Limit theorems of persistence diagrams for random cubical filtrations

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Acknowledgment

First of all, I would like to show my greatest appreciation to my supervisor Professor Yasuaki Hiraoka from the master courses to the present for seminars, discussions, and counseling sessions. When I started studying mathematics eight years ago, I didn't have any knowledge even essential terms in random topology, and he taught me the basics of this area. Without his persistent help and guidance, this thesis would not have been possible.

I would also like to acknowledge with appreciation Associate Professor Kenkichi Tsunoda and Dr. Shu Kanazawa for their suggestion and comment. They repeatedly give me the opportunity for discussion and the various knowledge of large deviation theory and random topology. Without their support, this thesis would not have been accomplished.

I would like to express my gratitude to Dr. Killian Meehan, Chenguang Xu, and Enhao Liu for their valuable comments and suggestions for English writting.

I would also like to thank all the current and past members of the Hiraoka laboratory at Kyushu University, Tohoku University, and Kyoto University for their kindness, support, and encouragement.

Finally, I wish to thank my family for their support and warm encouragement throughout my study.

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Chapter 1

Introduction

1.1 Background

In the last few decades, the growth of the power of computer processing has given us a new point of view on big data. Topological data analysis is a research field based on topological properties of data. In this thesis, we deal with *cubical sets*, inspired by imaging science and composed of various dimensional cubes such as lattice points, line segments, squares, and cubes in a Euclidean space. In digital image analysis, a *cubical set* is utilized for the representation of a digital image data for obtaining information about shapes. Especially, the *cubical homology* tells us the information about holes such as loops and cavities, and acts as a useful descriptor of topological features in digital images. (The definition of cubical set and cubical homology are written in Section 2.1.) From the practical view point, however, digital images usually contain measurement or quantization noise, and hence it is important to estimate the effects of randomness on cubical homology. Therefore there has been a growing interest in the study of *random cubical sets*, cubical sets with randomness. Following these interests, Hiraoka–Tsunoda [19] proved the strong law of large numbers for the Betti numbers of a class of random cubical sets. See also [9, 17, 23, 33] for other types of studies in this fields.

On the other hand, *persistent homology* [10, 35] is drawn much attention in the rapidly emerging field of topological data analysis. Comparing to (cubical) homology, the theory of persistent homology tells us not only the number of holes in a given data but also the robustness of those holes. Let us take a grayscale image for example. Since a grayscale image data can be regarded as a function representing the intensity of light on each elementary cube, we can construct a *cubical filtration*, an increasing family of cubical sets by considering the sublevel sets of the function at varying thresholds. Then, the persistent homology can extract the information about the appearing and vanishing times of holes of the cubical filtration. The above appearance time and vanishing time are called the birth time and the death time, respectively. The descriptor plotting all birth-death pairs, tuples of the birth time and the death time, into a 2-dimensional parameter space is called a *persistence diagram*. It shows the topological features at different scales in the grayscale image data. For instance, a birth-death pair whose position is far away from the diagonal line of persistence diagram is robust to a perturbation of data [3], and usually considered as a characterization of data. As an example, by using this property, the paper [21] was successful in the field of material science by fixing the coordinating number of a material in consideration of cubical complexes and recovering the nerve theorem which is not strictly valid when cubical complexes are considered. Further, there also exists the profitable result [26] in medical science. As mentioned above, however, grayscale image data usually contain some random noise, therefore it is important to examine the effect of randomness on persistence diagrams.

In this thesis, we focus on limit thoerems for random cubical filtration as evaluations of influence of randomness, those are the law of large numbers, the central limit theorem, and the large deviation principle. The law of large numbers and the central limit theorem tell us what is the typical value (or shape) of objects and how differences from typical one are distributed, respectively. While those two theorems concerning to the typical cases, the large deviation principle focuses on the rare value (or shape) of the objects. The details will be presented later, however for now large deviation principle shows us that the probability of rare event can be estimated by the exponential of a special function. The entire research field studying the asymptotic behaviors of the topological properties for random geometrical objects, including our random model, is called random topology, and those three are also major theorems in this field. For instance, Yogeshwaran, Subag, and Adler [34] study the law of large numbers and the central limit theorem for Betti numbers of random geometric complexes and show the concentration inequality of Betti numbers insted of the large deviation principle.

In addition, in the study of a law of large numbers, there are various approaches for proving it. For instance, Goel, Trinh, and Tsunoda [14] established a proof by taking a suitable partition of manifold, incorporating with the spatial independence property of Poisson point process and the finite additivity of Betti numbers, and converting the manifold setting to the Euclidean setting for random Cěch complexes. Another approach is using the local weak convergence of simplicial complexes to show the law of large numbers of Betti numbers for random simplical complexes or the empirical spectral distributions of their Laplacians proved by Kanazawa [25]. Similarly, in the study of a central limit thorem, there are also many methods such as using the extension of the central limit thoerm for Gibbsian random fields by Reedy, Vadlamani, and Yogeshwaran [29] or applying the central limit thoerem of some class of functionals on Poisson point process [18]. Thus a lot of studies have been conducted on the law of large numbers and the central limit theorem in random topology. In particular, the seminal work of Hiraoka, Shirai, and Trinh [18] is highly relevant to our study. We will apply their method of lifting the strong law of large number of persistent Betti numbers to persistence diagrams for random geometric complexes in our models.

We also remark on a few study of large deviation principles in random topology. As a relatively classic result, Chatterjee and Varadhan [1] obtained the large deviation principle for Erdös–Rényi random graph. Recently, Samorodnitsky and Owada [31] studied an upper tail large deviation estimates of Betti numbers for a random simplicial complex in the critical dimension. Moreover, Hirsch and Owada [20] proved the large deviation principle of the first persistent Betti numbers of a filtration of random geometric complex or alpha complex those are built over a homogeneous point process in 2-dimensional Euclidean space as an application of the large deviation principle for the counting measure with the configuration of a homogeneous Poisson point process in Euclidean space.

1.2 Contributions

In this section, we summarize our main results in this thesis. The precise definitions of some terminologies and symbols will be given in the later chapters. Let $d \in \mathbb{N}$ be the dimension of the state space $\mathbb{R}^d$ and fixed. For each $n \in \mathbb{N}$, we set a rectangular region

$$\Lambda^n = [-n, n]^d \subset \mathbb{R}^d.$$

Given $n \in \mathbb{N}$ and a random cubical filtration $\mathbb{X} = \{X(t)\}_{t \geq 0}$ in $\mathbb{R}^d$, we define a restricted random cubical filtration $\mathbb{X}^n = \{X^n(t)\}_{t \geq 0}$ by

$$X^n(t) \coloneqq X(t) \cap \Lambda^n$$

for every $t \geq 0$. Note that $\mathbb{X}^n = \{X^n(t)\}_{t \geq 0}$ is a random bounded cubical filtration (See Subsection 2.1.2 for details). In what follows, $|A|$ denotes the d-dimensional Lebesgue measure of a Borel subset $A \subset \mathbb{R}^d$. In particular, $|\Lambda^n| = (2n)^d$. Let $\mathcal{K}^d$ denote the set of all elementary cubes in $\mathbb{R}^d$. For a cubical filtration $\mathbb{X}$ in $\mathbb{R}^d$ and an elementary cube $Q \in \mathcal{K}^d$, define the birth time of Q in $\mathbb{X}$ by

$$t_Q^{\mathbb{X}} \coloneqq \inf\{t \geq 0 \mid Q \in X(t)\}.$$

Now, let C^d be the set of all cubical filtrations in $\mathbb{R}^d$ and $\mathcal{F}^d$ be the smallest σ-field such that the map $C^d \ni \mathbb{X} \mapsto t_Q^{\mathbb{X}} \in [0, \infty]$ is measurable for any $Q \in \mathcal{K}^d$. We call a random variable taking values in $(C^d, \mathcal{F}^d)$ a random cubical filtration in $\mathbb{R}^d$. Let $|| \cdot ||_{\max}$ be the max norm in $\mathbb{R}^d$. For any subsets $A, B \in \mathbb{R}^d$, define $d_{\max}(A, B) \coloneqq \inf\{||x - y||_{\max} \mid x \in A, y \in B\}$. We consider a random cubical filtration $\mathbb{X} = \{X(t)\}_{t \geq 0}$ in $\mathbb{R}^d$ satisfying the following two assumptions.

Assumption 2.1.6. For every $z \in \mathbb{Z}^d$, the $[0,\infty]^{\mathcal{K}^d}$-valued random variables $\{t_Q^{\mathbb{X}}\}_{Q\in\mathcal{K}^d}$ and $\{t_{z+Q}^{\mathbb{X}}\}_{Q\in\mathcal{K}^d}$ have the same probability distribution. Here, $z+Q := \{z+x \mid x \in Q\} \in \mathcal{K}^d$ for any $z \in \mathbb{Z}^d$ and $Q \in \mathcal{K}^d$.

Assumption 2.1.7. There exists an integer $R \geq 0$ such that for any subset $A, B \subset \mathbb{R}^d$ with $d_{\max}(A,B) > R$, the families $\{t_Q^{\mathbb{X}} \colon Q \in \mathcal{K}^d, Q \subset A\}$ and $\{t_Q^{\mathbb{X}} \colon Q \in \mathcal{K}^d, Q \subset B\}$ are independent.

The first result is the strong law of large numbers of the qth persistent Betti numbers $\beta_q^{\mathbb{X}^n}(s,t)$ for a random cubical filtration $\mathbb{X}^n$ satisfying Assumptions 2.1.6 and 2.1.7.

Theorem 1.2.1. *Let $\mathbb{X} = \{X(t)\}_{t\geq 0}$ be a random cubical filtration in $\mathbb{R}^d$ satisfying Assumptions 2.1.6 and 2.1.7. Fix an integer $0 \leq q < d$ and $0 \leq s \leq t < \infty$. Then, there exists a constant $\widehat{\beta}_q(s,t) \in [0,\infty)$, depending on q, s, and t, such that*

$$\frac{\mathbb{E}[\beta_q^{\mathbb{X}^n}(s,t)]}{|\Lambda^n|} \to \widehat{\beta}_q(s,t) \quad \textit{as } n \to \infty$$

and

$$\frac{\beta_q^{\mathbb{X}^n}(s,t)}{|\Lambda^n|} \to \widehat{\beta}_q(s,t) \quad \textit{almost surely as } n \to \infty.$$

Remark 1.2.2. Since $\beta_q^{\mathbb{X}^n}(t,t)$ coincides with the qth Betti number $\beta_q(X^n(t))$ of a cubical set $X^n(t)$ for every $t \geq 0$, Theorem 1.2.1 implies the strong law of large numbers of Betti numbers. This was first obtained by Hiraoka and Tsunoda [19, Theorem 2.8] with a slightly general setting, where the probability distribution of $\{u_Q\}_{Q\in\mathcal{K}^d}$ taken in Example 2.1.8 is ergodic with the canonical translation on $\mathbb{Z}^d$. See [19, Section 2] for more details.

Moreover, we also show the central limit theorem of persistent Betti numbers using the Penrose theorem (see [27]);

Theorem 1.2.3. *Let $0 \leq s \leq t < \infty$, and assume that the probability distribution is given as a product measure. Then there exists a constant σ^2 such that*

$$\frac{1}{|\Lambda_n|}\mathbb{E}[(\beta_q^{\mathbb{X}^n}(s,t) - \mathbb{E}[\beta_q^{\mathbb{X}^n}(s,t)])^2] \to \sigma^2 \quad \textit{as } n \to \infty, \tag{1.2.1}$$

and

$$\frac{1}{|\Lambda_n|^{1/2}}\left(\beta_q^{\mathbb{X}^n}(s,t) - \mathbb{E}[\beta_q^{\mathbb{X}^n}(s,t)]\right) \xrightarrow{law} \mathcal{N}(0,\sigma^2) \quad \textit{as } n \to \infty, \tag{1.2.2}$$

where $\xrightarrow{law}$ *means convergence in law and $\mathcal{N}(0,\sigma^2)$ denotes the Gaussian distribution with the mean 0 and variance σ^2.*

Remark 1.2.4. The proof is shown in Appendix D since this is mainly the result of the Master's degree.

The next result is the strong law of large numbers of the persistence diagrams for a random cubical filtration satisfying Assumptions 2.1.6 and 2.1.7. For every $n \in \mathbb{N}$ and $q \in \mathbb{Z}$, the mean measure $\mathbb{E}[\xi_q^{\mathbb{X}^n}]$ of the qth persistence diagram $\xi_q^{\mathbb{X}^n}$ of $\mathbb{X}^n$ is defined by $\mathbb{E}[\xi_q^{\mathbb{X}^n}](A) := \mathbb{E}[\xi_q^{\mathbb{X}^n}(A)]$ for any Borel set $A \subset \Delta$. Here, Δ is the upper half plane in which persistence diagrams are defined. Since the number of the qth birth-death pairs of $\mathbb{X}^n$ is bounded above by the number of elementary q-cubes in Λ^n, we have

$$\xi_q^{\mathbb{X}^n}(\Delta) \leq \#\mathcal{K}_q^d(\Lambda^n), \tag{1.2.3}$$

where $\#\mathcal{K}_q^d(\Lambda^n)$ denotes the number of elementary q-cubes in Λ^n. In particular, $\mathbb{E}[\xi_q^{\mathbb{X}^n}]$ is a Radon measure on Δ.

Theorem 1.2.5. *Let $\mathbb{X} = \{X(t)\}_{t\geq 0}$ be a random cubical filtration in $\mathbb{R}^d$ satisfying Assumptions 2.1.6 and 2.1.7. Fix an integer $0 \leq q < d$. Then, there exists a Radon measure $\widehat{\xi}_q$ on Δ such that*

$$\frac{\mathbb{E}[\xi_q^{\mathbb{X}^n}]}{|\Lambda^n|} \xrightarrow{v} \widehat{\xi}_q \quad \text{as } n \to \infty.$$

Here, $\xrightarrow{v}$ denotes the vague convergence of Radon measures on Δ. Furthermore,

$$\frac{\xi_q^{\mathbb{X}^n}}{|\Lambda^n|} \xrightarrow{v} \widehat{\xi}_q \quad \text{almost surely as } n \to \infty.$$

The principal aim in this thesis is to investigate the large deviation principle of the persistence diagrams for a random cubical filtration satisfying Assumptions 2.1.6 and 2.1.7. We start with the definition of the large deviation principle in a general setting.

Definition 1.2.6. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers tending to infinity as $n \to \infty$. Let $\mathcal{X}$ be a Hausdorff topological space equipped with the Borel σ-algebra $\mathcal{B}_{\mathcal{X}}$. An $\mathcal{X}$-valued process $\{S^n\}_{n\in\mathbb{N}}$, i.e., a sequence of $\mathcal{X}$-valued random variables, satisfies the (*Donsker–Varadhan type*) *large deviation principle* (*LDP*) with a speed a_n if there exists a lower semicontinuous function $I\colon \mathcal{X} \to [0,\infty]$ such that

- for any closed set $F \subset \mathcal{X}$,

$$\limsup_{n\to\infty} a_n^{-1} \log \mathbb{P}(S^n \in F) \leq -\inf_{x\in F} I(x), \tag{1.2.4}$$

- for any open set $G \subset \mathcal{X}$,

$$\liminf_{n\to\infty} a_n^{-1} \log \mathbb{P}(S^n \in G) \geq -\inf_{x\in G} I(x).$$

The function I is called a *rate function*. If, furthermore, the sublevel set $\Psi_I(\alpha) := \{x \in \mathcal{X} \mid I(x) \leq \alpha\}$ is compact for every $\alpha \in [0, \infty)$, then I is called a *good rate function*.

Remark 1.2.7. By taking $F = \mathcal{X}$ in (1.2.4), we have $\inf_{x\in\mathcal{X}} I(x) = 0$. When I is a good rate function, this implies that I has at least one (not necessarily unique) zero point.

The next result is the LDP for the tuples of the persistent Betti numbers for a random cubical filtration satisfying Assumptions 2.1.6 and 2.1.7. Before stating the result, we provide the basics of the Fenchel–Legendre transform. In what follows, $\langle\cdot,\cdot\rangle_{\mathbb{R}^h}$ denotes the canonical inner product in $\mathbb{R}^h$, and $\|\cdot\|_{\mathbb{R}^h}$ is its induced norm. Given a function $\varphi\colon \mathbb{R}^h \to [-\infty,\infty]$, its *Fenchel–Legendre transform* $\varphi^*\colon \mathbb{R}^h \to [-\infty,\infty]$ is defined by

$$\varphi^*(x) := \sup_{\lambda\in\mathbb{R}^h} \{\langle\lambda, x\rangle_{\mathbb{R}^h} - \varphi(\lambda)\} \tag{1.2.5}$$

for any $x \in \mathbb{R}^h$. Every Fenchel–Legendre transform is convex and lower semi-continuous since it is the supremum of affine functions. If $\varphi(0) = 0$, then we can check that $\varphi^*(x) \in [0,\infty]$ for every $x \in \mathbb{R}^h$ by taking $\lambda = 0$ in (1.2.5). In the large deviation theory, the Fenchel–Legendre transform appears as a natural candidate for rate functions of LDPs in a general setting (see, e.g., Theorem 4.5.3 (b) in [8]). We now state our first LDP result.

Theorem 1.2.8. *Let $\mathbb{X} = \{X(t)\}_{t\geq 0}$ be a random cubical filtration in $\mathbb{R}^d$ satisfying Assumptions 2.1.6 and 2.1.7. Fix an integer $0 \leq q < d$ and a finite family $\mathcal{P} = \{(s_i, t_i)\}_{i=1}^h$ with $0 \leq s_i \leq t_i < \infty$. Then, for every $\lambda = (\lambda_1, \ldots, \lambda_h) \in \mathbb{R}^h$, the limit*

$$\varphi_{q,\mathcal{P}}(\lambda) := \lim_{n\to\infty} |\Lambda^n|^{-1} \log \mathbb{E}\left[\exp\left(\sum_{i=1}^h \lambda_i \beta_q^{\mathbb{X}^n}(s_i, t_i)\right)\right]$$

exists in $\mathbb{R}$, and the $\mathbb{R}^h$-valued process

$$\left\{\left(\frac{\beta_q^{\mathbb{X}^n}(s_1,t_1)}{|\Lambda^n|}, \frac{\beta_q^{\mathbb{X}^n}(s_2,t_2)}{|\Lambda^n|}, \ldots, \frac{\beta_q^{\mathbb{X}^n}(s_h,t_h)}{|\Lambda^n|}\right)\right\}_{n\in\mathbb{N}}$$

satisfies the LDP with a speed $|\Lambda^n|$ and a good convex rate function $\varphi_{q,\mathcal{P}}^*\colon \mathbb{R}^h \to [0,\infty]$. Furthermore, $\varphi_{q,\mathcal{P}}^*(x) = 0$ if and only if $x = (\widehat{\beta}_q(s_1,t_1), \ldots, \widehat{\beta}_q(s_h,t_h))$.

Next, we state the LDP of the persistence diagrams for a random cubical filtration satisfying Assumptions 2.1.6 and 2.1.7. Let $C_c(\Delta)$ be the set of all real-valued continuous functions on Δ with compact support and $\mathcal{M}(\Delta)$ denote the set of all Radon measures on Δ.

Theorem 1.2.9. *Let $\mathbb{X} = \{X(t)\}_{t\geq 0}$ be a random cubical filtration in $\mathbb{R}^d$ satisfying Assumptions 2.1.6 and 2.1.7. Fix an integer $0 \leq q < d$. Then, for every $f \in C_c(\Delta)$, the limit*

$$\varphi_q(f) := \lim_{n\to\infty} |\Lambda^n|^{-1} \log \mathbb{E}\left[\exp\left(\int_\Delta f \, d\xi_q^{\mathbb{X}^n}\right)\right]$$

exists in $\mathbb{R}$, and the $\mathcal{M}(\Delta)$-valued process $\{\xi_q^{\mathbb{X}^n}/|\Lambda^n|\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed $|\Lambda^n|$ and a good convex rate function $I_q\colon \mathcal{M}(\Delta) \to [0,\infty]$ defined by

$$I_q(\xi) := \sup_{f\in C_c(\Delta)} \left\{\int_\Delta f \, d\xi - \varphi_q(f)\right\} \tag{1.2.6}$$

for any $\xi \in \mathcal{M}(\Delta)$. Furthermore, $I_q(\xi) = 0$ if and only if $\xi = \widehat{\xi}_q$.

For the proof, we establish a general method of lifting a large deviation principle for the tuples of persistent Betti numbers to that of persistence diagrams (Theorem 5.1.2). Our method relies on the technique of the exponentially good approximation in the large deviation theory (Theorem B.0.4 in Appendix B).

Organizations

This thesis is organized as follows:

In Chapter 2 we explain our model precisely (Section 2.1) and introduce some variations of large deviation principles (Section 2.2).

In Chapter 3 we review the *exponentially regular nearly additive process*, show the statement of the result (Section 3.1), and show the proof of it (Section 3.2). Moreover, we give proofs of some technical lemmas (Section 3.3).

In Chapter 4 we summarize properties of persistent Betti numbers (Section 4.1) and show limit theorems of persistent Betti numbers for random cubical filtrations. Additionally, we also show the tuple of persistent Betti numbers satisfites the large deviation principle (Section 4.2).

In Chapter 5 we review the limit theorems of persistence diagrams for random cubical filtrations (Section 5.1). We give a proof of the law of large numbers of persistence diagrams (Section 5.2) and a proof of the large deviation principle of persistence diagrams (Section 5.3).

The contents of Chapter 3, 4, and 5 and that of Appendix A, B, and C are based on the submitted paper[16], which is a joint work with Yasuaki Hiraoka, Shu Kanazawa, and Kenkichi Tsunoda.

Chapter 2

Preliminary

Throughout this thesis, we fix $d \in \mathbb{N}$ as the dimension of the state space $\mathbb{R}^d$ where cubical sets and cubical filtrations are considered. In this chapter, we introduce the random cubical filtration model and the weak large deviation principle, those are fundamental concepts of this thesis. In Section 2.1, we introduce our model precisely. In Subsection 2.1.1, we review the definitions of cubical sets and cubical homology concisely. We refer to Chapter 2 of [22] for more detailed descriptions. In Subsection 2.1.2, we introduce our random cubical filtration model. In Section 2.2, we state variations of large deviation principle. In Subsection 2.2.1, we review the weaker version of large deviation principle and explore the connection between the weaker and standard forms of large devitation principle. In Subsection 2.2.2, we note some variations of large deviation principle, which will be utilized implicitly or explicitly later in this thesis.

2.1 Random Cubical Filtration Model

2.1.1 Cubical Homology

An *elementary interval* is a closed interval $I \subset \mathbb{R}$ of the form $I = [l, l+1]$ or $I = \{l\}$ for some $l \in \mathbb{Z}$. Such elementary intervals $I = [l, l+1]$ and $I = \{l\}$ are said to be *nondegenerate* and *degenerate*, respectively. An *elementary cube in* $\mathbb{R}^d$ is a product set $I_1 \times I_2 \times \cdots \times I_d \subset \mathbb{R}^d$ of d elementary intervals $I_1, I_2, \ldots, I_d$. Let $\mathcal{K}^d$ denote the set of all elementary cubes in $\mathbb{R}^d$. Given an elementary cube $Q = I_1 \times I_2 \times \cdots \times I_d$ in $\mathbb{R}^d$, we define its *dimension* $\dim Q$ as the number of nondegenerate elementary intervals in $I_1, I_2, \ldots, I_d$. An elementary cube Q with $\dim Q = q$ is called an *elementary q-cube in* $\mathbb{R}^d$. For each $q \in \mathbb{Z}$, let $\mathcal{K}^d_q$ denote the set of all elementary q-cubes in $\mathbb{R}^d$. A *cubical set in* $\mathbb{R}^d$ is a union of elementary cubes in $\mathbb{R}^d$. Note here that the above union of elementary cubes is not necessarily a finite union unlike the definition in [22]. Instead, we call a finite union of elementary cubes in $\mathbb{R}^d$ a *bounded* cubical set in $\mathbb{R}^d$.

Let X denote a cubical set in $\mathbb{R}^d$. In what follows, we refer to an elementary cube Q such that $Q \subset X$ as an *elementary cube in* X. For $q \in \mathbb{Z}$, let $\mathcal{K}_q^d(X)$ be the set of all elementary q-cubes in X. The *qth cubical chain group* $C_q(X)$ is defined as the $\mathbb{R}$-vector space consisting of all formal linear combinations of finitely many elementary q-cubes in X with coefficients in $\mathbb{R}$. Each element is called a *cubical q-chain* and of the form $a_1Q_1 + a_2Q_2 + \cdots + a_mQ_m$ for some $a_i \in \mathbb{R}$ and $Q_i \in \mathcal{K}_q^d$ $(i = 1, 2, \ldots, m)$. Some authors use the notation $a_1\widehat{Q}_1 + a_2\widehat{Q}_2 + \cdots + a_m\widehat{Q}_m$ instead of $a_1Q_1 + a_2Q_2 + \cdots + a_mQ_m$ to stress that the elementary cubes are regarded as algebraic objects. Obviously, $C_q(X) = 0$ for $q < 0$ or $q > d$ since $\mathcal{K}_q^d(X) = \emptyset$ in such cases. Furthermore, $\mathcal{K}_q^d(X)$ forms the canonical basis of $C_q(X)$ whenever $\mathcal{K}_q^d(X) \neq \emptyset$. For $q \in \mathbb{Z}$, the *qth cubical boundary map* $\partial_q^X : C_q(X) \to C_{q-1}(X)$ is defined as the linear extension of

$$\partial_q^X Q := \sum_{j=1}^{q} (-1)^{j-1} (Q_j^+ - Q_j^-) \in C_{q-1}(X)$$

for any $Q = I_1 \times I_2 \times \cdots \times I_d \in C_q(X)$. Here, $Q_j^+ \in C_{q-1}(X)$ and $Q_j^- \in C_{q-1}(X)$ are defined by degenerating the jth nondegenerate elementary interval in $I_1, I_2, \ldots, I_d$ upward and downward, respectively. More precisely, letting

$$I_{i_1} = [l_1, l_1 + 1], I_{i_2} = [l_2, l_2 + 1], \ldots, I_{i_q} = [l_q, l_q + 1]$$

be the nondegenerate elementary intervals in $I_1, I_2, \ldots, I_d$, we define

$$Q_j^+ := I_1 \times \cdots \times I_{i_j - 1} \times \{l_j + 1\} \times I_{i_j + 1} \times \cdots \times I_d$$

and

$$Q_j^- := I_1 \times \cdots \times I_{i_j - 1} \times \{l_j\} \times I_{i_j + 1} \times \cdots \times I_d.$$

Example 2.1.1. Set $d = 2$, and consider a cubical set $X = [0, 1]^2$ in $\mathbb{R}^2$.

(1) Let $Q = \{0\} \times \{0\} = \{(0, 0)\} \in \mathcal{K}_0^d(X)$. Then,

$$\partial_0^X Q = 0 \in C_{-1}(X).$$

(2) Let $Q = [0, 1] \times \{0\} \in \mathcal{K}_1^d(X)$. Then,

$$\partial_1 Q = \{1\} \times \{0\} - \{0\} \times \{0\} = (1, 0) - (0, 0) \in C_0(X).$$

(3) Let $Q = [0, 1] \times [0, 1] \in \mathcal{K}_2^d(X)$. Then,

$$\begin{aligned} \partial_2^X Q &= (\{1\} \times [0, 1] - \{0\} \times [0, 1]) - ([0, 1] \times \{1\} - [0, 1] \times \{0\}) \\ &= [0, 1] \times \{0\} + \{1\} \times [0, 1] - [0, 1] \times \{1\} - \{0\} \times [0, 1] \in C_1(X). \end{aligned}$$

For $q \in \mathbb{Z}$, define subspaces $Z_q(X) \coloneqq \ker \partial_q^X$ and $B_q(X) \coloneqq \operatorname{Im} \partial_{q+1}^X$ of $C_q(X)$, which are called the *qth cubical cycle group* and the *qth cubical boundary group*, respectively. A simple calculation shows that $\partial_q^X \circ \partial_{q+1}^X = 0$ for all $q \in \mathbb{Z}$, that is, $Z_q(X) \supset B_q(X)$. The *qth cubical homology group* $H_q(X) = H_q(X; \mathbb{R})$ with coefficients in $\mathbb{R}$ is defined as the quotient $\mathbb{R}$-vector space $Z_q(X)/B_q(X)$. When X is a bounded cubical set in $\mathbb{R}^d$, the dimension of $H_q(X)$ is called the *qth Betti number* of X, denoted by $\beta_q(X)$.

2.1.2 Persistent Homology for Cubical Filtrations

In this subsection, we review the definition of persistence diagram of an increasing family of cubical sets in $\mathbb{R}^d$. A *right-continuous cubical filtration in* $\mathbb{R}^d$ is an increasing family $\mathbb{X} = \{X(t)\}_{t \geq 0}$ of cubical sets $X(t)$ in $\mathbb{R}^d$ such that $X(t) = \bigcap_{t' > t} X(t')$ for every $t \geq 0$. In what follows, we omit the word "right-continuous", and simply call $\mathbb{X}$ a cubical filtration. We say that a cubical filtration $\mathbb{X} = \{X(t)\}_{t \geq 0}$ in $\mathbb{R}^d$ is *bounded* if $\bigcup_{t \geq 0} X(t)$ is bounded. Note that if $\mathbb{X} = \{X(t)\}_{t \geq 0}$ is a bounded cubical filtration in $\mathbb{R}^d$, then $X(t)$ differs from $\bigcup_{t' < t} X(t')$ only finitely many t's.

Let $\mathbb{R}[\{z^t : t \geq 0\}]$ denote an $\mathbb{R}$-vector space of formal linear combinations of finitely many monomials z^t ($t \geq 0$), where z is an indeterminate. The product of two elements in $\mathbb{R}[\{z^t : t \geq 0\}]$ is defined by the linear extension of $az^t \cdot bz^{t'} \coloneqq abz^{t+t'}$ ($a, b \in \mathbb{R}$, $t, t' \geq 0$). This operation equips $\mathbb{R}[\{z^t : t \geq 0\}]$ with a graded ring structure.

Let $\mathbb{X} = \{X(t)\}_{t \geq 0}$ be a bounded cubical filtration in $\mathbb{R}^d$. For each $q \in \mathbb{Z}$, the *qth persistent homology group* $H_q(\mathbb{X})$ of $\mathbb{X}$ is defined by

$$H_q(\mathbb{X}) \coloneqq \bigoplus_{t \geq 0} H_q(X(t)).$$

We define the action of monomial z^u ($u \geq 0$) on $H_q(\mathbb{X})$ by

$$z^u \cdot (c_t + B_q(X(t)))_{t \geq 0} \coloneqq (c'_t + B_q(X(t)))_{t \geq 0}, \text{ where } c'_t \coloneqq \begin{cases} c_{t-u} & \text{if } t \geq u, \\ 0 & \text{if } t < u. \end{cases}$$

By the linear extension of the above action of monomials, $H_q(\mathbb{X})$ has a graded module structure over the graded ring $\mathbb{R}[\{z^t : t \geq 0\}]$. The following theorem, which is often called the *structure theorem of persistent homology group*, is crucial for defining the persistence diagram for $\mathbb{X} = \{X(t)\}_{t \geq 0}$.

Theorem 2.1.2 ([35, Theorem 2.1])**.** *Let $q \in \mathbb{Z}$ be fixed. There exists a finite family $\{(b_i, d_i)\}_{i=1}^p$ with $0 \leq b_i < d_i \leq \infty$ such that the following graded module isomorphism holds*:

$$H_q(\mathbb{X}) \simeq \bigoplus_{i=1}^{p} ((z^{b_i})/(z^{d_i})).$$

Here, (z^t) expresses an ideal in $\mathbb{R}[\{z^t : t \geq 0\}]$ generated by the monomial z^t, and (z^∞) is regarded as the zero ideal. Furthermore, $\{(b_i, d_i)\}_{i=1}^p$ is uniquely determined as a multiset.

In the above theorem, $\{b_i\}_{i=1}^p$ and $\{d_i\}_{i=1}^p$ are called the *qth birth times* and *death times*, respectively, and each pair (b_i, d_i) is called the *qth birth-death pair* of the cubical filtration $\mathbb{X} = \{X(t)\}_{t\geq 0}$. Intuitively speaking, each birth-death pair (b_i, d_i) corresponds to a q-dimensional hole that appears at time b_i, persists over the time-interval $[b_i, d_i)$, and disappears at time d_i. We note that the number of the qth birth-death pairs of the cubical filtration $\mathbb{X} = \{X(t)\}_{t\geq 0}$ is trivially bounded above by the number of elementary q-cubes in $\bigcup_{t\geq 0} X(t)$.

Now, let us write

$$\Delta = \{(s,t) \in [0,\infty]^2 \mid 0 \leq s < t \leq \infty\},$$

which is naturally homeomorphic to $\{(x,y) \mid 0 \leq x < y \leq 1\}$ equipped with the usual topology. For $q \in \mathbb{Z}$, we define the *qth persistence diagram* $\xi_q^{\mathbb{X}}$ of the cubical filtration $\mathbb{X} = \{X(t)\}_{t\geq 0}$ as a counting measure

$$\xi_q^{\mathbb{X}} := \sum_{i=1}^p \delta_{(b_i,d_i)}$$

on Δ, where $\delta_{(b_i,d_i)}$ is the Dirac measure at (b_i, d_i), i.e., for any Borel set $A \subset \Delta$,

$$\delta_{(b_i,d_i)}(A) := \begin{cases} 1 & \text{if } (b_i, d_i) \in A, \\ 0 & \text{if } (b_i, d_i) \notin A. \end{cases}$$

In order to deal with the convergence of persistence diagrams, we will regard each persistence diagram as an element of the space $\mathcal{M}(\Delta)$ of Radon measures on Δ defined below. A Borel measure ξ on Δ is called a *Radon measure* if $\xi(K) < \infty$ for any compact set $K \subset \Delta$. Let $\mathcal{M}(\Delta)$ be the set of all Radon measures on Δ. We equip $\mathcal{M}(\Delta)$ with the vague topology, i.e., the weakest topology such that for any $f \in C_c(\Delta)$, the map $\mathcal{M}(\Delta) \ni \xi \mapsto \int_\Delta f\, d\xi \in \mathbb{R}$ is continuous. Here, $C_c(\Delta)$ denotes the set of all real-valued continuous functions on Δ with compact support. Note that for a sequence $\{\xi^n\}_{n\in\mathbb{N}}$ in $\mathcal{M}(\Delta)$ and $\xi \in \mathcal{M}(\Delta)$, the Radon measure ξ^n converges vaguely to ξ as $n \to \infty$ if and only if $\lim_{n\to\infty} \int_\Delta f\, d\xi^n = \int_\Delta f\, d\xi$ for any $f \in C_c(\Delta)$.

Next, we review the notion of persistent Betti number. Let $0 \leq s \leq t < \infty$. We denote by ι_s^t the inclusion map from $X(s)$ to $X(t)$, and by $(\iota_s^t)_* \colon H_q(X(s)) \to H_q(X(t))$ the induced linear map of ι_s^t. We call the rank of the map $(\iota_s^t)_*$ the *qth persistent Betti number of* $\mathbb{X}$ *at* (s,t), and denote it by $\beta_q^{\mathbb{X}}(s,t)$. The notion of persistent Betti number is a generalization of Betti number. Indeed, $\beta_q^{\mathbb{X}}(t,t) = \beta_q(X(t))$ holds for every $t \geq 0$. Since

$$\begin{aligned}\operatorname{Im}(\iota_s^t)_* &\simeq \frac{H_q(X(s))}{\ker(\iota_s^t)_*} \\ &= \frac{Z_q(X(s))/B_q(X(s))}{(Z_q(X(s)) \cap B_q(X(t)))/B_q(X(s))} \simeq \frac{Z_q(X(s))}{Z_q(X(s)) \cap B_q(X(t))},\end{aligned}$$

we have

$$\beta_q^{\mathbb{X}}(s,t) = \dim \frac{Z_q(X(s))}{Z_q(X(s)) \cap B_q(X(t))}. \tag{2.1.1}$$

Intuitively speaking, $\beta_q^{\mathbb{X}}(s,t)$ expresses the number of q-dimensional holds that appear before time s and persist to time t in the filtration $\mathbb{X} = \{X(t)\}_{t\geq 0}$. The following relationship between the persistence diagram and the persistent Betti number is particularly important, which is called the k-triangle lemma in [3, 10].

Theorem 2.1.3 ([3, 10]). *Let $q \in \mathbb{Z}$ and $0 \leq s \leq t < \infty$ be fixed. Then, it holds that*

$$\xi_q^{\mathbb{X}}([0,s] \times (t,\infty]) = \beta_q^{\mathbb{X}}(s,t). \tag{2.1.2}$$

Remark 2.1.4. The persistence diagram $\xi_q^{\mathbb{X}}$ is in fact characterized as the unique counting measure on Δ satisfying (2.1.2) for any $0 \leq s \leq t < \infty$.

The following is an immediate corollary of Theorem 2.1.3 together with the inclusion-exclusion principle.

Corollary 2.1.5. *Let $q \in \mathbb{Z}$ and $0 \leq s_1 \leq s_2 \leq t_1 \leq t_2 < \infty$ be fixed. Then,*

$$\xi_q^{\mathbb{X}}([0,s_2] \times (t_1,t_2]) = \beta_q^{\mathbb{X}}(s_2,t_1) - \beta_q^{\mathbb{X}}(s_2,t_2)$$

and

$$\xi_q^{\mathbb{X}}((s_1,s_2] \times (t_1,t_2]) = \beta_q^{\mathbb{X}}(s_2,t_1) - \beta_q^{\mathbb{X}}(s_2,t_2) + \beta_q^{\mathbb{X}}(s_1,t_2) - \beta_q^{\mathbb{X}}(s_1,t_1).$$

2.1.3 Random Cubical Filtration Model

For a cubical filtration $\mathbb{X} = \{X(t)\}_{t\geq 0}$ in $\mathbb{R}^d$ and an elementary cube $Q \in \mathcal{K}^d$, the *birth time of Q in $\mathbb{X}$* is defined by

$$t_Q^{\mathbb{X}} := \inf\{t \geq 0 \mid Q \in X(t)\}.$$

By convention, we regard $t_Q^{\mathbb{X}} := \infty$ if $Q \notin \bigcup_{t\geq 0} X(t)$. Obviously, $Q' \subset Q \in \mathcal{K}^d$ implies $t_{Q'}^{\mathbb{X}} \leq t_Q^{\mathbb{X}}$. Conversely, given a family $\{t_Q\}_{Q\in\mathcal{K}^d}$ in $[0,\infty]$ satisfying that

$$Q' \subset Q \in \mathcal{K}^d \Rightarrow t_{Q'} \leq t_Q, \tag{2.1.3}$$

we can define a cubical filtration $\mathbb{X} = \{X(t)\}_{t\geq 0}$ in $\mathbb{R}^d$ so that $t_Q^{\mathbb{X}} = t_Q$ for any $Q \in \mathcal{K}^d$. Indeed, we may simply set

$$X(t) = \bigcup \{Q \in \mathcal{K}^d \mid t_Q \leq t\}$$

for every $t \geq 0$. We call such $\mathbb{X} = \{X(t)\}_{t\geq 0}$ the *cubical filtration in $\mathbb{R}^d$ corresponding to $\{t_Q\}_{Q\in\mathcal{K}^d}$*.

Now, let C^d denote the set of all cubical filtrations in $\mathbb{R}^d$, and let $\mathcal{F}^d$ denote the smallest σ-field such that the map $C^d \ni \mathbb{X} \mapsto t_Q^{\mathbb{X}} \in [0, \infty]$ is measurable for any $Q \in \mathcal{K}^d$. In other words, $\mathcal{F}^d$ is the σ-field generated by the maps $\{C^d \ni \mathbb{X} \mapsto t_Q^{\mathbb{X}} \in [0, \infty] \colon Q \in \mathcal{K}^d\}$. We call a random variable taking values in the measurable space $(C^d, \mathcal{F}^d)$ a *random cubical filtration in $\mathbb{R}^d$*.

Next, we introduce our random cubical filtration model. For any subsets $A, B \subset \mathbb{R}^d$, define $d_{\max}(A, B) := \inf\{\|x - y\|_{\max} \mid x \in A, y \in B\}$, where $\|\cdot\|_{\max}$ denotes the max norm in $\mathbb{R}^d$. In this thesis, we consider a random cubical filtration $\mathbb{X} = \{X(t)\}_{t\geq 0}$ in $\mathbb{R}^d$ satisfying the following two assumptions.

Assumption 2.1.6 (Stationarity)**.** For every $z \in \mathbb{Z}^d$, the $[0, \infty]^{\mathcal{K}^d}$-valued random variables $\{t_Q^{\mathbb{X}}\}_{Q\in\mathcal{K}^d}$ and $\{t_{z+Q}^{\mathbb{X}}\}_{Q\in\mathcal{K}^d}$ have the same probability distribution. Here, $z + Q := \{z + x \mid x \in Q\} \in \mathcal{K}^d$ for any $z \in \mathbb{Z}^d$ and $Q \in \mathcal{K}^d$. In such case, we say that $\mathbb{X}$ is *stationary*.

Assumption 2.1.7 (Local dependence)**.** There exists an integer $R \geq 0$ such that for any subsets $A, B \subset \mathbb{R}^d$ with $d_{\max}(A, B) > R$, the families $\{t_Q^{\mathbb{X}} \colon Q \in \mathcal{K}^d, Q \subset A\}$ and $\{t_Q^{\mathbb{X}} \colon Q \in \mathcal{K}^d, Q \subset B\}$ are independent. In such case, we say that $\mathbb{X}$ is *R-dependent*.

As typical random cubical filtration models that satisfy Assumptions 2.1.6 and 2.1.7, we introduce the upper and lower random cubical filtrations.

Example 2.1.8. Let $\{F_q\}_{q=0}^d$ be a family of probability distribution functions on $[0, \infty]$, i.e., F_q is a right-continuous function on $[0, \infty]$ with $F_q(\infty) = 1$ (while not necessarily $F_q(0) = 0$ or $\lim_{x\to\infty} F_q(x) = 1$) for each $0 \leq q \leq d$. To each elementary cube $Q \in \mathcal{K}^d$, we assign a $[0, \infty]$-valued random variable u_Q with probability distribution function $F_{\dim Q}$ independently. For each $Q \in \mathcal{K}^d$, we set

$$\overline{t}_Q = \min\{u_{Q'} \mid Q' \in \mathcal{K}^d, Q' \supset Q\} \quad \text{and} \quad \underline{t}_Q = \max\{u_{Q'} \mid Q' \in \mathcal{K}^d, Q' \subset Q\}.$$

Noting that both the families $\{\overline{t}_Q\}_{Q\in\mathcal{K}^d}$ and $\{\underline{t}_Q\}_{Q\in\mathcal{K}^d}$ satisfy (2.1.3), we define $\overline{\mathbb{X}} = \{\overline{X}(t)\}_{t\geq 0}$ and $\underline{\mathbb{X}} = \{\underline{X}(t)\}_{t\geq 0}$ as the random cubical filtrations in $\mathbb{R}^d$ corresponding to $\{\overline{t}_Q\}_{Q\in\mathcal{K}^d}$ and $\{\underline{t}_Q\}_{Q\in\mathcal{K}^d}$, respectively. Obviously, $\overline{\mathbb{X}}$ and $\underline{\mathbb{X}}$ are stationary, also 1- and 0-dependent, respectively. We call $\overline{\mathbb{X}}$ and $\underline{\mathbb{X}}$ the *upper* and *lower random cubical filtrations*, respectively, with probability distribution functions $\{F_q\}_{q=0}^d$.

Remark 2.1.9. The word “upper” and “lower” in Examples 2.1.8 derives from the upper and lower random simplicial complex model, extensively studied in [4, 6, 7, 5, 12, 13, 15]. In fact, for every $t \geq 0$, the random cubical sets $\overline{X}(t)$ and $\underline{X}(t)$ can be regarded as the cubical versions of the upper and lower random simplicial complex with parameters $\{F_q(t)\}_{q=0}^{d}$, respectively.

We additionally introduce other random cubical filtration models that satisfy Assumptions 2.1.6 and 2.1.7, where the birth times of elementary cubes are given in more geometric ways. For lattice points $z, z' \in \mathbb{Z}^d$, we say that z and z' are *adjacent* if $\|z - z'\|_{L^1} = 1$. Here, $\| \cdot \|_{L^1}$ is the L^1-norm in $\mathbb{R}^d$.

Example 2.1.10. Let μ be a probability measure on $\mathbb{R}^d$. Let $\{\varepsilon_z\}_{z\in\mathbb{Z}^d}$ be independent and identically distributed (i.i.d.) random variables drawn from μ, and define $x_z := z + \varepsilon_z$ for every $z \in \mathbb{Z}^d$. For each $Q \in \mathcal{K}^d$, we set

$$t_Q = \inf\{t \geq 0 \mid \|x_z - x_{z'}\|_{\mathbb{R}^d} \leq t \text{ for any adjacent lattice points } z \text{ and } z' \text{ in } Q\}.$$

Noting that the family $\{t_Q\}_{Q\in\mathcal{K}^d}$ satisfies (2.1.3), we define $\mathbb{X} = \{X(t)\}_{t\geq 0}$ as the random cubical filtration in $\mathbb{R}^d$ corresponding to $\{t_Q\}_{Q\in\mathcal{K}^d}$. Obviously, $\mathbb{X}$ is stationary and 0-dependent.

Example 2.1.11. Let μ be a probability measure on $\mathbb{R}^d$ with compact support. Let $\{\varepsilon_z\}_{z\in\mathbb{Z}^d}$ and $\{x_z\}_{z\in\mathbb{Z}^d}$ be the same as in Example 2.1.10. For each $Q \in \mathcal{K}^d$, we set

$$t_Q = \inf\left\{t \geq 0 \;\middle|\; Q \subset \bigcup_{z\in\mathbb{Z}^d} \bar{B}(x_z, t)\right\}.$$

Here, $\bar{B}(x_z, t)$ is the closed ball of radius t centered at x_z. Noting again that the family $\{t_Q\}_{Q\in\mathcal{K}^d}$ satisfies (2.1.3), we define $\mathbb{X} = \{X(t)\}_{t\geq 0}$ as the random cubical filtration in $\mathbb{R}^d$ corresponding to $\{t_Q\}_{Q\in\mathcal{K}^d}$. Obviously, $\mathbb{X}$ is stationary. Furthermore, $\mathbb{X}$ is locally dependent since μ has a compact support.

2.2 Large Deviation Principle

In this section, we review the LDP and their variations. The definition of LDP have been given in Definition 1.2.6 previously, yet here are equivalent definitions. Let $\mathcal{X}$ be a Hausdorff topological space equipped with the Borel σ-algebra $\mathcal{B}_{\mathcal{X}}$. For any set $\Gamma \subseteq \mathcal{X}$, Γ^c denotes the complement of Γ. Moreover the closure and the interior of Γ are denoted by $\overline{\Gamma}$ and Γ°, respectively. For a rate function I, the set of points in X of finite rate is denoted by $\mathcal{D}_I := \{x \in \mathcal{X}, I(x) < \infty\}$. Other notations are the same as Definition 1.2.6.

The first one is the standard notation in the book [8] and the second one is introduced as a more useful formulation to prove a LDP in the same book.

Definition 2.2.1 ([8])**.** An $\mathcal{X}$-valued process $\{S^n\}_{n\in\mathbb{N}}$ is said to satisfy the large deviation principle with a speed a_n and a rate function I, if for all measurable sets $\Gamma \in \mathcal{B}_{\mathcal{X}}$,

$$-\inf_{x\in\Gamma^\circ} I(x) \leq \liminf_{n\to\infty} a_n^{-1}\log \mathbb{P}(S^n \in \Gamma) \leq \limsup_{n\to\infty} a_n^{-1}\log \mathbb{P}(S^n \in \Gamma) \leq -\inf_{x\in\overline{\Gamma}} I(x).$$

Let notations be the same as Definition 1.2.6 and 2.2.1.

Definition 2.2.2 ([8])**.** An $\mathcal{X}$-valued process $\{S^n\}_{n\in\mathbb{N}}$ satisfies the large deviation principle with a speed a_n and a rate function I, if the following bounds hold:

(Upper bound) For every $\alpha < \infty$ and every measurable set Γ with $\overline{\Gamma} \subset \Psi_I(\alpha)^c$,

$$\limsup_{n\to\infty} a_n^{-1}\log \mathbb{P}(S^n \in \Gamma) \leq -\alpha. \tag{2.2.1}$$

(Lower bound) For any $x \in \mathcal{D}_I$ and any measurable set Γ with $x \in \Gamma^\circ$,

$$-I(x) \leq \liminf_{n\to\infty} a_n^{-1}\log \mathbb{P}(S^n \in \Gamma).$$

2.2.1 Weak Large Deviation Principles

LDPs are also useful for understanding the probability of events in which the process does not have the mean value, but sometimes checking the upper bound is difficult. There is a weaker version of LDP to deal with such a problem.

Definition 2.2.3 (Weak large deviation principles)**.** An $\mathcal{X}$-valued process $\{S^n\}_{n\in\mathbb{N}}$ is said to satisfy the *weak large deviation principle* with a speed a_n and a rate function I if the upper bound

$$\limsup_{n\to\infty} \frac{1}{a_n}\log \mathbb{P}\left(S^n \in K\right) \leq -\alpha, \tag{2.2.2}$$

holds for every $\alpha < \infty$ and all compact subsets K of $\Psi_I\left(\alpha\right)^c$, and the lower bound

$$-I(x) \leq \liminf_{n\to\infty} \frac{1}{a_n}\log \mathbb{P}\left(S^n \in \Gamma\right).$$

holds for any $x \in \mathcal{D}_I$ and any measurable set Γ with $x \in \Gamma^\circ$.

We call the usual LDP as the full LDP in contrast with the weak LDP. The difference between the weak LDP and the full LDP is upper bound (compare (2.2.1) and (2.2.2)), and if we obtain the following tightness, the weak LDP becomes the full LDP.

Definition 2.2.4. An $\mathcal{X}$-valued process $\{S^n\}_{n\in\mathbb{N}}$ is said to be *exponentially tight* (with a speed a_n) if for any $\alpha > 0$, there exists a compact set $K \subset \mathcal{X}$ such that

$$\limsup_{n\to\infty} a_n{}^{-1} \log \mathbb{P}(S^n \notin K) \leq -\alpha.$$

Lemma 2.2.5 ([8, Lemma 1.2.18])**.** *Let an $\mathcal{X}$-valued process $\{S^n\}_{n\in\mathbb{N}}$ satisfy the weak large deviation principle with a speed a_n and a rate function I. If $\{S^n\}_{n\in\mathbb{N}}$ is exponentially tight, then $\{S^n\}_{n\in\mathbb{N}}$ satisfies the full large deviation principle with a speed a_n and a rate function I. Moreover, I is a good rate function.*

2.2.2 Variations of Large Deviation Principle

In this subsection, we introduce the basic methods of LDPs. The following theorem of the LDP for i.i.d. random vectors in $\mathbb{R}^h$ is fundamental and called *Cramér's theorem*.

Theorem 2.2.6 ([8, Theorem 2.2.30, Corollary 6.1.6])**.** *Let $n \in \mathbb{N}$ be fiexed and X^n be an $\mathbb{R}^h$-valued random varibale indexed by n. Let $\{Y_i\}_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed $\mathbb{R}^h$-valued random variables with the same distibution as X^n. For each integer m, S_m denotes the emprical mean $S_m := \sum_{i=1}^m Y_i$. Then $\{S_m/m\}_{m\in\mathbb{N}}$ satisfies the weak large deviation principle with a speed m and a rate function*

$$J_n(x) = \sup_{\lambda\in\mathbb{R}^h} \left\{\langle\lambda, x\rangle - \log \mathbb{E}[\exp(\langle\lambda, X^n\rangle)]\right\}.$$

Moreover, if $\log \mathbb{E}[\exp(\langle\lambda, X^n\rangle)] < \infty$ in the neighborhood of zero point, then the LDP holds and J_n is a good rate function.

The above theorem is limited to i.i.d. case, however there is a similar theorem of non- i.i.d. case with some additional assumption. That theorem is called the *Gärtner-Ellis theorem*, but it is not used for main results of this thesis. Therefore we mention the statement of that theorem in $\mathbb{R}^h$ here in order to simplify the explanation of future work in Conclusion. Consider a sequence of $\mathbb{R}^h$-valued random vectors $\{Z_n\}_{n\in\mathbb{N}}$ and moment generating function $\Lambda_n(\lambda) := \log \mathbb{E}[\exp\langle\lambda, Z_n\rangle]$.

Assumption 2.2.7 ([8, Assumption 2.3.2])**.** The limit of $a_n^{-1}\Lambda_n(a_n\lambda)$ exists as an extended real number for all $\lambda \in \mathbb{R}^h$, that is,

$$\Lambda(\lambda) := \lim_{n\to\infty} a_n^{-1}\Lambda_n(a_n\lambda) \in [-\infty, \infty].$$

Furthermore, the origin belongs to the interior of $\mathcal{D}_\Lambda := \left\{\lambda \in \mathbb{R}^h \;\middle|\; \Lambda(\lambda) < \infty\right\}$.

To state the theorem, we need two more definitions. Let Λ^* be a Fenchel-Legendre transformation of $\Lambda(\cdot)$, with $\mathcal{D}_{\Lambda^*} := \left\{x \in \mathbb{R}^h \,\middle|\, \Lambda^*(x) < \infty\right\}$.

Definition 2.2.8 ([8, Definition 2.3.3])**.** A vector $y \in \mathbb{R}^h$ is called an *exposed point* of Λ^* if for some $\lambda \in \mathbb{R}^h$ and all $x \neq y$,

$$\langle \lambda, y\rangle - \Lambda^*(y) > \langle \lambda, x\rangle - \Lambda^*(x). \tag{2.2.3}$$

Here, λ in (2.2.3) is called an *exposing hyperplane*.

Definition 2.2.9 ([8, Definition 2.3.5])**.** A convex function $\Lambda\colon \mathbb{R}^h \to (-\infty, \infty]$ is *essentially smooth* if:

(1) ${\mathcal{D}_\Lambda}^\circ$ is non-empty.

(2) $\Lambda(\cdot)$ is differentiable in ${\mathcal{D}_\Lambda}^\circ$.

(3) $\Lambda(\cdot)$ is steep, namely, $\lim_{n\to\infty} |\nabla\Lambda(\lambda_n)| = \infty$ whenever $\{\lambda_n\}$ is a sequence in ${\mathcal{D}_\Lambda}^\circ$ converging to a boundary point of ${\mathcal{D}_\Lambda}^\circ$, where $\nabla\Lambda$ denotes the gradient of Λ.

Theorem 2.2.10 ([8, Theorem 2.3.6])**.** *Let Assumption 2.2.7 hold. Then $\{Z_n\}_{n\in\mathbb{N}}$ satisfies the large deviation principle as follows.*

(1) *For any closed set F,*

$$\limsup_{n\to\infty} a_n^{-1} \log \mathbb{P}(Z_n \in F) \leq -\inf_{x\in F} \Lambda^*(x).$$

(2) *For any open set G,*

$$-\inf_{x\in G\cap\mathcal{F}} \Lambda^*(x) \leq \liminf_{n\to\infty} a_n^{-1} \log \mathbb{P}(Z_n \in G),$$

where $\mathcal{F}$ is the set of exposed points of Λ^* whose exposing hyperplane belongs to ${\mathcal{D}_\Lambda}^\circ$.

(3) *If Λ is an essentially smooth, lower semicontinuous function, then the LDP holds with the good rate function Λ^*.*

We also refer to transformations of LDPs in Appendix B. They are basic methods for transferring LDP of a topological space to that of the another space. The *contraction principle* (Theorem B.0.1) shows us that an LDP is preserved under continuous maps, and the Theorem B.0.4 is the deduction theorem of the LDP from LDPs for the approximation sequence called *exponential good approximation*.

We also treat some methods of the large deviation principles in Appendix C. The *Dawson–Gärtner theorem* (Theorem C.0.4) has a crucial role in this chapter. It is a useful tool for making a LDP in a large space, called *projective limit*, from the aggregation of LDPs in small spaces. We also use the Lemma C.0.6, a basic lemma in the large deviation theory, whichreduces the LDP to the LDP in the subspace of our interests.

Chapter 3

Large deviation principle for exponentially regular nearly additive processes

In this chapter, we develop a general LDP result for exponentially regular nearly additive vector-valued processes, which is crucial for the proof of Theorem 1.2.8 in Chapter 4. In Section 3.1, we define the notions of exponentially near additivity and exponential regularity, and state the general LDP result (Theorem 3.1.4). Section 3.2 presents the proof of Theorem 3.1.4. The proofs of technical lemmas needed in the proof of Theorem 3.1.4 is deferred to Section 3.3.

3.1 Statement of result

Throughout this section, we fix $h \in \mathbb{N}$, and consider an $\mathbb{R}^h$-valued process $\{S^n\}_{n\in\mathbb{N}}$, i.e., a sequence of $\mathbb{R}^h$-valued random variables. In the applications dealt with in Section 4, S^n is taken to be a random vector associated to the rectangular region $\Lambda^n = [-n, n]^d$, and $|\Lambda^n|$ indicates its d-dimensional Lebesgue measure. However, we here regard S^n and $|\Lambda^n| := (2n)^d$ as just a random vector indexed by n and a scaling factor, respectively.

The following notions are crucial for stating the main theorem in this section (Theorem 3.1.4).

Definition 3.1.1. Let $r \geq 0$ be an integer. We say that an $\mathbb{R}^h$-valued process $\{S^n\}_{n\in\mathbb{N}}$ is *exponentially r-nearly additive* if there exist $\mathbb{R}^h$-valued random variables $\{S^{n,z}\}_{n\in\mathbb{N},z\in\mathbb{Z}^d}$ such that the following conditions are satisfied:

- $\{S^{n,z}\}_{z\in\mathbb{Z}^d}$ are independent copies of S^n for every $n \in \mathbb{N}$;

- for any $\varepsilon > 0$ and $C > 0$, there exists an integer $K > r$ such that

  $$\mathbb{P}\left(\left\| S^{(2m+1)k} - \sum_{z \in \mathbb{Z}^d \cap [-m,m]^d} S^{k-r,z} \right\|_{\mathbb{R}^h} > \varepsilon |\Lambda^{(2m+1)k}|\right) \leq \exp(-C|\Lambda^{(2m+1)k}|)$$

  for all $k \geq K$ and $m \in \mathbb{N}$.

We also say that an $\mathbb{R}^h$-valued process $\{S^n\}_{n\in\mathbb{N}}$ is *exponentially nearly additive* if there exists an integer $r \geq 0$ such that $\{S^n\}_{n\in\mathbb{N}}$ is exponentially r-nearly additive.

Remark 3.1.2. The exponentially near additivity in Definition 3.1.1 with $h = 1$ and $r = 0$ corresponds to the definition of the near additivity in Assumption 2.1 of [32].

Definition 3.1.3. We say that an $\mathbb{R}^h$-valued process $\{S^n\}_{n\in\mathbb{N}}$ is *exponentially regular* if the following property holds for each fixed $k \in \mathbb{N}$: if m_n is taken as the unique integer satisfying that $(2m_n + 1)k \leq n < (2m_n + 3)k$ for each $n \in \mathbb{N}$, then for any $\varepsilon > 0$ and $C > 0$, there exists $N \in \mathbb{N}$ such that

$$\mathbb{P}(\|S^n - S^{(2m_n+1)k}\|_{\mathbb{R}^h} > \varepsilon|\Lambda^n|) \leq \exp(-C|\Lambda^n|)$$

for all $n \geq N$.

Let $\{S^n\}_{n\in\mathbb{N}}$ be an exponentially regular nearly additive $\mathbb{R}^h$-valued process consisting of integrable random variables. If $\sup_{n\in\mathbb{N}} \|\mathbb{E}[S^n]\|_{\mathbb{R}^h}/|\Lambda^n| < \infty$, then $\{S^n\}_{n\in\mathbb{N}}$ satisfies a strong law of large numbers, i.e., the limit

$$\widehat{S} := \lim_{n\to\infty} \frac{\mathbb{E}[S^n]}{|\Lambda^n|} \tag{3.1.1}$$

exists in $\mathbb{R}^h$, and

$$\frac{S^n}{|\Lambda^n|} \to \widehat{S} \quad \text{almost surely as } n \to \infty.$$

See Appendix A for the proof under a weaker assumption.

The following is a large deviation principle for exponentially regular nearly additive processes.

Theorem 3.1.4. *Let $\{S^n\}_{n\in\mathbb{N}}$ be an exponentially regular nearly additive $\mathbb{R}^h$-valued process satisfying that*

$$\sup_{n\in\mathbb{N}} |\Lambda^n|^{-1} \log \mathbb{E}[\exp(\langle \lambda, S^n\rangle_{\mathbb{R}^h})] < \infty \tag{3.1.2}$$

for any $\lambda \in \mathbb{R}^h$. Then, for every $\lambda \in \mathbb{R}^h$, the limit

$$\varphi(\lambda) := \lim_{n\to\infty} |\Lambda^n|^{-1} \log \mathbb{E}[\exp(\langle \lambda, S^n\rangle_{\mathbb{R}^h})]$$

exists in $\mathbb{R}$, and the $\mathbb{R}^h$-valued process $\{S^n/|\Lambda^n|\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed $|\Lambda^n|$ and good convex rate function $\varphi^*\colon \mathbb{R}^h \to [0,\infty]$. Furthermore, $\varphi^*(x) = 0$ if and only if $x = \widehat{S}$, defined by (3.1.1).

Remark 3.1.5. In the above theorem, we note that if (3.1.2) holds for any $\lambda \in \mathbb{R}^h$, then each S^n is integrable and $\sup_{n\in\mathbb{N}} \|\mathbb{E}[S^n]\|_{\mathbb{R}^h}/|\Lambda^n| < \infty$ holds from an elementary calculation. Hence, the limit $\widehat{S}$ defined in (3.1.1) exists.

Remark 3.1.6. The above theorem can be regarded as a generalization of Theorem 2.1 in [32], where exponentially regular 0-nearly additive real-valued processes are considered.

Combining Theorem 3.1.4 and the preceding discussion on the strong law of large numbers, we immediately obtain the following useful corollary.

Corollary 3.1.7. *Let $\{S^n\}_{n\in\mathbb{N}}$ be an exponentially regular nearly additive $\mathbb{R}^h$-valued process satisfying that*

$$\sup_{n\in\mathbb{N}} |\Lambda^n|^{-1} \log \mathbb{E}[\exp(\langle \lambda, S^n\rangle_{\mathbb{R}^h})] < \infty$$

for any $\lambda \in \mathbb{R}^h$. Then, the following statements hold.

(1) *The limit*

$$\widehat{S} := \lim_{n\to\infty} \frac{\mathbb{E}[S^n]}{|\Lambda^n|}$$

exists in $\mathbb{R}^h$, and

$$\frac{S^n}{|\Lambda^n|} \to \widehat{S} \quad \text{almost surely as } n \to \infty.$$

(2) *For every $\lambda \in \mathbb{R}^h$, the limit*

$$\varphi(\lambda) := \lim_{n\to\infty} |\Lambda^n|^{-1} \log \mathbb{E}[\exp(\langle \lambda, S^n\rangle_{\mathbb{R}^h})]$$

exists in $\mathbb{R}$, and the $\mathbb{R}^h$-valued process $\{S^n/|\Lambda^n|\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed $|\Lambda^n|$ and good convex rate function $\varphi^*\colon \mathbb{R}^h \to [0,\infty]$. Furthermore, $\varphi^*(x) = 0$ if and only if $x = \widehat{S}$.

3.2 Proof of Theorem 3.1.4

The proof of Theorem 3.1.4 relies mainly on two theorems in the large deviation theory. The first one shows the existence of an LDP for the $\mathbb{R}^h$-valued process

$\{S^n/|\Lambda^n|\}_{n\in\mathbb{N}}$ with a (not necessarily convex) rate function $I\colon \mathbb{R}^h \to [0,\infty]$ (Theorem 3.2.1). The second one guarantees that if the rate function I is convex, then I is given as the Fenchel–Legendre transform of the limiting logarithmic moment generating function (Theorem 3.2.2).

For these theorems, we first review basic notions. An $\mathbb{R}^h$-valued process $\{S^n\}_{n\in\mathbb{N}}$ is said to be *exponentially tight* (with a speed $|\Lambda^n|$) if for any $\alpha > 0$, there exists a compact set $K \subset \mathbb{R}^h$ such that

$$\limsup_{n\to\infty} |\Lambda^n|^{-1} \log \mathbb{P}(S^n \notin K) \le -\alpha.$$

A function $I\colon \mathbb{R}^h \to [0,\infty]$ is said to be *convex* if for any $x_1, x_2 \in \mathbb{R}^h$ and $t \in (0,1)$, it holds that

$$tI(x_1) + (1-t)I(x_2) \ge I(tx_1 + (1-t)x_2). \tag{3.2.1}$$

In what follows in this section, we use the following notation: for any Borel function $f\colon \mathbb{R}^h \to \mathbb{R}$ and $n \in \mathbb{N}$,

$$\Gamma_n(f) := \log \mathbb{E}\left[\exp\left(|\Lambda^n| f\left(\frac{S^n}{|\Lambda^n|}\right)\right)\right] \in (-\infty, \infty].$$

Furthermore, let $\mathscr{F}(\mathbb{R}^h)$ be the class of Lipschitz continuous and concave real-valued functions on $\mathbb{R}^h$. Here, a real-valued function f on $\mathbb{R}^h$ is said to be *concave* if $-f$ satisfies (3.2.1) for any $x_1, x_2 \in \mathbb{R}^h$ and $t \in (0,1)$. The class $\mathscr{F}(\mathbb{R}^h)$ is well-separating in the sense that

- $\mathscr{F}(\mathbb{R}^h)$ contains the constant functions,
- $\mathscr{F}(\mathbb{R}^h)$ is closed under finite pointwise minima, i.e., $f_1, f_2 \in \mathscr{F}(\mathbb{R}^h)$ implies $f_1 \wedge f_2 \in \mathscr{F}(\mathbb{R}^h)$,
- $\mathscr{F}(\mathbb{R}^h)$ separates points in $\mathbb{R}^h$, i.e., for any two points $x \ne y$ in $\mathbb{R}^h$ and $a, b \in \mathbb{R}$, there exists a function $f \in \mathscr{F}(\mathbb{R}^h)$ such that both $f(x) = a$ and $f(y) = b$ hold.

The following theorem is a special case of Theorem 4.4.10 in [8] with the state space $\mathbb{R}^h$ and the well-separating class $\mathscr{F}(\mathbb{R}^h)$.

Theorem 3.2.1 ([8, Theorem 4.4.10]). *Let $\{S^n\}_{n\in\mathbb{N}}$ be an $\mathbb{R}^h$-valued process. Suppose that the $\mathbb{R}^h$-valued process $\{S^n/|\Lambda^n|\}_{n\in\mathbb{N}}$ is exponentially tight, and that the limit $\lim_{n\to\infty} |\Lambda^n|^{-1}\Gamma_n(f)$ exists in $[-\infty,\infty)$ for any $f \in \mathscr{F}(\mathbb{R}^h)$. Then, for every $f \in C_b(\mathbb{R}^h)$, the limit*

$$\Gamma(f) := \lim_{n\to\infty} |\Lambda^n|^{-1}\Gamma_n(f)$$

also exists in $\mathbb{R}$, and the $\mathbb{R}^h$-valued process $\{S^n/|\Lambda^n|\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed $|\Lambda^n|$ and a good rate function $I\colon \mathbb{R}^h \to [0,\infty]$ defined by

$$I(x) \coloneqq \sup_{f\in C_b(\mathbb{R}^h)} \{f(x) - \Gamma(f)\}. \tag{3.2.2}$$

The rate function defined by (3.2.2) is not necessarily convex unlike the Fenchel–Legendre transform. The following theorem in the large deviation theory identifies the good convex rate function for an LDP as the Fenchel–Legendre transform of the limiting logarithmic moment generating function (see also Theorem C.0.9 in Appendix C for a more general statement in the setting of a topological vector space instead of $\mathbb{R}^h$).

Theorem 3.2.2 ([8, Theorem 4.5.10])**.** *Let $\{S^n\}_{n\in\mathbb{N}}$ be an $\mathbb{R}^h$-valued process. Suppose that the $\mathbb{R}^h$-valued process $\{S^n/|\Lambda^n|\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed $|\Lambda^n|$ and a good convex rate function $I\colon \mathbb{R}^h \to [0,\infty]$, and also that*

$$\sup_{n\in\mathbb{N}} |\Lambda^n|^{-1} \log \mathbb{E}[\exp(\langle \lambda, S^n\rangle_{\mathbb{R}^h})] < \infty$$

for any $\lambda \in \mathbb{R}^h$. Then, for every $\lambda \in \mathbb{R}^h$, the limit

$$\varphi(\lambda) \coloneqq \lim_{n\to\infty} |\Lambda^n|^{-1} \log \mathbb{E}[\exp(\langle \lambda, S^n\rangle_{\mathbb{R}^h})]$$

exists in $\mathbb{R}$, and $I = \varphi^*$ holds.

In order to prove Theorem 3.1.4, what remains to be shown are the following: to check that the assumption of Theorem 3.2.1 is satisfied, to verify the convexity of the good rate function obtained via Theorem 3.2.1, and to characterize the zero point of the good rate function. Those are accomplished by the following three lemmas in this order. These proofs are deferred to Section 3.3.

Lemma 3.2.3. *Suppose that $\{S^n\}_{n\in\mathbb{N}}$ is an exponentially regular nearly additive $\mathbb{R}^h$-valued process satisfying that*

$$\sup_{n\in\mathbb{N}} |\Lambda^n|^{-1} \log \mathbb{E}[\exp(\langle \lambda, S^n\rangle_{\mathbb{R}^h})] < \infty$$

for any $\lambda \in \mathbb{R}^h$. Then, $\{S^n/|\Lambda^n|\}_{n\in\mathbb{N}}$ is exponentially tight, and the limit $\lim_{n\to\infty} |\Lambda^n|^{-1} \Gamma_n(f)$ exists in $[-\infty,\infty)$ for any $f \in \mathscr{F}(\mathbb{R}^h)$.

For the next lemma, we introduce a rate function associated with the empirical means of i.i.d. random variables by Cramér's large deviation theorem (Theorem 2.2.6). For each $k \in \mathbb{N}$, let $\psi_k\colon \mathbb{R}^h \to (-\infty,\infty]$ be the logarithmic moment generating function of $S^k/|\Lambda^k|$, i.e.,

$$\psi_k(\lambda) \coloneqq \log \mathbb{E}[\exp(\langle \lambda, S^k/|\Lambda^k|\rangle_{\mathbb{R}^h})]$$

for any $\lambda \in \mathbb{R}^h$, and let J_k denote the Fenchel–Legendre transform of ψ_k. Then, it is well known that, if we set $\{W_i\}_{i=1}^{\infty}$ as independent copies of $S^k/|\Lambda^k|$, then

$$\limsup_{m\to\infty} m^{-1}\log \mathbb{P}\biggl(\frac{1}{m}\sum_{i=1}^{m} W_i \in F\biggr) \leq -\inf_{x\in F} J_k(x)$$

for any closed set $F \subset \mathbb{R}^h$, and

$$\liminf_{m\to\infty} m^{-1}\log \mathbb{P}\biggl(\frac{1}{m}\sum_{i=1}^{m} W_i \in G\biggr) \geq -\inf_{x\in G} J_k(x)$$

for any open set $G \subset \mathbb{R}^h$ (see, e.g., [8, Theorem 2.2.30]).

Lemma 3.2.4. *Let $\{S^n\}_{n\in\mathbb{N}}$ be an exponentially nearly additive $\mathbb{R}^h$-valued process. Suppose that the $\mathbb{R}^h$-valued process $\{S^n/|\Lambda^n|\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed $|\Lambda^n|$ and a rate function $I\colon \mathbb{R}^h \to [0,\infty]$. Then, the following hold.*

(1) *For every $x \in \mathbb{R}^h$,*

$$I(x) \leq \liminf_{k\to\infty} \frac{J_k(x)}{|\Lambda^k|}. \tag{3.2.3}$$

(2) *I is convex.*

Lemma 3.2.5. *Let $\{S^n\}_{n\in\mathbb{N}}$ be an exponentially nearly additive $\mathbb{R}^h$-valued process. Suppose that the $\mathbb{R}^h$-valued process $\{S^n/|\Lambda^n|\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed $|\Lambda^n|$ and a rate function $I\colon \mathbb{R}^h \to [0,\infty]$. Suppose also that the limit*

$$\widehat{S} := \lim_{n\to\infty} \frac{\mathbb{E}[S^n]}{|\Lambda^n|} \tag{3.2.4}$$

exists in $\mathbb{R}^h$, and that $S^n/|\Lambda^n|$ converges to $\widehat{S}$ in probability as $n \to \infty$. Then, $I(x) = 0$ if and only if $x = \widehat{S}$.

Combining these lemmas with the above two theorems and the strong law of large numbers for $\{S^n\}_{n\in\mathbb{N}}$, we can immediately prove Theorem 3.1.4.

Proof of Theorem 3.1.4. From the assumption, Lemma 3.2.3 implies that the sequences $\{S^n/|\Lambda^n|\}_{n\in\mathbb{N}}$ is exponentially tight, and the limit $\lim_{n\to\infty}|\Lambda^n|^{-1}\Gamma_n(f)$ exists in $[-\infty,\infty)$ for any $f \in \mathscr{F}(\mathbb{R}^h)$. By Theorem 3.2.1, for every $f \in C_b(\mathbb{R}^h)$, the limit

$$\Gamma(f) := \lim_{n\to\infty} |\Lambda^n|^{-1}\Gamma_n(f)$$

also exists in $\mathbb{R}$, and the $\mathbb{R}^h$-valued process $\{S^n/|\Lambda^n|\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed $|\Lambda^n|$ and a good rate function $I\colon \mathbb{R}^h \to [0,\infty]$ defined by

$$I(x) := \sup_{f\in C_b(\mathbb{R}^h)} \{f(x) - \Gamma(f)\}.$$

Furthermore, I is convex by Lemma 3.2.4. Therefore, it follows from Theorem 3.2.2 that for every $\lambda \in \mathbb{R}^h$, the limit

$$\varphi(\lambda) := \lim_{n\to\infty} |\Lambda^n|^{-1} \log \mathbb{E}[\exp(\langle \lambda, S^n\rangle_{\mathbb{R}^h})]$$

exists in $\mathbb{R}$, and $I = \varphi^*$ holds. Lastly, combining Lemma 3.2.5 with the strong law of large numbers for $\{S^n\}_{n\in\mathbb{N}}$ discussed before Theorem 3.1.4, we conclude that $\varphi^*(x) = 0$ if and only if $x = \widehat{S}$, which completes the proof. □

3.3 Proofs of Lemmas 3.2.3, 3.2.4, and 3.2.5

In this section, we prove Lemmas 3.2.3, 3.2.4, and 3.2.5 in this order.

Proof of Lemma 3.2.3. We set $A(\lambda) = \sup_{n\in\mathbb{N}} |\Lambda^n|^{-1} \log \mathbb{E}[\exp(\langle \lambda, S^n\rangle_{\mathbb{R}^h})] < \infty$ for any $\lambda \in \mathbb{R}^h$, and write $S^n = (S^n_1, S^n_2, \dots, S^n_h)$. For any $\alpha \geq 0$, the Markov inequality after exponentiating yields

$$|\Lambda^n|^{-1} \log \mathbb{P}\left(\sum_{i=1}^h \left|S^n_i\right| \geq \alpha|\Lambda^n|\right) \leq |\Lambda^n|^{-1} \log \mathbb{E}\left[\exp\left(\sum_{i=1}^h \left|S^n_i\right|\right)\right] - \alpha.$$

Furthermore,

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\sum_{i=1}^h \left|S^n_i\right|\right)\right] &= \sum_{J\subset[h]} \mathbb{E}\left[\exp\left(\sum_{i=1}^h \left|S^n_i\right|\right); S^n_j \geq 0 \text{ for } j \in J, S^n_j < 0 \text{ for } j \notin J\right] \\
&\leq \sum_{J\subset[h]} \mathbb{E}[\exp(\langle \lambda_J, S^n\rangle)] \\
&\leq \sum_{J\subset[h]} \exp(A(\lambda_J)|\Lambda^n|), \qquad (3.3.1)
\end{aligned}$$

where $[h] := \{1, 2, \dots, h\}$ and $\lambda_J \in \mathbb{R}^h$ is a vector whose jth element is 1 if $j \in J$, otherwise -1. Combining the above estimates, we obtain

$$\limsup_{n\to\infty} |\Lambda^n|^{-1} \log \mathbb{P}\left(\sum_{i=1}^h \left|S^n_i\right| \geq \alpha|\Lambda^n|\right) \leq \max_{J\subset[h]} A(\lambda_J) - \alpha,$$

which immediately implies the exponential tightness of $\{S^n/|\Lambda^n|\}_{n\in\mathbb{N}}$.

Let $f \in \mathscr{F}(\mathbb{R}^h)$ be fixed, and set

$$\begin{aligned}\bar{\Gamma}(f) &:= \limsup_{n\to\infty} |\Lambda^n|^{-1}\Gamma_n(f)\\ &= \limsup_{n\to\infty} |\Lambda^n|^{-1} \log \mathbb{E}\left[\exp\left(|\Lambda^n| f\left(\frac{S^n}{|\Lambda^n|}\right)\right)\right] \in [-\infty, \infty].\end{aligned}$$

If $\bar{\Gamma}(f) = -\infty$, then there is nothing to prove. Hence, we assume $\bar{\Gamma}(f) > -\infty$. Writing the Lipschitz constant of f by $\|f\|_{\mathrm{Lip}}$, we have

$$\begin{aligned}\exp\left(\beta|\Lambda^n| f\left(\frac{S^n}{|\Lambda^n|}\right)\right) &\le \exp(\beta|\Lambda^n| f(0) + \beta\|f\|_{\mathrm{Lip}}\|S^n\|_{\mathbb{R}^h})\\ &\le \exp(\beta|\Lambda^n| f(0)) \exp\left(\beta\|f\|_{\mathrm{Lip}} \sum_{i=1}^h |S_i^n|\right)\end{aligned}$$

for any $\beta \ge 0$. Therefore, it follows from a similar calculation to (3.3.1) that for any $\beta \ge 0$, there exists a constant $A_\beta \ge 0$ such that

$$\mathbb{E}\left[\exp\left(\beta|\Lambda^n| f\left(\frac{S^n}{|\Lambda^n|}\right)\right)\right] \le \exp(A_\beta|\Lambda^n|) \tag{3.3.2}$$

for all $n \in \mathbb{N}$. In particular, $\bar{\Gamma}(f) \le A_1 < \infty$ by taking $\beta = 1$.

Now, we let $\varepsilon > 0$ and show that

$$\liminf_{n\to\infty} |\Lambda^n|^{-1}\Gamma_n(f) \ge \bar{\Gamma}(f) - \varepsilon. \tag{3.3.3}$$

Set $\varepsilon_0 := \varepsilon/(2\|f\|_{\mathrm{Lip}} + 1)$ and take a sufficiently large $C_0 \ge 0$ satisfying that $(A_2 - C_0)/2 < \bar{\Gamma}(f) - \varepsilon$. By the exponentially near additivity of $\{S^n\}_{n\in\mathbb{N}}$, we can take an integer $r \ge 0$, random variables $\{S^{n,z}\}_{n\in\mathbb{N}, z\in\mathbb{Z}^d}$, and an integer $K > r$ such that $\{S^{n,z}\}_{z\in\mathbb{Z}^d}$ are independent copies of S^n for every $n \in \mathbb{N}$ and

$$\mathbb{P}\left(\left\|S^{(2m+1)k} - \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} S^{k-r,z}\right\|_{\mathbb{R}^h} > \varepsilon_0|\Lambda^{(2m+1)k}|\right) \le \exp(-C_0|\Lambda^{(2m+1)k}|) \tag{3.3.4}$$

holds for all $k \ge K$ and $m \in \mathbb{N}$. We may choose the integer K large enough so that

$$\frac{A_2 - C_0}{2} < \left(1 - \frac{r}{K}\right)^d \bar{\Gamma}(f) - \varepsilon. \tag{3.3.5}$$

Now, we fix $k \ge K$ such that

$$|\Lambda^{k-r}|^{-1}\Gamma_{k-r}(f) \ge \bar{\Gamma}(f) - \varepsilon_0. \tag{3.3.6}$$

Let m_n be the unique integer satisfying that $(2m_n+1)k \le n < (2m_n+3)k$ for each $n \in \mathbb{N}$. Then, from the exponential regularity of $\{S^n\}_{n\in\mathbb{N}}$, we can take $N \in \mathbb{N}$ such that

$$\mathbb{P}(\|S^n - S^{(2m_n+1)k}\|_{\mathbb{R}^h} > \varepsilon_0|\Lambda^n|) \le \exp(-C_0|\Lambda^n|) \tag{3.3.7}$$

for all $n \ge N$. By the Lipschitzness of f, we have

$$\begin{aligned}
f\left(\frac{S^n}{|\Lambda^n|}\right) &\ge f\left(\frac{1}{|\Lambda^n|}\sum_{z\in\mathbb{Z}^d\cap[-m_n,m_n]^d} S^{k-r,z}\right) - \frac{\|f\|_{\mathrm{Lip}}}{|\Lambda^n|}\left\|S^n - \sum_{z\in\mathbb{Z}^d\cap[-m_n,m_n]^d} S^{k-r,z}\right\|_{\mathbb{R}^h} \\
&\ge f\left(\frac{1}{|\Lambda^n|}\sum_{z\in\mathbb{Z}^d\cap[-m_n,m_n]^d} S^{k-r,z}\right) - \frac{\|f\|_{\mathrm{Lip}}}{|\Lambda^n|}(R_1(n)+R_2(n)),
\end{aligned}$$

where

$$R_1(n) := \|S^n - S^{(2m_n+1)k}\|_{\mathbb{R}^h} \quad \text{and} \quad R_2(n) := \left\|S^{(2m_n+1)k} - \sum_{z\in\mathbb{Z}^d\cap[-m_n,m_n]^d} S^{k-r,z}\right\|_{\mathbb{R}^h}.$$

Furthermore, the concavity of f yields

$$\begin{aligned}
&f\left(\frac{1}{|\Lambda^n|}\sum_{z\in\mathbb{Z}^d\cap[-m_n,m_n]^d} S^{k-r,z}\right) \\
&= f\left(\frac{(2m_n+1)^d|\Lambda^{k-r}|}{|\Lambda^n|}\frac{1}{(2m_n+1)^d}\sum_{z\in\mathbb{Z}^d\cap[-m_n,m_n]^d}\frac{S^{k-r,z}}{|\Lambda^{k-r}|}\right) \\
&\ge \frac{(2m_n+1)^d|\Lambda^{k-r}|}{|\Lambda^n|} f\left(\frac{1}{(2m_n+1)^d}\sum_{z\in\mathbb{Z}^d\cap[-m_n,m_n]^d}\frac{S^{k-r,z}}{|\Lambda^{k-r}|}\right) \\
&\qquad + \left(1 - \frac{(2m_n+1)^d|\Lambda^{k-r}|}{|\Lambda^n|}\right) f(0) \\
&\ge \frac{|\Lambda^{k-r}|}{|\Lambda^n|}\sum_{z\in\mathbb{Z}^d\cap[-m_n,m_n]^d} f\left(\frac{S^{k-r,z}}{|\Lambda^{k-r}|}\right) + \left(1 - \frac{(2m_n+1)^d|\Lambda^{k-r}|}{|\Lambda^n|}\right) f(0).
\end{aligned}$$

For the first inequality, we note that $(2m_n+1)^d|\Lambda^{k-r}| \le |\Lambda^{(2m_n+1)k}| \le |\Lambda^n|$. Combining the above estimates, we have

$$\begin{aligned}
&|\Lambda^n| f\left(\frac{S^n}{|\Lambda^n|}\right) - \{|\Lambda^n| - (2m_n+1)^d|\Lambda^{k-r}|\} f(0) \\
&\ge |\Lambda^{k-r}|\sum_{z\in\mathbb{Z}^d\cap[-m_n,m_n]^d} f\left(\frac{S^{k-r,z}}{|\Lambda^{k-r}|}\right) - \|f\|_{\mathrm{Lip}}(R_1(n)+R_2(n)).
\end{aligned}$$

By exponentiating and taking expectation,

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(|\Lambda^n| f\left(\frac{S^n}{|\Lambda^n|}\right)\right)\right] \cdot \exp(-\{|\Lambda^n| - (2m_n+1)^d |\Lambda^{k-r}|\} f(0)) \\
&\geq \mathbb{E}\left[\exp\left(|\Lambda^{k-r}| \sum_{z \in \mathbb{Z}^d \cap [-m_n, m_n]^d} f\left(\frac{S^{k-r,z}}{|\Lambda^{k-r}|}\right)\right) \cdot \exp(-\|f\|_{\mathrm{Lip}}(R_1(n) + R_2(n)))\right] \\
&\geq \mathbb{E}\left[\exp\left(|\Lambda^{k-r}| \sum_{z \in \mathbb{Z}^d \cap [-m_n, m_n]^d} f\left(\frac{S^{k-r,z}}{|\Lambda^{k-r}|}\right)\right)\right] \cdot \exp(-\|f\|_{\mathrm{Lip}} \varepsilon_0 (|\Lambda^n| + |\Lambda^{(2m_n+1)k}|)) \\
&\quad - \mathbb{E}\left[\exp\left(|\Lambda^{k-r}| \sum_{z \in \mathbb{Z}^d \cap [-m_n, m_n]^d} f\left(\frac{S^{k-r,z}}{|\Lambda^{k-r}|}\right)\right); R_1(n) > \varepsilon_0 |\Lambda^n|\right] \\
&\quad - \mathbb{E}\left[\exp\left(|\Lambda^{k-r}| \sum_{z \in \mathbb{Z}^d \cap [-m_n, m_n]^d} f\left(\frac{S^{k-r,z}}{|\Lambda^{k-r}|}\right)\right); R_2(n) > \varepsilon_0 |\Lambda^{(2m_n+1)k}|\right] \\
&=: I_1 - I_2 - I_3. \qquad (3.3.8)
\end{aligned}
$$

For the following calculations, we note that $\{S^{n,z}\}_{z \in \mathbb{Z}^d}$ are independent copies of S^n for every $n \in \mathbb{N}$. Note also that $(2m_n+1)^d |\Lambda^{k-r}| \leq |\Lambda^{(2m_n+1)k}| \leq |\Lambda^n|$. By (3.3.6),

$$
\begin{aligned}
I_1 &= \mathbb{E}\left[\exp\left(|\Lambda^{k-r}| f\left(\frac{S^{k-r}}{|\Lambda^{k-r}|}\right)\right)\right]^{(2m_n+1)^d} \cdot \exp(-\|f\|_{\mathrm{Lip}} \varepsilon_0 (|\Lambda^n| + |\Lambda^{(2m_n+1)k}|)) \\
&\geq \exp((2m_n+1)^d |\Lambda^{k-r}| (\bar{\Gamma}(f) - \varepsilon_0)) \cdot \exp(-\|f\|_{\mathrm{Lip}} \varepsilon_0 (|\Lambda^n| + |\Lambda^{(2m_n+1)k}|)) \\
&\geq \exp((2m_n+1)^d |\Lambda^{k-r}| \bar{\Gamma}(f) - (2\|f\|_{\mathrm{Lip}} + 1) \varepsilon_0 |\Lambda^n|) \\
&= \exp\left(\left(\frac{(2m_n+1)^d |\Lambda^{k-r}|}{|\Lambda^n|} \bar{\Gamma}(f) - \varepsilon\right) |\Lambda^n|\right).
\end{aligned}
$$

By the Cauchy–Schwarz inequality, (3.3.7), and (3.3.2) with $\beta = 2$,

$$
\begin{aligned}
I_2 &\leq \mathbb{E}\left[\exp\left(2|\Lambda^{k-r}| \sum_{z \in \mathbb{Z}^d \cap [-m_n, m_n]^d} f\left(\frac{S^{k-r,z}}{|\Lambda^{k-r}|}\right)\right)\right]^{1/2} \cdot \mathbb{P}(R_1(n) > \varepsilon_0 |\Lambda^n|)^{1/2} \\
&= \mathbb{E}\left[\exp\left(2|\Lambda^{k-r}| f\left(\frac{S^{k-r}}{|\Lambda^{k-r}|}\right)\right)\right]^{(2m_n+1)^d/2} \cdot \exp(-C_0 |\Lambda^n|/2) \\
&\leq \exp\left(\frac{A_2 - C_0}{2} |\Lambda^n|\right)
\end{aligned}
$$

for all $n \geq N$. Similarly, by the Cauchy–Schwarz inequality, (3.3.4), and (3.3.2) with $\beta = 2$,

$$
I_3 \leq \exp(A_2 |\Lambda^n|/2 - C_0 |\Lambda^{(2m_n+1)k}|/2) = \exp\left(\frac{1}{2}\left(A_2 - C_0 \frac{|\Lambda^{(2m_n+1)k}|}{|\Lambda^n|}\right) |\Lambda^n|\right)
$$

for all $n \geq N$. Since

$$\lim_{n\to\infty} \frac{(2m_n+1)^d|\Lambda^{k-r}|}{|\Lambda^n|} = \left(1-\frac{r}{k}\right)^d \quad \text{and} \quad \lim_{n\to\infty} \frac{|\Lambda^{(2m_n+1)k}|}{|\Lambda^n|} = 1,$$

it follows from (3.3.5) that

$$\frac{A_2 - C_0}{2} < \frac{1}{2}\left(A_2 - C_0\frac{|\Lambda^{(2m_n+1)k}|}{|\Lambda^n|}\right) < \frac{(2m_n+1)^d|\Lambda^{k-r}|}{|\Lambda^n|}\bar{\Gamma}(f) - \varepsilon$$

for sufficiently large n. Therefore, an elementary calculation yields

$$\liminf_{n\to\infty} |\Lambda^n|^{-1}\log(I_1 - I_2 - I_3) \geq \left(1-\frac{r}{k}\right)^d \bar{\Gamma}(f) - \varepsilon.$$

Thus, combining this estimate with (3.3.8), we obtain

$$\liminf_{n\to\infty} |\Lambda^n|^{-1}\Gamma_n(f) - \left(1-\left(1-\frac{r}{k}\right)^d\right)f(0) \geq \left(1-\frac{r}{k}\right)^d \bar{\Gamma}(f) - \varepsilon.$$

Taking $k \to \infty$ in the above inequality along a suitable subsequence so that (3.3.6) is satisfied yields (3.3.3), which completes the proof since $\varepsilon > 0$ is arbitrary. □

Next, we turn to prove Lemma 3.2.4.

Proof of Lemma 3.2.4. (1) If $\liminf_{k\to\infty} J_k(x)/|\Lambda^k| = \infty$, then there is nothing to prove. Hence, we assume that $\liminf_{k\to\infty} J_k(x)/|\Lambda^k| < \infty$. Let $c < I(x)$ be fixed. By the lower semicontinuity of the rate function I, we take $\varepsilon > 0$ such that $y \in \bar{B}(x,\varepsilon)$ implies $I(y) \geq c$. We set $C := \liminf_{k\to\infty} J_k(x)/|\Lambda^k| + 1 < \infty$. From the exponentially near additivity of $\{S^n\}_{n\in\mathbb{N}}$, we can take an integer $r \geq 0$, random variables $\{S^{n,z}\}_{n\in\mathbb{N},z\in\mathbb{Z}^d}$, and an integer $K > r$ such that $\{S^{n,z}\}_{z\in\mathbb{Z}^d}$ are independent copies of S^n for every $n \in \mathbb{N}$ and

$$\mathbb{P}\left(\left\|S^{(2m+1)k} - \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} S^{k-r,z}\right\|_{\mathbb{R}^h} > \frac{\varepsilon}{2}|\Lambda^{(2m+1)k}|\right) \leq \exp(-C|\Lambda^{(2m+1)k}|) \tag{3.3.9}$$

holds for all $k \geq K$ and $m \in \mathbb{N}$. We may choose the integer K large enough so that

$$\frac{|\Lambda^{K-r}|}{|\Lambda^K|}x \in B(x,\varepsilon/2). \tag{3.3.10}$$

Now, we fix $k \geq K$ satisfying that

$$\frac{J_{k-r}(x)+1}{|\Lambda^{k-r}|} < C. \tag{3.3.11}$$

By the large deviation principles for $\{S^n/|\Lambda^n|\}_{n\in\mathbb{N}}$ controlled by I, we have

$$
\begin{aligned}
-c &\geq -\inf_{y\in\bar{B}(x,\varepsilon)} I(y) \\
&\geq \limsup_{m\to\infty} \frac{1}{|\Lambda^{(2m+1)k}|} \log \mathbb{P}\left(\frac{S^{(2m+1)k}}{|\Lambda^{(2m+1)k}|} \in \bar{B}(x,\varepsilon)\right) \\
&\geq \limsup_{m\to\infty} \frac{1}{|\Lambda^{(2m+1)k}|} \log\left\{\mathbb{P}\left(\frac{1}{|\Lambda^{(2m+1)k}|} \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} S^{k-r,z} \in B(x,\varepsilon/2)\right)\right. \\
&\quad \left. - \mathbb{P}\left(\left\| S^{(2m+1)k} - \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} S^{k-r,z}\right\|_{\mathbb{R}^h} > \frac{\varepsilon}{2}|\Lambda^{(2m+1)k}|\right)\right\} \\
&\geq \limsup_{m\to\infty} \frac{1}{|\Lambda^{(2m+1)k}|} \log\left\{\mathbb{P}\left(\frac{1}{(2m+1)^d} \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} \frac{S^{k-r,z}}{|\Lambda^{k-r}|} \in \frac{|\Lambda^k|}{|\Lambda^{k-r}|} B(x,\varepsilon/2)\right)\right. \\
&\quad \left. - \exp(-C|\Lambda^{(2m+1)k}|)\right\}. \qquad (3.3.12)
\end{aligned}
$$

The fourth inequality was obtained by (3.3.9). In order to estimate the right-hand side of (3.3.12), we will use the lower bound of Cramér's large deviation theorem (Theorem 2.2.6) for

$$
\left\{\frac{1}{(2m+1)^d} \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} \frac{S^{k-r,z}}{|\Lambda^{k-r}|}\right\}_{m\in\mathbb{N}}
$$

controlled by J_{k-r}:

$$
\begin{aligned}
&\liminf_{m\to\infty} \frac{1}{(2m+1)^d} \log \mathbb{P}\left(\frac{1}{(2m+1)^d} \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} \frac{S^{k-r,z}}{|\Lambda^{k-r}|} \in \frac{|\Lambda^k|}{|\Lambda^{k-r}|} B(x,\varepsilon/2)\right) \\
&\geq -\inf\left\{J_{k-r}(y) \;\middle|\; y \in \frac{|\Lambda^k|}{|\Lambda^{k-r}|} B(x,\varepsilon/2)\right\}.
\end{aligned}
$$

Since the right-hand side of the above inequality is bounded below by $-J_{k-r}(x)$ from (3.3.10), we obtain

$$
\begin{aligned}
&\mathbb{P}\left(\frac{1}{(2m+1)^d} \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} \frac{S^{k-r,z}}{|\Lambda^{k-r}|} \in \frac{|\Lambda^k|}{|\Lambda^{k-r}|} B(x,\varepsilon/2)\right) \\
&\geq \exp(-\{J_{k-r}(x)+1\}(2m+1)^d) \qquad (3.3.13)
\end{aligned}
$$

for sufficiently large m. Combining (3.3.12) and (3.3.13),

$$
\begin{aligned}
&-c \\
&\geq \limsup_{m\to\infty} \frac{1}{|\Lambda^{(2m+1)k}|} \log\Big\{\exp(-\{J_{k-r}(x)+1\}(2m+1)^d) - \exp(-C|\Lambda^{(2m+1)k}|)\Big\} \\
&= \limsup_{m\to\infty} \frac{1}{|\Lambda^{(2m+1)k}|} \log\Big\{\exp\Big(-\frac{J_{k-r}(x)+1}{|\Lambda^k|}|\Lambda^{(2m+1)k}|\Big) - \exp(-C|\Lambda^{(2m+1)k}|)\Big\} \\
&= -\frac{J_{k-r}(x)+1}{|\Lambda^k|}.
\end{aligned}
$$

For the last line, we note (3.3.11). Letting $k \to \infty$ in the above inequality along a suitable subsequence so that (3.3.11) is satisfied, we conclude that $c \leq \liminf_{k\to\infty} J_k(x)/|\Lambda^k|$, which completes the proof since $c < I(x)$ is arbitrary.

(2) Let $x_1 \neq x_2 \in \mathbb{R}^h$ and $t \in (0,1)$ be fixed, and set $x := tx_1 + (1-t)x_2$. We will show

$$
tI(x_1) + (1-t)I(x_2) \geq I(x). \tag{3.3.14}
$$

We may assume that $I(x_1) < \infty$ and $I(x_2) < \infty$. We first take an integer $r \geq 0$ and random variables $\{S^{n,z}\}_{n\in\mathbb{N},z\in\mathbb{Z}^d}$ in the definition of the exponentially near additivity of $\{S^n\}_{n\in\mathbb{N}}$. Let $l \in \mathbb{N}$ be fixed. We set $\delta > 0$ such that $ty_1 + (1-t)y_2 \in \bar{B}(x, l^{-1})$ for any $y_1 \in \bar{B}(x_1,\delta)$ and $y_2 \in \bar{B}(x_2,\delta)$. From the exponentially near additivity of $\{S^n\}_{n\in\mathbb{N}}$, letting $C > \max\{I(x_1), I(x_2)\}$, we can take an integer $K_l > r$ such that

$$
\mathbb{P}\Bigg(\Bigg\|S^{(2m+1)k} - \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} S^{k-r,z}\Bigg\|_{\mathbb{R}^h} > \frac{\delta}{3}|\Lambda^{(2m+1)k}|\Bigg) \leq \exp(-C|\Lambda^{(2m+1)k}|) \tag{3.3.15}
$$

holds for all $k \geq K_l$ and $m \in \mathbb{N}$. We may choose the integer K_l large enough so that for $i = 1, 2$,

$$
\frac{|\Lambda^{K_l}|}{|\Lambda^{K_l - r}|} B(x_i, 2\delta/3) \subset \bar{B}(x_i, \delta). \tag{3.3.16}
$$

Then, by the large deviation principle for $\{S^n/|\Lambda^n|\}_{n\in\mathbb{N}}$ controlled by I and Cramér's large deviation theorem (Theorem 2.2.6) for

$$
\left\{\frac{1}{(2m+1)^d} \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} \frac{S^{k-r,z}}{|\Lambda^{k-r}|}\right\}_{m\in\mathbb{N}}
$$

controlled by J_{k-r}, for any $k \geq K_l$ and $i = 1, 2$,

$$
\begin{aligned}
&- I(x_i) \\
&\leq - \inf_{y \in B(x_i, \delta/3)} I(y) \\
&\leq \liminf_{m\to\infty} \frac{1}{|\Lambda^{(2m+1)k}|} \log \mathbb{P}\left(\frac{S^{(2m+1)k}}{|\Lambda^{(2m+1)k}|} \in B(x_i, \delta/3) \right) \\
&\leq \limsup_{m\to\infty} \frac{1}{|\Lambda^{(2m+1)k}|} \log \left[\mathbb{P}\left(\frac{1}{(2m+1)^d} \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} \frac{S^{k-r,z}}{|\Lambda^{k-r}|} \in \frac{|\Lambda^k|}{|\Lambda^{k-r}|} B(x_i, 2\delta/3) \right) \right. \\
&\qquad\qquad \left. + \mathbb{P}\left(\left\| S^{(2m+1)k} - \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} S^{k-r,z} \right\|_{\mathbb{R}^h} > \frac{\delta}{3} |\Lambda^{(2m+1)k}| \right) \right] \\
&\leq \limsup_{m\to\infty} \frac{1}{|\Lambda^{(2m+1)k}|} \log \left[\mathbb{P}\left(\frac{1}{(2m+1)^d} \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} \frac{S^{k-r,z}}{|\Lambda^{k-r}|} \in \bar{B}(x_i, \delta) \right) \right. \\
&\qquad\qquad \left. + \exp(-C|\Lambda^{(2m+1)k}|) \right] \\
&\leq \limsup_{m\to\infty} \frac{1}{|\Lambda^{(2m+1)k}|} \log \left[\exp\left(- \left\{ \inf_{y\in\bar{B}(x_i,\delta)} J_{k-r}(y) - 1 \right\} (2m+1)^d \right) \right. \\
&\qquad\qquad \left. + \exp(-C|\Lambda^{(2m+1)k}|) \right] \\
&= - \min\left\{ \frac{\inf_{y\in\bar{B}(x_i,\delta)} J_{k-r}(y) - 1}{|\Lambda^k|}, C \right\}.
\end{aligned}
$$

The fourth inequality follows from (3.3.15) and (3.3.16). Since $C > \max\{I(x_1), I(x_2)\}$, we obtain

$$
I(x_i) \geq \frac{\inf_{y\in\bar{B}(x_i,\delta)} J_{k-r}(y) - 1}{|\Lambda^k|} \tag{3.3.17}
$$

for any $k \geq K_l$ and $i = 1, 2$. Therefore, it follows from the convexity of J_{k-r} and the setting of δ that

$$
\begin{aligned}
tI(x_1) + (1-t)I(x_2) &\geq \frac{t \inf_{y\in\bar{B}(x_1,\delta)} J_{k-r}(y) + (1-t)\inf_{y\in\bar{B}(x_2,\delta)} J_{k-r}(y) - 1}{|\Lambda^k|} \\
&\geq \frac{\inf_{y\in\bar{B}(x,\varepsilon)} J_{k-r}(y) - 1}{|\Lambda^k|}
\end{aligned}
$$

for any $k \geq K_l$. From the lower semicontinuity of J_{k-r}, there exists $y_l \in \bar{B}(x, l^{-1})$ such that

$$
J_{k-r}(y_l) = \inf_{y\in\bar{B}(x,\varepsilon)} J_{k-r}(y).
$$

From the above discussion, there exists a sequence $\{K_l\}_{l\in\mathbb{N}}$ of integers $K_l > r$ and $\{y_l\}_{l\in\mathbb{N}} \subset \mathbb{R}^h$ such that $\lim_{l\to\infty} y_l = x$ and

$$t I(x_1) + (1-t) I(x_2) \geq \frac{J_{k-r}(y_l) - 1}{|\Lambda^k|}$$

for any $l \in \mathbb{N}$ and $k \geq K_l$. By taking $k \to \infty$ in the above inequality and using (3.2.3), we have

$$t I(x_1) + (1-t) I(x_2) \geq \liminf_{k\to\infty} \frac{J_k(y_l)}{|\Lambda^k|} \geq I(y_l)$$

for any $l \in \mathbb{N}$. Letting $l \to \infty$ in the above inequality, we obtain (3.3.14) from the lower semicontinuity of I, which completes the proof. □

Lastly, we prove Lemma 3.2.5.

Proof of Lemma 3.2.5. We fist show $I(\widehat{S}) = 0$. Let $\varepsilon > 0$ be fixed. From the lower semicontinuity of I, there exists $\delta > 0$ such that $x \in \bar{B}(\widehat{S}, \delta)$ implies $I(x) \geq I(\widehat{S}) - \varepsilon$. Then, by the assumption,

$$0 = \limsup_{n\to\infty} \log \mathbb{P}\left(\frac{S^n}{|\Lambda^n|} \in \bar{B}(\widehat{S}, \delta) \right) \leq - \inf_{x \in \bar{B}(\widehat{S},\delta)} I(x) \leq -I(\widehat{S}) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $I(\widehat{S}) = 0$.

Next, suppose that $I(x) = 0$. Let $\delta > 0$ be fixed. By almost the same calculation to obtain (3.3.17), we can conclude that there exists $K \in \mathbb{N}$ such that

$$0 = I(x) \geq \frac{\inf_{y \in \bar{B}(x,\delta)} J_{k-r}(y) - \rho}{|\Lambda^k|}$$

for any $k \geq K$ and $\rho > 0$. Since $\rho > 0$ is arbitrary, for any $k \geq K$,

$$\inf_{y\in\bar{B}(x,\delta)} J_{k-r}(y) = 0,$$

which implies that $\mathbb{E}[S^{k-r}]/|\Lambda^{k-r}| \in \bar{B}(x, \delta)$ since $\mathbb{E}[S^{k-r}]/|\Lambda^{k-r}|$ is a unique zero point of the rate function J_{k-r}. Using (3.2.4), we obtain $\widehat{S} \in \bar{B}(x, \delta)$. Since $\delta > 0$ is arbitrary, it must be $x = \widehat{S}$. □

Chapter 4

Limit theorems of persistent Betti numbers

In this chapter, we prove Theorems 1.2.1 and 1.2.8. In Section 4.1, we estimate the difference of the persistent Betti numbers of two cubical filtrations in $\mathbb{R}^d$. In Section 4.2, we check the exponential regularity and exponentially near additivity of persistent Betti numbers using the estimate, and prove Theorems 1.2.1 and 1.2.8 by applying Corollary 3.1.7. As previously mentioned, the proof of Theorem 1.2.3 is carried out in Appendix D.

4.1 Properties of persistent Betti number

We start with the simple bound of the persistent Betti number of a cubical filtration in $\mathbb{R}^d$.

Proposition 4.1.1. *Let $\mathbb{X} = \{X(t)\}_{t\geq 0}$ be a bounded cubical filtration in $\mathbb{R}^d$. Fix an integer $0 \leq q < d$ and $0 \leq s \leq t < \infty$. Then,*

$$\beta_q^{\mathbb{X}}(s,t) \leq \beta_q(X(s)) \leq \#\mathcal{K}_q^d(X(s)). \tag{4.1.1}$$

Proof. Since $Z_q(X(s)) \cap B_q(X(t)) \supset Z_q(X(s)) \cap B_q(X(s)) = B_q(X(s))$, it follows from (2.1.1) that

$$\beta_q^{\mathbb{X}}(s,t) = \dim \frac{Z_q(X(s))}{Z_q(X(s)) \cap B_q(X(t))} \leq \dim \frac{Z_q(X(s))}{B_q(X(s))} = \beta_q(X(s)).$$

Furthermore,

$$\beta_q(X(s)) = \dim \frac{Z_q(X(s))}{B_q(X(s))} \leq \dim Z_q(X(s)) \leq \dim C_q(X(s)) = \#\mathcal{K}_q^d(X(s)). \quad \square$$

Next, we estimate the difference of the persistent Betti numbers of two cubical filtrations in $\mathbb{R}^d$. We will use the following basic fact in linear algebra.

Lemma 4.1.2. *Let*

$$D = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

be a block matrix, and let l be the number of columns in B and C. Then,

$$\operatorname{rank} A \le \operatorname{rank} D \le \operatorname{rank} A + l \quad \textit{and} \quad \dim\ker A \le \dim\ker D \le \dim\ker A + l.$$

Proof. Since the rank coincides with the number of linearly independent columns, the first conclusion is trivial. The second conclusion follows immediately from the first conclusion with the rank-nullity theorem: letting k be the number of columns in A,

$$\operatorname{rank} D + \dim\ker D = k + l \quad \text{and} \quad \operatorname{rank} A + \dim\ker A = k. \qquad \square$$

The following is a generalization of Lemma 3.1 of [19] to persistent Betti numbers as well as an analogous result of Lemma 2.11 in [18], where they consider filtrations of simplicial complexes.

Proposition 4.1.3. *Let $\mathbb{X} = \{X(t)\}_{t\ge 0}$ and $\mathbb{Y} = \{Y(t)\}_{t\ge 0}$ be bounded cubical filtrations in $\mathbb{R}^d$ with $X(t) \subset Y(t)$ for every $t \ge 0$. Fix an integer $0 \le q < d$ and $0 \le s \le t < \infty$. Then,*

$$|\beta_q^{\mathbb{Y}}(s,t) - \beta_q^{\mathbb{X}}(s,t)| \le \#(\mathcal{K}_q^d(Y(s)) \setminus \mathcal{K}_q^d(X(s))) + \#(\mathcal{K}_{q+1}^d(Y(t)) \setminus \mathcal{K}_{q+1}^d(X(t))). \tag{4.1.2}$$

Proof. From (2.1.1),

$$\begin{aligned}
\beta_q^{\mathbb{Y}}(s,t) - \beta_q^{\mathbb{X}}(s,t) &= \{\dim(Z_q(Y(s))) - \dim(Z_q(Y(s)) \cap B_q(Y(t)))\} \\
&\quad - \{\dim(Z_q(X(s))) - \dim(Z_q(X(s)) \cap B_q(X(t)))\} \\
&= \dim\left(\frac{Z_q(Y(s))}{Z_q(X(s))}\right) - \dim\left(\frac{Z_q(Y(s)) \cap B_q(Y(t))}{Z_q(X(s)) \cap B_q(X(t))}\right).
\end{aligned}$$

Since

$$\dim\left(\frac{Z_q(Y(s)) \cap B_q(Y(t))}{Z_q(X(s)) \cap B_q(X(t))}\right) \le \dim\left(\frac{Z_q(Y(s))}{Z_q(X(s))}\right) + \dim\left(\frac{B_q(Y(t))}{B_q(X(t))}\right)$$

from an elementary calculation, we obtain

$$|\beta_q^{\mathbb{Y}}(s,t) - \beta_q^{\mathbb{X}}(s,t)| \le \dim\left(\frac{Z_q(Y(s))}{Z_q(X(s))}\right) + \dim\left(\frac{B_q(Y(t))}{B_q(X(t))}\right).$$

Using Lemma 4.1.2, we have

$$\dim\left(\frac{Z_q(Y(s))}{Z_q(X(s))}\right) = \dim\ker\partial_q^{Y(s)} - \dim\ker\partial_q^{X(s)} \leq \#(\mathcal{K}_q^d(Y(s)) \setminus \mathcal{K}_q^d(X(s)))$$

and

$$\dim\left(\frac{B_q(Y(t))}{B_q(X(t))}\right) = \operatorname{rank}\partial_{q+1}^{Y(t)} - \operatorname{rank}\partial_{q+1}^{X(t)} \leq \#(\mathcal{K}_{q+1}^d(Y(t)) \setminus \mathcal{K}_{q+1}^d(X(t))).$$

Combining the above estimates, the conclusion follows. □

The following is useful when we estimate the right-hand sides of (4.1.1) and (4.1.2). This is an easy consequence of the fact that each elementary d-cube contains exactly $\binom{d}{q}2^{d-q}$ number of elementary q-cubes for each integer $0 \leq q \leq d$.

Lemma 4.1.4 ([19, Lemma 3.2]). *Let X and Y be bounded cubical sets with $X \subset Y$. Fix an integer $0 \leq q \leq d$. Suppose that the subset $Y \setminus X \subset \mathbb{R}^d$ is covered by v number of elementary d-cubes. Then, $\#(\mathcal{K}_q^d(Y) \setminus \mathcal{K}_q^d(X)) \leq \binom{d}{q}2^{d-q}v$. In particular, $\#(\mathcal{K}_q^d(Y) \setminus \mathcal{K}_q^d(X)) \leq \#(\mathcal{K}^d(Y) \setminus \mathcal{K}^d(X)) \leq 3^d v$.*

4.2 Proofs of Theorems 1.2.1 and 1.2.8

In this section, let $\mathbb{X} = \{X(t)\}_{t\geq 0}$ be a random cubical filtration in $\mathbb{R}^d$ satisfying Assumptions 2.1.6 and 2.1.7. We fix an integer $0 \leq q < d$ and a finite family $\{(s_i, t_i)\}_{i=1}^h$ with $0 \leq s_i \leq t_i < \infty$, and write $S^n = (\beta_q^{\mathbb{X}^n}(s_1, t_1), \dots, \beta_q^{\mathbb{X}^n}(s_h, t_h))$ for each $n \in \mathbb{N}$. We herein prove Theorems 1.2.1 and 1.2.8 by applying Corollary 3.1.7.

Proofs of Theorems 1.2.1 and 1.2.8. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
|\Lambda^n|^{-1}\log\mathbb{E}[\exp(\langle\lambda, S^n\rangle_{\mathbb{R}^h})] &\leq |\Lambda^n|^{-1}\log\mathbb{E}\left[\exp\left(\|\lambda\|_{\mathbb{R}^h}\left(\sum_{i=1}^h \beta_q^{\mathbb{X}^n}(s_i,t_i)^2\right)^{1/2}\right)\right] \\
&\leq \sqrt{h}\|\lambda\|_{\mathbb{R}^h}\frac{\#\mathcal{K}_q^d(\Lambda^n)}{|\Lambda^n|} \\
&\leq 3^d\sqrt{h}\|\lambda\|_{\mathbb{R}^h}
\end{aligned}$$

for any $\lambda = (\lambda_1, \dots, \lambda_h) \in \mathbb{R}^h$. Here, the second inequality follows from Proposition 4.1.1. For the third inequality, we used Lemma 4.1.4. In order to apply Corollary 3.1.7 to $\{S^n\}_{n\in\mathbb{N}}$, we additionally require the exponential regularity and exponentially near additivity of $\{S^n\}_{n\in\mathbb{N}}$.

We first show the exponential regularity of $\{S^n\}_{n\in\mathbb{N}}$. Let $k \in \mathbb{N}$ be fixed, and let m_n be the unique integer satisfying $(2m_n + 1)k \leq n < (2m_n + 3)k$ for each $n \in \mathbb{N}$. From Proposition 4.1.3 and Lemma 4.1.4, it holds that

$$\begin{aligned}
\|S^n - S^{(2m_n+1)k}\|_{\mathbb{R}^h} &= \left(\sum_{i=1}^{h}\{\beta_q^{\mathbb{X}^n}(s_i,t_i) - \beta_q^{\mathbb{X}^{(2m_n+1)k}}(s_i,t_i)\}^2\right)^{1/2} \\
&\leq \left(\sum_{i=1}^{h}\{\#(\mathcal{K}_q^d(X^n(s_i)) \setminus \mathcal{K}_q^d(X^{(2m_n+1)k}(s_i))\right. \\
&\qquad\qquad \left. + \#(\mathcal{K}_{q+1}^d(X^n(t_i)) \setminus \mathcal{K}_{q+1}^d(X^{(2m_n+1)k}(t_i)))\}^2\right)^{1/2} \\
&\leq \sqrt{h}\left\{\binom{d}{q}2^{d-q} + \binom{d}{q+1}2^{d-(q+1)}\right\}(|\Lambda^n| - |\Lambda^{(2m_n+1)k}|) \\
&\leq 3^d\sqrt{h}(|\Lambda^n| - |\Lambda^{(2m_n+1)k}|).
\end{aligned}$$

Therefore, we obtain

$$|\Lambda^n|^{-1}\|S^n - S^{(2m_n+1)k}\|_{\mathbb{R}^h} \leq 3^d\sqrt{h}\{1 - ((2m_n+1)k/n)^d\}.$$

The right-hand side of the above inequality converges to zero as $n \longrightarrow \infty$, which immediately implies the exponential regularity of $\{S^n\}_{n\in\mathbb{N}}$.

We turn to prove the exponentially near additivity of $\{S^n\}_{n\in\mathbb{N}}$. We take the integer $R \geq 0$ as in Assumption 2.1.7, and choose an integer r such that $2r > R$. For each $n \in \mathbb{N}$ and $z \in \mathbb{Z}^d$, define

$$\Lambda^{n,z} := 2(n+r)z + \Lambda^n,\ \mathbb{X}^{n,z} := \{X(t) \cap \Lambda^{n,z}\}_{t\geq 0},$$

and

$$S^{n,z} := (\beta_q^{\mathbb{X}^{n,z}}(s_1,t_1), \ldots, \beta_q^{\mathbb{X}^{n,z}}(s_h,t_h)).$$

Note that $d_{\max}(\Lambda^{n,z}, \Lambda^{n,z'}) \geq 2r > R$ for any distinct $z, z' \in \mathbb{Z}^d$. Therefore, $\{S^{n,z}\}_{z\in\mathbb{Z}^d}$ are independent copies of S^n for every $n \in \mathbb{N}$ from Assumptions 2.1.6 and 2.1.7. For each $k, m \in \mathbb{N}$, we also define

$$\widetilde{\Lambda}^{(2m+1)k} := \bigsqcup_{z\in\mathbb{Z}^d\cap[-m,m]^d} \Lambda^{k-r,z} \quad \text{and} \quad \widetilde{\mathbb{X}}^{(2m+1)k} := \{X(t) \cap \widetilde{\Lambda}^{(2m+1)k}\}_{t\geq 0}.$$

Since $\widetilde{\Lambda}^{(2m+1)k}$ is a disjoint union of $\Lambda^{k-r,z}$'s, we have

$$\beta_q^{\widetilde{\mathbb{X}}^{(2m+1)k}}(s_i,t_i) = \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} \beta_q^{\mathbb{X}^{k-r,z}}(s_i,t_i)$$

for every $i = 1, 2, \dots, h$. Therefore, again from Proposition 4.1.3 and Lemma 4.1.4, it follows that

$$\begin{aligned}
&\left\| S^{(2m+1)k} - \sum_{z \in \mathbb{Z}^d \cap [-m,m]^d} S^{k-r,z} \right\|_{\mathbb{R}^h} \\
&= \Big\| (\beta_q^{\mathbb{X}^{(2m+1)k}}(s_1,t_1), \dots, \beta_q^{\mathbb{X}^{(2m+1)k}}(s_h,t_h)) \\
&\qquad - (\beta_q^{\widetilde{\mathbb{X}}^{(2m+1)k}}(s_1,t_1), \dots, \beta_q^{\widetilde{\mathbb{X}}^{(2m+1)k}}(s_h,t_h)) \Big\|_{\mathbb{R}^h} \\
&= \left(\sum_{i=1}^h \left\{ \beta_q^{\mathbb{X}^{(2m+1)k}}(s_i,t_i) - \beta_q^{\widetilde{\mathbb{X}}^{(2m+1)k}}(s_i,t_i) \right\}^2 \right)^{1/2} \\
&\leq \sqrt{h} \left\{ \binom{d}{q} 2^{d-q} + \binom{d}{q+1} 2^{d-(q+1)} \right\} (|\Lambda^{(2m+1)k}| - |\widetilde{\Lambda}^{(2m+1)k}|) \\
&\leq 3^d \sqrt{h} (|\Lambda^{(2m+1)k}| - |\widetilde{\Lambda}^{(2m+1)k}|).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
|\Lambda^{(2m+1)k}|^{-1} \left\| S^{(2m+1)k} - \sum_{z \in \mathbb{Z}^d \cap [-m,m]^d} S^{k-r,z} \right\|_{\mathbb{R}^h} &\leq 3^d \sqrt{h} \left(1 - \frac{|\widetilde{\Lambda}^{(2m+1)k}|}{|\Lambda^{(2m+1)k}|} \right) \\
&\leq 3^d \sqrt{h} \left(1 - \left(1 - \frac{r}{k} \right)^d \right).
\end{aligned}$$

Since the right-hand side of the above inequality converges to zero as $k \to \infty$, the exponentially near additivity of $\{S^n\}_{n \in \mathbb{N}}$ follows.

Consequently, applying Corollary 3.1.7 with $S^n = (\beta_q^{\mathbb{X}^n}(s_1,t_1), \dots, \beta_q^{\mathbb{X}^n}(s_h,t_h))$, we complete the proofs of both Theorems 1.2.1 and 1.2.8. □

Chapter 5

Limit theorems of persistence diagrams

In this chapter, we prove Theorems 1.2.5 and 1.2.9. In Section 5.1, we briefly describe a method of lifting the strong law of large numbers for persistent Betti numbers to persistence diagrams, developed in [18]. Moreover, we develop a general method of lifting an LDP for the tuples of persistent Betti numbers to persistence diagrams (Theorem 5.1.2). Applying those methods, we prove Theorems 1.2.5 and 1.2.9 in Section 5.2. The proof of Theorem 5.1.2 is deferred to Section 5.3.

5.1 Statement of result

Before proceeding to our LDP result, we introduce a method of lifting the strong law of large numbers for persistent Betti numbers to persistence diagrams, developed in [18, Section 3 and Appendix A]. The following theorem is immediately obtained by combining [18, Proposition 3.4] and [18, Corollary A.3] together with the inclusion-exclusion principle.

Theorem 5.1.1. *Let $\{\xi^n\}_{n\in\mathbb{N}}$ be an $\mathcal{M}(\Delta)$-valued process. Assume that $\mathbb{E}[\xi^n] \in \mathcal{M}(\Delta)$ for all $n \in \mathbb{N}$ and that for any $0 \le s < t < \infty$, the limit*

$$c_{s,t} := \lim_{n\to\infty} \mathbb{E}[\xi^n([0, s] \times (t, \infty])]$$

exists in $[0, \infty)$. Then, there exists a Radon measure $\widehat{\xi} \in \mathcal{M}(\Delta)$ such that $\mathbb{E}[\xi^n]$ converges vaguely to $\widehat{\xi}$ as $n \to \infty$. Assume further that for any $0 \le s < t < \infty$,

$$\xi^n([0, s] \times (t, \infty]) \to c_{s,t} \quad \text{almost surely as } n \to \infty.$$

Then, ξ^n converges vaguely to $\widehat{\xi}$ almost surely as $n \to \infty$.

Next, we state our LDP result, which is useful to lift an LDP for the tuples of persistent Betti numbers to persistence diagrams. In what follows in this section, let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers tending to infinity as $n \to \infty$. The following is the main result in this section.

Theorem 5.1.2. *Let $\{\xi^n\}_{n\in\mathbb{N}}$ be an $\mathcal{M}(\Delta)$-valued process. Assume that*

$$\sup_{n\in\mathbb{N}} a_n^{-1} \log \mathbb{E}[\exp(a_n \lambda \xi^n(\Delta))] < \infty \tag{5.1.1}$$

for any $\lambda > 0$. Assume further that for any finite family $\mathcal{P} = \{(s_i, t_i)\}_{i=1}^h$ with $0 \le s_i < t_i < \infty$, the $\mathbb{R}^h$-valued process

$$\{(\xi^n([0, s_1] \times (t_1, \infty]), \xi^n([0, s_2] \times (t_2, \infty]), \dots, \xi^n([0, s_h] \times (t_h, \infty]))\}_{n\in\mathbb{N}}$$

satisfies the LDP with a speed a_n and a good rate function $I_{\mathcal{P}} \colon \mathbb{R}^h \to [0, \infty]$. Then, the $\mathcal{M}(\Delta)$-valued process $\{\xi^n\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function $I \colon \mathcal{M}(\Delta) \to [0, \infty]$. Furthermore, the following statements hold.

(1) *Suppose that there exists a constant $K > 0$ such that $\xi^n(\Delta) \le K$ for all $n \in \mathbb{N}$. If $I_{\mathcal{P}}$ has a unique zero point for each finite family $\mathcal{P} = \{(s_i, t_i)\}_{i=1}^h$ with $0 \le s_i < t_i < \infty$, then so does I.*

(2) *If $I_{\mathcal{P}}$ is convex for each finite family $\mathcal{P} = \{(s_i, t_i)\}_{i=1}^h$ with $0 \le s_i < t_i < \infty$, then for every $f \in C_c(\Delta)$, the limit*

$$\varphi(f) := \lim_{n\to\infty} a_n^{-1} \log \mathbb{E}\left[\exp\left(a_n \int_\Delta f \, d\xi^n\right)\right]$$

exists in $\mathbb{R}$, and it holds that

$$I(\xi) = \sup_{f\in C_c(\Delta)} \left\{ \int_\Delta f \, d\xi - \varphi(f) \right\}$$

for any $\xi \in \mathcal{M}(\Delta)$.

5.2 Proofs of Theorems 1.2.5 and 1.2.9

In this section, we apply Theorems 5.1.1 and 5.1.2 to prove Theorems 1.2.5 and 1.2.9, respectively.

Proof of Theorem 1.2.5. Note first that $\xi_q^{\mathbb{X}^n}([0,s]\times(t,\infty]) = \beta_q^{\mathbb{X}^n}(s,t)$ for every $0 \le s \le t < \infty$ from Theorem 2.1.3. Therefore, Theorem 1.2.1 implies that for any $0 \le s < t < \infty$,

$$\lim_{n\to\infty} \frac{\mathbb{E}[\xi_q^{\mathbb{X}^n}([0,s]\times(t,\infty])]}{|\Lambda^n|} = \widehat{\beta}_q(s,t)$$

and

$$\frac{\xi_q^{\mathbb{X}^n}([0,s]\times(t,\infty])}{|\Lambda^n|} \to \widehat{\beta}_q(s,t) \quad \text{almost surely as } n\to\infty.$$

Note also that $\mathbb{E}[\xi_q^{\mathbb{X}^n}]/|\Lambda^n| \in \mathcal{M}(\Delta)$. Therefore, applying Theorem 5.1.1 with $\xi^n = \xi_q^{\mathbb{X}^n}/|\Lambda^n|$ and $a_n = |\Lambda^n|$, we obtain the conclusion. $\square$

Proof of Theorem 1.2.9. We fix a finite family $\mathcal{P} = \{(s_i,t_i)\}_{i=1}^h$ with $0 \le s_i < t_i < \infty$. Since $\xi_q^{\mathbb{X}^n}([0,s]\times(t,\infty]) = \beta_q^{\mathbb{X}^n}(s,t)$ from Theorem 2.1.3, it follows from Theorem 1.2.8 that the $\mathbb{R}^h$-valued process

$$\left\{\left(\frac{\xi_q^{\mathbb{X}^n}([0,s_1]\times(t_1,\infty])}{|\Lambda^n|}, \frac{\xi_q^{\mathbb{X}^n}([0,s_2]\times(t_2,\infty])}{|\Lambda^n|}, \dots, \frac{\xi_q^{\mathbb{X}^n}([0,s_h]\times(t_h,\infty])}{|\Lambda^n|}\right)\right\}_{n\in\mathbb{N}}$$

satisfies the LDP with a speed $|\Lambda^n|$ and a good convex rate function that has a unique zero point. Furthermore, combining (1.2.3) and Lemma 4.1.4, we have $\xi_q^{\mathbb{X}^n}(\Delta) \le \#\mathcal{K}_q^d(\Lambda^n) \le 3^d|\Lambda^n|$. Therefore, (5.1.1) with $\xi^n = \xi_q^{\mathbb{X}^n}/|\Lambda^n|$ and $a_n = |\Lambda^n|$ is satisfied for any $\lambda > 0$. Consequently, Theorem 5.1.2 implies that the $\mathcal{M}(\Delta)$-valued process $\{\xi_q^{\mathbb{X}^n}/|\Lambda^n|\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed $|\Lambda^n|$ and a good convex rate function I_q, defined in (1.2.6), that has a unique zero point. Furthermore, the unique zero point of I_q must be $\widehat{\xi}_q$ in Theorem 1.2.5. $\square$

5.3 Proof of Theorem 5.1.2

In this section, we will prove Theorem 5.1.2. We first introduce the notion of histogram of a given measure $\xi \in \mathcal{M}(\Delta)$, which is useful for the proof of Theorem 5.1.2. For $l \in \mathbb{N}$, let $\mathcal{I}_l$ be the set of all disjoint rectangular regions I of the form either

$$I = \left[0, \frac{1}{2^{l+1}}\right] \times \left(\frac{j-1}{2^{l+1}}, \frac{j}{2^{l+1}}\right] \text{ for } j \in \mathbb{N} \text{ with } 3 \le j \le l\cdot 2^{l+1} \tag{5.3.1}$$

or

$$I = \left(\frac{i-1}{2^{l+1}}, \frac{i}{2^{l+1}}\right] \times \left(\frac{j-1}{2^{l+1}}, \frac{j}{2^{l+1}}\right]$$
$$\text{for } (i,j) \in \mathbb{N}^2 \text{ with } 2 \le i \le j \le l\cdot 2^{l+1} \text{ and } j-i \ge 2. \tag{5.3.2}$$

Given $I \in \bigcup_{l=1}^{\infty} \mathcal{I}_l$, we denote by $\mathrm{LR}(I)$, $\mathrm{UR}(I)$, $\mathrm{UL}(I)$, and $\mathrm{LL}(I)$ the lower-right, upper-right, upper-left, and lower-left corners of I, respectively. Note that every $I \in \bigcup_{l=1}^{\infty} \mathcal{I}_l$ is a relatively compact set in Δ since $\mathrm{LR}(I) \in \Delta$. Therefore, for any $\xi \in \mathcal{M}(\Delta)$ and $I \in \mathcal{I}_l$, it holds that $\xi(I) < \infty$. Note also that

$$\bigcup_{l=1}^{\infty} \bigcup_{I \in \mathcal{I}_l} I = \Delta. \tag{5.3.3}$$

Given $\xi \in \mathcal{M}(\Delta)$ and $l \in \mathbb{N}$, we define the *histogram of ξ with fineness degree l* by

$$\mathrm{HIST}_l(\xi) := (\xi(I))_{I \in \mathcal{I}_l} \in \mathbb{R}^{\mathcal{I}_l}.$$

Lemma 5.3.1. *Let $\{\xi^n\}_{n\in\mathbb{N}}$ be an $\mathcal{M}(\Delta)$-valued process, and let $l \in \mathbb{N}$ be fixed. Assume that for any finite family $\mathcal{P} = \{(s_i, t_i)\}_{i=1}^h$ with $0 \le s_i < t_i < \infty$, the $\mathbb{R}^h$-valued process*

$$\{(\xi^n([0, s_1] \times (t_1, \infty]), \xi^n([0, s_2] \times (t_2, \infty]), \dots, \xi^n([0, s_h] \times (t_h, \infty]))\}_{n\in\mathbb{N}}$$

satisfies the LDP with a speed a_n and a good rate function $I_{\mathcal{P}} \colon \mathbb{R}^h \to [0, \infty]$. Then, the $\mathbb{R}^{\mathcal{I}_l}$-valued process $\{\mathrm{HIST}_l(\xi^n)\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function $I_l \colon \mathbb{R}^{\mathcal{I}_l} \to [0, \infty]$. Furthermore, the following statements hold.

(1) *If $I_{\mathcal{P}}$ has a unique zero point for each finite family $\{(s_i, t_i)\}_{i=1}^h$ with $0 \le s_i < t_i < \infty$, then so does I_l.*

(2) *Suppose that $I_{\mathcal{P}}$ is convex for each finite family $\{(s_i, t_i)\}_{i=1}^h$ with $0 \le s_i < t_i < \infty$, and also that*

$$\sup_{n\in\mathbb{N}} a_n^{-1} \log \mathbb{E}\left[\exp\left(a_n \sum_{I \in \mathcal{I}_l} \lambda_I \xi^n(I)\right)\right] < \infty$$

for any $\lambda = (\lambda_I)_{I \in \mathcal{I}_l} \in \mathbb{R}^{\mathcal{I}_l}$. Then, for every $\lambda = (\lambda_I)_{I \in \mathcal{I}_l} \in \mathbb{R}^{\mathcal{I}_l}$, the limit

$$\varphi_l(\lambda) := \lim_{n\to\infty} a_n^{-1} \log \mathbb{E}\left[\exp\left(a_n \sum_{I \in \mathcal{I}_l} \lambda_I \xi^n(I)\right)\right]$$

exists in $\mathbb{R}$, and $I = \varphi_l^*$ holds.

Proof. Set $\mathcal{P}_l = \bigcup_{I \in \mathcal{I}_l} \{\mathrm{LR}(I), \mathrm{UR}(I), \mathrm{UL}(I), \mathrm{LL}(I)\}$. We define a linear map $F_l \colon \mathbb{R}^{\mathcal{P}_l} \to \mathbb{R}^{\mathcal{I}_l}$ by

$$(F_l(\beta))(I) := \begin{cases} \beta(\mathrm{LR}(I)) - \beta(\mathrm{UR}(I)) & \text{if } I \text{ is of the form (5.3.1)}, \\ \beta(\mathrm{LR}(I)) - \beta(\mathrm{UR}(I)) + \beta(\mathrm{UL}(I)) - \beta(\mathrm{LL}(I)) & \text{if } I \text{ is of the form (5.3.2)} \end{cases}$$

for any $\beta = (\beta(p))_{p \in \mathcal{P}_l} \in \mathbb{R}^{\mathcal{P}_l}$ and $I \in \mathcal{I}_l$. Note that F_l is continuous. Now, for $p = (s,t) \in \mathbb{R}^2$ with $0 \leq s \leq t < \infty$, we write $\beta^n(p) = \xi^n([0,s] \times (t,\infty])$ for convenience. Then, by the inclusion-exclusion principle, we have

$$\mathrm{HIST}_l(\xi^n) = F_l((\beta^n(p))_{p \in \mathcal{P}_l}).$$

Since the $\mathbb{R}^{\mathcal{P}_l}$-valued process $\{(\beta^n(p))_{p \in \mathcal{P}_l}\}_{n \in \mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function $I_{\mathcal{P}_l}$ from the assumption, it follows from Theorem B.0.1 that the $\mathbb{R}^{\mathcal{I}_l}$-valued process $\{\mathrm{HIST}_l(\xi^n)\}_{n \in \mathbb{N}}$ also satisfies the LDP with a speed a_n and a good rate function $I_l \colon \mathbb{R}^{\mathcal{I}_l} \to [0,\infty]$ defined by

$$I_l(H) := \inf_{\beta \in F_l^{-1}(\{H\})} I_{\mathcal{P}_l}(\beta)$$

for any $H \in \mathbb{R}^{\mathcal{I}_l}$. Statement (1) follows immediately from Remark B.0.2 (1). Furthermore, combining Remark B.0.2 (2) with Theorem 3.2.2 (with a speed a_n instead of $|\Lambda^n|$) yields Statement (2). □

Next, we prove the following lemma using the technique of exponentially good approximation (see Appendix B). In what follows, for $\xi \in \mathcal{M}(\Delta)$ and $f \in C_c(\Delta)$, we write $\xi f = \int_\Delta f \, d\xi \in \mathbb{R}$ for simplicity.

Lemma 5.3.2. *Let $\{\xi^n\}_{n \in \mathbb{N}}$ be an $\mathcal{M}(\Delta)$-valued process, and let $m \in \mathbb{N}$ and $f_1, f_2, \ldots, f_m \in C_c(\Delta)$ be fixed. Assume that (5.1.1) holds for any $\lambda > 0$. Assume further that for each fixed $l \in \mathbb{N}$, the $\mathbb{R}^{\mathcal{I}_l}$-valued process $\{\mathrm{HIST}_l(\xi^n)\}_{n \in \mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function $I_l \colon \mathbb{R}^{\mathcal{I}_l} \to [0,\infty]$. Then, the $\mathbb{R}^m$-valued process $\{(\xi^n f_1, \xi^n f_2, \ldots, \xi^n f_m)\}_{n \in \mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function $I_{f_1,f_2,\ldots,f_m} \colon \mathbb{R}^m \to [0,\infty]$. Furthermore, the following statements hold.*

(1) *Suppose that there exists $K > 0$ such that $\xi^n(\Delta) \leq K$ for all $n \in \mathbb{N}$. If I_l has a unique zero point for each $l \in \mathbb{N}$, then so does $I_{f_1,f_2,\ldots,f_m}$.*

(2) *If I_l is convex for each $l \in \mathbb{N}$, then for every $\lambda = (\lambda_j)_{j=1}^m \in \mathbb{R}^m$, the limit*

$$\varphi_{f_1,f_2,\ldots,f_m}(\lambda) := \lim_{n \to \infty} a_n^{-1} \log \mathbb{E}\left[\exp\left(a_n \sum_{j=1}^m \lambda_j (\xi^n f_j)\right)\right]$$

exists in $\mathbb{R}$, and $I = \varphi^*_{f_1,f_2,\ldots,f_m}$ holds.

Proof. We first note that (5.1.1) implies that $\{\xi^n(\Delta)\}_{n \in \mathbb{N}}$ is exponentially tight with a speed a_n. Indeed, for any $K \geq 0$, the Markov inequality after multiplying a_n and exponentiating yields

$$a_n^{-1} \log \mathbb{P}(\xi^n(\Delta) > K) \leq a_n^{-1} \log \frac{\mathbb{E}[\exp(a_n \xi^n(\Delta))]}{\exp(a_n K)} = a_n^{-1} \log \mathbb{E}[\exp(a_n \xi^n(\Delta))] - K,$$

which immediately implies the exponential tightness of $\{\xi^n(\Delta)\}_{n\in\mathbb{N}}$ with a speed a_n.

Now, for $i \in \{1,2,\dots,m\}$ and $l \in \mathbb{N}$, define a piecewise constant function $f_i^{(l)} \colon \Delta \to \mathbb{R}$ by

$$f_i^{(l)} := \sum_{I\in\mathcal{I}_l} f_i(\mathrm{UR}(I))\mathbb{1}_I.$$

For each $n \in \mathbb{N}$ and $l \in \mathbb{N}$, we set

$$Z^n = (\xi^n f_1, \xi^n f_2, \dots, \xi^n f_m) \quad \text{and} \quad Z^{n,l} = (\xi^n f_1^{(l)}, \xi^n f_2^{(l)}, \dots, \xi^n f_m^{(l)}).$$

In order to apply Theorem B.0.4, we first show that $\{Z^{n,l}\}_{n,l\in\mathbb{N}}$ is an exponentially good approximation of $\{Z^n\}_{n\in\mathbb{N}}$ with a speed a_n. Let $\delta > 0$ and $\alpha > 0$. By the exponential tightness of $\{\xi^n(\Delta)\}_{n\in\mathbb{N}}$ with a speed a_n, there exists a constant $K > 0$ such that

$$\limsup_{n\to\infty} a_n^{-1} \log \mathbb{P}(\xi^n(\Delta) > K) \le -\alpha. \tag{5.3.4}$$

Since $f_1, f_2, \dots, f_m \in C_c(\Delta)$ are uniformly continuous, we can take $\rho > 0$ such that

$$|f_i(x) - f_i(y)| \le \frac{\delta}{\sqrt{m}K}$$

for any $i \in \{1,2,\dots,m\}$ and $x, y \in \Delta$ with $d_\Delta(x,y) < \rho$. Here, d_Δ is a metric that induces the topology on Δ. Noting that (5.3.3) and the compactness of $\bigcup_{i=1}^m \mathrm{supp}(f_i)$, choose $L \in \mathbb{N}$ so that

$$\bigcup_{i=1}^m \mathrm{supp}(f_i) \subset \bigcup_{I\in\mathcal{I}_L} I \quad \text{and} \quad \max_{\substack{I\in\mathcal{I}_L \\ I\cap\bigcup_{i=1}^m \mathrm{supp}(f_i)\neq\emptyset}} \mathrm{diam}_\Delta(I) < \rho.$$

Then, for $l \ge L$ and $i \in \{1,2,\dots,m\}$,

$$\begin{aligned}
\sup_{x\in\Delta} |f_i(x) - f_i^{(l)}(x)| &= \sup_{x\in\bigcup_{I\in\mathcal{I}_L} I} |f_i(x) - f_i^{(l)}(x)| \\
&= \max_{\substack{I\in\mathcal{I}_L \\ I\cap\bigcup_{i=1}^m \mathrm{supp}(f_i)\neq\emptyset}} \max_{\substack{J\in\mathcal{I}_l \\ J\subset I}} \sup_{x\in J} |f_i(x) - f_i^{(l)}(x)| \\
&= \max_{\substack{I\in\mathcal{I}_L \\ I\cap\bigcup_{i=1}^m \mathrm{supp}(f_i)\neq\emptyset}} \max_{\substack{J\in\mathcal{I}_l \\ J\subset I}} \sup_{x\in J} |f_i(x) - f_i(\mathrm{UR}(J))| \\
&\le \frac{\delta}{\sqrt{m}K}.
\end{aligned}$$

Therefore, for $l \ge L$,

$$\|Z^n - Z^{n,l}\|_{\mathbb{R}^m} \le \left(\sum_{i=1}^m \left(\int_\Delta |f_i - f_i^{(l)}|\, d\xi^n \right)^2 \right)^{1/2} \le \frac{\xi^n(\Delta)}{K}\delta,$$

which implies that

$$\limsup_{n\to\infty} a_n^{-1} \log \mathbb{P}(\|Z^n - Z^{n,l}\|_{\mathbb{R}^m} > \delta) \le \limsup_{n\to\infty} a_n^{-1} \log \mathbb{P}(\xi^n(\Delta) > K) \le -\alpha.$$

Thus, we obtain

$$\limsup_{l\to\infty} \limsup_{n\to\infty} a_n^{-1} \log \mathbb{P}(\|Z^n - Z^{n,l}\|_{\mathbb{R}^m} > \delta) \le -\alpha.$$

Since $\alpha > 0$ is arbitrary,

$$\limsup_{l\to\infty} \limsup_{n\to\infty} a_n^{-1} \log \mathbb{P}(\|Z^n - Z^{n,l}\|_{\mathbb{R}^m} > \delta) = -\infty, \tag{5.3.5}$$

which means that $\{Z^{n,l}\}_{n,l\in\mathbb{N}}$ is an exponentially good approximation of $\{Z^n\}_{n\in\mathbb{N}}$ with a speed a_n.

Next, we fix $l \in \mathbb{N}$, and prove that the $\mathbb{R}^m$-valued process $\{Z^{n,l}\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a rate function. For each $i \in \{1, 2, \ldots, m\}$, we define a linear map $G_i^{(l)} : \mathbb{R}^{\mathcal{I}_l} \to \mathbb{R}$ by

$$G_i^{(l)}(H) := \sum_{I\in\mathcal{I}_l} f_i(\mathrm{UR}(I))H(I)$$

for any $H = (H(I))_{I\in\mathcal{I}_l} \in \mathbb{R}^{\mathcal{I}_l}$, and also define a linear map $G^{(l)} : \mathbb{R}^{\mathcal{I}_l} \to \mathbb{R}^m$ by

$$G^{(l)} := (G_1^{(l)}, G_2^{(l)}, \ldots, G_m^{(l)}).$$

Since

$$G_i^{(l)}(\mathrm{HIST}_l(\xi^n)) = \sum_{I\in\mathcal{I}_l} f_i(\mathrm{UR}(I))\xi^n(I) = \xi^n f_i^{(l)}$$

for all $i \in \{1, 2, \ldots, m\}$, we have $G^{(l)}(\mathrm{HIST}_l(\xi^n)) = Z^{n,l}$. Therefore, it follows from the assumption and Theorem B.0.1 that $\{Z^{n,l}\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function.

Furthermore, since $|\xi^n f_i| \le (\sup_{x\in\Delta} |f_i(x)|)\xi^n(\Delta)$, the exponential tightness of $\{Z^n\}_{n\in\mathbb{N}}$ with a speed a_n follows immediately from that of $\{\xi^n(\Delta)\}_{n\in\mathbb{N}}$. Thus, by Theorem B.0.4, the $\mathbb{R}^m$-valued process $\{Z^n\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function.

For Statement (1), suppose that there exists $K > 0$ such that $\xi^n(\Delta) \le K$ for all $n \in \mathbb{N}$. Then, we can replace α in (5.3.4) to ∞. Hence, instead of (5.3.5), we obtain

$$\limsup_{n\to\infty} a_n^{-1} \log \mathbb{P}(\|Z^n - Z^{n,l}\|_{\mathbb{R}^m} > \delta) = -\infty$$

for any $l \ge L$. Therefore, the conclusion follows from Remark B.0.5 (1).

Lastly, we prove Statement (2). By the Cauchy–Schwarz inequality,

$$
\begin{aligned}
&\sup_{n\in\mathbb{N}} a_n^{-1} \log \mathbb{E}\left[\exp\left(a_n \sum_{j=1}^{m} \lambda_j (\xi^n f_j)\right)\right] \\
&\le \sup_{n\in\mathbb{N}} a_n^{-1} \log \mathbb{E}\left[\exp\left(a_n \|\lambda\|_{\mathbb{R}^m} \left(\sum_{j=1}^{m} (\xi^n f_j)^2\right)^{1/2}\right)\right] \\
&\le \sup_{n\in\mathbb{N}} a_n^{-1} \log \mathbb{E}\left[\exp\left(a_n \|\lambda\|_{\mathbb{R}^m} \left(\sum_{j=1}^{m} \sup_{x\in\Delta} f_j(x)^2\right)^{1/2} \xi^n(\Delta)\right)\right]
\end{aligned}
$$

for $\lambda = (\lambda_j)_{j=1}^m \in \mathbb{R}^m$. The right-hand side of the above inequality is finite by (5.1.1). Thus, combining Remark B.0.5 (2) with Theorem 3.2.2 (with a speed a_n instead of $|\Lambda^n|$) yields Statement (2). □

Finally, we prove Theorem 5.1.2 using Theorem C.0.3, which is a general statement to ensure an LDP for random measures (see Appendix C).

Proof of Theorem 5.1.2. Combining Lemmas 5.3.1 and 5.3.2, we conclude that the $\mathbb{R}^m$-valued process $\{(\xi^n f_1, \xi^n f_2, \dots, \xi^n f_m)\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function for any $m \in \mathbb{N}$ and $f_1, f_2, \dots, f_m \in C_c(\Delta)$. Therefore, Theorem C.0.3 implies that the $\mathcal{M}(\Delta)$-valued process $\{\xi^n\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function.

For Statement (1), suppose that there exists $K > 0$ such that $\xi^n(\Delta) \le K$ for all $n \in \mathbb{N}$. Then, the conclusion is an immediate consequence of combining Statements (1) of Lemmas 5.3.1 and 5.3.2 and Theorem C.0.3.

Lastly, we prove Statement (2). We first note that from (5.1.1),

$$
\begin{aligned}
\sup_{n\in\mathbb{N}} a_n^{-1} \log \mathbb{E}\left[\exp\left(a_n \sum_{I\in\mathcal{I}_l} \lambda_I \xi^n(I)\right)\right] &\le \sup_{n\in\mathbb{N}} a_n^{-1} \log \mathbb{E}\left[\exp\left(a_n \max_{I\in\mathcal{I}_l} |\lambda_I| \cdot \xi^n(\Delta)\right)\right] \\
&< \infty
\end{aligned}
$$

for any $\lambda = (\lambda_I)_{I\in\mathcal{I}_l} \in \mathbb{R}^{\mathcal{I}_l}$, and

$$
\sup_{n\in\mathbb{N}} a_n^{-1} \log \mathbb{E}[\exp(a_n \xi^n f)] \le \sup_{n\in\mathbb{N}} a_n^{-1} \log \mathbb{E}\left[\exp\left(a_n \sup_{x\in\Delta} |f(x)| \cdot \xi^n(\Delta)\right)\right] < \infty
$$

for any $f \in C_c(\Delta)$. Therefore, Statement (2) follows immediately by combining Statements (2) of Lemmas 5.3.1 and 5.3.2 and Theorem C.0.3. □

Chapter 6

Conclusion

In order to obtain limit theorems of persistence diagrams for random cubical filtrations, firstly we extended the large deviation principles for regular nearly additive process to vector valued one and applied it to persistent histograms. Secondly, we have shown the law of large numbers and large deviation principle of persistent Betti numbers in the framework of exponentially regular nearly additive process. Thirdly, we have proved the law of large numbers and the large deviation principle of persistence diagrams. For the law of large numbers we used the vague convergence theory developed by Duy et al [18]. For the large deviation principle, we considered the projective system of linear functionals that corresponds to the persistence diagram as a measure and has a good approximation sequence of the persistent histograms, using inheritance theorems such as the contraction principle and Dowson-Gärtner theorem. In the conclusion of this thesis, we will mention the central limit theorem of persistence diagrams and discuss future prospects of random cubical models.

The central limit theorem of persistence diagrams is important but, still unproven. Recently, Fernós et al showed the central limit theorem for random walks on CAT(0) cubical complex, that is simply connected and its vertex links are flag [11]. Moreover, Chen and Epstein also revealed the relation between random walks and the limit distribution of central limit theorem for the random variables whose joint distribution is described by a set of measure [2]. From these papers, we may need to reinterpret our results in terms of links and random walks.

As a future work, we are also interested in the large deviation principles for the Gibbs measure. The book [28] is a good introduction to Gibbs measure. Briefly speaking, a Gibbs measure is a canonical ensemble in statistical mechanics, and one of the simplest examples is a product measure. While we can re-parameterize configurations using d-dimensional lattice points $x \in \mathbb{Z}^d$ and the set of all elementary cubes whose left endpoint is equal to the origin, denoted by $\mathcal{N}$ (See

Appendix D) as follow;

$$(\omega_Q)_{Q\in\mathcal{K}^d} = \{(\omega_{N+x})_{N\in\mathcal{N}}\}_{x\in\mathbb{Z}^d}\,. \tag{6.0.1}$$

The right hand side of (6.0.1) looks like vector valued configurations on the lattice points in $\mathbb{Z}^d$. Therefore, considering the proper function which corresponding to the potential, our result may extend to Gibbs measures since our model looks like a special case of them.

There is also another extension of our model using a *mixing condition*. Let K be a convex compact set in $\mathbb{R}^d$, and $\{X_i\}_{i\in\mathbb{N}}$ be a stationary process taking values in K. For each integers $0 \leq m < n$, let

$$\hat{S}_n^m := \frac{1}{n-m}\sum_{i=m+1}^{n} X_i,$$

with $\hat{S}_n = \hat{S}_n^0$ and μ_n which denotes the law of $\hat{S}_n$. The following assumption can be defined:

Assumption 6.0.1 ([8, Assumption 6.4.1])**.** Let ℓ be an integer. For any continuous function $f\colon K \to [0,1]$, there exist real numbers $\alpha(\ell) \geq 1$, $\beta(\ell) \geq 0$ depending on ℓ and $\gamma > 0$ such that

$$\lim_{\ell\to\infty}\beta(\ell) = 0, \quad \limsup_{\ell\to\infty}\,(\alpha(\ell)-1)\,\ell\,(\log\ell)^{1+\gamma} < \infty,$$

and when ℓ and $n+m$ are large enough,

$$\begin{aligned}\mathbb{E}\left[f(\hat{S}_n)^n f(\hat{S}_{n+m+\ell}^{n+\ell})^m\right] \geq \mathbb{E}\left[f(\hat{S}_n)^n\right]\mathbb{E}\left[f(\hat{S}_m)^m\right]\\ -\beta(\ell)\left\{\mathbb{E}\left[f(\hat{S}_n)^n\right]\mathbb{E}\left[f(\hat{S}_m)^m\right]\right\}^{1/\alpha(\ell)}.\end{aligned}$$

Replacing Theorem 2.2.10's smoothness to this mixing assumption, we obtain the LDP for $\{\mu_n\}$ with a good rate function as the Fenchel–Legendre transformation of $\Lambda(\lambda) = \lim_{n\to\infty} n^{-1}\log\mathbb{E}[\exp(n\langle\lambda,\hat{S}_n\rangle)]$([8, Theorem 6.4.4]). Assumption 6.0.1 holds when the $\{X_i\}_{i\in\mathbb{N}}$ is a bounded and Ψ-mixing process, which is a more general model than ours. Moreover, there is also the LDP for quite a general class of processes ([8, Theorem 6.4.14]). Unfortunately, our subjects fail the conditions of these theorems.

Appendix A

Strong law of large numbers for strongly regular nearly additive processes

In this chapter, we prove the strong law of large numbers under a weaker assumption than that in Chapter 3. In what follows, let $h \in \mathbb{N}$ be fixed.

Definition A.0.1. Let $r \geq 0$ be an integer. We say that an $\mathbb{R}^h$-valued process $\{S^n\}_{n\in\mathbb{N}}$ is *strongly r-nearly additive* if there exist $\mathbb{R}^h$-valued random variables $\{S^{n,z}\}_{n\in\mathbb{N},z\in\mathbb{Z}^d}$ such that the following conditions are satisfied:

- $\{S^{n,z}\}_{z\in\mathbb{Z}^d}$ are independent copies of S^n for every $n \in \mathbb{N}$;
- it holds that

$$\sup_{m\in\mathbb{N}} |\Lambda^{(2m+1)k}|^{-1} \left\| S^{(2m+1)k} - \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} S^{k-r,z} \right\|_{\mathbb{R}^h} \xrightarrow[k\to\infty]{\text{a.s.}} 0. \tag{A.0.1}$$

We also say that an $\mathbb{R}^h$-valued process $\{S^n\}_{n\in\mathbb{N}}$ is *strongly nearly additive* if there exists an integer $r \geq 0$ such that $\{S^n\}_{n\in\mathbb{N}}$ is strongly r-nearly additive.

Definition A.0.2. We say that an $\mathbb{R}^h$-valued process $\{S^n\}_{n\in\mathbb{N}}$ is *strongly regular* if the following property holds for each fixed $k \in \mathbb{N}$: if m_n is taken as the unique integer satisfying that $(2m_n + 1)k \leq n < (2m_n + 3)k$ for each $n \in \mathbb{N}$, then

$$|\Lambda^n|^{-1} \|S^n - S^{(2m_n+1)k}\|_{\mathbb{R}^h} \xrightarrow[n\to\infty]{\text{a.s.}} 0. \tag{A.0.2}$$

Remark A.0.3. By a standard Borel–Cantelli argument, we can easily verify that the exponential regularity and exponentially near additivity implies the strong regularity and strongly near additivity, respectively.

Theorem A.0.4. *Let $\{S^n\}_{n\in\mathbb{N}}$ be a strongly regular nearly additive $\mathbb{R}^h$-valued process consisting of integrable random variables. Suppose that $\sup_{n\in\mathbb{N}} \|\mathbb{E}[S^n]\|_{\mathbb{R}^h}/|\Lambda^n| < \infty$. Then, the limit*

$$\widehat{S} := \lim_{n\to\infty} \frac{\mathbb{E}[S^n]}{|\Lambda^n|} \tag{A.0.3}$$

exists in $\mathbb{R}^h$, and

$$\frac{S^n}{|\Lambda^n|} \to \widehat{S} \quad \textit{almost surely as } n \to \infty.$$

Proof. Since $\sup_{n\in\mathbb{N}} \|\mathbb{E}[S^n]\|_{\mathbb{R}^h}/|\Lambda^n| < \infty$ from the assumption, there exists an accumulation point $\widehat{S} \in \mathbb{R}^h$ of the sequence $\{\mathbb{E}[S^n]/|\Lambda^n|\}_{n\in\mathbb{N}}$. From the strongly near additivity of $\{S^n\}_{n\in\mathbb{N}}$, we can take an integer $r \geq 0$ and $\mathbb{R}^h$-valued random variables $\{S^{n,z}\}_{n\in\mathbb{N},z\in\mathbb{Z}^d}$ such that $\{S^{n,z}\}_{z\in\mathbb{Z}^d}$ are independent copies of S^n for every $n \in \mathbb{N}$ and

$$\sup_{m\in\mathbb{N}} \frac{1}{|\Lambda^{(2m+1)k}|} \left\| S^{(2m+1)k} - \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} S^{k-r,z} \right\|_{\mathbb{R}^h} \xrightarrow[k\to\infty]{\text{a.s.}} 0. \tag{A.0.4}$$

Now, let $k \in \mathbb{N}$ be fixed, and let m_n be the unique integer satisfying that $(2m_n + 1)k \leq n < (2m_n + 3)k$ for each $n \in \mathbb{N}$. Then, by the triangle inequality,

$$\begin{aligned}
&\left\| \frac{S^n}{|\Lambda^n|} - \widehat{S} \right\|_{\mathbb{R}^h} \\
&\leq \frac{1}{|\Lambda^n|} \|S^n - S^{(2m_n+1)k}\|_{\mathbb{R}^h} \\
&\quad + \frac{|\Lambda^{(2m_n+1)k}|}{|\Lambda^n|} \cdot \frac{1}{|\Lambda^{(2m_n+1)k}|} \left\| S^{(2m_n+1)k} - \sum_{z\in\mathbb{Z}^d\cap[-m_n,m_n]^d} S^{k-r,z} \right\|_{\mathbb{R}^h} \\
&\quad + \frac{(2m_n+1)^d |\Lambda^{k-r}|}{|\Lambda^n|} \left\| \frac{1}{(2m_n+1)^d} \sum_{z\in\mathbb{Z}^d\cap[-m_n,m_n]^d} \frac{S^{k-r,z}}{|\Lambda^{k-r}|} - \frac{\mathbb{E}[S^{k-r}]}{|\Lambda^{k-r}|} \right\|_{\mathbb{R}^h} \\
&\quad + \frac{(2m_n+1)^d |\Lambda^{k-r}|}{|\Lambda^n|} \left\| \frac{\mathbb{E}[S^{k-r}]}{|\Lambda^{k-r}|} - \widehat{S} \right\|_{\mathbb{R}^h} + \left(1 - \frac{(2m_n+1)^d |\Lambda^{k-r}|}{|\Lambda^n|}\right) \|\widehat{S}\|_{\mathbb{R}^h} \\
&\leq \frac{1}{|\Lambda^n|} \|S^n - S^{(2m_n+1)k}\|_{\mathbb{R}^h} \\
&\quad + \sup_{m\in\mathbb{N}} \frac{1}{|\Lambda^{(2m+1)k}|} \left\| S^{(2m+1)k} - \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} S^{k-r,z} \right\|_{\mathbb{R}^h} \\
&\quad + \left\| \frac{1}{(2m_n+1)^d} \sum_{z\in\mathbb{Z}^d\cap[-m_n,m_n]^d} \frac{S^{k-r,z}}{|\Lambda^{k-r}|} - \frac{\mathbb{E}[S^{k-r}]}{|\Lambda^{k-r}|} \right\|_{\mathbb{R}^h} \\
&\quad + \left\| \frac{\mathbb{E}[S^{k-r}]}{|\Lambda^{k-r}|} - \widehat{S} \right\|_{\mathbb{R}^h} + \left(1 - \frac{(2m_n+1)^d |\Lambda^{k-r}|}{|\Lambda^n|}\right) \|\widehat{S}\|_{\mathbb{R}^h}
\end{aligned} \tag{A.0.5}$$

for any $n \in \mathbb{N}$. For the second inequality, we also used $(2m_n + 1)^d |\Lambda^{k-r}| \leq |\Lambda^{(2m_n+1)k}| \leq |\Lambda^n|$. The first and third terms in the right-hand side of (A.0.5) converges to zero almost surely as $n \to \infty$ because of the strongly regularity of $\{S^n\}_{n\in\mathbb{N}}$ and the strong law of large numbers, respectively. Noting also that

$$\lim_{n\to\infty} \frac{(2m_n + 1)^d |\Lambda^{k-r}|}{|\Lambda^n|} = \left(1 - \frac{r}{k}\right)^d,$$

we take $n \to \infty$ of both sides of (A.0.5) to obtain

$$\limsup_{n\to\infty} \left\| \frac{S^n}{|\Lambda^n|} - \widehat{S} \right\|_{\mathbb{R}^h} \leq \sup_{m\in\mathbb{N}} \frac{1}{|\Lambda^{(2m+1)k}|} \left\| S^{(2m+1)k} - \sum_{z\in\mathbb{Z}^d\cap[-m,m]^d} S^{k-r,z} \right\|_{\mathbb{R}^h} + \left\| \frac{\mathbb{E}[S^{k-r}]}{|\Lambda^{k-r}|} - \widehat{S} \right\|_{\mathbb{R}^h} + \left(1 - \left(1 - \frac{r}{k}\right)^d\right) \|\widehat{S}\|_{\mathbb{R}^h}.$$

By letting $k \to \infty$ in the above inequality (choose a suitable subsequence in k if necessary), we conclude from (A.0.4) that $S^n/|\Lambda^n|$ converges to $\widehat{S}$ almost surely as $n \to \infty$. In particular, $\widehat{S}$ is a unique accumulation point of the sequence $\{\mathbb{E}[S^n]/|\Lambda^n|\}_{n\in\mathbb{N}}$. Finally, the uniqueness of the accumulation point $\widehat{S}$ together with $\sup_{n\in\mathbb{N}} \|\mathbb{E}[S^n]\|_{\mathbb{R}^h}/|\Lambda^n| < \infty$ implies (A.0.3). □

Remark A.0.5. The *weakly near additivity* and *weak regularity* of an $\mathbb{R}^h$-valued process $\{S^n\}_{n\in\mathbb{N}}$ are defined by replacing the almost sure convergence in (A.0.1) and (A.0.2) to the convergence in probability. For a weakly regular nearly additive $\mathbb{R}^h$-valued process $\{S^n\}_{n\in\mathbb{N}}$ consisting of integrable random variables with $\sup_{n\in\mathbb{N}} \|\mathbb{E}[S^n]\|_{\mathbb{R}^h}/|\Lambda^n| < \infty$, the weak law of large numbers holds: the limit

$$\widehat{S} := \lim_{n\to\infty} \frac{\mathbb{E}[S^n]}{|\Lambda^n|}$$

exists in $\mathbb{R}^h$, and

$$\frac{S^n}{|\Lambda^n|} \to \widehat{S} \quad \text{in probability as } n \to \infty.$$

Since the proof is almost the same, we omit the proof.

Appendix B

Transformations of large deviation principles

We here review basic methods to move around LDPs between different spaces. In the following, let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers tending to infinity as $n \to \infty$.

We first state the *contraction principle*, which states that an LDP is preserved under continuous maps.

Theorem B.0.1 ([8, Theorem 4.2.1])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hausdorff topological spaces, and let $F\colon \mathcal{X} \to \mathcal{Y}$ be a continuous function. Assume that an $\mathcal{X}$-valued process $\{Z^n\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function $I\colon \mathcal{X} \to [0,\infty]$. Then, the $\mathcal{Y}$-valued process $\{F(Z^n)\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function $I'\colon \mathcal{Y} \to [0,\infty]$ defined by*

$$I'(y) := \inf_{x\in F^{-1}(\{y\})} I(x) \tag{B.0.1}$$

for any $y \in \mathcal{Y}$. Here, the infimum over the empty set is regarded as ∞ by convention.

Remark B.0.2. We remark on the uniqueness of the zero point and the convexity of the rate function I' in Theorem B.0.1.

(1) (Uniqueness of the zero point) If the good rate function I has a unique zero point, then so does I'. Indeed, let $\widehat{x} \in \mathcal{X}$ be the unique zero point of I. Then, $I'(F(\widehat{x})) = 0$ from the definition of I'. Furthermore, if $I'(y) = 0$, then there exists $x \in F^{-1}(\{y\})$ such that $I(x) = 0$ by the goodness of the rate function I together with the continuity of F. Thus, $x = \widehat{x}$ from the uniqueness of the zero point of I, hence necessarily $y = F(\widehat{x})$.

(2) (Convexity) Suppose that $\mathcal{X} = \mathbb{R}^h$ and $\mathcal{Y} = \mathbb{R}^{h'}$ for some $h, h' \in \mathbb{N}$ and that f is a linear map. We can easily verify from (B.0.1) that if the rate function I is convex, then so is I'.

Next, we review the notion of exponentially good approximation, and state a technical result, which deduces the LDP from LDPs for approximation sequences.

Definition B.0.3. Let $(\mathcal{Y}, d_{\mathcal{Y}})$ be a metric space, and let $\{Z^n\}_{n\in\mathbb{N}}$ and $\{Z^{n,l}\}_{n,l\in\mathbb{N}}$ be $\mathcal{Y}$-valued random variables. $\{Z^{n,l}\}_{n,l\in\mathbb{N}}$ is called an *exponentially good approximation* of $\{Z^n\}_{n\in\mathbb{N}}$ with a speed a_n if for any $\delta > 0$,

$$\lim_{l\to\infty} \limsup_{n\to\infty} a_n^{-1} \log \mathbb{P}(d_{\mathcal{Y}}(Z^n, Z^{n,l}) > \delta) = -\infty.$$

Combining [8, Lemma 1.2.18] and [8, Theorem 4.2.16], we immediately have the following.

Theorem B.0.4. *Let $(\mathcal{Y}, d_{\mathcal{Y}})$ be a metric space, and let $\{Z^n\}_{n\in\mathbb{N}}$ and $\{Z^{n,l}\}_{n,l\in\mathbb{N}}$ be $\mathcal{Y}$-valued random variables. Assume that the following three conditions are satisfied.*

- *$\{Z^{n,l}\}_{n,l\in\mathbb{N}}$ is an exponentially good approximation of $\{Z^n\}_{n\in\mathbb{N}}$ with a speed a_n.*
- *For each fixed $l \in \mathbb{N}$, the $\mathcal{Y}$-valued process $\{Z^{n,l}\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a rate function $I_l : \mathcal{Y} \to [0, \infty]$.*
- *$\{Z^n\}_{n\in\mathbb{N}}$ is exponentially tight with a speed a_n: for any $\alpha > 0$, there exists a compact set $K \subset \mathcal{Y}$ such that*

$$\limsup_{n\to\infty} a_n^{-1} \log \mathbb{P}(Z^n \notin K) \leq -\alpha.$$

Then, the $\mathcal{Y}$-valued process $\{Z^n\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function $I : \mathcal{Y} \to [0, \infty]$ defined by

$$I(y) := \sup_{\delta>0} \limsup_{l\to\infty} \inf_{y'\in\bar{B}(y,\delta)} I_l(y') \tag{B.0.2}$$

for any $y \in \mathcal{Y}$.

Remark B.0.5. We remark on the uniqueness of the zero point and the convexity of the rate function I in Theorem B.0.4.

(1) (Uniqueness of the zero point) Suppose that $\{Z^{n,l}\}_{n,l\in\mathbb{N}}$ satisfies the following slightly stronger condition than the usual exponentially good approximation condition: for any $\delta > 0$, there exists $L_\delta \in \mathbb{N}$ such that

$$\limsup_{n\to\infty} a_n^{-1} \log \mathbb{P}(\|Z^n - Z^{n,l}\|_{\mathbb{R}^m} > \delta) = -\infty$$

for any $l \geq L_\delta$. In this case, by a simple modification of the proof of Theorem B.0.4, the rate function I is given by

$$I(y) = \sup_{\delta>0} \sup_{l \geq L_\delta} \inf_{y' \in \bar{B}(y,\delta)} I_l(y'). \tag{B.0.3}$$

From (B.0.3), it is not hard to verify that if I_l has a unique zero point for each $l \in \mathbb{N}$, then so does I. In fact, the unique zero point of I is given as the limit of the unique zero point of I_l with respect to l.

(2) (Convexity) Suppose that $\mathcal{X} = \mathbb{R}^h$ and $\mathcal{Y} = \mathbb{R}^{h'}$ for some $h, h' \in \mathbb{N}$. Then, (B.0.2) implies that if the rate function I_l is convex for each $l \in \mathbb{N}$, then so is I.

Appendix C

Large deviation principle for random measures

In this chapter, we provide a sufficient condition for an LDP for random measures. Theorem C.0.3 is the main statement in this chapter, which will be used for proving the LDP for persistence diagrams in Chapter 5.

Let Δ be a (general) locally compact Hausdorff space with countable base, hence necessarily Δ is a complete and separable metric space, the so-called Polish space. Let $C_c(\Delta)$ be the set of all real-valued continuous functions on Δ with compact support. A Borel measure ξ on Δ is called a *Radon measure* if $\xi(K) < \infty$ for every compact set $K \subset \Delta$. Let $\mathcal{M}(\Delta)$ denote the set of all Radon measures on Δ. We equip $\mathcal{M}(\Delta)$ with the vague topology, i.e., the weakest topology such that for every $f \in C_c(\Delta)$, the map $\mathcal{M}(\Delta) \ni \xi \mapsto \xi f := \int_\Delta f \, d\xi \in \mathbb{R}$ is continuous. In fact, $\mathcal{M}(\Delta)$ with the vague topology is a Polish space (see, e.g., Lemma 4.6 in [24]).

Given a Radon measure $\xi \in \mathcal{M}(\Delta)$, we define a linear functional L_ξ on $C_c(\Delta)$ by $L_\xi(f) := \xi f$ for any $f \in C_c(\Delta)$. Note that L_ξ is positive in the sense that $L_\xi(f) \geq 0$ for any $f \in C_c(\Delta)$ with $f \geq 0$. It is well known that there exists a one-to-one correspondence between $\mathcal{M}(\Delta)$ and the set of all positive linear functionals on $C_c(\Delta)$, and the correspondence is given by $\xi \mapsto L_\xi$.

Theorem C.0.1 (Riesz–Markov–Kakutani representation theorem)**.** *Let Δ be a locally compact Hausdorff space with countable base. Then, for any positive linear functional L on $C_c(\Delta)$, there exists a unique Radon measure ξ on Δ such that $L(f) = \xi f := \int_\Delta f \, d\xi$ for any $f \in C_c(\Delta)$.*

Remark C.0.2. Let $C_c(\Delta)'$ be the set of all linear functionals on $C_c(\Delta)$, and let $C_c(\Delta)'_+$ denote the subset of $C_c(\Delta)'$ consisting of all positive linear functionals. It follows from Theorem C.0.1 and the preceding discussion that the map $\Phi \colon \mathcal{M}(\Delta) \ni \xi \to L_\xi \in C_c(\Delta)'_+$ is bijective. We equip $C_c(\Delta)'$ with the weak-* topology, i.e., the weakest topology such that for every $f \in C_c(\Delta)$, the evaluation map $\pi_f\colon C_c(\Delta)' \ni L \mapsto L(f) \in \mathbb{R}$ is continuous. In other words, the weak-* topology on $C_c(\Delta)'$ is generated by all the sets of the form $\pi_f^{-1}(A)$ for some $f \in C_c(\Delta)$ and open set $A \subset \mathbb{R}$. Hence, we can easily verify that $C_c(\Delta)'_+$ is a closed set of $C_c(\Delta)'$ and that the map $\Phi\colon \mathcal{M}(\Delta) \to C_c(\Delta)'_+$ is homeomorphism with respect to the vague topology on $\mathcal{M}(\Delta)$ and the relative topology on $C_c(\Delta)'_+$ induced from the weak-* topology on $C_c(\Delta)'$.

Our aim in this chapter is to prove the following theorem. In what follows, let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers tending to infinity as $n \to \infty$.

Theorem C.0.3. *Let Δ be a locally compact Hausdorff space with countable base. Let $\{\xi^n\}_{n\in\mathbb{N}}$ be an $\mathcal{M}(\Delta)$-valued process. Assume that for any $m \in \mathbb{N}$ and $f_1, f_2, \dots, f_m \in C_c(\Delta)$, the $\mathbb{R}^m$-valued process $\{(\xi^n f_1, \xi^n f_2, \dots, \xi^n f_m)\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function $I_{f_1,f_2,\dots,f_m}\colon \mathbb{R}^m \to [0,\infty]$. Then, the $\mathcal{M}(\Delta)$-valued process $\{\xi^n\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function $I\colon \mathcal{M}(\Delta) \to [0,\infty]$ defined by*

$$I(\xi) \coloneqq \sup_{m\in\mathbb{N}} \sup_{f_1,f_2,\dots,f_m\in C_c(\Delta)} I_{f_1,f_2,\dots,f_m}((\xi f_1, \xi f_2, \dots, \xi f_m)) \tag{C.0.1}$$

for any $\xi \in \mathcal{M}(\Delta)$. Furthermore, the following statements hold.

(1) *If I_f has a unique zero point for every $f \in C_c(\Delta)$, then so does I.*

(2) *Suppose that $I_{f_1,f_2,\dots,f_m}$ is convex for any $m \in \mathbb{N}$ and $f_1, f_2, \dots, f_m \in C_c(\Delta)$, and also that*

$$\sup_{n\in\mathbb{N}} a_n^{-1} \log \mathbb{E}[\exp(a_n \xi^n f)] < \infty$$

for any $f \in C_c(\Delta)$. Then, for every $f \in C_c(\Delta)$, the limit

$$\varphi(f) \coloneqq \lim_{n\to\infty} a_n^{-1} \log \mathbb{E}[\exp(\xi^n f)]$$

exists in $\mathbb{R}$, and it holds that

$$I(\xi) = \sup_{f\in C_c(\Delta)} \{\xi f - \varphi(f)\} \tag{C.0.2}$$

for any $\xi \in \mathcal{M}(\Delta)$.

A key for the proof is the Dawson–Gärtner theorem, which is a useful tool to lift a collection of LDPs in relatively small spaces into an LDP in a larger space identified as their projective limit. We begin by reviewing the notion of projective system and projective limit. Let $(J, \leq)$ be a partially ordered set. Assume that for any $i, j \in J$, there exists $k \in J$ such that both $i \leq k$ and $j \leq k$ hold. Let

$\{\mathcal{Y}_j\}_{j\in J}$ be a family of Hausdorff spaces, and let $\{p_{ij}\colon \mathcal{Y}_j \to \mathcal{Y}_i\}_{i\le j\in J}$ be a family of continuous maps satisfying that $p_{ij}\circ p_{jk} = p_{ik}$ for any $i \le j \le k$ and that p_{jj} is the identity map on $\mathcal{Y}_j$ for any $j \in J$. A pair $(\mathcal{Y}_j, p_{ij})_{i\le j\in J}$ is called a *projective system*. The *projective limit* of a projective system $(\mathcal{Y}_j, p_{ij})_{i\le j\in J}$ is defined as

$$\widetilde{\mathcal{Y}} = \varprojlim_{j\in J} \mathcal{Y}_j := \left\{ x = (y_j)_{j\in J} \in \prod_{j\in J} \mathcal{Y}_j \;\middle|\; y_i = p_{ij}(y_j) \text{ for any } i \le j \in J \right\},$$

equipped with the relative topology induced from the product topology of $\prod_{j\in J}\mathcal{Y}_j$. For each $j \in J$, let $p_j\colon \widetilde{\mathcal{Y}} \to \mathcal{Y}_j$ denote the canonical projection that maps $(y_j)_{j\in J} \in \widetilde{\mathcal{Y}}$ to $y_j \in \mathcal{Y}_j$.

Theorem C.0.4 (Dawson–Gärtner theorem)**.** *Let $\widetilde{\mathcal{Y}}$ be the projective limit of a projective system $(\mathcal{Y}_j, p_{ij})_{i\le j\in J}$, and let $\{S^n\}_{n\in\mathbb{N}}$ be a $\widetilde{\mathcal{Y}}$-valued process. Suppose that for every $j \in J$, the $\mathcal{Y}_j$-valued process $\{p_j(S^n)\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function $I_j\colon \mathcal{Y}_j \to [0,\infty]$. Then, the $\widetilde{\mathcal{Y}}$-valued process $\{S^n\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function $I\colon \widetilde{\mathcal{Y}} \to [0,\infty]$ defined by*

$$I(x) := \sup_{j\in J} I_j(p_j(x))$$

for any $x \in \widetilde{\mathcal{Y}}$.

An important application of Theorem C.0.4 is the case where the projective limit is identified as an algebraic dual, equipped with the weak-* topology, of an infinite dimensional real vector space as follows. Given a real vector space W, let W' denote its *algebraic dual*, i.e., the set of all linear functionals on W. We define a topological space $\mathcal{X}$ as the algebraic dual W' with the weak-* topology, i.e., the weakest topology such that for every $w \in W$, the evaluation map $\pi_w\colon \mathcal{X} \ni x \mapsto x(w) \in \mathbb{R}$ is continuous. Then, $\mathcal{X}$ can be regarded as a projective limit in the following way. First, let $\mathcal{V}$ be the set of all finite dimensional linear subspaces of W, equipped with a partial order $\le$ simply given by the inclusion. Next, for each $V \in \mathcal{V}$, we define $\mathcal{Y}_V$ as the algebraic dual of V equipped with the weak-* topology. This makes $\mathcal{Y}_V$ a Hausdorff space. Also, for any $V \le U \in \mathcal{V}$, we define a continuous map $p_{VU}\colon \mathcal{Y}_U \to \mathcal{Y}_V$ by the restriction: $p_{VU}(L) := L|_U$ for any $L \in \mathcal{Y}_V$. Then, obviously, $(\mathcal{Y}_V, p_{VU})_{V\le U\in\mathcal{V}}$ is a projective system. Let $\widetilde{\mathcal{Y}}$ denote the projective limit of $(\mathcal{Y}_V, p_{VU})_{V\le U\in\mathcal{V}}$. Finally, we define a map $\Psi\colon \mathcal{X} \to \widetilde{\mathcal{Y}}$ by $\Psi(L) := (L|_V)_{V\in\mathcal{V}}$ for any $L \in \mathcal{X}$. One can show that the map Ψ is in fact homeomorphism using the consistency condition: every $(y_V)_{V\in\mathcal{V}} \in \widetilde{\mathcal{Y}}$ satisfies that $y_V = p_{VU}(y_U)$ for any $V \le U \in \mathcal{V}$ (see Theorem 4.6.9 in [8] for details). Consequently, the problem of finding an LDP in the topological space $\mathcal{X}$ is transferred to that in the projective limit $\widetilde{\mathcal{Y}}$, which reduces to LDPs in finite dimensional linear subspaces of W by Theorem C.0.4. Such application of Theorem C.0.4 is summarized into the following useful theorem.

Theorem C.0.5 ([8, Theorem 4.6.9]). *Let W be a real vector space, and let $\mathcal{X}$ be its algebraic dual W' equipped with the weak-* topology. Let $\{x^n\}_{n\in\mathbb{N}}$ be an $\mathcal{X}$-valued process. Assume that for any $m \in \mathbb{N}$ and $w_1, w_2, \ldots, w_m \in W$, the $\mathbb{R}^m$-valued process $\{(x^n(w_1), x^n(w_2), \ldots, x^n(w_m))\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function $J_{w_1,w_2,\ldots,w_m}\colon \mathbb{R}^m \to [0,\infty]$. Then, the $\mathcal{X}$-valued process $\{x^n\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and the good rate function $J\colon \mathcal{X} \to [0,\infty]$ defined by*

$$J(x) := \sup_{m\in\mathbb{N}} \sup_{w_1,w_2,\ldots,w_m\in W} J_{w_1,w_2,\ldots,w_m}((x(w_1), x(w_2), \ldots, x(w_m)))$$

for any $x \in \mathcal{X}$.

In the proof of Theorem C.0.3, we also use the following basic lemma in the large deviation theory.

Lemma C.0.6 ([8, Lemma 4.1.5 (b)]). *Let $\mathcal{E}$ be a closed set of a topological space $\mathcal{X}$, and let $\{S^n\}_{n\in\mathbb{N}}$ be an $\mathcal{X}$-valued process such that $S^n \in \mathcal{E}$ for every $n \in \mathbb{N}$. If the $\mathcal{X}$-valued process $\{S^n\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function $I'\colon \mathcal{X} \to [0,\infty]$, then $\{S^n\}_{n\in\mathbb{N}}$ as an $\mathcal{E}$-valued process satisfies the LDP with a speed a_n and a good rate function $I'|_{\mathcal{E}}\colon \mathcal{E} \to [0,\infty]$.*

Combining Theorem C.0.5 with Remark C.0.2 and Lemma C.0.6, we can prove Theorem C.0.3 except for Statement (2).

Proof of Theorem C.0.3 except for Statement (2). Note first that $(\Phi(\xi^n))(f) = L_{\xi^n}(f) = \xi^n f$ for every $f \in C_c(\Delta)$ from the definition of Φ in Remark C.0.2. Therefore, it follows from the assumption and Theorem C.0.5 with $W = C_c(\Delta)$ that the $C_c(\Delta)'$-valued process $\{\Phi(\xi^n)\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and the good rate function $I'\colon C_c(\Delta)' \to [0,\infty]$ defined by

$$I'(L) := \sup_{m\in\mathbb{N}} \sup_{f_1,f_2,\ldots,f_m\in C_c(\Delta)} I_{f_1,f_2,\ldots,f_m}((L(f_1), L(f_2), \ldots, L(f_m)))$$

for any $L \in C_c(\Delta)'$. Since $C_c(\Delta)'_+ \subset C_c(\Delta)'$ is a closed set as mentioned in Remark C.0.2, Lemma C.0.6 implies that the $C_c(\Delta)'_+$-valued process $\{\Phi(\xi^n)\}_{n\in\mathbb{N}}$ also satisfies the LDP with a speed a_n and a good rate function $I'|_{C_c(\Delta)'_+}\colon C_c(\Delta)'_+ \to [0,\infty]$. Recalling that the map $\Phi\colon \mathcal{M}(\Delta) \to C_c(\Delta)'_+$ is homeomorphism, we can conclude that the $\mathcal{M}(\Delta)$-valued process $\{\xi^n\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and the good rate function defined by (C.0.1).

Next, we prove Statement (1). The existence of a zero point of I follows immediately from the goodness of I. For the uniqueness of the zero point of I, suppose that $I(\xi) = I(\xi') = 0$. Then, $I_f(\xi f) = I_f(\xi' f) = 0$ for any $f \in W$ by (C.0.1). The uniqueness of the zero point of I_f implies that $\xi f = \xi' f$, hence necessarily $\xi = \xi'$. $\square$

In order to prove Theorem C.0.3 (2), we use a generalization of Theorem 3.2.2 to the setting of topological vector spaces. We first review some notion of topological vector space. All vector spaces below are over the field of real numbers. A *topological vector space* $\mathcal{X}$ is a vector space equipped with a topology such that the vector space operations are continuous, i.e.,

- the addition $\mathcal{X} \times \mathcal{X} \ni (x, x') \mapsto x + x' \in \mathcal{X}$ is continuous,
- the scalar multiplication $\mathbb{R} \times \mathcal{X} \ni (\alpha, x) \mapsto \alpha x \in \mathcal{X}$ is continuous.

A topological vector space $\mathcal{X}$ is said to be *locally convex* if there exists a local base at 0 consisting of convex sets. Given a topological vector space $\mathcal{X}$, let $\mathcal{X}^* \subset \mathcal{X}'$ be the subspace consisting of all continuous linear functionals on $\mathcal{X}$. We refer to $\mathcal{X}^*$ as the *topological dual* of $\mathcal{X}$. For $x \in \mathcal{X}$ and $\lambda \in \mathcal{X}'$, define $\langle \lambda, x \rangle := \lambda(x)$ by convention. The following theorem is useful to obtain a topology on a vector space that makes it a locally convex topological vector space.

Theorem C.0.7 ([30, Theorem 3.10]). *Let $\mathcal{X}$ be a vector space, and let $\mathcal{H}$ be a separating subspace of $\mathcal{X}'$, i.e., $\mathcal{H} \subset \mathcal{X}'$ is a subspace satisfying that for any $0 \neq x \in \mathcal{X}$, there exists $\lambda \in \mathcal{H}$ such that $\langle \lambda, x \rangle \neq 0$. Then, the $\mathcal{H}$-topology makes $\mathcal{X}$ into a locally convex Hausdorff topological vector space with $\mathcal{X}^* = \mathcal{H}$. Here, $\mathcal{H}$-topology is the weakest topology on $\mathcal{X}$ such that every $\lambda \in \mathcal{H}$ is continuous.*

Corollary C.0.8. *Let W be a vector space, and let $\mathcal{X}$ be its algebraic dual W' equipped with the weak-* topology. Then, $\mathcal{X}$ is a locally convex Hausdorff topological vector space. Moreover, $\mathcal{X}^*$ and W are isomorphic as vector spaces.*

Proof. We define an injective linear map $\iota \colon W \to \mathcal{X}'$ by $\iota(w) := \pi_w$ for any $w \in W$. Recall here that π_w is the evaluation map. Set $\mathcal{H} := \mathrm{Im}(\iota)$. Then, the $\mathcal{H}$-topology on W' is nothing but the weak-* topology. Additionally, we can easily verify that $\mathcal{H}$ is a separating subspace of $\mathcal{X}'$. Therefore, $\mathcal{X}$ is a locally convex Hausdorff topological vector space by Theorem C.0.7. Furthermore, $\mathcal{X}^* = \mathcal{H}$, which together with the injectivity of ι implies that $\mathcal{X}^*$ and W are isomorphic as vector spaces. $\square$

The following theorem states that the good convex rate function for an LDP in a locally convex Hausdorff topological vector space is identified as the Fenchel–Legendre transform of the limiting logarithmic moment generating function (cf. Theorem 3.2.2).

Theorem C.0.9 ([8, Theorem 4.5.10]). *Let $\mathcal{X}$ be a locally convex Hausdorff topological vector space, and let $\{S^n\}_{n\in\mathbb{N}}$ be an $\mathcal{X}$-valued process. Suppose that the $\mathcal{X}$-valued process $\{S^n\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a good convex rate function $I\colon \mathcal{X} \to [0,\infty]$, and also that*

$$\sup_{n\in\mathbb{N}} a_n^{-1} \log \mathbb{E}[\exp(a_n\langle \lambda, S^n\rangle)] < \infty$$

for any $\lambda \in \mathcal{X}^*$. Then, for every $\lambda \in \mathcal{X}^*$, the limit

$$\varphi(\lambda) := \lim_{n\to\infty} a_n^{-1} \log \mathbb{E}[\exp(a_n\langle \lambda, S^n\rangle)]$$

exists in $\mathbb{R}$, and $I = \varphi^*$ holds. Here, $\varphi^*\colon \mathcal{X} \to [0,\infty]$ is the Fenchel–Legendre transform of $\varphi\colon \mathcal{X}^* \to [0,\infty]$:

$$\varphi^*(x) := \sup_{\lambda\in\mathcal{X}^*} \{\langle \lambda, x\rangle - \varphi(\lambda)\}$$

for any $x \in \mathcal{X}$.

Now, we are ready to prove Theorem C.0.3 (2).

Proof of Theorem C.0.3 (2). Let $\mathcal{X}$ denote the algebraic dual $C_c(\Delta)'$ with the weak-* topology. From Corollary C.0.8 with $W = C_c(\Delta)$, $\mathcal{X}$ is a locally convex Hausdorff topological vector space and $\mathcal{X}^* = \mathrm{Im}(\iota)$. Here, ι is the injective linear map defined in the proof of Corollary C.0.8. As mentioned in *Proof of Theorem C.0.3 except for (2)*, the $C_c(\Delta)'$-valued process $\{\Phi(\xi^n)\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and a good rate function $I'\colon C_c(\Delta)' \to [0,\infty]$ defined by

$$I'(L) := \sup_{m\in\mathbb{N}} \sup_{f_1,f_2,\ldots,f_m\in C_c(\Delta)} I_{f_1,f_2,\ldots,f_m}((L(f_1), L(f_2), \ldots, L(f_m)))$$

for any $L \in C_c(\Delta)'$.

We next claim that I' is a convex function from the assumption of Statement (2). Indeed, suppose that $L_1, L_2 \in C_c(\Delta)'$ and $t \in (0,1)$, and write $L' := tL_1 + (1-t)L_2$. Then, for any $m \in \mathbb{N}$ and $f_1, f_2, \ldots, f_m \in C_c(\Delta)$,

$$\begin{aligned}
&I_{f_1,f_2,\ldots,f_m}((L'(f_1), L'(f_2), \ldots, L'(f_m)))\\
&= I_{f_1,f_2,\ldots,f_m}(t(L_1(f_1), L_1(f_2), \ldots, L_1(f_m))\\
&\quad + (1-t)(L_2(f_1), L_2(f_2), \ldots, L_2(f_m)))\\
&\le tI_{f_1,f_2,\ldots,f_m}((L_1(f_1), L_1(f_2), \ldots, L_1(f_m)))\\
&\quad + (1-t)I_{f_1,f_2,\ldots,f_m}((L_2(f_1), L_2(f_2), \ldots, L_2(f_m)))\\
&\le tI'(L_1) + (1-t)I'(L_2),
\end{aligned}$$

which implies that $I'(L') \leq tI'(L_1) + (1-t)I'(L_2)$. Consequently, I' is convex. Furthermore, for any $\lambda = \iota(f) \in \mathrm{Im}(\iota) = \mathcal{X}^*$,

$$\sup_{n\in\mathbb{N}} a_n^{-1} \log \mathbb{E}[\exp(a_n\langle\lambda, \Phi(\xi^n)\rangle)] = \sup_{n\in\mathbb{N}} a_n^{-1} \log \mathbb{E}[\exp(a_n\xi^n f)] < \infty$$

from the assumption. Therefore, Theorem C.0.9 implies that for every $\lambda \in \mathcal{X}^*$, the limit

$$\varphi(\lambda) \coloneqq \lim_{n\in\mathbb{N}} a_n^{-1} \log \mathbb{E}[\exp(a_n\langle\lambda, \Phi(\xi^n)\rangle)]$$

exists in $\mathbb{R}$, and it holds that

$$I'(L) = \sup_{\lambda\in\mathcal{X}^*} \{\langle\lambda, L\rangle - \varphi(\lambda)\} = \sup_{f\in C_c(\Delta)} \{L(f) - \varphi(\lambda)\}.$$

for any $L \in \mathcal{X}$. Consequently, using Lemma C.0.6 with the closed set $C_c(\Delta)'_+ \subset C_c(\Delta)'$ and the fact that the map $\Phi\colon \mathcal{M}(\Delta) \to C_c(\Delta)'_+$ is homeomorphism, we can conclude that the $\mathcal{M}(\Delta)$-valued process $\{\xi^n\}_{n\in\mathbb{N}}$ satisfies the LDP with a speed a_n and the good rate function defined by (C.0.2). $\square$

Appendix D

Central limit theorem of persistent Betti numbers

In this chapter we review the proof of Theorem 1.2.3, the CLT for persistent Betti numbers, using the Penrose theorem. Before applying the Penrose theorem, we re-parameterize configurations by d-dimensional lattice points $x \in \mathbb{Z}^d$.

Definition D.0.1. The set of all elementary cubes whose left endpoint is equal to the origin

$$\mathcal{N}^d := \left\{ N \in \mathcal{K}^d : \min_{a_i \in I_i(N)} a_i = 0 \text{ for all } i = 1, 2, \ldots, d \right\}.$$

Then all elementary cubes $Q \in \mathcal{K}^d$ are uniquely expressed by $Q = x + N$ using $x \in \mathbb{Z}^d$ and $N \in \mathcal{N}^d$. Using this fact, we regard the $[0, \infty]^{\mathcal{K}^d}$-valued random variables $\left\{ \omega_Q : Q \in \mathcal{K}^d \right\}$ as

$$\left\{ \omega_x = \left(\omega_{x,N} : N \in \mathcal{N}^d \right) : x \in \mathbb{Z}^d \right\},$$

where $\omega_{x,N} := \omega_{x+N} = \omega_Q$. Moreover, replacing ω_Q to this $\omega_{x,N}$, a random cubical set is expressed by

$$X(t) := \bigcup \left\{ x + N : \omega_{x,N} \leq t, x \in \mathbb{Z}^d, N \in \mathcal{N}^d \right\}$$

Let $(\omega^*_{0,x} : N \in \mathcal{N}^d)$ be independent copies of $\omega_0 = \left(\omega_{0,N} : N \in \mathcal{N}^d \right)$, and for ω_x we define

$$\omega^*_x = \begin{cases} (\omega^*_{0,x} : N \in \mathcal{N}^d) & x = 0, \\ (\omega_{0,x} : N \in \mathcal{N}^d) & \text{otherwise.} \end{cases}$$

Then, as $X(t)$ for $\omega^* := \left(\omega_x^* : x \in \mathbb{Z}^d\right)$, the cubical set $X^*(t)$ can be defined. Here we put $\mathcal{B} := \left\{B \subset \mathbb{Z}^d : B = (x + \Lambda^n) \cap \mathbb{Z}^d, x \in \mathbb{Z}^d, n \in \mathbb{N}\right\}$ and $\preceq$ means the lexicographic ordering on $\mathbb{Z}^d$ and $\mathcal{F}_o := \sigma\left(\{\omega_x ; x \preceq O\}\right)$. Let H be the random variable depends on ω_x $(x \in B)$, and for $\{H(\omega; B)\}_{B \in \mathcal{B}}$ we define $(D_O H)(B) := H(\omega; B) - H(\omega^*; B)$. The following result is obtained by Penrose [27].

Theorem D.0.2. *Let the family of real valued random variables $\{H(\omega; B), B \in \mathcal{B}\}$ satisfy the following three conditions;*

(1) *translation invariant*
$H(\tau_x \omega; x + B) = H(\omega; B)$ for all $x \in \mathbb{Z}^d, B \in \mathcal{B}$.

(2) *weak stabilization*
There exists a random variable $D_H(\infty)$, and if $\{A_n; n \in \mathbb{N}\} \subset \mathcal{B}$ satisfies $\liminf A_n = \mathbb{Z}^d$, then $(D_O H)(A_n) \to D_H(\infty)$ in probability as n goes to ∞.

(3) *bounded moment condition*
There exists some constant $\gamma > 2$ and $\sup_{B \in \mathcal{B}} \mathbb{E}|(D_O H)(B)|^\gamma < \infty$.

If $\{A_n : n \in \mathbb{N}\} \subset \mathcal{B}$ satisfies $\liminf A_n = \mathbb{Z}^d$, then

$$\frac{1}{|\Lambda^n|}\mathbb{E}[(H(\omega; A_n) - \mathbb{E}H(\omega; A_n))^2] \to \sigma^2 \ (n \to \infty),$$
$$\frac{1}{|\Lambda^n|^{1/2}}\mathbb{E}[H(\omega; A_n) - \mathbb{E}H(\omega; A_n)] \overset{law}{\Longrightarrow} \mathcal{N}(0, \sigma) \ (n \to \infty)$$

where $\sigma^2 = \mathbb{E}[(\mathbb{E}[D_H(\infty)|\mathcal{F}_O])^2]$ and $\mathcal{N}(0, \sigma^2)$ means the Gauss distribution with the mean 0 and variance σ^2.

Now we check that persistent Betti numbers satisfy the above three conditions to apply this theorem. In particular, since it is non-trivial whether weak stabilization is satisfied, we show the following proposition. Here for a filtration $\mathbb{X}$ and $A \subset \mathbb{Z}^d$, $\mathbb{X}(A)$ means $\mathbb{X}$ with restriction by $\bigcup_{x \in A}\left(x + N : N \in \mathcal{N}^d\right)$.

Proposition D.0.3. *For two random filtrations $\mathbb{X}$ and $\mathbb{Y}$, let their symmetric difference $X(t) \triangle Y(t)$ be bounded for any time $t \in [0, \infty]$. If $\{A_n : n \in \mathbb{N}\}$ satisfies $\liminf A_n = \mathbb{Z}^d$, then there exists a constant θ_∞ and n_∞ such that*

$$\beta_q^{\mathbb{X}(A_n)}(s, t) - \beta_q^{\mathbb{Y}(A_n)}(s, t) = \theta_\infty$$

for $n \geq n_\infty$.

Proof. (1) Firstly, we consider the case restricted by Λ^n. Let the intersection of two filtrations $\mathbb{X}$ and $\mathbb{Y}$ with restricting window Λ^n be denoted by $\mathbb{U}^n :=$

$\{X(t) \cap Y(t) \cap \Lambda^n\}_{0 \le t \le 1}$. From the definition of a persistence Betti number, we obtain the following equation:

$$
\begin{aligned}
&\beta_q^{\mathbb{X}^n}(s,t) - \beta_q^{\mathbb{Y}^n}(s,t) \\
&= \left\{\beta_q^{\mathbb{X}^n}(s,t) - \beta_q^{\mathbb{U}^n}(s,t)\right\} - \left\{\beta_q^{\mathbb{Y}^n}(s,t) - \beta_q^{\mathbb{U}^n}(s,t)\right\} \\
&= \dim Z_q\left(X^n(s)\right) - \dim Z_q\left(U^n(s)\right) \qquad \text{(D.0.1)} \\
&\quad - \left\{\dim Z_q\left(Y^n(s)\right) - \dim Z_q\left(U^n(s)\right)\right\} \\
&\quad + \{\dim(Z_q\left(X^n(s)\right) \cap B_q\left(X^n(t)\right)) \\
&\qquad - \dim(Z_q\left(U^n(s)\right) \cap B_q\left(U^n(t)\right))\} \qquad \text{(D.0.2)} \\
&\quad - \left\{\dim(Z_q\left(Y^n(s)\right) \cap B_q\left(Y^n(t)\right)) \right. \\
&\qquad \left. - \dim(Z_q\left(U^n(s)\right) \cap B_q\left(U^n(t)\right))\right\}.
\end{aligned}
$$

For the sake of the similarity in the form of the formula, to show the convergence of this formula, we need only prove the boundedness and convergences of two terms (D.0.1) and (D.0.2).

- (D.0.1)

boundedness

From Lemma 4.1.1 and the boundedness of the symmetric difference we obtain

$$
\begin{aligned}
\dim Z_q\left(X^n(s)\right) - \dim Z_q\left(U^n(s)\right) &\le \#\mathcal{K}^d\left(X^n(s)\right) - \#\mathcal{K}^d\left(U^n(s)\right) \\
&\le 3^d |X^n(s) - U^n(s)| \\
&\le 3^d |X(s) \setminus U(s)| < \infty.
\end{aligned}
$$

It implies the boundedness of (D.0.1).

convergence

Take integer n, m with $n \le m$ and consider following map $f_{n,m}$:

$$
\begin{array}{cccc}
f_{n,m}: & \frac{Z_q(X^n(s))}{Z_q(U^n(s))} & \longrightarrow & \frac{Z_q(X^m(s))}{Z_q(U^m(s))} \\
& \Cup & & \Cup \\
& c + Z_q\left(U^n(s)\right) & \longmapsto & c + Z_q\left(U^m(s)\right).
\end{array}
$$

If $f_{n,m}([c]) = 0$ where $[*]$ means equivalence class, then $c \in Z_q\left(X^n(s)\right) \cap Z_q\left(U^m(s)\right)$. Here c is the element of $U^n(s)$ because $X^n(s) \cap Y^m(s) \setminus Y^n(s) = \emptyset$. It implies $c \in Z_q\left(U^n(s)\right)$ and $f_{n,m}$ is injection. Then

$$\dim \frac{Z_q\left(X^n(s)\right)}{Z_q\left(U^n(s)\right)} \leq \dim \frac{Z_q\left(X^m(s)\right)}{Z_q\left(U^m(s)\right)},$$

and it means the non-decreasing of $\dim \frac{Z_q(X^n(s))}{Z_q(U^n(s))}$ for window size n.

- (D.0.2)

  (a) $s = t$ case

  boundedness

  In this case (D.0.2) is $\frac{B_q(X^n(t))}{B_q(U^n(t))}$ and $U^n(t) \subset X^n(t)$. It implies $0 \leq \dim \frac{B_q(X^n(t))}{B_q(U^n(t))}$.

  convergence

  Here from assumption $X(r) \setminus U(r)$ each time r is bounded. If $N \in \mathbb{N}$ is large enough, then we can take $X(t) \setminus U(t) \subset \Lambda^N$. For $N \leq n \leq m$ considering map $g_{n,m}$ as follow:

$$\begin{array}{cccc} g_{n,m}: & \frac{B_q(X^n(t))}{B_q(U^n(t))} & \longrightarrow & \frac{B_q(X^m(t))}{B_q(U^m(t))} \\ & \Uparrow & & \Uparrow \\ & c + B_q\left(U^n(t)\right) & \longmapsto & c + B_q\left(U^m(t)\right). \end{array}$$

  Then for the element $c \in B_q\left(X^m(t)\right)$, there exists an element $d \in C_{q+1}(X^m(t))$ such that $\partial_q d = c$. Since we take n, m large enough, this d can be expressed by $d = d_1 + d_2$ using $d_1 \in C_{q+1}(X^n(t))$ and $d_2 \in C_{q+1}(U^m(t))$. By $\partial_q d_1 \in B_q(X^n(t))$ and $\partial_q d_2 \in B_q(U^m(t))$ considering $c = \partial_q d_1 + \partial_q d_2 =: c_1 + c_2$, we obtain $g_{n,m}([c_1]) = [c_1] = [c]$. It implies $g_{n,m}$ is surjective. Therefore, $\dim B_q(X^n(t)) - \dim B_q(U^m(t))$ is non-increasing and bounded below for n.

  (b) $s < t$ case

  In this case, by a fundamental homeomorphism theorem (D.0.2) is expressed by

$$\dim \frac{Z_q(X^n(s)) \cap B_q(X^n(t))}{Z_q(U^n(s)) \cap B_q(U^n(t))} = \dim \frac{Z_q(X^n(s)) \cap B_q(X^n(t))}{Z_q(U^n(s)) \cap B_q(X^n(t))} + \dim \frac{Z_q(U^n(s)) \cap B_q(X^n(t))}{Z_q(U^n(s)) \cap B_q(U^n(t))}.$$

boundedness Here, the first term is bounded since

$$\dim \frac{Z_q(X^n(s)) \cap B_q(X^n(t))}{Z_q(U^n(s)) \cap B_q(X^n(t))} \leq \dim \frac{Z_q(X^n(s))}{Z_q(U^n(s))},$$

and by the convergence of D.0.1 letting k be a non-negative integer which is the destination of convergence, then

$$\dim \frac{Z_q(X^n(s))}{Z_q(U^n(s))} \xrightarrow[n\to\infty]{} k.$$

As the first term letting l be non-negative integer that is the destination of $B_q(X^n(t))/B_q(U^n(t))$, we obtain

$$\dim \frac{Z_q(U^n(s)) \cap B_q(X^n(t))}{Z_q(U^n(s)) \cap B_q(U^n(t))} \leq \dim \frac{B_q(X^n(t))}{B_q(U^n(t))} \xrightarrow[n\to\infty]{} l.$$

Therefore (D.0.2) is bounded.

convergence

Recall that by fundamental homeomorphism theorem we discuss the convergence of each term of

$$\dim \frac{Z_q(X^n(s)) \cap B_q(X^n(t))}{Z_q(U^n(s)) \cap B_q(X^n(t))} + \dim \frac{Z_q(U^n(s)) \cap B_q(X^n(t))}{Z_q(U^n(s)) \cap B_q(U^n(t))}.$$

in this case.

i. $Z_q(X^n(s)) \cap B_q(X^n(t))/Z_q(U^n(s)) \cap B_q(X^n(t))$
Putting $B_n := Z_q(X^n(s))/Z_q(U^n(s))$, then $f_{n,m} \colon B_n \to B_m$ is injective for large enough n. Therefore by the cokernel $\operatorname{cok} f_{n,m} := B_m/\operatorname{Im} f_{n,m}$, there exists an exact sequence

$$0 \to B_n \xrightarrow{f_{n,m}} B_m \to \operatorname{cok} f_{n,m} \to 0$$

and $\dim B_m = \dim B_n + \dim \operatorname{cok} f_{n,m}$. Since $Z_q(X^n(s))/Z_q(U^n(s))$ converges, $\operatorname{cok} f_{n,m} = 0$ for large enough $n \leq m$. It implies $\dim B_n = \dim B_m$. Hence $f_{n,m}$ is an isomorphism. Now we consider

$$\begin{array}{cccc} \bar{f}_{n,m} \colon & \frac{Z_q(X^n(s))\cap B_q(X^n(t))}{Z_q(U^n(s))\cap B_q(X^n(t))} & \to & \frac{Z_q(X^m(s))\cap B_q(X^m(t))}{Z_q(U^m(s))\cap B_q(X^m(t))} \\ & \Psi & & \Psi \\ & [c]_n & \mapsto & [c]_m \end{array}$$

where $[c]_*$ means $c+Z_q(U_*(s)) \cap B_q(X_*(t))$. If $f_{n,m}([c]_n) = 0$ then $c \in Z_q(X^n(s)) \cap B_q(X^n(t)) \cap Z_q(U^m(s)) \cap B_q(X^m(t))$. Since by the taking way of n, m the morphism $f_{n,m}$ is an isomorphism, we obtain $c \in Z_q(U^m(s))$. Therefore $c \in Z_q(U^m(s)) \cap B_q(X^m(t))$ and $\bar{f}_{n,m}$ is an injection.

ii. $Z_q(U^n(s)) \cap B_q(X^n(t))/Z_q(U^n(s)) \cap B_q(U^n(t))$
As a first term putting $D_n := B_q(X^n(t))/B_q(U^n(t))$, then $g_{n,m}\colon D_n \to D_m$ is surjective for large enough n. Hence there exists an exact sequence

$$0 \to \mathrm{Ker} g_{n,m} \to D_n \xrightarrow{g_{n,m}} D_m \to 0$$

and $\dim D_n = \dim D_m + \mathrm{Ker} g_{n,m}$. Here $\dim \mathrm{Ker} g_{n,m} = 0$ because $B_q(X^n(t))/B_q(U^n(t))$ converges: $\dim D_n = \dim D_m$. Since $\mathrm{Ker} g_{n,m}$ is a vector space, $\mathrm{Ker} g_{n,m} = 0$. Therefore $g_{n,m}$ is an isomorphism. Now let us consider

$$\begin{array}{cccc} \bar{g}_{n,m}\colon & \frac{Z_q(U^n(s))\cap B_q(X^n(t))}{Z_q(U^n(s))\cap B_q(U^n(t))} & \to & \frac{Z_q(U^m(s))\cap B_q(X^m(t))}{Z_q(U^m(s))\cap B_q(U^m(t))} \\ & \Psi & & \Psi \\ & [c]_n & \mapsto & [c]_m, \end{array}$$

where $[c]_*$ means $c + Z_q(U_*(s)) \cap B_q(U_*(t))$. If $\bar{g}_{n,m}([c]_n) = 0$, then $c \in Z_q(U^n(s)) \cap B_q(X^n(t)) \cap Z_q(U^m(s)) \cap B_q(U^m(t))$. Since $g_{n,m}$ is an isomorphism, $c \in B_q(X^n(t)) \cap B_q(U^m(t))$ implies $c \in B_q(U^n(t))$. Hence $c \in Z_q(U^n(s)) \cap B_q(U^n(t))$ and $\bar{g}_{n,m}$ is an injection.

Now, let $n'_\infty \in \mathbb{N}$ be a number that satisfies $\beta_q^{\mathbb{X}^n}(s,t) - \beta_q^{\mathbb{Y}^n}(s,t) = \theta_\infty$ for $n > n'_\infty$.

(2) If $\liminf A_n = \mathbb{Z}^d$, then for any point in $\mathbb{Z}^d$ belongs to all A_n except finite A_n. Hence we take n_∞ which is bigger than n'_∞ such that $\Lambda^{n'_\infty} \subset A_n$ for any n larger than n_∞. Here we will show $\dim Z_q(X^n(s)) - \dim Z_q(U^n(s)) = \dim Z_q\left(X^{n'_\infty}(s)\right) - \dim Z_q\left(U^{n'_\infty}(s)\right)$. Take $l \in \mathbb{N}$ which depends on n such that $A_n \subset \Lambda^{n'_\infty + l}$, then we obtain the following injective sequence;

$$\frac{Z_q\left(X^{n'_\infty}\right)}{Z_q\left(W^{n'_\infty}\right)} \rightarrowtail \frac{Z_q\left(X(t) \cap A_n\right)}{Z_q\left(U(t) \cap A_n\right)} \rightarrowtail \frac{Z_q\left(X^{n'_\infty + l}\right)}{Z_q\left(U^{n'_\infty + l}\right)}.$$

From the first part of the proof, the rank of $Z_q\left(X^{n'_\infty}\right)/Z_q\left(U^{n'_\infty}\right)$ is coincident with $Z_q\left(X^{n'_\infty+l}\right)/Z_q\left(U^{n'_\infty+l}\right)$. Repeating this operation for each term, we get $\beta_q^{\mathbb{X}(A_n)}(s,t) - \beta_q^{\mathbb{Y}(A_n)}(s,t) = \theta_\infty$ for $n \geq n_\infty$.

□

Theorem 1.2.3. From the definition of persistent Betti number, translation invariance is satisfied. For $\mathbb{X}^{0,n} := \{X(t) \cap X^*(t) \cap \Lambda^n\}$ and Lemma 4.1.3

$$
\begin{aligned}
&|D_O \beta_q^{\mathbb{X}^n}(s,t)| \\
&\leq |\beta_q^{\mathbb{X}^n}(s,t) - \beta_q^{\mathbb{X}^{0,n}}(s,t)| + |\beta_q^{\mathbb{X}^{0,*}}(s,t) - \beta_q^{\mathbb{X}^{0,n}}(s,t)| \\
&\leq 4 \cdot \#\mathcal{N}^d = 2^{d+2},
\end{aligned}
$$

the moment condition also satisfied. Proposition D.0.3 implies weak stabilization. Therefore persistence Betti number satisfies the central limits theorem. □

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