing in the decomposition

$$L^2(\Gamma\backslash\mathcal{G}'(\mathbb{R})) \cong \widehat{\bigotimes_{\pi\in\widehat{\mathcal{G}'(\mathbb{R})}}} \mathrm{Int}_\Gamma(\pi)\otimes\pi.$$

Since π is an irreducible unitary representation, π decomposes as $\pi \cong \widehat{\otimes}_{i=1}^d \pi_i$. See [**51**, Theorem 1.2].

Then by the Künneth formula [**18**, Section 1.3], we have

$$H^{2e-1}(\mathfrak{g}',\mathcal{K}';\pi) \cong \bigoplus_{i_1+\cdots+i_d=2e-1} \bigotimes_{k=1}^d H^{i_k}(\mathfrak{g}'_k,\mathcal{K}'_k;\pi_k). \tag{6.3.1}$$

For $e+1\le i\le d$, we have $H^j(\mathfrak{g}'_i,\mathcal{K}'_i;\pi_i)=0$ for any $j\ge 1$ since $\mathrm{SO}(n+2)$ is compact. Therefore (6.3.1) can be written as follows:

$$\begin{aligned} &H^{2e-1}(\mathfrak{g}',\mathcal{K}';\pi) \\ &\cong \left(\bigoplus_{i_1+\cdots+i_e=2e-1}\bigotimes_{k=1}^e H^{i_k}(\mathfrak{g}'_k,\mathcal{K}'_k;\pi_k)\right)\otimes_{\mathbb{C}} \bigotimes_{k=e+1}^d H^0(\mathfrak{g}'_k,\mathcal{K}'_k;\pi_k). \end{aligned} \tag{6.3.2}$$

Lemma 6.3.1. *Assume $n\ge 3$. For $1\le i\le e$, if π_i is non-trivial, then we have*

$$H^j(\mathfrak{g}'_i,\mathcal{K}'_i;\pi_i)=0$$

for $j=0,1$.

Proof. See [**143**, Theorem 8.1] and the Kumaresan vanishing theorem [**98**, Section 3]. □

In the rest of this subsection, we assume $n\ge 3$. Then, by Lemma 6.3.1, we can write (6.3.2) as follows:

$$\begin{aligned} &H^{2e-1}(\mathfrak{g}',\mathcal{K}';\pi) \\ &\cong \left(\bigoplus_{\substack{i_1+\cdots+i_e=2e-1\\ 1\le\exists j\le e,\ \pi_j:\text{trivial}}}\bigotimes_{k=1}^e H^{i_k}(\mathfrak{g}'_k,\mathcal{K}'_k;\pi_k)\right)\otimes_{\mathbb{C}} \bigotimes_{k=e+1}^d H^0(\mathfrak{g}'_k,\mathcal{K}'_k;\pi_k). \end{aligned} \tag{6.3.3}$$

Here we need the following lemma.

Lemma 6.3.2. *Let L be a totally real number field, V a non-degenerate quadratic space of dimension $n+2$ over L, and $\pi\cong\otimes_v\pi_v$ an automorphic representation of $\mathrm{SO}(V)(\mathbb{A}_L)$. If there exists an archimedean place w such that $\mathrm{SO}(V)(L_w)\cong\mathrm{SO}(n,2)$ and the restriction of π_w to the identity component of $\mathrm{SO}(V)(L_w)$ is the trivial representation, then π_v is a character for any places v.*

Proof. See [**50**, Lemma 3.24]. □

The connected Lie group $\mathrm{SO}_0(n,2)$ is semisimple and has no compact factor. Hence π_i is the trivial representation for every $1\le i\le e$. See [**147**, Section 4.3.2, Example 4].

Then (6.3.3) becomes as follows:

$$H^{2e-1}(\mathfrak{g}',\mathcal{K}';\pi) \tag{6.3.4}$$

$$\cong \left(\bigoplus_{i_1+\cdots+i_e=2e-1} \bigotimes_{k=1}^{e} H^{i_k}(\mathfrak{g}'_k, \mathcal{K}'_k; 1) \right) \otimes_{\mathbb{C}} \bigotimes_{k=e+1}^{d} H^0(\mathfrak{g}'_k, \mathcal{K}'_k; \pi_k).$$

Finally by [**11**, Section 5.10], for $1 \leq i \leq e$, we have $H^s(\mathfrak{g}'_i, \mathcal{K}'_i; 1) = 0$ if s is odd. Thus,

$$H^{2e-1}(\mathfrak{g}', \mathcal{K}'; \pi) = 0.$$

Combining the above results, we get the following theorem.

Theorem 6.3.3. *Assume $n \geq 3$. Then we have*

$$H^{2e-1}(M_{K_f}, \mathbb{C}) = 0.$$

Corollary 6.3.4. *Assume $n \geq 3$. Assume moreover that Conjecture 6.1.3 holds for M_{K_f} and $m = e$. Then the cycle map tensored with $\mathbb{C}$*

$$cl^e_{\mathbb{C}} \colon \mathrm{CH}^e(M_{K_f})_{\mathbb{C}} \to H^{2e}(M_{K_f}, \mathbb{C})$$

is injective.

6.3.2. Proof of Theorem 6.1.5 (2) and Theorem 6.1.6 (2). If $n \geq 3$, by Corollary 6.3.4, the assertion follows from the results of Kudla [**95**, Section 5.3] and Rosu-Yott [**131**, Theorem 1.1].

If $n \leq 2$, we take a totally positive quadratic space W of dimension ≥ 3 over F. We embed V into $V \oplus W$. For any K_f-invariant Bruhat-Schwartz functions $\phi_f \in \mathbf{S}(\widehat{V})$ and $\phi'_f \in \mathbf{S}(\widehat{W})$, using the pull-back formula (Proposition 6.2.4), we get

$$i^*(Z_{\phi_f \otimes \phi'_f})(g') = Z_{\phi_f}(g')\theta_{\phi'_f}(g')$$

for any $g' \in \mathrm{Mp}_2(\mathbb{A}_{E_0})$. Since $Z_{\phi_f \otimes \phi'_f}(g')$ and $\theta_{\phi'_f}(g')$ are absolutely convergent and left $\mathrm{SL}_2(E_0)$-invariant, we conclude that $Z_{\phi_f}(g')$ is absolutely convergent and left $\mathrm{SL}_2(E_0)$-invariant.

The proof of Theorem 6.1.5 (2) and Theorem 6.1.6 (2) is complete.

6.4. Proof for the case of $r > 1$

6.4.1. Invariance under the Siegel parabolic subgroup. For $a \in \mathrm{GL}_r(E_0)$ and $u \in \mathrm{Sym}_r(E_0)$, we put $m(a) := \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}$ and $n(u) := \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$. The elements $m(a)$ and $n(u)$ generate the Siegel parabolic subgroup $P(E_0) \subset \mathrm{Sp}_{2r}(E_0)$.

For $g' \in \mathrm{Mp}_{2r}(\mathbb{A}_{E_0})$, its infinity component in $\mathrm{Mp}_{2r}(E_{0\infty}) \cong \prod_{i=1}^d \mathrm{Mp}_{2r}(\mathbb{R})$ is denoted by $g'_\infty = (g'_{\infty,1}, \ldots, g'_{\infty,d})$. For $1 \leq i \leq d$, we consider the Iwasawa decomposition of $g'_{\infty,i}$:

$$g'_{\infty,i} = \begin{pmatrix} 1 & s_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t_i & 0 \\ 0 & {}^t t_i^{-1} \end{pmatrix} k_i \quad (s_i \in \mathrm{Sym}_r(\mathbb{R}), t_i \in \mathrm{GL}_r^+(\mathbb{R}), k_i \in \widetilde{K}_\infty).$$

For $T \in \mathrm{Sym}_r(\mathbb{R})$, the degenerate Whittaker function satisfies the following formula:

$$W_T(g'_{\infty,i}) = |\det(s_i)|^{(n+2)/4} \exp(2\pi\sqrt{-1}(\mathrm{Tr}(\tau_i T))) \det(k_i)^{(n+2)/2}$$

where $\tau_i = s_i + i t_i {}^t t_i$.

By [**93**, Part I, Section 1], $n(u)$ acts as follows:

$$\omega_f(n(u)_f)\phi_f(x) = \psi_f(\mathrm{Tr}(u_f T(x)))\phi_f(x).$$

Thus, we have

$$\begin{aligned}
&\omega_f(n(u)_f g'_f)(\phi_f)(x)Z(x)_{K_f}\prod_{i=1}^{d} W_{T(x)}(n(u)_{\infty,i}g'_{\infty,i})\\
=\ &\psi_f(\mathrm{Tr}(u_f T(x_f)))\omega_f(g'_f)\phi_f(x)Z(x)_{K_f}\psi_\infty(\sum_{i=1}^{d}\mathrm{Tr}(u_{i,\infty}T(x)_{\infty,i}))\prod_{i=1}^{d} W_{T(x)}(g'_{\infty,i})\\
=\ &\psi(\mathrm{Tr}(uT(x)))\omega_f(g'_f)\phi_f(x)Z(x)_{K_f}\prod_{i=1}^{d} W_{T(x)}(g'_{\infty,i})\\
=\ &\omega_f(g'_f)(\phi_f)(x)Z(x)_{K_f}\prod_{i=1}^{d} W_{T(x)}(g'_{\infty,i}).
\end{aligned}$$

Therefore, we have the term-wise invariance under $n(u)$:

$$\omega_f(n(u)_f g'_f)(\phi_f)(x)Z(x)_{K_f}W_{T(x)}(n(u)_\infty g'_\infty) = \omega_f(g'_f)(\phi_f)(x)Z(x)_{K_f}W_{T(x)}(g'_\infty).$$

By the same way, we have

$$\omega_f(m(a)_f g'_f)(\phi_f)(x)Z(x)_{K_f}W_{T(x)}(m(a)_\infty g'_\infty) = \omega_f(g'_f)(\phi_f)(xa)Z(x)_{K_f}W_{T(xa)}(g'_\infty)$$

for any $a \in \mathrm{GL}_r(F)$.

On the other hand, we have $U(x) = U(xa)$, so $Z_{\phi_f}(x) = Z_{\phi_f}(xa)$. Therefore, combining the above calculation and the fact $Z_{\phi_f}(x) = Z_{\phi_f}(xa)$, we conclude that

$$\begin{aligned}
Z_{\phi_f}(\omega_f(m(a))g') &= \sum_{\substack{x\in K_f\backslash \widehat{V}^r\\ \text{admissible}}} \omega_f(g'_f)\phi_f(xa)Z(xa)_{K_f}W_{T(xa)}(g'_\infty)\\
&= \sum_{\substack{x\in K_f\backslash \widehat{V}^r\\ \text{admissible}}} \omega_f(g'_f)\phi_f(x)Z(x)_{K_f}W_{T(x)}(g'_\infty)\\
&= Z_{\phi_f}(g').
\end{aligned}$$

This shows $Z_{\phi_f}(g')$ is invariant under the action of the Siegel parabolic subgroup $P(F)$.

6.4.2. Invariance under w_1. By Proposition 6.2.4, we get the following expression. (For details, see [**151**].)

$$Z_{\phi_f}(\tau) = \sum_{\substack{y\in K\backslash \widehat{V}^{r-1}\\ \text{admissible}}}\ \sum_{x_2\in E_0 y}\ \sum_{\substack{x_1\in K_{f,y}\backslash y^\perp\\ \text{admissible}}} \phi_f(x_1+x_2,y)Z(x_1)_{K_{f,y}}q^{T(x_1+x_2,y)}$$

Thus we have

$$\begin{aligned}
Z_{\phi_f}(g') &= \sum_{\substack{y\in K_f\backslash \widehat{V}^{r-1}\\ \text{admissible}}}\ \sum_{x_2\in E_0 y}\ \sum_{\substack{x_1\in K_{f,y}\backslash y^\perp\\ \text{admissible}}} \omega_f(g'_f)\phi_f(x_1+x_2,y)Z(x_1)_{K_{f,y}}W_T(x_1+x_2,y)(g'_\infty)\\
&= \sum_{\substack{y\in K_f\backslash \widehat{V}^{r-1}\\ \text{admissible}}}\ \sum_{x_2\in E_0 y}\ \sum_{\substack{x_1\in K_{f,y}\backslash y^\perp\\ \text{admissible}}} \omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi_+^d)(x_1+x_2,y)Z(x_1)_{K_{f,y}}.
\end{aligned}$$

Here $\phi^x(x,y)$ is the partial Fourier transformation with respect to the first coordinate.

Now, by Theorem 6.1.5 (2) and Theorem 6.1.6 (2), we have

$$Z_{\phi_f}(w_1 g') = \sum_{\substack{y \in K_f \backslash \widehat{V}^{r-1} \\ \text{admissible}}} \sum_{x_2 \in E_0 y} \sum_{\substack{x_1 \in K_{f,y} \backslash y^{\perp} \\ \text{admissible}}} \omega_{\mathbb{A}}(g')(\phi_f \otimes \varphi_+^d)^{x_2}(x_1 + x_2, y) Z(x_1)_{K_{f,y}}.$$

Here we use the fact that

$$\omega_{\mathbb{A}}(w_1)(\phi_f \otimes \varphi_+^d)(x, y) = (\phi_f \otimes \varphi_+^d)^x(x, y).$$

By the Poisson summation formula, this equals to

$$\sum_{\substack{y \in K_f \backslash \widehat{V}^{r-1} \\ \text{admissible}}} \sum_{x_2 \in E_0 y} \sum_{\substack{x_1 \in K_{f,y} \backslash y^{\perp} \\ \text{admissible}}} \omega_{\mathbb{A}}(g')(\phi_f \otimes \varphi_+^d)(x_1 + x_2, y) Z(x_1)_{K_{f,y}},$$

which coincides with the definition of $Z_{\phi_f}(g')$. Therefore, we get

$$Z_{\phi_f}(w_1 g') = Z_{\phi_f}(g').$$

This shows the function $Z_{\phi_f}(g')$ is invariant under w_1.

The proof of Theorem 6.1.5 and Theorem 6.1.6 is complete.

CHAPTER 7

Modularity of the generating series of special cycles on unitary Shimura varieties

7.1. Introduction

We study the modularity of the generating series of special cycles on unitary Shimura varieties over CM-fields of degree $2d$ associated with a Hermitian form in $n+1$ variables whose signature is $(n,1)$ at e real places and $(n+1,0)$ at the remaining $d-e$ real places for $1 \leq e < d$. For $e = 1$, Liu proved the modularity, and Xia showed the absolute convergence of the generating series. On the other hand, Bruinier constructed regularized theta lifts on orthogonal groups over totally real fields and proved the modularity of special divisors on orthogonal Shimura varieties. By using Bruinier's result, we work on the problem for $e = 1$ and give another proof of Liu's theorem [**103**, Theorem 3.5]. For $e > 1$, we prove that the generating series of special cycles of codimension er in the Chow group is a Hermitian modular form of weight $n+1$ and genus r, assuming the Beilinson-Bloch conjecture for orthogonal Shimura varieties. Our result is a generalization of *Kudla's modularity conjecture*, solved by Liu unconditionally when $e = 1$.

Hirzebruch-Zagier [**67**] observed that the intersection number of special divisors on Hilbert modular surfaces generates a certain weight 2 elliptic modular form. Kudla-Millson generalized this work in [**97**], and they proved that special cycles on orthogonal (resp. unitary) Shimura varieties generate Siegel (resp. Hermitian) modular forms with coefficients in the cohomology group. Yuan-Zhang-Zhang [**151**] and Zhang [**154**] treated this problem in the Chow group in the case of orthogonal Shimura varieties and proved the modularity under a convergence assumption. Bruinier-Raum [**26**] showed the convergence. Kudla [**95**] and the author [**111**] generalized this problem for a certain orthogonal Shimura variety under the Beilinson-Bloch conjecture.

In this chapter, we shall work on the unitary case in the Chow group. Our problem is Conjecture 1.8.1. We give two solutions to this problem (Corollary 7.1.2 and Theorem 7.1.3). First, we prove Conjecture 1.8.1 for $e = 1$ unconditionally by using Bruinier's result [**23**]. On the other hand, for $e = 1$, Liu [**103**] solved Conjecture 1.8.1, i.e., proved the modularity of special cycles on unitary Shimura varieties in the Chow group, assuming the absolute convergence of the generating series. Recently, Xia [**148**] showed the modularity and absolute convergence of the generating series for $e = 1$. Our result in this chapter gives another proof of Liu's result [**103**, Theorem 3.5]. For $e = 1$ and $r = 1$, the modularity of special divisors is proved in Theorem 7.1.1. To treat higher codimensional cycles, we adopt the induction method [**151**]. Second, for $e > 1$, we show Conjecture 1.8.1 under the Beilinson-Bloch conjecture for orthogonal Shimura varieties. We reduce the problem to the orthogonal case ([**95**] and [**111**]), so we also need the Beilinson-Bloch conjecture for orthogonal Shimura varieties. We remark that we do not prove the absolute convergence of the generating series in this chapter.

7.1.1. Main results. For notations, see Subsection 1.2.3. We give two partial solutions to Conjecture 1.8.1 in this chapter.

Theorem 7.1.1 (Theorem 7.3.1). *Assume that $e = 1$ and $r = 1$. Then, $Z^{\mathcal{H}}_{\phi_f}(\tau)$ is a Hermitian modular form for $\mathrm{SU}(1,1)$ of weight $n+1$ under the assumption that the series converges absolutely.*

Theorem 7.1.1 generalizes [**68**, Theorem 10.1]. We can prove a stronger result by induction on r [**151**]. See Corollary 7.1.2. It does not follow immediately from [**68**] or Theorem 7.1.1 that $Z^{\mathcal{H}}_{\phi_f}(\tau)$ is a Hermitian modular form for $\mathrm{U}(1,1)$, i.e., Theorem 7.1.1 shows only the $\mathrm{SU}(1,1)$-modularity of $Z^{\mathcal{H}}_{\phi_f}(\tau)$. However, we can show the $\mathrm{U}(1,1)$-modularity of $Z^{\mathcal{H}}_{\phi_f}(\tau)$ by proving term-wise modularity. This means that we can show the modularity of $Z^{\mathcal{H}}_{\phi_f}(\tau)$ for the parabolic subgroup P_1 and a specific element w_1 defined in section 7.3. On the other hand, P_1 and w_1 generate $\mathrm{U}(1,1)$, and we already know the modularity for $w_1 \in \mathrm{SU}(1,1)$ from Theorem 7.1.1, so the problem reduces to proving the modularity for P_1. For the proof of the modularity for the parabolic subgroup P_1, see [**103**], [**111**], and [**151**]. By combining the above modularity and induction on r, we can prove the modularity of special cycles of a higher codimension.

Corollary 7.1.2 (Corollary 7.3.2). *Assume $e = 1$. Then, $Z^{\mathcal{H}}_{\phi_f}(\tau)$ is a Hilbert-Hermitian modular form for $\mathrm{U}(r,r)$ of weight $n+1$ under the assumption that the series converges absolutely.*

This gives another proof of Theorem 7.1.3 for the $e = 1$ case and [**103**, Theorem 3.5]. This is shown unconditionally differently from Theorem 7.1.3.

Now, we state the theorem for $e > 1$. Recall that $\mathcal{H} := \mathrm{Res}_{E_0/\mathbb{Q}}\,\mathrm{U}(V_E)$ is the unitary group associated with a Hermitian space V_E over a CM field E, and for a Bruhat-Schwartz function $\phi_f \in \mathbf{S}(V_E(\mathbb{A}_f)^r)^{K_f^{\mathcal{H}}}$, our generating series $Z^{\mathcal{H}}_{\phi_f}(\tau)$ is defined as follows with coefficients in $\mathrm{CH}^{er}(M_{K_f^{\mathcal{H}}})_{\mathbb{C}}$ in the variable $\tau = (\tau_1, \ldots, \tau_d) \in (\mathscr{H}_r)^d$:

$$Z^{\mathcal{H}}_{\phi_f}(\tau) := \sum_{x \in \mathcal{H}(\mathbb{Q})\backslash V_E^r} \; \sum_{g \in \mathcal{H}_x(\mathbb{A}_f)\backslash \mathcal{H}(\mathbb{A}_f)/K_f^{\mathcal{H}}} \phi_f(g^{-1}x) Z^{\mathcal{H}}(x,g)_{K_f^{\mathcal{H}}} q^{T(x)}.$$

Our main result in this chapter is as follows.

Theorem 7.1.3 (Theorem 7.4.1). *$Z^{\mathcal{H}}_{\phi_f}(\tau)$ is a Hilbert-Hermitian modular form for $\mathrm{U}(r,r)$ of weight $n+1$ under the Beilinson-Bloch conjecture for $m = e$ with respect to orthogonal Shimura varieties and the assumption that the series converges absolutely for $e > 1$.*

Remark 7.1.4. We assume the Beilinson-Bloch conjecture for $m = e$ for $N_{K_f^{\mathcal{G}}}$ when $2n \geq 3$, i.e., $n > 1$. When $n = 1$, we need to assume the Beilinson-Bloch conjecture for $m = e$ for a larger orthogonal Shimura variety $N'_{K_f^{\mathcal{G}}}$ including $N_{K_f^{\mathcal{G}}}$; see [**111**, Theorem 1.6]. For the precise statement of the Beilinson-Bloch conjecture, see [**111**, Section 1.2].

We can also restate the result using Kudla's modularity conjecture for orthogonal Shimura varieties as follows.

Corollary 7.1.5. *$Z^{\mathcal{H}}_{\phi_f}(\tau)$ is a Hilbert-Hermitian modular form for $\mathrm{U}(r,r)$ of weight $n+1$, assuming the modularity of the generating series of special cycles on orthogonal Shimura varieties for $r = 1$ and absolute convergence of the series $Z^{\mathcal{H}}_{\phi_f}(\tau)$ for $e > 1$.*

We explain in section 7.4.4, why we only assume the modularity for $r = 1$ on orthogonal Shimura varieties.

7.1.2. Outline of the proof of Theorem 7.1.1 and Theorem 7.1.3. As an application of the modularity of special cycles on orthogonal Shimura varieties proved by using regularized theta lifts, we can prove Theorem 7.1.1 and Corollary 7.1.2. This is another proof of [**103**, Theorem 3.5] for the special divisors case. Theorem 7.1.3 is reduced to the orthogonal case, [**95**] and [**111**], so we have to assume the Beilinson-Bloch conjecture for orthogonal Shimura varieties, and this is our solution to Conjecture 1.8.1.

7.1.3. Organization of this chapter. In section 2, we review the modularity of the generating series of special cycles on orthogonal Shimura varieties. In section 3, we prove the modularity for the $e = 1$ case. In section 4, we give the Hermitian modularity of special cycles for $e > 1$ under the Beilinson-Bloch conjecture for orthogonal Shimura varieties.

7.2. Modularity on orthogonal groups

In this section, we shall recall Bruinier's work [**23**]. He constructed regularized theta lifts on orthogonal groups and showed the modularity of special cycles on orthogonal Shimura varieties.

Throughout this section, let $L \subset V_{E_0}$ be an even $\mathscr{O}_{E_0}$-lattice and $L^\vee$ be the $\mathbb{Z}$-dual lattice of L with respect to $\mathrm{Tr}_{E_0/\mathbb{Q}}(\ ,\ )$. Let $\hat{\mathbb{Z}} := \prod_{p<\infty} \mathbb{Z}_p$, and we define $\hat{L} := L \otimes \hat{\mathbb{Z}}$. We have $L^\vee/L \cong \hat{L}^\vee/\hat{L}$, so for $\mu \in L^\vee/L$, let $1_\mu \in \mathbf{S}(V_{E_0}(\mathbb{A}_{E_0,f}))$ be the characteristic function associated with $\mu + \hat{L}$. In the current section, we assume that $r = 1$ and $n > 2$.

7.2.1. Regularized theta lifts on orthogonal groups. We review the results of [**23**]. Let

$$k := (k_1, k_2, \ldots, k_d) = (1-n, 1+n, \ldots, 1+n) \in \mathbb{Z}^d$$

and $s_0 := 1 - k_1 = n$. We call k weight and define the dual weight κ to be

$$\kappa := (2 - k_1, k_2 \ldots, k_d) = (1+n, 1+n, \ldots, 1+n) \in \mathbb{Z}^d.$$

We use Kummer's confluent hypergeometric function

$$M(a, b, z) := \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!},\ (a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}$$

for $a, b, z \in \mathbb{C}$. and Whittaker functions

$$M_{\nu,t}(z) := e^{-z/2} z^{1/2+t} M(1/2 + t - \nu, 1 + 2t, z) \quad (t, \nu \in \mathbb{C}),$$

$$\mathcal{M}_s(v_1) := |v_1|^{-k_1/2} M_{\mathrm{sgn}(v_1)k_1/2, s/2}(|v_1|) e^{-v_1/2} \quad (s \in \mathbb{C}, v_1 \in \mathbb{R}).$$

Now, we shall define the Whittaker forms

$$f_{m,\mu}(\tau, s) := C(m, k, s)\mathcal{M}_s(-4\pi m_1 v_1) \exp(-2\pi\sqrt{-1}\,\mathrm{Tr}(m\overline{\tau}))1_\mu \quad (m_i := \sigma_i(m)),$$

where $\mu \in L^\vee/L \cong \hat{L}^\vee/\hat{L}$ and 1_μ is the characteristic function associated with $\mu + \hat{L}$.

Here, $C(m, k, s)$ is a normalizing factor

$$C(m, k, s) := \frac{(4\pi m_2)^{k_2-1} \ldots (4\pi m_d)^{k_d-1}}{\Gamma(s+1)\Gamma(k_2-1)\ldots\Gamma(k_d-1)}.$$

We define for $\tau \in (\mathscr{H}_1)^d$, the function

$$
\begin{aligned}
f_{m,\mu}(\tau) &:= f_{m,\mu}(\tau, s_0)\\
&= C(m,k,s_0)\Gamma(2-k_1)\Big(1-\frac{\Gamma(1-k_1,4\pi m_1 v_1)}{\Gamma(1-k_1)}\Big)e^{4\pi m_1 v_1}\exp(-2\pi\sqrt{-1}\,\mathrm{Tr}(m\overline{\tau}))1_\mu.
\end{aligned}
$$

Here, for $m \in E_0$, $m >> 0$ means $m_i := \sigma_i(m) > 0$ for all i, and ∂_{E_0} denotes the different ideal of a totally real field E_0. Note that we consider a finite $\mathscr{O}_{E_0}$-module $L^\vee/L$ equipped with a quadratic form $(\ ,\ )/2$ which takes values in $E_0/\partial^{-1}\mathscr{O}_{E_0}$ since we assume that L is even.

Definition 7.2.1. A *Whittaker form of weight* k is a finite linear combination of the functions $f_{m,\mu}(\tau, s)$ for $\mu \in L^\vee/L, m \in (\mu,\mu)/2 + \partial^{-1}\mathscr{O}_{E_0}$ and $m >> 0$. A *harmonic Whittaker form of weight* k is a Whittaker form with $s = s_0$, i.e., a function which has the form

$$\sum_{\mu\in L^\vee/L}\sum_{m>>0} c(m,\mu)f_{m,\mu}(\tau)$$

for $c(m,\mu) \in \mathbb{C}$. Here, the second sum runs $m \in (\mu,\mu)/2 + \partial^{-1}\mathscr{O}_{E_0}$. Let $H_{k,\overline{\rho_L}}$ be the $\mathbb{C}$-vector space consisting of harmonic Whittaker forms of weight k.

Note that in the above definition, the weight k is used in the definition of the normalizing factor $C(m,k,s)$ and $s_0 := 1 - k_1$.

Remark 7.2.2. Here, ρ_L is a lattice model of the Weil representation of the metaplectic group $\mathrm{Mp}_2(\hat{\mathscr{O}}_{E_0})$, and $f_{m,\mu}$ satisfies a certain modularity condition on ρ_L and a certain differential equation. For details, see [**23**, Chapter 4].

Under our assumption on $n > 2$ and $\kappa_j \geq 2$ for all j, there is a surjective map $\xi_k\colon H_{k,\overline{\rho_L}} \to S_{\kappa,\rho_L}$ [**23**, Proposition 4.3]. Here, S_{κ,ρ_L} is the space of Hilbert modular forms of weight κ and type ρ_L. Let $M^!_{k,\overline{\rho_L}}$ be the kernel of this map, and we call elements of this space weakly holomorphic Whittaker forms of weight k.

Hence, there is an exact sequence,

$$0 \to M^!_{k,\overline{\rho_L}} \to H_{k,\overline{\rho_L}} \xrightarrow{\xi_k} S_{\kappa,\rho_L} \to 0.$$

This exact sequence and the following are analogs of classical ones. See Borcherds [**14**]. This pairing is non-degenerate, so a non-degenerate pairing is induced between $H_{k,\overline{\rho_L}}/M^!_{k,\overline{\rho_L}}$ and S_{k,ρ_L} defined by

$$\{g,f\} := (g, \xi_k(f))_{\mathrm{Pet}}$$

for the Petersson inner product on S_{k,ρ_L}. We recall the result [**23**, Proposition 4.5] that provides an explicit formula for the above non-degenerate pairing $\{\ ,\ \}$.

Proposition 7.2.3 ([**23**, Proposition 4.5])**.** *For* $g \in S_{\kappa,\rho_L}$ *and* $f \in H_{k,\overline{\rho_L}}$ *with Fourier expansions*

$$
\begin{aligned}
g &= \sum_{\nu\in L^\vee/L}\sum_{n>>0} b(n,\nu)\exp(2\pi\sqrt{-1}\,\mathrm{Tr}(n\tau))1_\nu,\\
f &= \sum_{\mu\in L^\vee/L}\sum_{m>>0} c(m,\mu)f_{m,\mu}(\tau),
\end{aligned}
$$

we have

$$\{g,f\} = \sum_{\mu\in L^\vee/L}\sum_{m>>0} c(m,\mu)b(m,\mu).$$

We remark that Whittaker forms are analogs of Maass forms. See [**23**, Section 4.1].

For $f = \sum_\mu \sum_m c(m,\mu) f_{m,\mu}(\tau) \in H_{k,\overline{\rho_L}}$, we define

$$Z(f) := \sum_\mu \sum_m c(m,\mu) Z^{\mathcal{G}}(m,\mu)_{K_f^{\mathcal{G}}}.$$

Let $I := \operatorname{Res}_{E_0/\mathbb{Q}} \mathrm{SL}_2$ and χ_V be a quadratic character of $\mathbb{A}_{E_0}^\times / E_0^\times$ associated with V given by

$$\chi_V(x) := (x, (-1)^{\ell(\ell-1)/2} \det(V))_{E_0} \quad (\ell := 2n+2).$$

We review the definition of the Eisenstein series [**23**, Section 6.2]. Let $Q \subset H$ be the parabolic subgroup consisting of upper triangular matrices, and let $s \in \mathbb{C}$. We take a standard section $\Phi \in I(s,\chi) := \operatorname{Ind}_Q^H \chi_V |\cdot|^s$. Now we have the Eisenstein series

$$E(g,s,\Phi) := \sum_{\gamma \in I(E_0) \backslash \mathcal{G}(E_0)} \Phi(\gamma g).$$

$$E(\tau, s, \ell; \Phi_f) := v^{-\ell/2} E(g_\tau, s, \Phi_f \otimes \Phi_\infty^\ell),$$

where $g_\tau \in \mathrm{Mp}_2(\mathbb{R})^d$ satisfies $g_\tau(\sqrt{-1}, \ldots, \sqrt{-1}) = \tau \in \mathscr{H}^d$, and Φ_∞^ℓ is defined in [**23**, Chapter 6]. Let 1_μ be the characteristic function associated with $\mu + \hat{L}$ for $\mu \in L^\vee/L \cong \hat{L}^\vee/\hat{L}$. Here, the Weil representation gives an intertwining operator between the space of Bruhat-Schwartz functions and the space of standard sections at $s = s_0$:

$$\lambda = \lambda \otimes \lambda_f \colon \mathbf{S}(V(\mathbb{A}_{E_0})) \to I(s_0, \chi_V).$$

We obtain a vector-valued Eisenstein series of weight ℓ with respect to ρ_L by taking

$$E_L(\tau, s, \ell) := \sum_{\mu \in L^\vee/L} E(\tau, s, \ell; \lambda_f(1_\mu)) 1_\mu.$$

Recall that $1_\mu \in \mathbf{S}(V_F(\mathbb{A}_{E_0,f}))$ is the characteristic function associated with $\mu + \hat{L}$ for $L^\vee/L \cong \hat{L}^\vee/\hat{L}$. We get the Fourier expansion of the Eisenstein series at ∞:

$$E_L(\tau,\kappa) := E_L(\tau, s_0, \kappa) = 1_0 + \sum_{\mu \in L^\vee/L} \sum_{m>>0} B(m,\mu) \exp(2\pi\sqrt{-1}\operatorname{Tr}(m\tau)) 1_\mu.$$

We define

$$B(f) := \sum_{\mu \in L^\vee/L} \sum_{m>>0} c(m,\mu) B(m,\mu)$$

for a harmonic Whittaker form $f = \sum_\mu \sum_m c(m,\mu) f_{m,\mu}$. Note that $B(f) = \{E_L(\tau,\kappa), f\}$.

The following theorem is the regularized theta lift over totally real fields, proved by Bruinier [**23**, Theorem 1.3].

Theorem 7.2.4 ([23, Theorem 6.8]). *Let $f \in M!_{k,\overline{\rho_L}}$ be a weakly holomorphic Whittaker form of weight k for $\Gamma = \mathrm{SL}_2(\mathscr{O}_{E_0}) \subset I(\mathbb{R}) = \operatorname{Res}_{E_0/\mathbb{Q}} \mathrm{SL}_2(\mathbb{R})$ whose coefficients $c(m,\mu)$ are integral. Then, there exists a meromorphic modular form $\Psi_f(\tau, g)$ for $\mathcal{G}(\mathbb{Q})$ of level $K_f^{\mathcal{G}}$ satisfying*

(1) The weight of Ψ is $-B(f)$,
(2) $\operatorname{div}\Psi = Z(f)$.

7.2.2. Modularity of special divisors on orthogonal groups. Now, we review the modularity of special divisors on orthogonal Shimura varieties. To state the theorem, we need to prepare the generating series for orthogonal Shimura varieties. From [**23**], recall that for x_0 taken in Remark 1.2.1 and for a totally real element $m = \langle x_0, x_0\rangle/2 >> 0$ in E_0, we define

$$\begin{aligned}
Z^{\mathcal{G}}(m,\phi_f)_{K_f^{\mathcal{G}}} &:= \sum_{h\in \mathcal{G}_{x_0}\backslash \mathcal{G}(\mathbb{A}_{E_0,f})/K_f^{\mathcal{G}}} \phi_f(h^{-1}x_0)Z^{\mathcal{G}}(x_0,h),\\
A^0(\tau) &:= \sum_{\mu\in L^\vee/L} -c_1(\mathscr{L})1_\mu + \sum_{\mu\in L^\vee/L}\sum_{m>>0}(Z^{\mathcal{G}}(m,1_\mu)_{K_f^{\mathcal{G}}} + B(m,\mu)c_1(\mathscr{L}))q^m 1_\mu,\\
A(\tau,\phi_f) &:= -c_1(\mathscr{L}) + \sum_{m>>0} Z^{\mathcal{G}}(m,\phi_f)_{K_f^{\mathcal{G}}}q^m,\\
A(\tau) &:= \sum_\mu A(\tau,1_\mu)1_\mu = \sum_{\mu\in L^\vee/L} -c_1(\mathscr{L})1_\mu + \sum_{\mu\in L^\vee/L}\sum_{m>>0} Z^{\mathcal{G}}(m,1_\mu)_{K_f^{\mathcal{G}}}q^m 1_\mu.
\end{aligned}$$

Note that since we treat the $r=1$ case, x_0 is an element of V_{E_0}, so that the notion "m is totally real" makes sense and corresponds to $\beta \geq 0$ in Remark 1.2.1.

We want to show the modularity of $Z_{\phi_f}(\tau)$, but first, we will prove the modularity of $A^0(\tau)$. See Remark 7.2.6. We remark that $A(\tau,\phi_f) = Z^{\mathcal{G}}_{\phi_f}(\tau)$. The following theorem was proved by Bruinier [**23**].

Theorem 7.2.5 ([**23**, Theorem 7.1, Proposition 7.3]). *For any $n > 0$,*

$$A^0(\tau) \in S_{\kappa,\rho_L} \otimes \mathrm{CH}^1(N_{K_f^{\mathcal{G}}}).$$

Remark 7.2.6. We know

$$A^0(\tau) = A(\tau) + c_1(\mathscr{L})E_L(\tau,\kappa)$$

by [**23**, Remark 6.5]. Hence, combining with Theorem 7.2.5, we also get

$$A(\tau) \in S_{\kappa,\rho_L} \otimes \mathrm{CH}^1(N_{K_f^{\mathcal{G}}}).$$

7.3. Modularity of special cycles on unitary groups for $e = 1$ case

7.3.1. Divisors case.

Theorem 7.3.1. *Assume that $e=1$ and $r=1$. Then, $Z^{\mathcal{H}}_{\phi_f}(\tau)$ is a Hermitian modular form for $\mathrm{SU}(1,1)$ of weight $n+1$ under the assumption that the series converges absolutely.*

Proof. First, we want to show the modularity of $Z^{\mathcal{H}}_{\phi_f}(\tau)$. Now, ϕ_f is a locally constant, compactly supported function, so we can factorize this as $\phi_f = \sum_{\mu\in L^\vee/L} e_\mu 1_\mu$ for some $e_\mu \in \mathbb{C}$ and $\mu \in L^\vee/L$. Recall that

$$S_L := \bigoplus_{\mu\in L^\vee/L} \mathbb{C}1_\mu \subset \mathbf{S}(V(\mathbb{A}_{E_0,f})),$$

so that we define the map

$$\begin{aligned}
\delta : S_L &\to \mathbb{C}\\
\sum_{\mu\in L^\vee/L} c_\mu 1_\mu &\mapsto \sum_{\mu\in L^\vee/L} c_\mu e_\mu.
\end{aligned}$$

Then we have

$$\delta\colon S_{\kappa,\rho_L} \otimes \mathrm{CH}^1(N_{K_f^{\mathcal{G}}})_{\mathbb{C}}[[q]] \subset S_L \otimes \mathrm{CH}^1(N_{K_f^{\mathcal{G}}})_{\mathbb{C}}[[q]] \to \mathrm{CH}^1(N_{K_f^{\mathcal{G}}})_{\mathbb{C}}[[q]]$$
$$\sum_{\mu \in L^\vee/L} \sum_m b(m,\mu) 1_\mu \otimes Z_{m,\mu} q^m \mapsto \sum_{\mu \in L^\vee/L} \sum_m b(m,\mu) e_\mu Z_{m,\mu} q^m,$$

where $\sum_{\mu\in L^\vee/L}\sum_m b(m,\mu)1_\mu q^m \in S_{\kappa,\rho_L}$ and $Z_{m.\mu} \in \mathrm{CH}^1(N_{K_f^{\mathcal{G}}})_{\mathbb{C}}$. Note that we consider these two spaces as formally defined, not assuming absolute convergence. Then, $\delta(A(\tau)) = Z^{\mathcal{G}}_{\phi_f}(\tau)$ because from the definition of the generating series and Remark 1.2.1,

$$Z^H_{\phi_f}(\tau) = \sum_{m>>0} Z^{\mathcal{G}}(m,\phi_f)_{K_f^{\mathcal{G}}} q^m,$$

where $q^m := \exp(2\pi\sqrt{-1}\,\mathrm{Tr}(m\tau))$ Hence, this is formally modular in the sense of Definition 1.2.2 for Theorem 7.2.5 and Remark 7.2.6. See also [**23**, Section 2.3].

On the other hand, [**103**, Corollary 3.4], we have $\iota^\star Z^{\mathcal{G}}_{\phi_f}(\tau) = Z^{\mathcal{H}}_{\phi_f}(\tau)$. Therefore, by the modularity of $Z^{\mathcal{G}}_{\phi_f}(\tau)$, the generating series $Z^{\mathcal{H}}_{\phi_f}(\tau)$ is a Hermitian modular form for $\mathrm{SU}(1,1)$ under the assumption that the series converges absolutely. Since the weight of $Z^H_{\phi_f}(\tau)$ is $n+1$, this finishes the proof. □

This gives proof of Theorem 7.1.1. Note that to prove the modularity of $Z^{\mathcal{H}}_{\phi_f}(\tau)$ for $n > 1$, we use the perfect pairing presented in Proposition 7.2.3. For $n = 1$, we use an embedding trick. For more details, see [**23**] or [**111**].

7.3.2. General r case. To show the Hermitian modularity, we reduce the problem to the generators of the associated unitary group. Now, the indefinite unitary group $\mathrm{U}(r,r)$ is generated by the parabolic subgroup $P_r(E_0) = M_r(E_0)N_r(E_0)$ and $w_{r,r-1}$, where

$$M_r(E_0) := \{m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t\bar{a}^{-1} \end{pmatrix} \mid a \in \mathrm{GL}_r(E)\}$$

$$N_r(E_0) := \{n(u) = \begin{pmatrix} 1_r & u \\ 0 & 1_r \end{pmatrix} \mid u \in \mathrm{Her}_r(E)\}$$

$$w_{r,r-1} := \begin{pmatrix} 1_{r-1} & 0 & 0_{r-1} & 0 \\ 0 & 0 & 0 & 1 \\ 0_{r-1} & 0 & 1_{r-1} & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

See [**103**, Proof of Theorem 3.5]. We put $w_1 := w_{1,0}$. By induction on r, we get the following result.

Corollary 7.3.2. *Assume $e = 1$. Then $Z^{\mathcal{H}}_{\phi_f}(\tau)$ is a Hilbert-Hermitian modular form for $\mathrm{U}(r,r)$ of weight $n+1$ under the assumption that the series converges absolutely.*

Proof. To prove that the generating series $Z^{\mathcal{H}}_{\phi_f}(\tau)$ is a Hermitian modular form for $r = 1$, we already know the modularity for $\mathrm{SU}(1,1)$ from Theorem 7.3.1. Therefore, in particular, we know the modularity for the element $w_1 \in \mathrm{SU}(1,1)$. Hence, it suffices to prove the modularity for the parabolic subgroup $P_1 \subset \mathrm{U}(1,1)$ because $\mathrm{U}(1,1)$ is generated by P_1 and $w_1 = w_{1,0}$. We can prove the invariance under P_1 in the same way as [**103**] or [**111**]. This finishes the proof of the corollary for the $r = 1$ case. For $r > 1$, we use induction on r. More specifically, for any r, we can prove the modularity for P_r, i.e.,

$$\omega_f(n(u)_f g'_f)(\phi_f \otimes \varphi_+^d)(x)Z^{\mathcal{H}}(x)_{K_f^{\mathcal{H}}} = \omega_f(g'_f)(\phi_f \otimes \varphi_+^d)(x)Z^{\mathcal{H}}(x)_{K_f^{\mathcal{H}}}$$
$$\omega_f(m(a)_f g'_f)(\phi_f \otimes \varphi_+^d)(x)Z^{\mathcal{H}}(x)_{K_f^{\mathcal{H}}} = \omega_f(g'_f)(\phi_f \otimes \varphi_+^d)(x)Z^{\mathcal{H}}(x)_{K_f^{\mathcal{H}}}$$

hold for any $u \in \mathrm{Her}_r(F)$ and $a \in \mathrm{GL}_r(F)$. This will also be done in more detail in section 7.4.2. By using the modularity for w_1 in the $r = 1$ case, we can prove the modularity for $w_{r,r-1}$ when $r > 1$ in the same way as in section 7.4.3, and we already know the w_1-modularity. We will show the induction step in section 7.4.3. $\square$

This shows the modularity of special cycles on a unitary Shimura variety for $e = 1$ (Theorem 7.1.2) and gives another proof of Liu's theorem [**103**, Theorem 3.5].

7.4. General e case

7.4.1. Weil representations. Let $\psi\colon E\backslash\mathbb{A}_E \to \mathbb{C}^\times$ be the composite of the trace map $E\backslash\mathbb{A}_E \to \mathbb{Q}\backslash\mathbb{A}$ and the usual additive character

$$\begin{aligned}&\mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times\\ &(x_v)_v \mapsto \exp(2\pi\sqrt{-1}(x_\infty - \sum_{v<\infty} \overline{x_v})),\end{aligned}$$

where $\overline{x_v}$ is the class of x_v in $\mathbb{Q}_p/\mathbb{Z}_p$.

Let $(W, (\ ,\ ))$ be a Hermitian space of dimension $2r$ over E whose signature is (r, r) so that $\mathrm{U}(W) = \mathrm{U}(r, r)$. Then, we get a symplectic vector space $\mathbb{W} := \mathrm{Res}_{E/E_0}(V_E \otimes_E W)$ with the skew-symmetric form $\mathrm{Tr}_{E/E_0}(\langle\ ,\ \rangle \otimes (\ ,\ ))$. Let $\mathrm{Sp}(\mathbb{W})$ be the symplectic group and $\mathrm{Mp}(\mathbb{W})$ be its metaplectic $\mathbb{C}^\times$ covering group. Then, we get the Weil representation ω_f and $\omega_{\mathbb{A}}$, the action of $\mathrm{Mp}(\mathbb{W})(\mathbb{A}_f)$ to $\mathbf{S}(V(\mathbb{A}_{E_0,f})^r)$ and $\mathrm{Mp}(\mathbb{W})(\mathbb{A})$ to $\mathbf{S}(V(\mathbb{A}_{E_0})^r)$.

Now, we state the second solution to Conjecture 1.8.1.

Theorem 7.4.1. *Assuming absolute convergence for* $e > 1$, $Z^{\mathcal{H}}_{\phi_f}(\tau)$ *is a Hilbert-Hermitian modular form for* $\mathrm{U}(r, r)$ *of weight* $n + 1$ *under the Beilinson-Bloch conjecture for orthogonal Shimura varieties for* $m = e$ *and the assumption that the series in the orthogonal case converges absolutely.*

We reduce Theorem 7.4.1 to the orthogonal case, so we have to assume the Beilinson-Bloch conjecture for orthogonal Shimura varieties. The strategy is as follows. For the general e case, we can prove the modularity for P_r for any r by direct calculation. We can also show the modularity for $w_{r,r-1}$ when $r > 1$, assuming the modularity for $w_1 = w_{1,0}$ in the $r = 1$ case. Hence, the problem is the modularity for w_1 for $r = 1$ and general e. We treat this problem by embedding unitary Shimura varieties into orthogonal varieties, studied in [**68**]. In the orthogonal cases, the modularity of the generating series is proved by [**95**] or [**111**] under the Beilinson-Bloch conjecture. We remark that when $e = 1$, the modularity for w_1 is solved by Corollary 7.3.2, followed by the modularity for $\mathrm{SU}(1, 1)$ using the regularized theta lifts. For the precise statement of the Beilinson-Bloch conjecture, see [**111**, Section 1.2].

From [**151**], we get the following expression for the generating series for the unitary group $\mathcal{H}$.

$$Z^{\mathcal{H}}_{\phi_f}(\tau) = \sum_{\substack{x \in K^{\mathcal{H}}_f \backslash \widehat{V}^{r-1}_E \\ \text{admissible}}} \sum_{y_2 \in Ex} \sum_{\substack{y_1 \in K^{\mathcal{H}}_{f,x} \backslash x^{\perp} \\ \text{admissible}}} \phi_f(x, y_i + y_2) Z^G(y_1)_{K_{f,x}} q^{T(x,y_1+y_2)},$$

where $K^{\mathcal{H}}_{f,x}$ is the stabilizer of x and let $\widehat{V}_E := V_E \otimes \mathbb{A}_f$, $q^T(x)$. Here, for the notion "admissible" and the definition of the special cycles $Z^{\mathcal{H}}(x)_{K_f}$, see [**103**, Lemma 3.1], [**111**, Lemma 2.1], or [**151**, Lemma 2.1]. Let $\varphi_+(x) = \exp(-\pi \operatorname{Tr} T(x))$ be the Gaussian. We extend the definition of $Z^{\mathcal{H}}_{\phi_f}(\tau)$ for $\tau \in (\mathscr{H}_r)^d$ to $Z^{\mathcal{H}}_{\phi_f}(g')$ for $g' \in \mathrm{U}(r,r)(\mathbb{A}_{E_0})$ defined by

$$\begin{aligned} Z^{\mathcal{H}}_{\phi_f}(g') &:= \sum_{x \in \mathcal{H}(\mathbb{Q}) \backslash \widehat{V}^r_E} \sum_{g \in \mathcal{H}_x(\mathbb{A}_f) \backslash \mathcal{H}(\mathbb{A}_f)/K^{\mathcal{H}}_f} \omega_{\mathbb{A}}(g')(\phi_f \otimes \varphi^d_+)(g^{-1}x) Z^{\mathcal{H}}(x,g)_{K^{\mathcal{H}}_f} \\ &= \sum_{\substack{x \in K^{\mathcal{H}}_f \backslash \widehat{V}^{r-1}_E \\ \text{admissible}}} \sum_{y_2 \in Ex} \sum_{\substack{y_1 \in K^{\mathcal{H}}_{f,x} \backslash x^{\perp} \\ \text{admissible}}} \omega_{\mathbb{A}}(g')(\phi_f \otimes \varphi^d_+)(x, y_1 + y_2) Z^{\mathcal{H}}(y_1)_{K^{\mathcal{H}}_{f,x}}. \end{aligned}$$

Remark 7.4.2. The modularity of the generating series $Z^{\mathcal{H}}_{\phi_f}(\tau)$ is equivalent to the left $\mathrm{U}(r,r)(E_0)$-invariance of the function $Z^{\mathcal{H}}_{\phi_f}(g')$ on $\mathrm{U}(r,r)(\mathbb{A})$.

Hence, in the following, we show the left $\mathrm{U}(r,r)$-invariance of $Z^{\mathcal{H}}_{\phi_f}(g')$. First, we show the P_r-invariance of $Z^{\mathcal{H}}_{\phi_f}(g')$ for any r. Second, for $r > 1$, we show the $w_{r,r-1}$-invariance of $Z^{\mathcal{H}}_{\phi_f}(g')$, assuming w_1-invariance for the $r = 1$ case. Finally, we show that $Z^{\mathcal{H}}_{\phi_f}(g')$ is w_1-invariant for the $r = 1$ case.

7.4.2. Invariance under the parabolic subgroup P_r. The elements $m(a)$ and $n(u)$ generate the parabolic subgroup $P_r(E_0) \subset \mathrm{U}(r,r)(E_0)$.

In the same way [**103**, Theorem 3.5 (1)] or [**111**, Section 4.1], we can show the following invariance under $n(u)_f$ and $m(a)_f$:

$$\begin{aligned} \omega_f(n(u)_f g'_f)(\phi_f \otimes \varphi^d_+)(x) Z^{\mathcal{H}}(x)_{K^{\mathcal{H}}_f} &= \omega_f(g'_f)(\phi_f \otimes \varphi^d_+)(x) Z^{\mathcal{H}}(x)_{K^{\mathcal{H}}_f} \\ \omega_f(m(a)_f g'_f)(\phi_f \otimes \varphi^d_+)(x) Z^{\mathcal{H}}(x)_{K^{\mathcal{H}}_f} &= \omega_f(g'_f)(\phi_f \otimes \varphi^d_+)(xa) Z^{\mathcal{H}}(x)_{K^{\mathcal{H}}_f} \end{aligned}$$

for any $u \in \mathrm{Her}_r(E_0)$ and $a \in \mathrm{GL}_r(E_0)$. The first equation shows the $n(u)$-invariance of $Z^{\mathcal{H}}_{\phi_f}(g')$. We shall prove that $Z^{\mathcal{H}}_{\phi_f}(g')$ is $m(a)$-invariant as follows. We have $U(x) = U(xa)$, so $Z^{\mathcal{H}}_{\phi_f}(x) = Z^{\mathcal{H}}_{\phi_f}(xa)$. Therefore, combining the above calculation and the fact that $Z^{\mathcal{H}}_{\phi_f}(x) = Z^{\mathcal{H}}_{\phi_f}(xa)$, we conclude that

$$\begin{aligned} Z^G_{\phi_f}(\omega_f(m(a))g') &= \sum_{\substack{x \in K^{\mathcal{H}}_f \backslash \widehat{V}^r_E \\ \text{admissible}}} \omega_f(g'_f)(\phi_f \otimes \varphi^d_+)(xa) Z^{\mathcal{H}}(xa)_{K^{\mathcal{H}}_f} \\ &= \sum_{\substack{x \in K^{\mathcal{H}}_f \backslash \widehat{V}^r_E \\ \text{admissible}}} \omega_f(g'_f)(\phi_f \otimes \varphi^d_+)(x) Z^{\mathcal{H}}(x)_{K^{\mathcal{H}}_f} \\ &= Z^{\mathcal{H}}_{\phi_f}(g'). \end{aligned}$$

This shows that $Z^{\mathcal{H}}_{\phi_f}(g')$ is invariant under the action of the parabolic subgroup P_r.

7.4.3. Invariance under $w_{r,r-1}$ for $r > 1$. For the following discussion, we use an induction method used in [**103**, Proof of Theorem 3.5] and [**151**, Section 4.2]. Recall that

$$Z^{\mathcal{H}}_{\phi_f}(g') = \sum_{\substack{x\in K^{\mathcal{H}}_f\backslash \widehat{V}^{r-1}_E \\ \text{admissible}}} \sum_{y_2\in Ex} \sum_{\substack{y_1\in K^{\mathcal{H}}_{f,x}\backslash x^{\perp} \\ \text{admissible}}} \omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi^d_+)(x, y_1+y_2) Z^{\mathcal{H}}(y_1)_{K^{\mathcal{H}}_{f,x}}.$$

Hence,

$$Z^{\mathcal{H}}_{\phi_f}(w_{r,r-1}g') = \sum_{\substack{x\in K^{\mathcal{H}}_f\backslash \widehat{V}^{r-1}_E \\ \text{admissible}}} \sum_{y_2\in Ex} \sum_{\substack{y_1\in K^{\mathcal{H}}_{f,x}\backslash x^{\perp} \\ \text{admissible}}} \omega_{\mathbb{A}}(w_{r,r-1})(\omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi^d_+))(x, y_1+y_2) Z^{\mathcal{H}}(y_1)_{K^{\mathcal{H}}_{f,x}}.$$

Now, from the definition of the Weil representation, we have

$$\omega_{\mathbb{A}}(w_{r,r-1})(\phi_f\otimes\varphi^d_+)(x,y) = (\phi_f\otimes\varphi^d_+)^y(x,y),$$

where $\phi^y(x,y)$ is the partial Fourier transformation with respect to the second coordinate. Applying this,

$$Z^{\mathcal{H}}_{\phi_f}(w_{r,r-1}g') = \sum_{\substack{x\in K^{\mathcal{H}}_f\backslash \widehat{V}^{r-1}_E \\ \text{admissible}}} \sum_{y_2\in Ex} \sum_{\substack{y_1\in K^{\mathcal{H}}_{f,x}\backslash x^{\perp} \\ \text{admissible}}} (\omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi^d_+))^{y_1,y_2}(x, y_1+y_2) Z^{\mathcal{H}}(y_1)_{K^{\mathcal{H}}_{f,x}}.$$

For fixed x, applying the $r = 1$ case (modularity of the generating series constructed by special divisors) to the special divisors $Z^{\mathcal{H}}(y_1)_{K^{\mathcal{H}}_{f,x}}$, we have

$$\begin{aligned}
&\sum_{\substack{y_1\in K^{\mathcal{H}}_{f,x}\backslash x^{\perp} \\ \text{admissible}}} \omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi^d_+)^{y_1,y_2}(x, y_1+y_2) Z^{\mathcal{H}}(y_1)_{K^{\mathcal{H}}_{f,x}} \\
= &\sum_{\substack{y_1\in K^{\mathcal{H}}_{f,x}\backslash x^{\perp} \\ \text{admissible}}} \omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi^d_+)^{y_2}(x, y_1+y_2) Z^{\mathcal{H}}(y_1)_{K^{\mathcal{H}}_{f,x}},
\end{aligned}$$

as a function of y_2. Note that $w_{1,0} = w_1$, and here we can use the w_1-modularity for the $r = 1$ case. Thus,

$$Z^{\mathcal{H}}_{\phi_f}(w_{r,r-1}g') = \sum_{\substack{x\in K^{\mathcal{H}}_f\backslash \widehat{V}^{r-1}_E \\ \text{admissible}}} \sum_{y_2\in Ex} \sum_{\substack{y_1\in K^{\mathcal{H}}_{f,x}\backslash x^{\perp} \\ \text{admissible}}} \omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi^d_+)^{y_2}(x, y_1+y_2) Z^{\mathcal{H}}(y_1)_{K^{\mathcal{H}}_{f,x}}.$$

Here, for fixed x and y_2, by the Poisson summation formula for the function $\omega_{\mathbb{A}}(g')(\phi_f \otimes \varphi^d_+)(x, y_1 + y_2)$ on $y_2 \in Ex \subset \mathbb{A}x$, we have

$$\begin{aligned}
&\sum_{y_2\in Ex} \omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi^d_+)^{y_2}(x, y_1+y_2) Z^{\mathcal{H}}(y_1)_{K^{\mathcal{H}}_{f,x}} \\
= &\sum_{y_2\in Ex} \omega_{\mathbb{A}}(g')(\phi_f\otimes\varphi^d_+)(x, y_1+y_2) Z^{\mathcal{H}}(y_1)_{K^{\mathcal{H}}_{f,x}}.
\end{aligned}$$

This leads to

$$Z_{\phi_f}^{\mathcal{H}}(w_{r,r-1}g') = \sum_{\substack{x \in K_f^{\mathcal{H}} \backslash \widehat{V}_E^{r-1} \\ \text{admissible}}} \sum_{y_2 \in Ex} \sum_{\substack{y_1 \in K_{f,x}^{\mathcal{H}} \backslash x^{\perp} \\ \text{admissible}}} \omega_{\mathbb{A}}(g')(\phi_f \otimes \varphi_+^d)(x, y_1 + y_2) Z^{\mathcal{H}}(y_1)_{K_{f,x}^{\mathcal{H}}},$$

which coincides with the definition of $Z_{\phi_f}^{\mathcal{H}}(g')$. Therefore, we get

$$Z_{\phi_f}^{\mathcal{H}}(w_{r,r-1}g') = Z_{\phi_f}^{\mathcal{H}}(g').$$

This shows that the function $Z_{\phi_f}^{\mathcal{H}}(g')$ is invariant under the action of the element $w_{r,r-1}$.

7.4.4. Invariance under w_1 for $r = 1$. We use Liu's proof [**103**, Theorem 3.5]. Now, $\mathrm{U}(1) \times \mathrm{U}(1)$ is the maximal compact subgroup of $\mathrm{U}(1,1)$, and $\mathrm{SL}_2(\mathbb{A}_{E_0,f})(\mathrm{U}(1) \times \mathrm{U}(1))(\mathbb{A}_{F,f}) = \mathrm{U}(1,1)(\mathbb{A}_{E_0,f})$. Therefore, we reduce the problem to proving that $Z_{\phi_f}^{\mathcal{H}}(w_1 g') = Z_{\phi_f}^{\mathcal{H}}(g')$ for all $g' \in \mathrm{SL}_2(\mathbb{A}_{E_0})$. By [**103**, Corollary 3.4] and the proof of [**103**, Lemma 3.6], it suffices to prove $Z_{\phi_f}^{\mathcal{G}}(w_1 g') = Z_{\phi_f}^{\mathcal{G}}(g')$. However, this follows from [**95**] or [**111**] under the Beilinson-Bloch conjecture for orthogonal Shimura varieties. This finishes the proof of Theorem 7.4.1.

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