Some inequalities between Ahlfors regular conformal dimension and spectral dimensions for resistance forms

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Abstract

Quasisymmetric maps are well-studied homeomorphisms between metric spaces preserving annuli, and the Ahlfors regular conformal dimension $\dim_{\mathrm{ARC}}(X, d)$ of a metric space (X, d) is the infimum over the Hausdorff dimensions of the Ahlfors regular images of the space by quasisymmetric transformations. For a given regular Dirichlet form with the heat kernel, the spectral dimension d_s is an exponent which indicates the short-time asymptotic behavior of the on-diagonal part of the heat kernel. In this paper, we consider the Dirichlet form induced by a resistance form on a set X and the associated resistance metric R. We prove $\dim_{\mathrm{ARC}}(X, R) \leq \overline{d_s} < 2$ for $\overline{d_s}$, a variation of d_s defined through the on-diagonal asymptotics of the heat kernel. We also give an example of a resistance form whose spectral dimension d_s satisfies the opposite inequality $d_s < \dim_{\mathrm{ARC}}(X, R) < 2$.

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1 Introduction and main results

The subject of this paper is an evaluation of a dimension of metric spaces, defined through the quasisymmetric transformations. We first recall the definition of quasisymmetry.

Definition 1.1 (Quasisymmetry)**.** Let X be a set and d, ρ be metrics on X. We say d is *quasisymmetric* to ρ, and write $d \sim_{\mathrm{QS}} \rho$, if there exists a homeomorphism $\theta : [0, \infty) \to [0, \infty)$ such that for any $x, y, z \in X$ with $x \neq z$,

$$\rho(x, y)/\rho(x, z) \leq \theta\big(d(x, y)/d(x, z)\big).$$

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Roughly speaking, this definition means that an annulus in (X, d) is comparable to one in (X, ρ). A typical example of a metric quasisymmetric to a given metric d is d^α for each $\alpha \in (0, 1)$. It is known that $\sim_{\mathrm{QS}}$ is an equivalence relation among metrics on X, and that if $d \sim_{\mathrm{QS}} \rho$ then ρ induces the same topology as d.

Quasisymmetry between general metric spaces was defined by Tukia and Väisälä in [31] as the analogy with the case of $\mathbb{R}$. Note that quasisymmetry on $\mathbb{R}$ was discovered as a characterization of the boundary values of quasiconformal mappings from the upper half-plane to itself, by Beurling and Ahlfors in [5], and named by Kelingos in [17]. Properties of quasisymmetry were well-studied in analysis on metric spaces (see [12, 29], for example). Quasisymmetry has been also used in various fields, such as heat kernel estimates (see [2, 4, 15, 20, 24], for example) and hyperbolic group theory (see [6, 7, 23, 25], for example).

The Ahlfors regular conformal dimension of a metric space (X, d) is defined as follows. We set $B_d(x, r) := \{y \in X \mid d(x, y) < r\}$ for $x \in X$ and $r > 0$, and $\mathrm{diam}(X, d) := \sup_{x,y \in X} d(x, y)$.

Definition 1.2 (Ahlfors regular conformal dimension)**.** For $\alpha \in (0, \infty)$, the metric d is called *α-Ahlfors regular* if the α-dimensional Hausdorff measure $\mathcal{H}_\alpha$ satisfies $C^{-1} r^\alpha \leq \mathcal{H}_\alpha(B_d(x, r)) \leq C r^\alpha$ for any $0 < r \leq \mathrm{diam}(X, d)$ and $x \in X$, for some $C > 1$. (Note that if (X, d) is α-Ahlfors regular then $\dim_H(X, d) = \alpha$, where $\dim_H$ is the Hausdorff dimension.) The *Ahlfors regular conformal dimension* $\dim_{\mathrm{ARC}}(X, d)$ of (X, d) is defined by

$$\dim_{\mathrm{ARC}}(X, d) = \inf\left\{ \alpha \in (0, \infty) \;\middle|\; \begin{array}{l} \text{there exists an } \alpha\text{-Ahlfors regular metric } \rho \\ \text{on } X \text{ with } d \sim_{\mathrm{QS}} \rho \end{array} \right\}.$$

This exponent implicitly appeared in Bourdon and Pajot's paper [7] and was named by Bonk and Kleiner in [6]. In the latter paper, it was related to Cannon's conjecture, which claims that every Gromov hyperbolic group whose boundary is homeomorphic to the 2-sphere has a discrete, cocompact and isometric action on the hyperbolic 3-space $\mathbb{H}^3$. $\dim_{\mathrm{ARC}}(X, d)$ was also characterized as a critical value related to the combinatorial p-modulus of a family of curves Γ in a graph (V, G) (approximating (X, d)), defined by

$$\mathrm{Mod}_p(\Gamma) = \inf\left\{ \sum_{v \in V} f(v)^p \;\middle|\; f : V \to [0, \infty), \sum_{v \in \gamma} f(v) \geq 1 \text{ for any } \gamma \in \Gamma \right\}$$

(see [8, 21]). This characterization of $\dim_{\mathrm{ARC}}(X, d)$ has been also used in recent studies on the construction of p-energies on fractals ([22, 30]).

In [21], Kigami introduced the notion of a partition satisfying the basic framework and used it to evaluate the Ahlfors regular conformal dimension of compact metric spaces. Roughly speaking, a partition satisfying the basic framework is a successive division of a given compact metric space with some good conditions. We explain this idea in the case of the Sierpiński carpet. Let $Q = \{z \mid \max\{|\mathrm{Re}(z)|, |\mathrm{Im}(z)|\} \leq 1/2\} \subset \mathbb{C}$ and p_j be the points on the boundary of Q with $\arg(z) = j\pi/4$ for $1 \leq j \leq 8$ (see Figure 1). We also

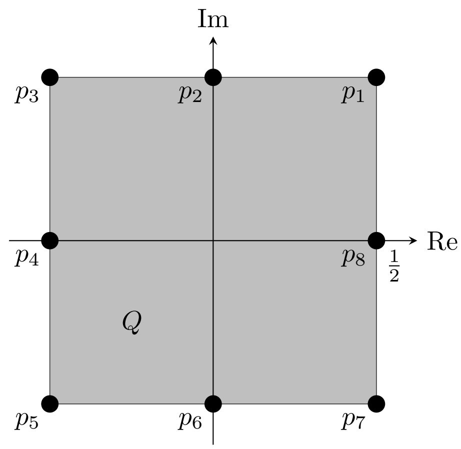

Figure 1: Q and $\{p_j\}_{j=1}^8$.

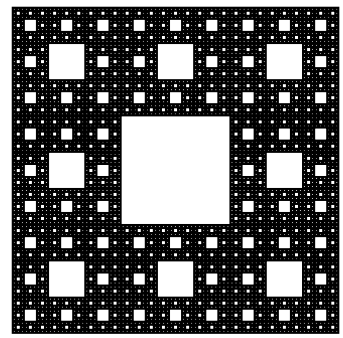

Figure 2: (Standard) Sierpiński carpet.

let $\varphi_j(z) = p_j + (z - p_j)/3$. The standard Sierpiński carpet SC is the unique nonempty compact subset of $\mathbb{C}$ satisfying $\mathrm{SC} = \cup_{j=1}^8 \varphi_j(\mathrm{SC})$ (see Figure 2). An example of a partition K of $(\mathrm{SC}, |\cdot|)$ satisfying the basic framework is a map from $\{\phi\} \cup \bigcup_{n\geq 1}\{1, ..., 8\}^n$ to the power set $\mathfrak{P}(\mathrm{SC})$, defined by

$$K(\phi) = \mathrm{SC} \text{ and } K(\{w_j\}_{j=1}^n) = \varphi_{w_1} \circ \cdots \circ \varphi_{w_n}(\mathrm{SC}).$$

In [21], Kigami considered the graph structure on $\{1, ..., 8\}^n$ for each n such that there is an edge between $w, v \in \{1, ..., 8\}^n$ if $K(w) \cap K(v) \neq \emptyset$ and $w \neq v$, and defined some potential theoretic exponents $\overline{d}_p^s(K)$, $\underline{d}_p^s(K)$ of this family of graphs, which he called the upper and lower p-spectral dimensions, for $p > 0$. See Definitions 3.4 and 3.7 for the precise definitions of a partition satisfying the basic framework and the p-spectral dimensions. For these exponents, Kigami showed the following result.

Theorem 1.3 ([21, Theorem 4.7.9] and [26, Theorem 3.9])**.** *Let (X, d) be a metric space with a partition K satisfying the basic framework. Then for $p > 0$,*

(1) if $p > \dim_{\mathrm{ARC}}(X, d)$ then $p > \overline{d}_p^s(K) \geq \underline{d}_p^s(K) \geq \dim_{\mathrm{ARC}}(X, d)$.

(2) If $p \leq \dim_{\mathrm{ARC}}(X, d)$ then $p \leq \underline{d}_p^s(K) \leq \overline{d}_p^s(K) \leq \dim_{\mathrm{ARC}}(X, d)$.

Note that the assumption in Theorem 1.3 is slightly different from that in [21, Theorem 4.7.9], but it is justified by [21, Theorem 4.7.6]. We also note that the contribution of [26] was an extension of the framework and the result to non-compact spaces. We emphasize that the p-spectral dimensions depend only on the given metric space and the partition, and do not have any stochastic characterization. However, it was pointed out in [21] that if (X, d) is the Sierpiński gasket or a generalized Sierpiński carpet with the Euclidean metric and K is the canonical partition as described above, then $\overline{d}_2^s(K)$ and $\underline{d}_2^s(K)$ coincide with the spectral dimension, defined as follows, of the standard Dirichlet form.

Definition 1.4 (Spectral dimension)**.** Let (X, d) be a locally compact separable metric space, μ be a Radon measure on X with full support, and $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X, \mu)$. We assume that $(\mathcal{E}, \mathcal{F})$ has the associated *heat kernel* (or *transition density*), namely, a (jointly) continuous function $p(t, x, y) : (0, \infty) \times X \times X \to [0, \infty)$ such that

$$T_t u(x) = \int_X p(t, x, y) u(y) d\mu(y) \quad \text{for } \mu\text{-a.e. } x \in X$$

for any $t \in (0, \infty)$ and any $u \in L^2(X, \mu)$, where $\{T_t\}_{t \in (0,\infty)}$ denotes the Markovian semigroup on $L^2(X, \mu)$ associated with $(\mathcal{E}, \mathcal{F})$. The limit

$$d_s(\mu, \mathcal{E}, \mathcal{F}) = d_s(X, \mu, \mathcal{E}, \mathcal{F}) := -2 \lim_{t \downarrow 0} \frac{\log p(t, x, x)}{\log t}$$

is called the *spectral dimension* of the regular Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{F})$, if it exists and is independent of a choice of $x \in X$.

In this paper, we prove an inequality similar to the case of $p = 2$ of Theorem 1.3 (1), between the Ahlfors regular conformal dimension and a variation of the spectral dimension defined through the on-diagonal asymptotics of the heat kernel, for the case where the Dirichlet form is induced by a resistance form, defined as follows.

Definition 1.5 (Resistance form)**.** Let X be a set, $\mathcal{F}$ be a linear subspace of the space $\ell(X)$ of $\mathbb{R}$-valued functions on X, and $\mathcal{E}$ be a nonnegative quadratic form on $\mathcal{F}$. The pair $(\mathcal{E}, \mathcal{F})$ is called a *resistance form* on X if it satisfies the following conditions.

- $1_X \in \mathcal{F}$, and $\mathcal{E}(u, u) = 0$ if and only if u is constant. (1.1)
- Define an equivalence relation $\sim$ as $u \sim v$ if and only if $u - v$ is constant, then $(\mathcal{F}/\sim, \mathcal{E})$ is a Hilbert space. (1.2)
- If $x \neq y$ then there exists $u \in \mathcal{F}$ with $u(x) \neq u(y)$. (1.3)
- $R(x, y) := \bigl(\inf\{\mathcal{E}(u, u) \mid u \in \mathcal{F},\ u(x) = 1,\ u(y) = 0\}\bigr)^{-1} < \infty$ if $x \neq y$. (1.4)
- If $u \in \mathcal{F}$ then $\hat{u} := \max\{0, \min\{1, u\}\} \in \mathcal{F}$ and $\mathcal{E}(\hat{u}, \hat{u}) \leq \mathcal{E}(u, u)$. (1.5)

We define $R(x, x) = 0$ for $x \in X$.

One of the most basic properties of a resistance form is that the infimum in (1.4) is attained and defines a metric R on X, called the *resistance metric* associated with the resistance form. The notion of resistance form was introduced in [18]. Typical examples of the Dirichlet forms induced by resistance forms are the standard Dirichlet forms on "low-dimensional" fractals, such as p.c.f.

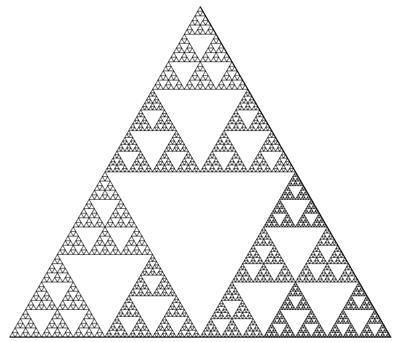

Figure 3: Inhomogeneous fractal, having a canonical resistance form.

self-similar fractals. This framework includes Dirichlet forms whose associated Hunt processes have jumps (see [20, Chapter 16], for example). Moreover, there are also examples of resistance forms on some spatially inhomogeneous fractals (see Figure 3 and [11], for example) and more general sets (see [9], for example).

In the remainder of this section except for Subsection 1.1, we make the following assumption, which is needed for our main theorem.

Assumption 1.6. $(\mathcal{E},\mathcal{F})$ is a resistance form on a set X, and the resistance metric R associated with $(\mathcal{E},\mathcal{F})$ is complete and satisfies $\dim_{\mathrm{ARC}}(X,R)<\infty$.

We note that the condition $\dim_{\mathrm{ARC}}(Y,\rho)<\infty$ for a metric space (Y,ρ) has simple geometric characterizations which may be easily checked (see Theorem 3.5). In particular, under Assumption 1.6, there exists a partition K of (X,R) satisfying the basic framework. Let us recall the definition of the volume doubling property.

Definition 1.7 (Volume doubling property)**.** A Borel measure μ on a metric space (Y,ρ) has the *volume doubling property* with respect to ρ if

$$0<\mu(B_\rho(x,2r))\le C\mu(B_\rho(x,r))<\infty$$

for any $x\in Y$ and $r>0$, for some $C>1$. Then we say μ is $(\mathrm{VD})_\rho$ for short. We write $\mathcal{M}_{(Y,\rho)}$ for the set of all Borel measures on (Y,ρ) that are $(\mathrm{VD})_\rho$.

For any $\mu\in\mathcal{M}_{(X,R)}$, we can check that the assumptions of [20, Theorems 9.4 and 10.4] are satisfied (see Proposition 2.6) and obtain the following lemma.

Lemma 1.8. *Let $\mu\in\mathcal{M}_{(X,R)}$. For $u,v\in\mathcal{F}\cap L^2(X,\mu)$, we define $\mathcal{E}_{\mu,1}(u,v)$ by*

$$\mathcal{E}_{\mu,1}(u,v)=\mathcal{E}(u,v)+\int_X uv\,d\mu,$$

then $(\mathcal{F}\cap L^2(X,\mu),\mathcal{E}_{\mu,1})$ is a Hilbert space. Let $\mathcal{D}_\mu$ be the closure of $\mathcal{F}\cap C_0(X)$ with respect to $\mathcal{E}_{\mu,1}$, and $\mathcal{E}_\mu=\mathcal{E}|_{\mathcal{D}_\mu\times\mathcal{D}_\mu}$, where $C_0(X)$ is the set of all continuous functions on (X,R) whose supports are compact. Then $(\mathcal{E}_\mu,\mathcal{D}_\mu)$ is a regular Dirichlet form on $L^2(X,\mu)$. Moreover, the associated heat kernel $p_\mu(t,x,y)$ exists.

The main theorem of this paper is the following.

Theorem 1.9. *Let $\mu \in \mathcal{M}_{(X,R)}$. Then the limit*

$$\overline{d_s}(\mu, \mathcal{E}_\mu, \mathcal{D}_\mu) := \lim_{t\to\infty} \sup_{x\in X, s\in(0,\mathrm{diam}(X,R))} 2\frac{\log\bigl(p_\mu(s/t,x,x)/p_\mu(s,x,x)\bigr)}{\log t} \tag{1.6}$$

exists and satisfies $\dim_{\mathrm{ARC}}(X,R) \le \overline{d_s}(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu) < 2$.

Note that if $\mathrm{diam}(X,R) < \infty$ then $\mathrm{diam}(X,R)$ in the right-hand side of (1.6) can be replaced by 1 because of Lemma 2.20 (5) and (6).

The following theorem is needed to prove Theorem 1.9, and it characterizes $\overline{d}^s_2$ if $(\mathcal{E},\mathcal{F})$ is local. Recall from [20, Definition 7.5] that $(\mathcal{E},\mathcal{F})$ is said to be *local* if $\mathcal{E}(u,v) = 0$ whenever $u, v \in \mathcal{F}$ and $\inf_{x,y\in X, u(x)v(y)\neq 0} R(x,y) > 0$ (see also Definition 2.15). We also recall that (X,R) has a partition satisfying the basic framework by Theorem 3.5.

Theorem 1.10. *Let K be a partition of (X,R) satisfying the basic framework. Then $\overline{d}^s_2(K) \le \overline{d_s}(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu)$ for any $\mu \in \mathcal{M}_{(X,R)}$. Moreover, if $(\mathcal{E},\mathcal{F})$ is local then $\inf_{\mu\in\mathcal{M}_{(X,R)}} \overline{d_s}(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu) = \overline{d}^s_2(K)$.*

If $(\mathcal{E},\mathcal{F})$ and μ are the standard resistance form and the standard measure on the Sierpiński gasket or a generalized Sierpiński carpet with $\dim_{\mathrm{ARC}}(X,d) < 2$ where d is the Euclidean metric on X, then $\overline{d_s}(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu)$ coincides with $d_s(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu)$. Therefore in these cases Theorem 1.9 yields $\dim_{\mathrm{ARC}}(X,d) \le d_s(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu) < 2$ because $d \sim_{\mathrm{QS}} R$, which recovers the result obtained in [21] as an application of Theorem 1.3 (1).

In general, $\overline{d_s}(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu)$ does not coincide with $d_s(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu)$ even if the latter exists. Moreover, the analog of the inequality in Theorem 1.9 is false in general for the latter, as the following theorem states.

Theorem 1.11. *There exist X, $(\mathcal{E},\mathcal{F})$ (satisfying Assumption 1.6) and $\mu \in \mathcal{M}_{(X,R)}$, such that $d_s(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu)$ exists and*

$$d_s(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu) < \dim_{\mathrm{ARC}}(X,R) < 2.$$

We briefly describe the set X on which we will construct the example of Theorem 1.11. Following a particular rule, we use either the cell subdivision rule of SC or that of the Vicsek set (that is, the unique nonempty compact subset VS of $\mathbb{C}$ with $\mathrm{VS} = \cup_{j=1,3,5,7}\varphi_j(\mathrm{VS}) \bigcup \frac{1}{3}\mathrm{VS})$ for each scale, and obtain X as the resulting set (see Section 5 for details). X has the full symmetry of the unit square, but is not exactly self-similar (see Figure 4). In this example, we also show that the resistance metric R associated with $(\mathcal{E},\mathcal{F})$ is quasisymmetric to the Euclidean metric on X (see Theorem 5.1).

The structure of this paper is as follows. Section 2 is devoted to proving inequalities of resistances used in the later sections. In Section 3 we introduce the precise definition of a partition satisfying the basic framework and show related inequalities. We prove Theorems 1.9 and 1.10 in Section 4, and Theorem 1.11 in Section 5. Appendix A is devoted to proving the equivalence between different formulations of the local property of a resistance form.

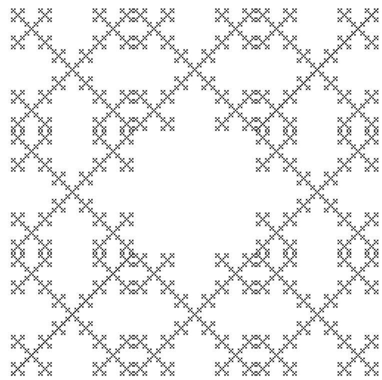

Figure 4: Example of Theorem 1.11.

1.1 Notation

Throughout this paper, we use the following notation.

- The letter # denotes the cardinality of sets, and $\mathfrak{P}$ denotes the power set of sets.
- For a set X, we denote by $\ell(X)$ the set of all $\mathbb{R}$-valued functions on X.
- $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ for $a, b \in \mathbb{R}$ (or $\mathbb{R}$-valued functions).
- By abuse of notation, we write x instead of $\{x\}$ if no confusion can arise. For example, we write $f^{-1}(x)$ instead of $f^{-1}(\{x\})$.
- For a set X and $A \subset X$, we write A^c instead of $X \setminus A$ if the whole set X is obvious.
- Let (X, d) be a metric space. For $A \subset X$, we will denote by $\mathrm{int}(A)$ the interior of A and by $\overline{A}$ the closure of A. Moreover, for any Borel measure μ on X, we write $V_{d,\mu}(x, r) = \mu(B_d(x, r))$ (where $B_d(x, r) = \{y \in X \mid d(x, y) < r\}$). We will omit subscripts of $V_{d,\mu}(x, r)$ and $B_d(x, r)$ if the metric and/or measure is obvious.
- Let X be a set and $f : X \to X$ be a map, then we set $f^k := \overbrace{f \circ \cdots \circ f}^{k}$ for $k > 0$ and $f^0 := \mathrm{id}_X$. Moreover, f^{-k} denotes $(f^k)^{-1}$ for $k > 0$.
- Let f, g be functions on a set X and $A \subset X$. We say $f(x) \lesssim g(x)$ (resp. $f(x) \gtrsim g(x)$) for any $x \in A$ if there exists $C > 0$ such that $f(x) \leq Cg(x)$ (resp. $f(x) \geq Cg(x)$) for any $x \in A$. We also write $f(x) \asymp g(x)$ (for any x) if $f(x) \lesssim g(x)$ and $f(x) \gtrsim g(x)$. Note that we will not use this notation when we want to stress the constant C.

- Let f be a function or variable defined by some type of maximum or minimum over a set of functions. Then we say g is the optimal function for f if g attains the maximum or minimum. For example, let R be the resistance metric associated with $(\mathcal{E},\mathcal{F})$, then the optimal function u for $R(x,y)$ is such that $u \in \mathcal{F}$, $u(x)=1$, $u(y)=0$ and $\mathcal{E}(u,u)=R(x,y)^{-1}$.

2 Resistance forms

In this section, we prove some properties of resistance forms and associated heat kernels, which we will use in the proof of Theorem 1.9 and related statements. We first note the difference between two types of resistances between subsets. Throughout the rest of this paper, $(\mathcal{E},\mathcal{F})$ denotes a resistance form on a set X and R denotes the associated resistance metric.

Lemma 2.1. *Let $A,B \subset X$ be nonempty. If*

$$\mathcal{F}_{A,B} := \{u \in \mathcal{F} \mid u|_A \equiv 1,\ u|_B \equiv 0\} \neq \emptyset,$$

then $\min_{u \in F_{A,B}} \mathcal{E}(u,u)$ exists and $u \in F_{A,B}$ attaining the minimum is unique.

Proof. Fix any $x \in B$, then we have

$$|u(y)-v(y)| = |(u-v)(x)-(u-v)(y)| \le \mathcal{E}(u-v,u-v)^{1/2}R(x,y)^{1/2}$$

for any $y \in X$ and $u,v \in \mathcal{F}$ with $u(x)=v(x)=0$. This shows that $\mathcal{F}_{A,B}$ is a closed convex subset of the Hilbert space $(\{u \in \mathcal{F} \mid u(x)=0\},\mathcal{E})$ and the claim follows. □

Definition 2.2 (Resistance between sets)**.** Let $\mathcal{R}(A,B)$ denote the reciprocal of $\min_{u \in F_{A,B}} \mathcal{E}(u,u)$ if $\mathcal{F}_{A,B} \neq \emptyset$ and 0 if $\mathcal{F}_{A,B} = \emptyset$. We also define $\mathcal{R}(A,B)=\infty$ if $A=\emptyset$ or $B=\emptyset$ for ease of notation. We call $\mathcal{R}(A,B)$ the *resistance between sets* A and B, associated with $(\mathcal{E},\mathcal{F})$.

On the other hand, we use the notation $R(A,B)$ for the resistance metric between sets, that is, $R(A,B)=\inf_{x \in A, y \in B} R(x,y)$. Note that $R(x,y)=\mathcal{R}(\{x\},\{y\})$ for any $x,y \in X$ and $\mathcal{R}(A,B) \le R(A,B)$ for any $A,B \subset X$ but $R(A,B) \neq \mathcal{R}(A,B)$ in general. Throughout this paper $\mathcal{R}$ denotes the resistance between sets associated with $(\mathcal{E},\mathcal{F})$, and more generally, the letter R is used for resistance metrics ("R_n" for example) and $\mathcal{R}$ is used for resistances between sets ("$\mathcal{R}_n$" for example).

Our next aim is to prove Lemma 1.8. For this purpose, we first introduce some notions of a metric space and a resistance form.

Definition 2.3 (Doubling, uniformly perfect)**.** Let (Y,ρ) be a metric space.

(1) (Y,ρ) is called *doubling* if there exists $N \in \mathbb{N}$ such that for any $x \in Y$ and $r>0$, there exist $\{x_i\}_{i=1}^N \subset Y$ with $B(x,2r) \subset \bigcup_{i=0}^N B(x_i,r)$.

(2) (Y, ρ) is called *uniformly perfect* if there exists $\gamma > 1$ such that $B(x, \gamma r) \setminus B(x, r) \neq \emptyset$ whenever $B(x, r) \neq Y$.

Remark. It is easy to see that a doubling metric space is separable.

Definition 2.4 (regular)**.** $(\mathcal{E}, \mathcal{F})$ is called *regular* if $\mathcal{F} \cap C_0(X, R)$ is dense in $C_0(X, R)$ with respect to the supremum norm.

Definition 2.5 (Annulus comparable condition)**.** We say that $(\mathcal{E}, \mathcal{F})$ satisfies the *annulus comparable condition*, (ACC) for short, if there exists $\alpha > 1$ such that $\mathcal{R}(x, B(x, r)^c) \gtrsim \mathcal{R}(x, B(x, r)^c \cap \overline{B(x, \alpha r)})$ for any $x \in X$ and $r > 0$.

Note that the inverse direction of the above inequality immediately follows from the inclusion of sets. It is easy to see that if (ACC) holds then (X, R) is uniformly perfect.

Hereafter, we make Assumption 1.6 to the end of Section 4. Note that by Theorem 3.5, both R and d are doubling and uniformly perfect.

Proposition 2.6. *$(\mathcal{E}, \mathcal{F})$ is regular and satisfies (ACC).*

For the proof of this proposition, we introduce some results.

Proposition 2.7 ([20, Theorem 8.4])**.** *Let Y be a nonempty subset of X. Define $\mathcal{F}|_Y = \{u|_Y \mid u \in \mathcal{F}\}$ and*

$$\mathcal{E}|_Y(u_*, u_*) = \inf\{\mathcal{E}(u, u) \mid u \in \mathcal{F},\ u_* = u|_Y\} \tag{2.1}$$

for any $u_* \in \mathcal{F}|_Y$, then the infimum of (2.1) *is attained. Moreover, there exists a unique extension of $\mathcal{E}|_Y$ to $\mathcal{F}|_Y \times \mathcal{F}|_Y$ such that $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ is a resistance form. In particular, if $\#(Y) < \infty$ then $\mathcal{F}|_Y = \ell(Y)$.*

Definition 2.8. $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ is called the *trace* of $(\mathcal{E}, \mathcal{F})$ on Y.

Remark. In [20, Theorem 8.4], R is assumed to be separable and complete, and it is so in our case. However, by the standard argument in Hilbert space theory, it is easy to show that Proposition 2.7 is also true for resistance forms whose associated resistance metric is not necessarily separable and complete (see [14, Theorem 2.29], for example).

Definition 2.9. We define the following terminologies for abbreviation.

(1) We say $\{V_n\}_{n \geq 0}$ is a *spread sequence* of a metric space (Y, ρ) if it is an increasing sequence of nonempty finite subsets satisfying $\overline{\cup_{n \geq 0} V_n} = Y$.

(2) Assume that V is a finite set and $(\mathcal{E}, \ell(V))$ is a resistance form on V. We call $\mu = \{\mu_{x,y}\}_{x,y \in V} \subset \mathbb{R}^{V \times V}$ the *resistance weights* associated with $(\mathcal{E}, \ell(V))$ if $\mu_{x,x} = -\sum_{z : z \neq x} \mu_{x,z}$ and $\mu_{x,y} = \mu_{y,x}$ for any $x, y \in V$, and

$$\mathcal{E}(u, v) = \frac{1}{2} \sum_{x,y \in V} (u(x) - u(y))(v(x) - v(y))\mu_{x,y} \text{ for any } u, v \in \ell(V).$$

Moreover, we write $\mathcal{R}_Y(A,B)$ (resp. $R_Y(A,B)$) instead of $\mathcal{R}_Y(A\cap Y, B\cap Y)$ (resp. $R_Y(A\cap Y, B\cap Y)$) for abbreviation, where $\mathcal{R}_Y$ (resp. R_Y) is the resistance between sets (resp. resistance metric) associated with the trace of $(\mathcal{E},\mathcal{F})$ on Y.

Remark. By [19, Proposition 2.1.3], for any $(\mathcal{E},\ell(V))$, the unique resistance weights associated with that exist. Moreover, $\mu_{x,y}\geq 0$ for any $x,y\in V$ with $x\neq y$ and $\mu_{x,x}<0$ for any $x\in V$ by the same proposition and (1.1).

Proposition 2.10 ([19, Section 2.3])**.** *Assume that $\{V_n\}_{n\geq 0}$ is a spread sequence, then*

$$\mathcal{F}=\{u \mid u\in C(X,R),\ \lim_{n\to\infty}\mathcal{E}|_{V_n}(u|_{V_n},u|_{V_n})<\infty\}.$$

Moreover, $\mathcal{E}(u,v)=\lim_{n\to\infty}\mathcal{E}|_{V_n}(u|_{V_n},v|_{V_n})$ for any $u,v\in\mathcal{F}$.

Remark. $\{\mathcal{E}|_{V_n}(u|_{V_n},u|_{V_n})\}_{n\geq 0}$ is an increasing sequence for any $u\in\mathcal{F}$ by definition.

Lemma 2.11. *Let $\{f_n\}_{n\geq 0}\subset\mathcal{F}$ with $\sum_{n\geq 0}\mathcal{E}(f_n,f_n)<\infty$.*

(1) If $\sup_{n\geq 0}f_n(x_*)<\infty$ for some $x_*\in X$, then $\bar{f}:=\sup_{n\geq 0}f_n\in\mathcal{F}$ and $\mathcal{E}(\bar{f},\bar{f})\leq\sum_{n\geq 0}\mathcal{E}(f_n,f_n)$.

(2) If $\inf_{n\geq 0}f_n(x_*)>-\infty$ for some $x_*\in X$, then $\underline{f}:=\inf_{n\geq 0}f_n\in\mathcal{F}$ and $\mathcal{E}(\underline{f},\underline{f})\leq\sum_{n\geq 0}\mathcal{E}(f_n,f_n)$.

Remark. Lemma 2.11 is essentially a special case of [14, Theorem 2.38 (1)], which is written in Japanese. We give a proof of Lemma 2.11 for the reader's convenience.

Proof. Replacing f_n by $-f_n$, we only need to show (1). We first note that

$$\begin{aligned}|\bar{f}(x)-\bar{f}(y)|\leq\sup_n|f_n(x)-f_n(y)|&\leq\sup_n(R(x,y)\mathcal{E}(f_n,f_n))^{1/2}\\&\leq R(x,y)^{1/2}(\textstyle\sum_{n\geq 0}\mathcal{E}(f_n,f_n))^{1/2}\end{aligned}$$

for any $x,y\in X$, so $\bar{f}(x)<\infty$ and $\bar{f}\in C(X,R)$. Let $\{V_m\}_{m\geq 1}$ be a spread sequence of (X,R) and μ_m be associated resistance weights with $(\mathcal{E}|_{V_m},\ell(V_m))$. Then

$$\begin{aligned}\mathcal{E}(\bar{f},\bar{f})&=\lim_{m\to\infty}\mathcal{E}|_{V_m}(\bar{f}|_{V_m},\bar{f}|_{V_m})\\&=\lim_{m\to\infty}\frac{1}{2}\sum_{x,y\in V_m:x\neq y}\big(\bar{f}(x)-\bar{f}(y)\big)^2(\mu_m)_{x,y}\\&\leq\lim_{m\to\infty}\frac{1}{2}\sum_{n\geq 0}\sum_{x,y\in V_m:x\neq y}\big(f_n(x)-f_n(y)\big)^2(\mu_m)_{x,y}\\&=\lim_{m\to\infty}\sum_{n\geq 0}\mathcal{E}|_{V_m}(f_n|_{V_m},f_n|_{V_m})=\sum_{n\geq 0}\mathcal{E}(f_n,f_n).\end{aligned}$$

(Note that any term of the sums in the above inequalities is nonnegative.) This with Proposition 2.10 proves the lemma. □

Proof of Proposition 2.6. By [20, Lemma 7.10], there exist $n \geq 0$ and $C > 0$ with

$$C^{-1}2^k \leq \mathcal{R}(x, B(x,2^k)^c \cap \overline{B(x,2^{k+N})}) \leq C2^k$$

for any $x \in X$ and $k \in \mathbb{Z}$ unless $B(x,2^k) = X$. Fix any $x \in X$ and set $A_k = B(x,2^k)^c \cap \overline{B(x,2^{k+N})}$. Let f_k be the optimal functions for $\mathcal{R}(x, A_k)$ if $B(x,2^k) \neq X$, and otherwise $f_k \equiv 1$. Then for any $a \in \mathbb{Z}$, $\sum_{k\geq a} \mathcal{E}(f_k, f_k) < C2^{-a+1}$ and so $g_a := \inf_{k\geq a} f_k \in \mathcal{F}$ by Lemma 2.11 (2). Since $g_a(x) = 1$ and $g_a|_{B(x,2^a)^c} \equiv 0$, it follows that

$$\mathcal{R}(x, B(x,2^a)^c) > C^{-1}2^{a-1} \text{ for any } a \in \mathbb{Z} \text{ with } B(x,2^a) \neq X, \tag{2.2}$$

which shows (ACC). Moreover, (2.2) also shows that for any nonempty $Y \subset X$ with $x \notin \overline{Y}$, there exists $f \in \mathcal{F}$ satisfying $f(x) = 1$ and $f|_Y \equiv 0$. Applying [20, Theorem 6.3], we conclude that $(\mathcal{E}, \mathcal{F})$ is regular. □

Now Lemma 1.8 immediately follows from [20, Theorems 9.4 and 10.4] with Proposition 2.6. (Note that the condition (ACC) is used later.)

We next give some properties of resistance forms, which will be needed in Section 3.

Lemma 2.12. *Let A_1, A_2 be nonempty subsets of X and $\{V_n\}_{n\geq 0}$ be a spread sequence. Suppose that $A_i \subset \overline{\cup_{n\geq 0}(A_i \cap V_n)}$ for $i = 1, 2$. Then $\mathcal{R}(A_1, A_2) = \lim_{n\to\infty} \mathcal{R}_n(A_1, A_2)$, where $\mathcal{R}_n$ is the resistance between sets associated with $(\mathcal{E}|_{V_n}, \mathcal{F}|_{V_n})$.*

Proof. By definition of $\mathcal{E}|_{Y_n}$, it suffices to show that

$$\mathcal{R}(A_1, A_2) \geq \lim_{n\to\infty} \mathcal{R}_n(A_1, A_2).$$

We may assume $\lim_{n\to\infty} \mathcal{R}_n(A_1, A_2) > 0$ and $A_i \cap V_0 \neq \emptyset$ for $i = 1, 2$ without loss of generality. Let $\{f_n\}_{n\geq 0} \subset \mathcal{F}$ be functions satisfying

$$\begin{aligned} &\mathcal{R}_n(A_1, A_2)^{-1} \\ =&\min\Big\{\min\{\mathcal{E}(f,f) \mid f \in \mathcal{F}, f|_{V_n} \equiv f_*\} \Big| f_* \in \ell(V_n), f_*|_{A_1\cap V_n} \equiv 1, f_*|_{A_2\cap V_n} \equiv 0\Big\} \\ =&\mathcal{E}(f_n, f_n), \end{aligned}$$

$f_n|_{A_1\cap V_n} \equiv 1$ and $f_n|_{A_2\cap V_n} \equiv 0$. Then by the convexity argument, we obtain $0 \leq \mathcal{E}(f_n - f_m, f_n - f_m) = \mathcal{E}(f_n, f_n) - \mathcal{E}(f_m, f_m)$ for any $n, m \in \mathbb{N}$ with $n > m$. Fix any $x \in A_1 \cap V_1$. Since $\lim_{n\to\infty} \mathcal{E}(f_n, f_n) = \lim_{n\to\infty} \mathcal{R}_n(A_1, A_2)^{-1} < \infty$, there exists $f \in \mathcal{F}$ such that $f(x) = 1$ and $\lim_{n\to\infty} \mathcal{E}(f - f_n, f - f_n) = 0$ by (1.2). Then for any $y \in \cup_{n\geq 0}(A_1 \cap V_n)$,

$$|f(y) - 1| = |(f - f_n)(y) - (f - f_n)(x)| \leq R(x,y)\mathcal{E}(f - f_n, f - f_n).$$

Hence $f|_{A_1} \equiv 1$ because the right hand side of the last inequality tends to 0 and $f \in C(X, R)$. In the same way, we obtain $f|_{A_2} \equiv 0$ and so

$$\mathcal{R}(A_1, A_2)^{-1} \leq \mathcal{E}(f,f) = \lim_{n\to\infty} \mathcal{E}(f_n, f_n) = \lim_{n\to\infty} \mathcal{R}_n(A_1, A_2)^{-1},$$

which completes the proof. □

Lemma 2.13. *Let* $\eta \in (0,1)$, *then* $\mathcal{R}(\overline{B(x,\eta r)}, B(x,r)^c) \asymp r$ *for any* $x \in X, r > 0$ *with* $B(x,r) \neq X$.

Proof. By Proposition 2.6 and [20, Theorem 7.12], there exists $C > 0$ such that $C^{-1}r \leq \mathcal{R}(x, B(x,r)^c) \leq Cr$ for any $x \in X, r > 0$ with $B(x,r) \neq X$. Hence we only need to show $\mathcal{R}(\overline{B(x,\eta r)}, B(x,r)^c) \gtrsim r$.
We first prove for the case $\eta \leq 1/2C$. Let $f_{x,r}$ be the optimal function of $\mathcal{R}(x, B(x,r)^c)$, then by [20, Theorems 4.1 and 4.3 and Lemma 4.5],

$$\begin{aligned} f_{x,r}(y) \geq &\frac{\mathcal{R}(x, B(x,r)^c) + \mathcal{R}(y, B(x,r)^c) - \mathcal{R}(x,y)}{2\mathcal{R}(x, B(x,r)^c)} \\ \geq &(C^{-1}r - \eta r)C^{-1}/2r \geq C^{-2}/4 \end{aligned}$$

for any $y \in B(x,\eta r)$. Hence let $g_{x,r} := \big((4C^2 f_{x,r} \wedge 1) \vee 0\big)$, then $g_{x,r}|_{B(x,\eta r)} \equiv 1$, $g_{x,r}|_{B(x,r)^c} \equiv 0$ and $\mathcal{E}(g_{x,r}, g_{x,r}) \leq 16C^4 \mathcal{E}(f_{x,r}, f_{x,r}) \leq 16C^3/r$. This proves the statement for this case.
We now turn to the case $1/2C < \eta < 1$. Since (X,R) is doubling, there exists $N = N_\eta \in \mathbb{N}$ such that for any $x \in X$, there exists $\{x_i\}_{i=1}^N \subset X$ satisfying $B(x,\eta r) \subset \cup_{i=1}^N B(x_i, (1-\eta)r/2C)$. Let $g = \max_{1 \leq i \leq N}(g_{x_i,(1-\eta)r})$, where $g_{x_i,(1-\eta)r}$ is same as above, then $g \in \mathcal{F}$ by Lemma 2.11 (1). Moreover, $g|_{B(x,\eta r)} \equiv 1$ and $g|_{B(x,r)^c} \equiv 0$ because $R(x_i,y) \geq R(x,y) - R(x,x_i) \geq (1-\eta)r$ for any $y \in B(x,r)^c$ and $1 \leq i \leq N$. Therefore

$$\begin{aligned} \mathcal{R}(\overline{B(x,\eta r)}, B(x,r)^c)^{-1} \leq &\mathcal{E}(g,g) \leq \sum_{i=1}^N \mathcal{E}(g_{x_i,(1-\eta)r}, g_{x_i,(1-\eta)r}) \\ \leq &16N_\eta C^3/(1-\eta)r, \end{aligned}$$

which proves the lemma. □

Corollary 2.14. *Let* $A_1, A_2 \subset X$ *be nonempty subsets. If* A_1 *is bounded and* $R(A_1,A_2) > 0$, *then* $\mathcal{R}(A_1,A_2) > 0$.

Proof. Since (X,R) is doubling, there exist $N \in \mathbb{N}$ and $\{x_i\}_{i=1}^N \subset A_1$ with $A_1 \subset \cup_{i=1}^N B(x_i, R(A_1,A_2)/2)$. Thus the proof is straightforward by Lemmas 2.11 (1) and 2.13. □

Definition 2.15 (Local)**.** $(\mathcal{E},\mathcal{F})$ is called *local* if it satisfies $\mathcal{E}(u,v) = 0$ whenever $u,v \in \mathcal{F}$ and $R(\{x \mid u(x) \neq 0\}, \{x \mid v(x) \neq 0\}) > 0$.

Under Assumption 1.6, for each $\mu \in \mathcal{M}_{(X,d)}$, $(\mathcal{E},\mathcal{F})$ is a local resistance form if and only if $(\mathcal{E}_\mu, \mathcal{D}_\mu)$ is a local Dirichlet form. See Appendix A for details.

Proposition 2.16. *Let* A_i $(i = 1,2)$ *be nonempty subsets of* (X,R) *with* $R(A_1,A_2) > 0$ *and* $\operatorname{diam}(A_1) < \infty$, $\{V_n\}_{n\geq 0}$ *be a spread sequence and* μ_n *be the resistance weights associated with* $(\mathcal{E}|_{V_n}, \ell(V_n))$. *Assume* $(\mathcal{E},\mathcal{F})$ *to be local, then*

$$\lim_{n\to\infty} \sum_{(x,y)\in D_n} (\mu_n)_{x,y} = 0, \text{ where } D_n = (A_1 \times A_2 \cup A_2 \times A_1) \cap V_n \times V_n.$$

Proof. Let

$$A_1^* = \overline{\{x \mid R(x, A_1) \geq R(A_1, A_2)/3\}} \text{ and } A_2^* = \overline{\{x \mid R(x, A_1) \leq 2R(A_1, A_2)/3\}}.$$

By Corollary 2.14, we can take the optimal functions $f_i \in \mathcal{F}$ for $\mathcal{R}(A_i, A_i^*)$ and $i = 1, 2$. Then,

$$\begin{aligned} &\mathcal{E}|_{V_n}(f_1|_{V_n}, f_1|_{V_n}) + \mathcal{E}|_{V_n}(f_2|_{V_n}, f_2|_{V_n}) - \mathcal{E}|_{V_n}(f_1 + f_2|_{V_n}, f_1 + f_2|_{V_n}) \\ = &\sum_{(x,y)\in D_n} (\mu_n)_{x,y} \\ &+\frac{1}{2} \sum_{\substack{(x,y)\in V_n\times V_n\setminus D_n \\ :x\neq y}} \big((f_1(x) - f_1(y))^2 + (f_2(x) - f_2(y))^2 \\ &\qquad - ((f_1 + f_2)(x) - (f_1 + f_2)(y))^2\big)(\mu_n)_{x,y}. \end{aligned}$$

Since $0 \leq f_1, f_2 \leq 1$ and $supp(f_1) \cap supp(f_2) = \emptyset$, we have

$$\begin{aligned} |(f_1 + f_2)(x) - (f_1 + f_2)(y)| =&|(f_1 \vee f_2)(x) - (f_1 \vee f_2)(y)| \\ \leq&|f_1(x) - f_1(y)| \vee |f_2(x) - f_2(y)| \end{aligned}$$

for any $x, y \in X$. Therefore

$$\begin{aligned} 0 =&\mathcal{E}(f_1, f_1) + \mathcal{E}(f_2, f_2) - \mathcal{E}(f_1 + f_2, f_1 + f_2) \\ =& \lim_{n\to\infty} \mathcal{E}|_{V_n}(f_1|_{V_n}, f_1|_{V_n}) + \mathcal{E}|_{V_n}(f_2|_{V_n}, f_2|_{V_n}) - \mathcal{E}|_{V_n}(f_1 + f_2|_{V_n}, f_1 + f_2|_{V_n}) \\ \geq& \lim_{n\to\infty} \sum_{(x,y)\in D_n} (\mu_n)_{x,y} \geq 0 \end{aligned}$$

because $(\mathcal{E}, \mathcal{F})$ is local, which completes the proof. □

Proposition 2.17. *There exists $\alpha > 1$ with $R_{V,B(x,\alpha R(x,y))}(x, y) \leq 2R(x, y)$ for any nonempty finite subset $V \subset X$ and $x, y \in V$, where $\mu_{x,y}$ are the resistance weights associated with $(\mathcal{E}|_V, \ell(V))$ and*

$$R_{V,A}(x, y)^{-1} = \min\{\frac{1}{2} \sum_{x,y\in A} (f(x) - f(y))^2 \mu_{x,y} \mid f(x) = 1,\ f(y) = 0\}.$$

Remark. The idea and the proof of Proposition 2.17 essentially come from [3, Lemma 2.5].

Proof. We first note that since Lemma 2.13 holds,

$$\mathcal{R}_V(x, B(x, r)^c) \geq \mathcal{R}_V(\overline{B(x, r/2)}, B(x, r)^c) \geq \mathcal{R}(\overline{B(x, r/2)}, B(x, r))^c \gtrsim r$$

for any $x \in X$ and $r > 0$, where $\mathcal{R}_V$ is the resistance between sets associated with $(\mathcal{E}|_V, \ell(V))$. Thus we can find $\alpha > 1$ such that

$$\mathcal{R}_V(x, B(x, (\alpha/2)r)^c) \vee \mathcal{R}_V(\overline{B(x, (\alpha/2)r)}, B(x, \alpha r)^c) \geq 4r$$

for any x and r. To shorten notation, we write B_β instead of $B(x,B(x,\beta R(x,y)))$. Let f_1 (resp. f_2, f_3) be the optimal function for $R_{V,B_\alpha}(x,y)$ (resp. $\mathcal{R}_V(x,B^c_{\alpha/2})$, $\mathcal{R}_V(\overline{B_{\alpha/2}},B^c_\alpha)$). We define $f \in \ell(V)$ by

$$f(x) = \begin{cases} f_1(x) \wedge f_2(x) \wedge f_3(x) & (x \in B_\alpha) \\ f_2(x) \wedge f_3(x) & (\text{otherwise}) \end{cases}.$$

Then, $f(x) = 1$, $f(y) = 0$, $f|_{B^c_{2/\alpha}} \equiv 0$ and

$$|f(x) - f(y)| \le \begin{cases} \sum_{i=1}^3 |f_i(x) - f_i(y)| & (\text{if } x, y \in B_\alpha) \\ 0 & (\text{if } x, y \notin B_{\alpha/2}) \\ |f_3(x) - f_3(y)| = 1 & (\text{if } x \in B_{\alpha/2} \text{ and } y \notin B_\alpha, \\ & \ \text{or } y \in B_{\alpha/2} \text{ and } x \notin B_\alpha). \end{cases}$$

Therefore

$$\begin{aligned} (R(x,y))^{-1} &\le \mathcal{E}(f,f) \\ &\le (R_{V,B_\alpha}(x,y))^{-1} + (\mathcal{R}_V(x,B^c_{\alpha/2}))^{-1} + (\mathcal{R}_V(\overline{B_{\alpha/2}},B^c_\alpha))^{-1} \\ &\le (R_{V,B_\alpha}(x,y))^{-1} + \frac{1}{2}(R(x,y))^{-1} \end{aligned}$$

and $R_{V,B_\alpha}(x,y) \le 2R(x,y)$ as claimed. □

The remainder of this section is devoted to the proof of Proposition 2.19 below, which gives one of the key inequalities to prove Theorem 1.10. For the rest of this section, we assume d to be a metric on X with $d \sim_{\mathrm{QS}} R$. Then by Assumption 1.6, it is easily shown that $\mathcal{M}_{(X,d)} = \mathcal{M}_{(X,R)}$ (see [20, Corollary 12.4], for example).

Definition 2.18. Set

$$\overline{R}_d(x,r) := \sup_{y \in B_d(x,r)} R(x,y) \quad \text{and} \quad h_{d,\mu}(x,r) := V_{d,\mu}(x,r)\overline{R}_d(x,r).$$

We write $h_d(x,r)$ instead of $h_{d,\mu}(x,r)$ when no confusion can arise.

Proposition 2.19. *The limit* $\overline{d_s}(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu)$ *exists for any* $\mu \in \mathcal{M}_{(X,d)}$. *Moreover,*

$$\overline{d_s}(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu) = 2 \limsup_{s\to\infty} \sup_{x \in X, r \in (0,\mathrm{diam}(X,d))} \frac{\log\bigl(V_{d,\mu}(x,r)/V_{d,\mu}(x,r/s)\bigr)}{\log\bigl(h_{d,\mu}(x,r)/h_{d,\mu}(x,r/s)\bigr)}. \tag{2.3}$$

Remark. We only use the case $d = R$ for the proof of Theorem 1.10 (recall that $R \sim_{\mathrm{QS}} R$). However, we prove general case for future works.

We introduce some basic facts for the proof of Proposition 2.19.

Lemma 2.20. *Assume* $\mu \in \mathcal{M}_{(X,d)}$, *then the following statements are true.*

(1) *There exists* $\gamma_1 > 1$ *such that* $V_d(x, r/\gamma_1) \leq V_d(x,r)/2$ *for any* $x \in X$ *and* $r \in (0, \mathrm{diam}(X,d))$.

(2) $h_d(x, 2r) \lesssim h_d(x,r)$ *for any* $x \in X$ *and* $r > 0$.

(3) *There exists* $\gamma_2 > 1$ *such that* $h_d(x, r/\gamma_2) \leq h_d(x,r)/2$ *for any* $x \in X$ *and* $r \in (0, \mathrm{diam}(X,d))$.

(4) *For any* $C > 0$, *there exists* $\gamma_C > 1$ *such that for any* $t \in (0, C)$ *and* $x \in X$, *there exists* $r \in (0, \mathrm{diam}(X,d))$ *with* $t/\gamma_C \leq h_d(x,r) \leq t$

(5) *For any* $x \in X$, $p_\mu(\cdot, x, x) : t \mapsto p_\mu(t,x,x)$ *is a decreasing function of* t.

(6) *Fix any* $C' > 0$. *Then* $p_\mu(t/2, x, x) \lesssim p_\mu(t,x,x)$ *for any* $x \in X$ *and* $t \in (0, C')$.

Proof. (1) It is well-known and easily follows from the volume doubling and uniformly perfect conditions (see [12, Excersise 13.1] for example).

(2), (3) Since $d \sim_{\mathrm{QS}} R$ and both d and R are uniformly perfect, it is easy to check that there exists $\gamma' > 1$ such that $\overline{R}_d(x, 2r) \lesssim \overline{R}_d(x,r)$ and $\overline{R}_d(x, r/\gamma') \leq \overline{R}_d(x,r)/2$ for any $x \in X$ and $r \in (0, \mathrm{diam}(X,d))$. This with (1) and the volume doubling condition shows (2) and (3).

(4) Since $C/\sup_{r \in (0, \mathrm{diam}(X,d))} h_d(x,r) \leq 2C/\mathrm{diam}(X,d)\mu(X)$ for any $x \in X$, it follows from (2) and (3).

(5) By the proof of [20, Theorem 10.4 and Lemma 10.7], $\lim_{n\to\infty} p_n(t,x,y) = p_\mu(t,x,y)$ where $p_n(t,x,y) : (0,\infty) \times X \times X \to \mathbb{R}$ is of the form

$$p_n(t,x,y) = \sum_{k \geq 1} \exp(-\lambda_{n,k} t) \varphi_{n,k}(x) \varphi_{n,k}(y)$$

for some $\lambda_{n,k} > 0$ and $\varphi_{n,k} : X \to \mathbb{R}$. Hence (5) is clear.

(6) Recall that by Proposition 2.6, (X, R) satisfies (ACC). Thus this follows from (3)-(5), [20, Theorem 15.6] and the fact that $p_\mu(t,x,x) \geq \mu(X)^{-1}$ for any $t > 0$, which follows from the Chapman-Kolmogorov equation. □

Remark. The condition of Lemma 2.20 (1) is called reverse volume doubling condition (e.g. [10, 15]).

For the existence of $\overline{d_s}(\mu, \mathcal{E}_\mu, \mathcal{D}_\mu)$, we use the classical result for subadditive functions. For a proof, see [13, Proof of Theorem 7.6.1] for instance.

Lemma 2.21. *Let* $f : (0,\infty) \to \mathbb{R}$ *be subadditive, that is,* $f(t+s) \leq f(t) + f(s)$ *for any* $t, s \in (0,\infty)$. *Assume that* $\sup_{t \in I} f(t) < \infty$ *for any bounded interval* I, *then* $\lim_{t\to\infty} f(t)/t = \inf_{t>0} f(t)/t < \infty$.

Proof of Proposition 2.19. Let

$$f(\tau) = \log\Big(\sup_{x \in X, s \in (0, \mathrm{diam}(X,d))} p_\mu(s/e^\tau, x, x)/p_\mu(s,x,x) \Big)$$

for $\tau > 0$, then f is subadditive by definition. By Lemma 2.20 (5) and (6), $\sup_{\tau\in I} f(\tau) < \infty$ for any bounded interval I and $\inf_{\tau>0} f(\tau)/\tau \geq 0$. Hence $\lim_{\tau\to\infty} f(\tau)/\tau$ and the limit $\overline{d_s}(\mu, \mathcal{E}_\mu, \mathcal{D}_\mu)$ exist because $\tau \to \infty$ as $t = e^\tau \to \infty$.

Our next goal is to prove (2.3). To this end, let

$$u(s) = \sup_{\substack{x\in X, \alpha\in[s,\infty),\\ r\in(0,\mathrm{diam}(X,d))}} \frac{\log\bigl(p_\mu(h_d(x,r/\alpha),x,x)/p_\mu(h_d(x,r),x,x)\bigr)}{\log\bigl(h_d(x,r)/h_d(x,r/\alpha)\bigr)} \text{ and}$$

$$v(s) = \sup_{\substack{x\in X, \alpha\in[s,\infty),\\ r\in(0,\mathrm{diam}(X,d))}} \frac{\log\bigl(p_\mu(t/\alpha,x,x)/p_\mu(t,x,x)\bigr)}{\log\alpha}.$$

By [20, Theorem 15.6], Proposition 2.6 and Lemma 2.20(3), the right hand side of (2.3) equals $\lim_{s\to\infty} u(s)$, hence it is sufficient to show $\lim_{s\to\infty} u(s) = \lim_{s\to\infty} v(s)$. We proceed to show the following claim.

Claim. *For any $\epsilon > 0$, there exists $s_0(\epsilon)$ such that for any $s > s_0(\epsilon)$, there exists $s_*(s,\epsilon)$ satisfying $(1+\epsilon)(\epsilon + u(s)) \geq v(s_*(s,\epsilon))$*

This claim implies $\lim_{s\to\infty} u(s) \geq \lim_{s\to\infty} v(s)$ because both u and v are decreasing.

Proof. For any $\epsilon > 0$ and $s > 1$, we can take $C_1, ..., C_4 > 1$ satisfying the following conditions by Lemma 2.20 (2), (4)-(6).

- If $x \in X$, $r \in (0, \mathrm{diam}(X,d))$ and $\beta > 0$ satisfy $h_d(x,r)/h_d(x,r/\beta) > C_1$, then $\beta > s$.
- For any $x \in X$ and $t \in (0, \mathrm{diam}(X,d))$, there exists $r \in (0, \mathrm{diam}(X,d))$ with $t/C_2 \leq h_d(x,r) \leq t$.
- Any $x \in X$ and $t_1, t_2 \in (0, \mathrm{diam}(X,d))$ with $t_1 \leq C_2 t_2$ satisfy $C_3 \geq \log(p_\mu(t_2,x,x)/p_\mu(t_1,x,x))$.
- $C_4 > C_2$, $C_3/\log C_4 < \epsilon$ and $\log C_4/(\log C_4 - \log C_2) < 1+\epsilon$.

Let $x \in X$, $t \in (0, \mathrm{diam}(X,d))$ and $\alpha > C_2(C_1 \vee C_4) =: s_*$ We take $r_1, r_2 \in (0, \mathrm{diam}(X,d))$ such that $t/C_2 \leq h_d(x,r_1) \leq t$ and $t/C_2\alpha \leq h_d(x,r_2) \leq t/\alpha$. Then $C_2\alpha > h_d(x,r_1)/h_d(x,r_2) > C_1 \vee C_4$ and so for

$$l_1 := \log\left(\frac{p_\mu(h_d(x,r_2),x,x)}{p_\mu(h_d(x,r_1),x,x)}\right) \quad \text{and} \quad l_2 := \log\left(\frac{p_\mu(h_d(x,r_2),x,x)}{p_\mu(h_d(x,r_1),x,x)}\right),$$

it follows that

$$\begin{aligned}
&\log\bigl(p_\mu(t/\alpha,x,x)/p_\mu(t,x,x)\bigr)/\log\alpha \\
\leq&\bigl(l_1 + \log\bigl(p_\mu(h_d(x,r_1),x,x)/p_\mu(t,x,x)\bigr)\bigr)/\bigl(l_2 - \log C_2\bigr) \\
\leq&\bigl((l_1/l_2) + (C_3/l_2)\bigr)l_2/\bigl(l_2 - \log C_2\bigr) \\
\leq&\bigl((l_1/l_2) + (C_3/\log C_4)\bigr)\log C_4/\bigl(\log C_4 - \log C_2\bigr).
\end{aligned}$$

Therefore $v(s_*) \leq (1+\epsilon)(\epsilon + u(s))$ by $r_1/r_2 > s$, and the claim follows. ☐

For $\lim_{s\to\infty} u(s) \leq \lim_{s\to\infty} v(s)$, we consider the case $\mathrm{diam}(X,d) < \infty$ and fix any $\epsilon > 0$. Then by Lemma 2.20 (5) and (6), there exists $s > 1$ with

$$\sup_{\substack{x\in X,\\ r\in(0,\mathrm{diam}(X,d))}} \log\left(\frac{p_\mu(\mathrm{diam}(X,d),x,x)}{p_\mu(h_d(x,r),x,x)}\right) < \epsilon \log s$$

because

$$\sup_{x\in X, r\in(0,\mathrm{diam}(X,d))} \big(h_d(x,r)/\mathrm{diam}(X,d)\big) \leq \mu(X) < \infty. \tag{2.4}$$

By (2.4) and Lemma 2.20 (3), we can take $s' > 1$ such that if $\alpha > s'$ then $\big(h_d(x,r) \wedge \mathrm{diam}(X,d)\big)/h_d(x,r/\alpha) > s$ for any $x \in X$ and $r \in (0, \mathrm{diam}(X,d))$. Thus we obtain

$$\begin{aligned}
&\log\left(\frac{p_\mu(h_d(x,r/\alpha),x,x)}{p_\mu(h_d(x,r),x,x)}\right) \Big/ \log\left(\frac{h_d(x,r)}{h_d(x,r/\alpha)}\right)\\
\leq &\left(\log\left(\frac{p_\mu(h_d(x,r/\alpha),x,x)}{p_\mu(h_d(x,r)\wedge \mathrm{diam}(X,d),x,x)}\right) \Big/ \log\left(\frac{h_d(x,r)\wedge \mathrm{diam}(X,d)}{h_d(x,r/\alpha)}\right)\right) + \epsilon
\end{aligned}$$

and $u(s') < v(s) + \epsilon$. This shows $\lim_{s\to\infty} u(s) \leq \lim_{s\to\infty} v(s)$ in the same way as the inverse direction. The proof for the case $\mathrm{diam}(X,d) = \infty$ is similar, and the proposition follows. □

3 Partition satisfying basic framework

In the former part of this section, we introduce the notion and related results of the *partition satisfying the basic framework,* which is defined in [21] for the bounded case and extended to unbounded cases in [26]. In the latter part of this section, we show some related resistance estimates. Note that we continue to make Assumptions 1.6.

Definition 3.1 (Tree with a reference point)**.** Let T be a countable set and $\pi : T \to T$ be a map such that the following conditions hold.

- Let $F_\pi = \{w \mid \pi^n(w) = w \text{ for some } n \geq 1\}$, then $\#F_\pi \leq 1$.
- For any $w, v \in T$, there exist $n, m \geq 0$ such that $\pi^n(w) = \pi^m(v)$.

Let $\phi \in F_\pi$ if $F_\pi \neq \emptyset$, otherwise we fix any $\phi \in T$. We call the triplet (T, π, ϕ) a *tree with a reference point.*

The above definition is justified as follows.

Lemma 3.2 ([28, Lemma 3.2])**.**

(1) Let $b(w,v) = \min\{n \geq 0 | \pi^n(w) = \pi^m(v)$ for some $m \geq 0\}$ for $w, v \in T$, then $\pi^{b(w,v)}(w) = \pi^{b(v,w)}(v)$.

(2) Let $\mathcal{A} = \{(w,v) \mid \pi(w) = v$ or $\pi(v) = w\} \setminus \{(\phi,\phi)\}$, then $(T,\mathcal{A})$ is a tree.

From now on we assume (T,π,ϕ) to be a tree with a reference point. We define $[w] = b(w,\phi) - b(\phi,w)$, $T_n = \{w \in T \mid [w] = n\}$ for any $w \in T$ and $n \in \mathbb{Z}$. By abuse of notation, we write $\acute{\pi}^{-k}(w)$ instead of $\pi^{-k}(w) \cap T_{[w]+k}$. Note that $\acute{\pi}^{-k}(w) \neq \pi^{-k}(w)$ if and only if $F_\pi \neq \emptyset$, $w = \phi$ and $k \geq 1$. We also define $T^w = \cup_{k\geq 0}\acute{\pi}^{-k}(w)$.

Definition 3.3 (Partition)**.** Let (Y,ρ) be a (σ-compact) metric space without isolated points. We say $K : T \to \mathfrak{P}(Y)$ is a *partition* of (Y,ρ) parametrized by (T,π,ϕ) if the following conditions hold.

- For any $w \in T$, $K(w)$ is a compact set, neither a single point nor empty.
- $\displaystyle\bigcup_{w\in(T)_0} K(w) = Y$ and for any $w \in T$, $\displaystyle\bigcup_{v\in\acute{\pi}^{-1}(w)} K(v) = K(w)$.
- If $(w_k)_{k\in\mathbb{Z}} \subset T$ satisfies $\pi(w_{k+1}) = w_k$ for any $k \in \mathbb{Z}$, then $\cap_{k\in\mathbb{Z}} K(w_k)$ is a single point.

Hereafter, we write K_w instead of $K(w)$ for simplicity.

Remark. The condition that K_w has no isolated points, assumed in [21, Definition 2.2.1], follows from Definition 3.3 (see [28, Lemma 3.6]).

Definition 3.4 (Basic framework)**.** Let (T,π,ϕ) be a tree with a reference point satisfying $\sup_{w\in T}\#(\acute{\pi}^{-1}(w)) < \infty$, and K be a partition of a metric space (Y,ρ) parametrized by (T,π,ϕ). Let

$$E_n = \{(w,v) \in T_n \times T_n \mid K_w \cap K_v \neq \emptyset,\ w \neq v\}$$

and let l_n denote the graph distance of (T_n, E_n) allowing $l_n(w,v) = \infty$. We say K satisfies the *basic framework* if the following conditions hold.

- $\mathrm{int}(K_w) \cap \mathrm{int}(K_v) = \emptyset$ for any $w, v \in T$ with $[w] = [v]$ and $w \neq v$. (3.1)
- There exists $\zeta \in (0,1)$ such that $\mathrm{diam}_\rho(K_w) \asymp \zeta^{[w]}$ for any $w \in T$. (3.2)
- There exists $\xi > 0$ such that for each $w \in T$, $B_\rho(x_w, \xi\zeta^{[w]}) \subset K_w$ for some $x_w \in K_w$. (3.3)
- Let $\Delta_m(x,y) = \sup\{n \mid x \in K_w, y \in K_v$ and $l_n(w,v) \leq m$ for some $w, v \in T_n\}$ then $\rho(x,y) \asymp \zeta^{\Delta_{M_*}(x,y)}$ for any $x, y \in X$, for some $M_* \in \mathbb{N}$. (3.4)
- $L_* := \sup_{w\in T}\#(\{v \mid (w,v) \in E_{[w]}\}) < \infty$ (3.5)

Remark. (1) The formulation in Definition 3.4 differs from the original one in [21, Section 4.3], for the reader's convenience. However it follows from (3.2) and [21, Proposition 3.2.1] that the above definition is equivalent to the original one.

(2) By (3.2), $\mathrm{diam}(X,d) < \infty$ if $\pi(\phi) = \phi$ and otherwise $\mathrm{diam}(X,d) = \infty$.

For the existence of a partition of the given metric space satisfying the basic framework, there is the following result.

Theorem 3.5 ([28, Theorem 3.12])**.** *Let* (Y, ρ) *be a complete metric space. Then the following conditions are equivalent.*

(1) $\dim_{\mathrm{ARC}}(Y, \rho) < \infty$.

(2) (Y, ρ) *is doubling and uniformly perfect.*

(3) There exist a tree with a reference point (T, π, ϕ) *and a partition* K *of* (Y, ρ) *such that* K *satisfies the basic framework with respect to* ρ.

Remark. (1) In [26, 27, 28], the definition of the Ahlfors regular conformal dimension was slight different in order to consider that of a discrete metric space. This difference required the additional assumption that $((Y, \rho)$ is) "without isolated points" in the original statement of [28, Theorem 3.12].

(2) The equivalence between (1) and (2) was well-known (see [12, Theorem 13.3 and Corollary 14.15], for example).

We also note that we can choose $\{x_w\}_{w\in T_n}$ as an increasing sequence of sets.

Lemma 3.6. *Let* K *be a partition of* (Y, ρ)*, parametrized by* (T, π, ϕ) *satisfying the basic framework. Then there exist* $\{x_w\}_{w\in T}$ *satisfying* (3.3) *and for any* $n \le m$, $\cup_{w\in T_n}\{x_w\} \subset \cup_{w\in T_m}\{x_w\}$.

Remark. It is obvious that $\{x_w \mid w \in T_n \cap T^{\pi^n(\phi)}\}_{n\ge 0}$ is a spread sequence.

Proof. Let $\{x_w\}_{w\in T}$ be points satisfying (3.3). By (3.2), there exists $k \ge 1$ such that $g_\rho(K_w) \le \xi\zeta^n/2$ for any $w \in T_{n+k}$. We can define $f : \cup_n T_{kn} \to \cup_n T_{kn}$ such that $f(w) \in T_{[w]+k}$ and $x_w \in K_{f(w)}$. For $w \in \cup_n T_{kn}$, let y_w be the unique point with $y_w \in \cap_{n\ge 0} K_{f^n(w)}$. Then $\cup_{w\in T_{kn}}\{y_w\} \subset \cup_{w\in T_{km}}\{y_w\}$ for $n \le m$ and

$$B_\rho(y_w, \frac{\xi}{2}\zeta^{[w]}) \subset B_\rho(x_w, \xi\zeta^{[w]}) \subset K_w$$

for $w \in \cup_n T_{kn}$. For $w \in \cup_n T_{kn-m}$ $(k > m > 0)$, we define y_w by induction on m. Let $y_w = y_v$ for some $v \in \acute{\pi}^{-1}(w)$ such that $v = \pi^{m-1} \circ f \circ \pi^{k-m}(w)$ whenever $w = \pi^m \circ f \circ \pi^{k-m}(w)$. Then we obtain

$$\cup_{w\in T_{kn-(m-1)}}\{y_w\} \supset \cup_{w\in T_{kn-m}}\{y_w\} \supset \cup_{w\in T_{k(n-1)}}\{y_w\}$$

and

$$B_\rho(y_w, \frac{\xi\zeta^k}{2}\zeta^{[w]}) \subset B_\rho(y_v, \frac{\xi}{2}\zeta^{[v]}) \subset K_v \subset K_w$$

for some $v \in \acute{\pi}^{-m}(w)$. This shows $\{y_w\}_{w\in T}$ is the desired set of points. □

We are now able to introduce the precise definitions of $\overline{d}_p^s$ and $\underline{d}_p^s$. For the rest of this section, we assume that K denotes a partition of a metric space (Y, ρ) parametrized by (T, π, ϕ) satisfying the basic framework, with points $\{x_w\}_{w\in T}$ satisfying (3.3) such that for any $n \le m$, $\cup_{w\in T_n}\{x_w\} \subset \cup_{w\in T_m}\{x_w\}$.

Definition 3.7 (p-spectral dimensions)**.** Let

$$\begin{aligned}
N_* &= \limsup_{k\to\infty}\Big(\sup_{w\in T}\#(\acute{\pi}^{-k}(w))\Big)^{1/k},\\
\mathcal{E}_n^p(f) &= \frac{1}{2}\sum_{(x,y)\in E_n}|f(x)-f(y)|^p,\\
\mathcal{C}_{w,k} &= \{v\in T_{[w]+k}\mid l_{[w]}(w,\pi^k(v))>M_*\}\text{ and}\\
\mathcal{E}_{p,k,w} &= \inf\{\mathcal{E}_n^p(f)\mid f\in\ell(T_{[w]+k}),\ f|_{\acute{\pi}^{-k}(w)}\equiv 1,\ f|_{\mathcal{C}_{w,k}}\equiv 0\}
\end{aligned}$$

for any n, $f\in\ell(T_n)$, $w\in T$ and $p>0$. (In particular for $p=2$, $\mathcal{E}_{2,k,w}=(\Omega_{[w]+k}(\acute{\pi}^{-k}(w),\mathcal{C}_{w,k}))^{-1}$ where Ω_n is the standard graph resistance of (T_n,E_n).) We define the *upper p-spectral dimensions* of the partition K for $p>0$ by

$$\overline{d}_p^s(K)=p\bigg(1-\frac{\limsup_{k\to\infty}\frac{1}{k}\big(\sup_{w\in T}\log\mathcal{E}_{p,k,w}\big)}{\log N_*}\bigg)^{-1}\tag{3.6}$$

and the *lower p-spectral dimensions* $\underline{d}_p^s(K)$ for $p>0$ by (3.6) but replacing $\limsup$ by $\liminf$.

Remark. In the same way as in the proof of Proposition 2.19, we have

$$N_*=\lim_{k\to\infty}\big(\sup_{w\in T}\#(\acute{\pi}^{-k}(w))\big)=\inf_{k\geq 0}\big(\sup_{w\in T}\#(\acute{\pi}^{-k}(w))\big)\tag{3.7}$$

because $\big(\sup_{w\in T}\#(\acute{\pi}^{-j}(w))\big)\big(\sup_{w\in T}\#(\acute{\pi}^{-k}(w))\big)\geq\big(\sup_{w\in T}\#(\acute{\pi}^{-(j+k)}(w))\big)$ for any $j,k\geq 0$.

The following is the main result of [21], which leads to Theorem 1.3.

Theorem 3.8 ([21, Theorems 4.6.9] and [26, Theorem 3.9])**.**

$$\begin{aligned}
\dim_{\mathrm{ARC}}(Y,\rho) &= \inf\{p\mid\liminf_{k\to\infty}(\sup_{w\in T}\mathcal{E}_{p,k,w})=0\}\\
&= \inf\{p\mid\limsup_{k\to\infty}(\sup_{w\in T}\mathcal{E}_{p,k,w})=0\}.
\end{aligned}$$

In the reminder of this section, we assume $(Y,\rho)=(X,R)$ and prove some inequalities of indices of the partition, which are necessary for the proof of Theorem 1.10.

Let $V_n=\{x_w\mid w\in T_n\}$ and $A_w=\cup\{K_v\mid v\in\mathcal{C}_{w,0}\}$. We will denote by $\mathcal{R}_n$ (resp. R_n, $\mu_{x,y,n}$) the resistance between sets (resp. metric, weights) associated with $(\mathcal{E}|_{V_n},\ell(V_n))$.

Lemma 3.9. *$\mathcal{R}(K_w,A_w)\asymp\zeta^{[w]}$ for any $w\in T$ with $A_w\neq\emptyset$.*

Proof. Since (3.2) and (3.3) hold, $\mathcal{R}(K_w,A_w)\lesssim\zeta^{[w]}$ follows from Lemma 2.13. On the other hand, there exists $\iota\in(0,1)$ satisfying $R(K_w,A_w)>\iota\zeta^{[w]}$ for any $w\in T$ because (3.4) holds. Since (X,R) is doubling each K_w is covered by N balls of radius $\iota\zeta^{[w]}/2$, for some $N\geq 0$. Therefore $\mathcal{R}(K_w,A_w)\gtrsim\zeta^{[w]}$ by Lemmas 2.11 (1) and 2.13, similarly to the latter part of the proof of Lemma 2.13. □

Proposition 3.10. *(1)* $\mathcal{E}_{2,k,w} \lesssim \zeta^k$ *for any* $w \in T$.

(2) If $(\mathcal{E}, \mathcal{F})$ *is local then* $\mathcal{E}_{2,k,w} \gtrsim \zeta^k$ *for any* $w \in T$.

For proving Proposition 3.10 (1), we use the argument of flow.

Definition 3.11 (Unit flow)**.** Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a finite set V. For $A, B \subset V$ with $A \cap B = \emptyset$, $f : V \times V \to R$ is called a *unit flow* from A to B if it satisfies

- $f(x, y) = -f(y, x)$ for any $x, y \in V$.
- $\sum_{y \in V} f(x, y) = 0$ for any $x \notin A \cup B$,
- $\sum_{x \in A} \sum_{y \in V} f(x, y) = 1$ and $\sum_{x \in B} \sum_{y \in V} f(x, y) = -1$.

Let $\{\mu_{x,y}\}, \mathcal{R}$ be the associated resistance weight and resistance between subsets, then it is known that

$$\mathcal{R}(A, B) = \min\{\frac{1}{2} \sum_{x,y \in V} \frac{f(x, y)^2}{\mu_{x,y}} \mid f \text{ is a unit flow from } A \text{ to } B\}. \tag{3.8}$$

We say f is the *optimal flow* for $\mathcal{R}(A, B)$, or optimal flow from A to B if f is the optimal function for the right hand side of (3.8).

Proof of Proposition 3.10. (1) Fix any $w \in T$ and $k \geq 0$. Let τ be the optimal flow for $\Omega_{[w]+k}(\acute{\pi}^{-k}(w), \mathcal{C}_{w,k})$, $\alpha > 1$ be the constant appeared in Proposition 2.17 and $f_{u,v}$ be the optimal flow for $R_{V_{[w]+k}, B(x, \alpha R(x_u, x_v))}(x_u, x_v)$ for $u, v \in T_{[w]+k}$. We define $f : V_{[w]+k} \times V_{[w]+k} \to \mathbb{R}$ by

$$f(p, q) = \frac{1}{2} \sum_{(u,v) \in E_{[w]+k}} \tau(u, v) f_{u,v}(p, q),$$

then f is a unit flow from $K_w \cap V_{[w]+k}$ to $A_w \cap V_{[w]+k}$ on $(\mathcal{E}|_{V_n}, \ell(V_n))$. Note that $f_{u,v}(p, q) = 0$ if

$$R(x_u, p) \vee R(x_v, q) \geq \alpha R(x_u, x_v) \geq \alpha \xi \zeta^{[w]+k}.$$

Since R is doubling, there exists $N > 0$ such that

$$\sup_{x \in X,\ r > 0} \{\#(Y) \mid Y \subset B(x, \alpha r), R(y, z) \geq r \text{ for any } y, z \in Y \text{ with } y \neq z\} \leq N$$

Therefore

$$\begin{aligned}
\mathcal{R}_{[w]+k}(K_w, A_w) \leq& \frac{1}{2} \sum_{p,q \in V_{[w]+k}} \frac{f(p, q)^2}{\mu_{p,q}} \\
\leq& \frac{N}{8} \sum_{(u,v) \in E_{[w]+k}} \tau(u, v)^2 \sum_{p,q \in V_{[w]+k}} \frac{f_{u,v}(p, q)^2}{\mu_{p,q}} \\
\leq& \frac{N}{2} \Big(\sup_{(u,v) \in E_{[w]+k}} 2R(x_u, x_v) \Big) \sum_{(u,v) \in E_{[w]+k}} \tau(u, v)^2 \\
\leq& 2N\xi\zeta^{[w]+k} (\mathcal{E}_{2,k,w})^{-1}.
\end{aligned}$$

Since $\zeta^{[w]} \lesssim \mathcal{R}(K_w, A_w) \le \mathcal{R}_{[k]+w}(K_w, A_w)$ for any $w \in T$, the claim holds.

(2) We first note that there exists $\beta > 0$ such that $\Delta_{M_*}(x, y) \ge n$ whenever $R(x, y) \le \beta\zeta^n$ by (3.4). Fix any $w \in T$ and $k \ge 0$. Since $T_{[w]+k} \setminus \mathcal{C}_{w,k}$ is a finite set, there exists $n \ge [w] + k$ such that

$$\sum_{v \in T_{[w]+k} \setminus \mathcal{C}_{w,k}} \sum_{\substack{x \in K_v, \\ y \notin B(x, \beta\zeta^{[w]+k})}} \mu_{x,y,n} < \frac{1}{3}\mathcal{R}(K_w, A_w)^{-1}$$

and $\mathcal{R}_n(K_w, A_w) \le 2\mathcal{R}(K_w, A_w)$, by Lemma 2.12 and Proposition 2.16.
Let f be the optimal function for $\Omega_{[w]+k}(\acute{\pi}^{-k}(w), \mathcal{C}_{w,k})$ and τ_v be the optimal functions for $R_n(K_v, A_v)$ and $v \in T_{[w]+k}$. We also let

$$\overline{f}(v) = 2\max\bigl\{|f(v) - f(u)| \bigm| l_{[w]+k}(u, v) \le 2M_*\bigr\}.$$

Our next goal is to construct a suitable function τ on V_n with $\tau|_{K_w \cap V_n} \equiv 1$ and $\tau|_{A_w \cap V_n} \equiv 0$ with the above functions. Set

$$\begin{aligned} P(x) = \bigl\{\{x_i\}_{i=0}^m \subset V_n \bigm| & m \in \{0\} \cup \mathbb{N},\ x_0 \in A_w,\ x_m = x, \\ & R(x_i, x_{i-1}) \le \beta\zeta^{[w]+k} \text{ for any } i\bigr\} \end{aligned}$$

for $x \in V_n$. We define $\tau : V_n \to \mathbb{R}$ as

$$\tau(x) = 1 \wedge \inf\left\{\sum_{i=1}^m \sup_{v \in T_{[w]+k}} \overline{f}(v)|\tau_v(x_i) - \tau_v(x_{i-1})| \;\middle|\; \{x_i\}_{i=0}^m \in P(x)\right\}$$

where $\sum_{i=1}^0 \infty := 0$. Then it clearly holds $0 \le \tau \le 1$ and $\tau|_{A_w \cap V_n} \equiv 0$.

Claim. $\tau|_{K_w \cap V_n} \equiv 1$.

Proof. Fix any $x \in K_w \cap V_n$ and $\{x_i\}_{i=0}^m \in P(x)$. We inductively choose $i_0 = 0$, $v_j \in T_{[w]+k}$ such that $x_{i_j} \in K_{v_j}$, and $i_{j+1} = \min\{i \mid x_j \in A_{v_j}\}$. Note that since $\Delta_{M_*}(x_{i_j}, x_{(i_j)-1}) \ge [w] + k$, $l_{[w]+k}(v_j, v_{j-1}) \le 2M_*$. Let $\iota \ge 0$ and $v_* \in T_{[w]+k}$ be such that $x \notin A_{v_\iota}$, $v_* \in T_{[w]+k}$ and $l_{[w]+k}(v_*, v_\iota) \le M_*$, then

$$\begin{aligned} &\sum_{i=1}^m \sup_{v \in T_{[w]+k}} \overline{f}(v)|\tau_v(x_i) - \tau(x_{i-1})| \\ \ge &\sum_{j=0}^{\iota-1} \overline{f}(v_j) \sum_{i=i_j}^{i_{(j-1)}-1} |\tau_{v_j}(x_{i+1}) - \tau(x_i)| \\ \ge &\sum_{j=0}^{\iota-1} \overline{f}(v_j) \ge |f(v_*) - f(v_\iota)| + \sum_{j=0}^{\iota-1} |f(v_{i+1}) - f(v_i)| \ge 1. \end{aligned}$$

This shows the claim. □

Next we evaluate $\mathcal{E}|_{V_n}(\tau,\tau)$:

$$\begin{aligned}\frac{1}{2}(\mathcal{R}(K_w,A_w))^{-1} \leq&\mathcal{E}|_{V_n}(\tau,\tau)\\ \leq&\frac{1}{2}\sum_{\substack{x,y\in V_n\\ :(x,y)\notin A_w\times A_w}}(\tau(x)-\tau(y))^2\mu_{x,y,n}\\ \leq&\frac{1}{3}(\mathcal{R}(K_w,A_w))^{-1}+\frac{1}{2}\sum_{\substack{x,y\in V_n\\ :R(x,y)\leq\beta\zeta^{[w]+k}}}(\tau(x)-\tau(y))^2\mu_{x,y,n}.\end{aligned}$$

If $R(x,y)\leq\beta\zeta^{[w]+k}$, then $|\tau(x)-\tau(y)|\leq\sup_{v\in T_{[w]+k}}\overline{f}(v)|\tau_v(x)-\tau_v(y)|$. Moreover, since $\tau|_{A_v}\equiv 0$, $R(x_v,x_u)\geq\xi\zeta^{[w]+k}$ for $u,v\in T_{[w]+k}$ with $u\neq v$ and R is doubling, there exists $J>0$ such that

$$\sup_{\substack{w\in T,k\geq 0,\\ x,y\in V_n:R(x,y)\leq\beta\zeta^{[w]+k}}}\#\{v\in T_{[w]+k}\mid\tau_v(x)\vee\tau_v(y)\geq 0\}\leq J<\infty.$$

Therefore

$$\begin{aligned}\frac{1}{6}(\mathcal{R}(K_w,A_w))^{-1}\leq&\frac{J}{2}\sum_{v\in T_{[w]+k}}\overline{f}(v)^2\sum_{x,y\in V_n}(\tau_v(x)-\tau_v(y))^2\mu_{x,y,n}\\ \leq&\frac{J}{\xi}\zeta^{-[w]-k}\sum_{v\in T_{[w]+k}}\overline{f}(v)^2.\end{aligned}$$

Since $\overline{f}(v)\leq\sum_{i=1}^{2M_*}|f(v_i)-f(v_{i-1})|$ for some $\{v_i\}_{i=0}^{2M_*}$ satisfying $v_0=v$ and $(v_i,v_{i-1})\in E_{[w]+k}$,

$$\sum_{v\in T_{[w]+k}}\overline{f}(v)^2\leq 4L_*^{2M_*-1}\sum_{u,v\in E_{[w]+k}}(f(u)-f(v))^2=8L_*^{2M_*-1}\mathcal{E}_{2,k,w}.$$

Thus we have

$$\mathcal{E}_{2,k,w}\geq\frac{\xi}{48JL_*^{2M_*-1}}\zeta^{[w]+k}(\mathcal{R}(K_w,A_w))^{-1}.$$

This with Lemma 3.9 shows the proposition. □

Proposition 3.12.

(1) $$\sup_{x\in X,r\in(0,\mathrm{diam}(X,R))}\frac{V_\mu(x,r)}{V_\mu(x,\zeta^k r)}\gtrsim N_*^k \text{ for any } \mu\in\mathcal{M}_{(X,R)} \text{ and } k\geq 0.$$

(2) *For any $\epsilon>0$, there exists $\mu\in\mathcal{M}_{(X,R)}$ with*
$$\sup_{x\in X,r\in(0,\mathrm{diam}(X,R))}\frac{V_\mu(x,r)}{V_\mu(x,\zeta^k r)}\lesssim(N_*+\epsilon)^k \text{ for any } k\geq 0.$$

Proof. (1) We have

$$\mu(K_w) \geq \sum_{v \in \acute{\pi}^{-k}(w)} V_\mu(x_v, \xi\zeta^k) \geq \min_{v \in \acute{\pi}^{-k}(w)} V_\mu(x_v, \xi\zeta^k) \#(\acute{\pi}^{-k(w)}),$$

which leads to

$$\sup_{\substack{w \in T,\\ v \in \acute{\pi}^{-k}(w)}} \frac{V_\mu(x_v, 2\mathrm{diam}(K_w))}{V_\mu(x_v, \xi\zeta^{[v]})} \geq \sup_{w \in T} \#(\acute{\pi}^{-k(w)}) \geq N_*^k$$

for any $k \geq 0$. Since μ is $(\mathrm{VD})_R$, $\mathrm{diam}(K_w) \lesssim \zeta^{[w]}$ for any $w \in T$ and (3.7), the claim follows.

(2) Fix any $\epsilon > 0$. By (3.7) and [21, Proposition 4.3.5], we can choose $k \geq 1$ such that $\sup_{w \in T} \#(\acute{\pi}^{-k}(w)) \leq (N_* + \epsilon)^k$ and for any $w \in T$, there exists $v(w) \in \acute{\pi}^{-k}(w)$ with $K_{v(w)} \subset \mathrm{int}(K_w)$.
Let $\tilde{T} = \cup_{n \in \mathbb{Z}} T_{kn}$, then it is easily seen that $K|_{\tilde{T}}$ is a partition of (X, R) parametrized by $(\tilde{T}, \pi^k, \phi)$, satisfying the basic framework. We define $\varphi, \psi : \tilde{T} \to \mathbb{R}$ as

$$\varphi(w) = \begin{cases} \left(1 - \frac{\#(\acute{\pi}^{-k}(\pi^k(w)))-1}{(N_*+\epsilon)^k}\right) & (\text{if } w = v(\pi^k(w))) \\ \frac{1}{(N_*+\epsilon)^k} & (\text{otherwise}) \end{cases}$$

and $\psi(w) = \left(\prod_{i=0}^{\tilde{b}(w,\phi)} \varphi(\pi^{ik}(w))\right) / \left(\prod_{i=0}^{\tilde{b}(\phi,w)} \varphi(\pi^{ik}(\phi))\right)$, where

$$\tilde{b}(w,u) = b_{\pi^k}(w,u) := \min\{i \geq 0 \mid \pi^{ik}(w) \text{ for some } j \geq 0\}$$

for $w, u \in \tilde{T}$. Note that

$$\frac{1}{(N_* + \epsilon)^k} \leq \varphi(w) \leq \left(1 - \frac{1}{(N_* + \epsilon)^k}\right) \quad \text{and} \quad \sum_{v \in \acute{\pi}^{-k}(w)} \varphi(w) = 1.$$

We next claim that

$$((N_* + \epsilon)^k - 1)\psi(w) \geq \max\{\psi(u) \mid (w,u) \in E_{[w]}\} \text{ for any } w \in \tilde{T}. \tag{3.9}$$

Recall that $\tilde{b}(w,u) = \tilde{b}(u,w)$ by $[w] = [u]$. If $i < \tilde{b}(w,u) - 1$ then we have $\pi^{ik}(w) \neq v(\pi^{(i+1)k}(w))$ because

$$\phi \neq K_{\pi^{ik}(w)} \cap K_{\pi^{ik}(u)} \subset K_{\pi^{(i+1)k}(w)} \cap K_{\pi^{(i+1)k}(u)} \not\subset \mathrm{int}(K_{\pi^{(i+1)k}(w)}).$$

This shows

$$\frac{\psi(u)}{\psi(w)} = \frac{\prod_{i=0}^{\tilde{b}(w,u)} \varphi(\pi^{ik}(w))}{\prod_{i=0}^{\tilde{b}(u,w)} \varphi(\pi^{ik}(u))} = \frac{\pi^{(\tilde{b}(w,u)-1)k}(w)}{\pi^{(\tilde{b}(u,w)-1)k}(u)} \leq ((N_* + \epsilon)^k - 1).$$

We now prove the proposition for the case of $\mathrm{diam}(X, R) < \infty$.
Let $\mu_n = \sum_{w \in T_{nk}} \psi(w)\delta_{x_w}$, where δ_{x_w} is the Dirac measure on x_w. Then by

Prokhorov's theorem, there exists a Borel probability measure μ_* such that $\mu_{n_m} \to \mu_*$ weakly as $m \to \infty$, for some subsequence $\{\mu_{n_m}\}_{m\geq 0}$. We have

$$\mu_*(K_w) \geq \limsup_{m\to\infty} \mu_{n_m}(K_w) = \psi(w)$$

for any $w \in T$. Moreover, since $K_w \subset U_w := \mathrm{int}(\cup_{u:(w,u)\in E_{[w]}} K_u)$,

$$\mu_*(K_w) \leq \mu_*(U_w) \leq \liminf_{m\to\infty} \mu_{n_m}(U_w) \leq (L_*((N_* + \epsilon)^k - 1) + 1)\psi(w).$$

By (3.2) and (3.4), there exists $m \geq 0$ such that for any $x \in X$, $r \in (0, \mathrm{diam}(X, R))$ and $n \geq 0$, we can choose some $w \in \tilde{T}$ with

$$K_w \subset B(x, \zeta^{nk} r) \quad \text{and} \quad B(x, r) \subset X \setminus A_{\pi^{(m+n)k}(w)}.$$

Since

$$\begin{aligned}\frac{\mu_*(X \setminus A_{\pi^{(m+n)k}(w)})}{\mu_*(K_w)} &\lesssim \frac{\sum^{u\in T_{[w]-(m+n)k}}_{:l_{[u]}(u,\pi^{(m+n)k})\leq M_*} \psi(u)}{\psi(w)} \\ &\lesssim \frac{\psi(\pi^{(m+n)k}(w))}{\psi(w)} \leq (N_* + \epsilon)^{(m+n)k}\end{aligned}$$

for any $n \geq 0$ and $w \in \tilde{T}$ by (3.9), the claim holds for the bounded case.

We now turn to the case of $\mathrm{diam}(X, R) = \infty$. We can choose μ_u for each $u \in T_0$ such that $\mu_u(X \setminus K_u) = 0$ and $\psi(w) \leq \mu_u(K_w) \leq L_*(N_* + \epsilon)^k \psi(w)$ for any $w \in \tilde{T}^u$, by the former case. Let $\mu_* = \sum_{u\in T_0} \mu_u$, then it is clear that $\psi(w) \leq \mu_*(K_w)$ and

$$\begin{aligned}\mu_*(K_w) &\leq \begin{cases} \sum_{u:(u,w)\in E_{[w]}} \sum_{q\in T^u\cap T_0} \mu_q(K_u) & (\text{if } [w] < 0) \\ \sum_{u:(u,w)\in E_{[w]}} \mu_{\pi^{[w]}(u)}(K_u) & (\text{otherwise }) \end{cases} \\ &\leq \sum_{u:(u,w)\in E_{[w]}} L_*(N_* + \epsilon)^k \psi(u) \leq L_*^2 (N_* + \epsilon)^{2k} \psi(w)\end{aligned}$$

for any $w \in \tilde{T}$. Therefore the same proof as the bounded case works for the present case. □

Remark. (1) The idea of this proof comes from [21, Theorems 4.2.2 and 4.5.1].

(2) We can show that $-\log N_*/\log \zeta$ coincides with the *Assouad dimension* $\dim_A(X, R)$, where

$$\begin{aligned}\dim_A(X, R) = \inf\{t > 0 \mid\ & B(x, r) \text{ is covered by} \lfloor C(r/s)^t \rfloor \text{ bolls} \\ & \text{of radius } s \text{ for any } 0 < s < r \text{ and } x \in X, \text{ for some } C > 0\}.\end{aligned}$$

Thus we can also deduce Proposition 3.12 (2) from [12, Theorem 13.5].

4 Proof of main results

In this section we prove Theorems 1.9 and 1.10.

Proof of Theorem 1.10. By Proposition 2.19,

$$
\begin{aligned}
&\frac{\overline{d_s}(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu)}{2}\\
&=\limsup_{s\to\infty}\sup_{x\in X, r\in(0,\operatorname{diam}(X,R))}\left(1+\frac{\log s}{\log\bigl(V_\mu(x,r)/V_\mu(x,r/s)\bigr)}\right)^{-1}\\
&=\limsup_{k\to\infty}\sup_{x\in X, r\in(0,\operatorname{diam}(X,R))}\left(1+\frac{\log \zeta^{-k}}{\log\bigl(V_\mu(x,r)/V_\mu(x,\zeta^k r)\bigr)}\right)^{-1}\\
&=\left(1+\frac{\log\zeta^{-1}}{\limsup_{k\to\infty}\sup_{x\in X, r\in(0,\operatorname{diam}(X,R))}\frac{1}{k}\log\bigl(V_\mu(x,r)/V_\mu(x,\zeta^k r)\bigr)}\right)^{-1}
\end{aligned}
\tag{4.1}
$$

because R is uniformly perfect and μ is $(\mathrm{VD})_R$.

Since $\limsup_{k\to\infty}(\sup_w \mathcal{E}_{2,k,w})^{1/k}\le\zeta<1$ by Proposition 3.10 (1),

$$
\frac{\overline{d}^s_2(K)}{2}\le\left(1+\frac{\log\zeta^{-1}}{\log N_*}\right)^{-1}, \tag{4.2}
$$

so $\overline{d}^s_2(K)\le\overline{d_s}(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu)$ holds by Proposition 3.12 (1). Moreover, if $(\mathcal{E},\mathcal{F})$ is local, the equality in (4.2) holds by Proposition 2.19. Since Proposition 3.12 (2) holds, for any $\epsilon>0$ there exists $\mu\in\mathcal{M}_{(X,R)}$ such that

$$
\frac{\overline{d_s}(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu)}{2}\le\left(1+\frac{\log\zeta^{-1}}{\log(N_*+\epsilon)}\right)^{-1},
$$

which shows $\inf_{\mu\in\mathcal{M}_{(X,R)}}\overline{d_s}(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu)\le\overline{d}^s_2(K)$ for the local case. □

Proof of Theorem 1.9. We have a partition of (X,R) satisfying the basic framework by Theorem 3.5, and obtain $\overline{d}^s_2(K)\le\overline{d_s}(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu)<2$ by (4.1) and Theorem 1.10. This and Theorem 1.3 (1) with $p=2$ prove the theorem. □

5 Example with $d_s(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu)<\dim_{\mathrm{ARC}}(X,R)<2$

In this section we prove Theorem 1.11. In other words, we give an example with the inequality $d_s(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu)<\dim_{\mathrm{ARC}}(X,R)<2$. The results in this section are the continuous version of [27, Section 3], and many of the resistance estimates used in this section come from that preprint. We also note that the techniques used for showing these resistance inequalities was originated in [1]. The main difficulty in the continuous case is to construct the desired resistance form. We overcome this difficulty by using the results of [16, 19] in the proof of Corollary 5.4.

Recall that $Q = \{z \mid |\mathrm{Re}(z)| \vee |\mathrm{Im}(z)| \leq 1/2\}$. Let

$$p_i = \begin{cases} 0 & (i=0) \\ \frac{1}{\sqrt{2}}\left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)^i & (i=1,3,5,7) \\ \frac{1}{2}\left(\sqrt{-1}\right)^{i/2} & (i=2,4,6,8) \end{cases}, \qquad \varphi_i(z) = \frac{1}{3}(z-p_i)+z,$$

$$F(n) = \begin{cases} 1 & (\text{if } k^2(k-1) < n \leq k^3 \text{ for some } k \in \mathbb{N}) \\ 0 & (\text{otherwise}) \end{cases},$$

$$\Phi_n(\mathcal{S}) = \begin{cases} \bigcup_{i=1}^{8} \varphi_i(\mathcal{S}) & (\text{if } F(n)=1) \\ \bigcup_{i=0,1,3,5,7} \varphi_i(\mathcal{S}) & (\text{if } F(n)=0) \end{cases} \quad \text{for } \mathcal{S} \in \mathfrak{P}(\mathbb{C})$$

and $X = \bigcap_{n \geq 1} \Phi_1 \circ \Phi_2 \circ \cdots \circ \Phi_n(I)$. It is easy to see that (X,d) is a complete, doubling, uniformly perfect metric space, where d is the Euclidean metric on X given by $d(z,w) = |z-w|$. We also let

$$T_n = \begin{cases} \{\phi\} & (n=0) \\ \{(w_i)_{i=1}^n \mid w_i \in \{0,1,3,5,7\} \text{ if } F(i)=1, \\ \qquad\qquad w_i \in \{1,...,8\} \text{ if } F(i)=0\} & (\text{otherwise}) \end{cases}$$

and $T = \bigsqcup_{n \geq 0} T_n$. For any $w \in T$, we define

$$\varphi_w = \begin{cases} \mathrm{id}_{\mathbb{C}} & (\text{if } w = \phi) \\ \varphi_{w_1} \circ \cdots \circ \varphi_{w_n} & (\text{otherwise}) \end{cases}, \qquad K_w = \varphi_w(Q) \cap X$$

$$\text{and } \pi(w) = \begin{cases} \phi & (\text{if } w \in T_0 \sqcup T_1) \\ (w_i)_{i=1}^{n-1} & (\text{if } w = (w_i)_{i=1}^n \text{ for some } n > 1). \end{cases}$$

Then it is easily seen that K is a partition of (X,d) parametrized by (T,π,ϕ), satisfying the basic framework. Moreover, in the same way as [27, Proposition 3.11], we have $\dim_{\mathrm{ARC}}(X,d) = \dim_{\mathrm{ARC}}(\mathrm{SC},d)$, where SC is the standard Sierpiński carpet (recall Figure 2). It is also routine work to show that there exists a Borel measure μ such that

$$\mu(K_w) = 3^{-n}(5/3)^{-\#\{k \leq n \mid F(k)=1\}} \qquad \text{for every } n \in \mathbb{N} \text{ and } w \in T_n,$$

and so μ is $(\mathrm{VD})_d$.

By the fact $2\log 5/(\log 3+\log 5) < 1.5 < 1+(\log 2/\log 3) \leq \dim_{\mathrm{ARC}}(\mathrm{SC},d) < 2$ (see [32, 33] for the proof of the last two inequalities) and the above argument, for the proof of Theorem 1.11 it suffices to prove the following theorem.

Theorem 5.1. *There exists a resistance form $(\mathcal{E},\mathcal{F})$ on X such that the associated resistance metric R is quasisymmetric to the Euclidean metric d, the limit $d_s(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu)$ exists and is independent of x, and $d_s(\mu,\mathcal{E}_\mu,\mathcal{D}_\mu) = 2\log 5/(\log 3 + \log 5)$.*

Remark. (1) $\dim_{\mathrm{ARC}}(X,d) = \dim_{\mathrm{ARC}}(X,R)$ because $R \sim_{\mathrm{QS}} d$.

(2) $2\log 5/(\log 3+\log 5)$ equals the spectral dimension of the standard Dirichlet form on the Vicsek set (recall that it is the unique nonempty compact subset VS of $\mathbb{C}$ with $\mathrm{VS} = \cup_{j=0,1,3,5,7}\varphi_j(\mathrm{VS})$).

Let

$$V_0 = \{p_1, p_3, p_5, p_7\}, \quad G_0 = \{(x, y) \in V_0 \times V_0 \mid |x - y| = 1\},$$
$$V_{n,m} = \begin{cases} \Phi_{m+1} \circ \cdots \circ \Phi_n(V_0) & (\text{if } m < n) \\ V_0 & (\text{if } m = n) \end{cases}$$
$$G_{n,m} = \{(x, y) \in V_{n,m} \times V_{n,m} \mid \text{ for some } w \in T_n, \text{ there exists } (x', y') \in G_0 \text{ such that } \varphi_w(x') = \varphi_{\pi^{n-m}(w)}(x),\ \varphi_w(y') = \varphi_{\pi^{n-m}(w)}(y)\}$$

for any $n \geq 0$ and $0 \leq m \leq n$. We also define $\mathcal{E}_{n,m} : \ell(V_{n,m}) \times \ell(V_{n,m}) \to \mathbb{R}$ by

$$\mathcal{E}_{n,m}(u, v) = \frac{1}{2} \sum_{(x,y)\in G_{n,m}} (u(x) - u(y))(v(x) - v(y)),$$

then it is clear that $(\mathcal{E}_{n,m}, \ell(V_{n,m}))$ are resistance forms. Here $R_{n,m}$ denotes the associated resistance metric and $\mathcal{R}_{n,m}$ denotes the resistance between sets. For simplicity of notation, we write

$$(\mathrm{TB})_{n,m} = \mathcal{R}_{n,m}(\{z \in V_{n,m} \mid \mathrm{Im}(z) = \frac{1}{2}\}, \{z \in V_{n,m} \mid \mathrm{Im}(z) = \frac{-1}{2}\})$$
$$\text{and } (\mathrm{Pt})_{n,m} = R_{n,m}(p_1, p_5).$$

Moreover, we write n instead of $n, 0$ if no confusion may occur. For example, we write V_n instead of $V_{n,0}$. In the same way as [1, Section 4] and [27, Theorem 3.2], we have the following inequalities.

Lemma 5.2. *There exists $C > 0$, satisfying the following conditions for any $n \geq m \geq 0,\ x, y \in V_m$ and $w \in T_n$.*

(1) $R_n(x, y) \leq CR_m(x, y)(\mathrm{Pt})_{n,m}$ *and* $CR_n(x, y) \geq R_m(x, y)(\mathrm{TB})_{n,m}$.

(2) $C(\mathrm{TB})_{n,m} \geq (\mathrm{Pt})_{n,m} \geq (\mathrm{TB})_{n,m}$.

(3) Let $A = \{z \in \mathbb{C} \mid |\mathrm{Re}(z)| \vee |\mathrm{Im}(z)| \geq 3/2\}$, *then* $C\mathcal{R}_n(\varphi_w(Q) \cap V_n, \varphi_w(A) \cap V_n) \geq \mathcal{R}_m(\varphi_w(Q) \cap V_m, \varphi_w(A) \cap V_m)(\mathrm{TB})_{n,m}$

In particular,

$$(\mathrm{Pt})_n \leq C(\mathrm{Pt})_m(\mathrm{Pt})_{n,m} \leq C^2(\mathrm{Pt})_m(\mathrm{TB})_{n,m} \leq C^3(\mathrm{Pt})_n \tag{5.1}$$

for any $n \geq m \geq 0$, which follows from (1) and (2).

Remark. C only depends on the structure of the standard 3-adic squares and resistance estimates for the (graphical) standard Sierpiński carpet, so does not depend on m and n.

For the construction of the desired resistance form, we use the following proposition which implicitly appeared and is proved in [16, Proof of Theorem 5.1].

Proposition 5.3. *Let V be a finite set and $(\mathcal{E}_n, \ell(V))$ be a resistance form on V with the associated resistance metric R_n for every $n \geq 0$. If $R(x,y) := \lim_{n\to\infty} R_n(x,y) > 0$ exists for any $x, y \in V$ with $x \neq y$, then there exists a resistance form $(\mathcal{E}, \ell(V))$ such that the associated resistance metric coincides with R.*

Corollary 5.4. *Let $\{V_n\}$ be a sequence of increasing nonempty finite sets and $(\mathcal{E}_n, \ell(V_n))$ be a resistance form on V_n with the associated resistance metric R_n for every $n \geq 0$. If $R(x,y) := \lim_{n\to\infty} R_n(x,y) > 0$ exists for any $x, y \in \cup_{n\geq 0} V_n$ with $x \neq y$, there exists a resistance form on V_* such that the associated resistance metric coincides with R_*, where (V_*, R_*) is the completion of $(\cup_{n\geq 0} V_n, R)$.*

Proof. Applying Proposition 5.3 to $\{(\mathcal{E}_m|_{V_n}, \ell(V_n))\}_{m\geq n}$, we obtain the resistance form $(\mathcal{L}_n, \ell(V_n))$ such that the associated resistance metric coincides with $R|_{V_n\times V_n}$. By [19, Theorem 2.1.12] and [20, Theorem 3.13], the claim follows. □

Since (5.1) and Lemma 5.2 (1) and (2) holds, we have

$$C^{-2}\frac{R_m(x,y)}{(\mathrm{Pt})_m} \leq \frac{R_n(x,y)}{(\mathrm{Pt})_n} \leq C^3\frac{R_m(x,y)}{(\mathrm{Pt})_m} \tag{5.2}$$

for any $0 \leq m \leq n$ and $x, y \in V_m$, so by Corollary 5.4 with the diagonal sequence argument, we obtain a resistance form $(\mathcal{E}, \mathcal{F})$ on V_*, such that for some $\{n_j\}_{j\in\mathbb{N}}$ the associated resistance metric R_* satisfies

$$R_*(x,y) = \lim_{j\to\infty}\frac{R_{n_j}(x,y)}{(\mathrm{Pt})_{n_j}}$$

for any $x, y \in \cup_{n\geq 0} V_n$.

In order to show $V_* = X$ and $R \sim_{\mathrm{QS}} d$, and to calculate $d_s(\mu, \mathcal{E}_\mu, \mathcal{D}_\mu)$, we need more detailed evaluation as in [27].

Lemma 5.5. *(1) There exist $M \geq 0$ such that $(\mathrm{Pt})_{n+M} \geq 2(\mathrm{Pt})_n \gtrsim (\mathrm{Pt})_{n+1}$ for any $n \geq 0$.*

(2) For each $x, y \in \cup_{n\geq 0} V_n$, let

$$\Delta(x,y) = \min\{n \mid x \in \varphi_w(I) \text{ and } y \in \varphi_w(A) \text{ for some } w \in T_n\},$$

then $R_*(x,y) \asymp ((\mathrm{Pt})_{\Delta(x,y)})^{-1}$ for any $x, y \in \cup_{n\geq 0} V_n$.

Proof. (1) Let $k_1(n,m)$ denote $\#\{j \mid m < j \leq n, f_*(j) = 1\}$ and $k_2(n,m)$ denote $\#\{j \mid m < j < n, f_*(j) = 1, f_*(j+1) = 0\}$. Then in the same way as [27, Theorem 3.2 (1)] but induction of $(n-k)$ for any fixed n, it follows that there exist $C_a, C_b > 0$ and $\rho > 1$ such that

$$\rho^{k_1(n,m)}3^{n-m-k_1(n,m)}C_a^{k_2(n,m)} \lesssim (\mathrm{Pt})_{n,m} \lesssim \rho^{k_1(n,m)}3^{n-m-k_1(n,m)}C_b^{k_2(n,m)} \tag{5.3}$$

for any $0 \le m \le n$, because constants do not depend on n and m. Therefore the lemma follows from (5.1) and (5.3).

(2) We first note that

$$R_n(x,y) \asymp 1 \text{ for any } n \in \mathbb{N} \text{ and } x,y \in V_n \text{ with } (x,y) \in G_n, \tag{5.4}$$
$$\mathcal{R}_n(\varphi_w(I), \varphi_w(A)) \asymp 1 \text{ for any } n \in \mathbb{N} \text{ and } w \in T_n. \tag{5.5}$$

By definition of $\Delta(x,y)$, for any n with $x,y \in V_n$, we have $\{x_i\}_{i=\Delta(x,y)-2}^n$ and $\{y_i\}_{i=\Delta(x,y)-2}^n$ such that $x_{\Delta(x,y)-2} = y_{\Delta(x,y)-2}$, $x_n = x$, $y_n = y$ and for any $\Delta(x,y) - i \le i \le n$, $(x_{i-1}, x_i) \in \varphi_{w_i}(V_0)$ and $(y_{i-1}, y_i) \in \varphi_{u_i}(V_0)$ for some $w_i, u_i \in T_i$. Then by (1) and (5.2) and (5.4),

$$\begin{aligned}
\frac{R_n(x,y)}{(\mathrm{Pt})_n} &\le \frac{1}{(\mathrm{Pt})_n} \sum_{i=\Delta(x,y)-1}^{n} (R_n(x_{i-1}, x_i) + R_n(y_{i-1}, y_i)) \\
&\lesssim \sum_{i=\Delta(x,y)-1}^{n} \frac{1}{(\mathrm{Pt})_i} \\
&\lesssim \sum_{i=0}^{\infty} \frac{\sum_{j=1}^{M} (\mathrm{Pt})_{\Delta(x,y)-1+iM+j, \Delta(x,y)-1+iM}}{(\mathrm{Pt})_{\Delta(x,y)-1+iM}} \\
&\lesssim ((\mathrm{Pt})_{\Delta(x,y)-1})^{-1} \lesssim ((\mathrm{Pt})_{\Delta(x,y)})^{-1}
\end{aligned}$$

for any $n \in \mathbb{N}$ and $x,y \in V_n$.
On the other hand, let $w \in T_{\Delta(x,y)}$ be a vertex appeared in the definition of $\Delta(x,y)$, then by (5.5) and Lemma 5.2 (3),

$$\begin{aligned}
\frac{R_n(x,y)}{(\mathrm{Pt})_n} &\ge \frac{\mathcal{R}_n(\varphi_w(I), \varphi_w(A))}{(\mathrm{Pt})_n} \\
&\gtrsim \frac{(\mathrm{Pt})_{n,\Delta(x,y)}}{(\mathrm{Pt})_n} \gtrsim ((\mathrm{Pt})_{\Delta(x,y)})^{-1},
\end{aligned}$$

which completes the proof.

□

Proof of Theorem 5.1. We first prove $d|_{\cup_{n\ge0} V_n} \sim_{\mathrm{QS}} R_*|_{\cup_{n\ge0} V_n}$. By Lemma 5.5 (1) and (2), there exist $\alpha, \tau > 1$ such that

- if $\Delta(x,y) - \Delta(x,z) \ge Ma$ for $a \ge 0$ then $\dfrac{R_*(x,y)}{R_*(x,z)} \le \alpha 2^{-a}$.
- If $\Delta(x,z) - \Delta(x,y) \le a$ for $a \ge 0$ then $\dfrac{R_*(x,y)}{R_*(x,z)} \le \alpha \tau^a$.

Since $\Delta(x,z) - \Delta(x,y) \le \log(6\sqrt{2} d(x,y)/d(x,z))/\log 3$ and the above inequalities hold, there exist t_1, t_2 with $0 < t_1 < t_2$ such that

- if $d(x,y)/d(x,z) \le t_1$ then $R_*(x,y)/R_*(x,z) \le \theta_1(d(x,y)/d(x,z))$,
- $R_*(x,y)/R_*(x,z) \le \theta_2\big((d(x,y)/d(x,z)) \vee t_2\big)$

for x, y, z with $x \neq z$, where

$$\theta_1(t) = \alpha 2^{(\log 6\sqrt{2}t/M \log 3)+1}, \quad \theta_2(t) = \alpha \tau^{(\log 6\sqrt{2}t/\log 3)+1}.$$

It is obvious that there exists a homeomorphism $\theta : [0, \infty) \to [0, \infty)$ satisfying $\theta_1(t) \leq \theta(t)$ for $t \leq t_1$ and $\theta_2(t \vee t_2) \leq \theta(t)$ for $t > t_1$, which proves the desired quasisymmetry. This also shows a sequence in $\cup_{n\geq 0} V_n$ is d-Cauchy if and only if R_*-Cauchy, therefore $V_* = X$ and $d \sim_{\mathrm{QS}} R_*$. In other words, $(\mathcal{E}, \mathcal{F})$ is the desired resistance form.

It remains to calculate $d_s(\mu, \mathcal{E}_\mu, \mathcal{D}_\mu)$. By Proposition 2.6 we can apply [20, Theorem 15.6] for d and obtain

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{-\log p_\mu(1/t, x, x)}{\log t} \\
=&\limsup_{n\to\infty} \frac{\log V_d(x, 3^{-n})}{\log h_d(x, 3^{-n})} \\
\leq&\limsup_{n\to\infty} \frac{\frac{k_1(n)}{n}\log 8 + (1 - \frac{k_1(n)}{n})\log 5}{\frac{k_1(n)}{n}(\log 8 + \log \rho) + (1 - \frac{k_1(n)}{n})(\log 5 + \log 3) + \frac{k_2(n)}{n}\log C_a} \\
=&\frac{\log 5}{\log 5 + \log 3}
\end{aligned}$$

for any $x \in X$. (Note that the first equation follows from Lemma 2.20.) Similarly we have

$$\liminf_{t\to\infty} \frac{-\log p_\mu(1/t, x, x)}{\log t} \geq \frac{\log 5}{\log 5 + \log 3},$$

and the proof is complete. $\square$

A Equivalence of local properties

Let us recall that X is a set, $(\mathcal{E}, \mathcal{F})$ is a resistance form on X and R is the resistance metric associated with $(\mathcal{E}, \mathcal{F})$. In this appendix we discuss the relation between the local property of $(\mathcal{E}, \mathcal{F})$ and that of the Dirichlet form induced by $(\mathcal{E}, \mathcal{F})$. We also recall that the local property of a Dirichlet form is defined as follows.

Definition A.1 (Local)**.** Let (Y, ρ) be a locally compact separable metric space and ν be a Radon measure on Y with full support. A Dirichlet form (E, D) on $L^2(Y, \nu)$ is called *local* if $E(u, v) = 0$ whenever $u, v \in D$ have disjoint compact supports, where the *support* $\mathrm{supp}(u)$ of $u \in L^2(Y, \nu)$ is defined as the support of the measure $u d\nu$ on (Y, ρ).

By [20, Theorem 9.4], if $(\mathcal{E}, \mathcal{F})$ is a regular resistance form satisfying (ACC), then for each Radon measure μ on X with full support, $(\mathcal{E}_\mu, \mathcal{D}_\mu)$, defined in the same way as Lemma 1.8, is a regular Dirichlet form on $L^2(X, \mu)$. Here we remark that $\mathrm{supp}(u) = \overline{\{x \in X \mid u(x) \neq 0\}}$ because $\mathcal{F} \subset C(X, R)$. Therefore by the definition of $\mathcal{D}_\mu$, $(\mathcal{E}_\mu, \mathcal{D}_\mu)$ is a local Dirichlet form (over (X,R)) if $(\mathcal{E},\mathcal{F})$ is a local resistance form. In this appendix we prove that, under Assumption 1.6, the converse direction is also true. Indeed, the following holds.

Proposition A.2. *Assume that R is complete and doubling. Then for any $u \in \mathcal{F}$, there exists $\{u_n\}_{n\geq 0} \subset \mathcal{F} \cap C_0(X,R)$ such that* $\mathrm{supp}(u_n) \subset \mathrm{supp}(u)$ *for any n and* $\lim_{n\to\infty} \mathcal{E}(u-u_n, u-u_n) = 0$.

Corollary A.3. *We make the Assumption 1.6 and let μ be a Radon measure on X with full support. Then the following conditions are equivalent.*

(1) $\mathcal{E}(u,v) = 0$ *if* $u, v \in \mathcal{F}$ *and* $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$.

(2) $(\mathcal{E},\mathcal{F})$ *is a local resistance form.*

(3) $(\mathcal{E}_\mu, \mathcal{D}_\mu)$ *is a local Dirichlet form.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is obvious. (3) $\Rightarrow$ (1) follows from Theorem 3.5 and Proposition A.2. □

In the remainder of this appendix, we assume that R is complete and doubling, and prove Proposition A.2. In the same way as the proof of Lemma 2.13, the following inequality holds without the uniform perfectness condition.

Lemma A.4. $\mathcal{R}(\overline{B(x,r)}, B(x,2r)^c) \gtrsim r$ *for any $x \in X$ and $r > 0$.*

The following Proposition A.5, Corollary A.6 and Lemma A.7 were proved in [14] for a general resistance form whose associated resistance metric is not necessarily doubling. Here we give proofs for the same reason as we did for Lemma 2.11.

Proposition A.5 (cf. [14, Theorem 2.38 (2)])**.** *Let $u \in \mathcal{F}$ and $\{u_n\}_{n\geq 0} \subset \mathcal{F}$. Then* $\lim_{n\to\infty} \mathcal{E}(u-u_n, u-u_n) = 0$ *if and only if* $\limsup_{n\to\infty} \mathcal{E}(u_n,u_n) \leq \mathcal{E}(u,u)$ *and* $\lim_{n\to\infty}(u-u_n)(x)$ *exists in $\mathbb{R}$ and is constant on X.*

Proof. The necessity is clear by the triangle inequality of $\mathcal{E}^{1/2}$ and (1.4). For the sufficiency, let $\{V_m\}_{m\geq 0}$ be a spread sequence of (X,R) then

$$\begin{aligned}\mathcal{E}(u,u) &= \lim_{m\to\infty} \mathcal{E}|_{V_m}(u|_{V_m}, u|_{V_m}) \\ &= \lim_{m\to\infty}\lim_{n\to\infty} \mathcal{E}|_{V_m}(u_n|_{V_m}, u_n|_{V_m}) \leq \liminf_{n\to\infty} \mathcal{E}(u_n,u_n),\end{aligned}$$

which proves $\lim_{n\to\infty} \mathcal{E}(u_n,u_n) = \mathcal{E}(u,u)$. Let $u_n^* := (u+u_n)/2$, then by the triangle inequality of $\mathcal{E}^{1/2}$, $\{u_n^*\}_{n\geq 0}$ also satisfies the same condition as $\{u_n\}_{n\geq 0}$. Therefore $\lim_{n\to\infty} \mathcal{E}(u_n^*, u_n^*) = \mathcal{E}(u,u)$ and

$$\lim_{n\to\infty} \mathcal{E}(u-u_n, u-u_n) = 2\mathcal{E}(u,u) + \lim_{n\to\infty}(2\mathcal{E}(u_n,u_n) - \mathcal{E}(2u_n^*, 2u_n^*)) = 0.$$

□

Corollary A.6 (cf. [14, Corollary 2.39 (4)]). *$\lim_{n\to\infty} \mathcal{E}(u - \hat{u}_n, u - \hat{u}_n) = 0$ for any $u \in \mathcal{F}$, where $\hat{u}_n = (u \wedge n) \vee (-n)$.*

Proof. It immediately follows from Proposition A.5 and (1.5). □

Lemma A.7 (cf. [14, Corollary 2.39 (3)]). *Let $u, v \in \mathcal{F}$ be bounded. Then $uv \in \mathcal{F}$ and $\mathcal{E}(uv, uv)^{1/2} \le \|u\|_\infty \mathcal{E}(v, v)^{1/2} + \|v\|_\infty \mathcal{E}(u, u)^{1/2}$, where $\|u\|_\infty = \sup_{x\in X} |u(x)|$.*

Proof. This follows from Proposition 2.10 with easy calculation. □

Proof of Proposition A.2. Since Corollary A.6 holds, we only need to show the case that $u \in \mathcal{F}$ is bounded and $\mathrm{diam}(X, R) = \infty$. Fix some $x \in X$ and let f_n be the optimal function for $\mathcal{R}(\overline{B(x, 2^n)}, B(x, 2^{n+1})^c)$ for each $n \ge 0$. Let $u_n = f_n u$, then $u_n \in C_0(X, R)$ and $\mathrm{supp}(u_n) \subset \mathrm{supp}(u)$. Moreover, $u_n \in \mathcal{F}$ and

$$\limsup_{n\to\infty} \mathcal{E}(u_n, u_n) \le \left(\mathcal{E}(u, u)^{1/2} + \|u\|_\infty \lim_{n\to\infty} \mathcal{E}(f_n, f_n)^{1/2}\right)^2 = \mathcal{E}(u, u)$$

because of Lemmas A.4 and A.7. Therefore $\lim_{n\to\infty} \mathcal{E}(u - u_n, u - u_n) = 0$ by Proposition A.5, which completes the proof. □

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