Gradient Flow Exact Renormalization Group for Scalar Field Theories

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Abstract

The renormalization group method is a fundamental tool in modern physics, which enables us to study the dynamics of various physical systems with a flow that connects the effective infrared descriptions and the underlying ultraviolet theories.

Exact Renormalization Group has been developed as a quantitative formulation of the renormalization group method. It has been utilized to investigate the phase structure of quantum field theories and the properties of critical statistical systems. It has revealed both perturbative and non-perturbative aspects of quantum field theories.

Recently, a new framework, *Gradient Flow Exact Renormalization Group* (GFERG), was proposed to define the Wilsonian effective action with coarse-graining based on a diffusion equation. One of the novel features of GFERG is that it can be constructed to preserve gauge symmetries or global structure of the field space, despite introducing an ultraviolet cutoff. This feature makes GFERG a promising approach to studying the dynamics of theories with gauge symmetry or nonlinearity in a non-perturbative manner.

In this thesis, based on Ref. [1], we investigate the fixed point structure of the GFERG equation associated with the general polynomial diffusion equation of scalar field theories. Then, we show that it has an almost identical form to the conventional Wilson-Polchinski equation. We further discuss that the GFERG equation has a similar renormalization group flow structure around a fixed point to the Wilson-Polchinski equation.

List of papers included in this thesis

Parts of this thesis have been published in the following journal articles:

[1] Y. Abe, Y. Hamada and J. Haruna, "Fixed point structure of the gradient flow exact renormalization group for scalar field theories," PTEP **2022**, No.3, 033B03 (2022) doi:10.1093/ptep/ptac021 [arXiv:2201.04111 [hep-th]].
(Chapter 4 in the thesis)

Contents

Chapter 1

Introduction

Renormalization group (RG) is a fundamental and powerful framework to study various properties of physical systems using a coarse-graining procedure, namely, varying the energy scale. It plays an essential role in modern physics and emphasizes the importance of the system's typical energy scale, such as particle mass, correlation length between particles, or lattice spacing of a lattice system.

In particular, it has established the notion of "hierarchy" in physical theories. This notion states that even though we do not know the microscopic (ultraviolet, high-energy) theory, we can construct effective theories on macroscopic (or low-energy) dynamics. Quantum Field Theories (QFTs) and Classical Field Theories (such as Classical Electromagnetism and General Relativity) are good examples to illustrate this notion. We know that the former is valid at all scales, while the latter is just a low-energy effective theory of QFT. However, we can make predictions on the dynamics of macroscopic objects by Classical Field Theories even before the establishment of QFTs. As is seen from this example, if we focus on the low-energy behavior of a system, we can make a practical description with adequate accuracy independently of its microscopic physics. The renormalization group serves as a bridge between effective infrared descriptions and underlying ultraviolet theories.

In QFTs, a framework called *Exact Renormalization Group* (ERG) [1] has been developed as a quantitative formulation of the RG method [2–6]. It introduces an ultraviolet (UV) cutoff to the theory and studies the flow of the Wilsonian effective action S_Λ, which effectively describes the physics at the energy scale Λ. It is helpful to investigate critical phenomena and phase structures of various systems in relativistic quantum field theories, statistical physics, and condensed matter physics. Phase structures of strongly coupled theory have also been investigated via this framework. ERG has also been applied to formulating quantum gravity, by studying non-trivial UV fixed points and the continuum limit of gravitational field theories.

[1] ERG is also called Functional Renormalization Group or Non-perturbative Renormalization Group.

Although ERG has succeeded in revealing both perturbative and non-perturbative aspects of scalar or fermionic systems, its treatment of theories with local symmetries has not been completely established yet. It is intuitively because an explicit UV cutoff conflicts with the local symmetries. Recent studies [7, 8] proposed a new framework to define the Wilsonian effective action with coarse-graining based on a diffusion equation, called *Gradient Flow Exact Renormalization Group* (GFERG). In this framework, the Wilsonian effective action $S_\Lambda[\phi]$ at the energy scale Λ is schematically given by the following path integral:

$$e^{S_\Lambda[\phi]} \sim \int [D\phi']\, \delta(\phi - \varphi[\phi'])e^{S_{\Lambda_0}[\phi']} \,, \tag{1.0.1}$$

where Λ_0 is the bare scale and $\varphi[\phi']$ is the solution to the diffusion equation with the initial condition $\varphi\big|_{\Lambda=\Lambda_0} = \phi'$. In GFERG, coarse-graining based on the diffusion equation is regarded as a block spin transformation via diffusion. A novel characteristic of the GFERG flow is that it preserves gauge symmetries even though it effectively has a UV cutoff [9, 10]. From this gauge invariance, GFERG is a promising approach to studying gauge theories or quantum gravity non-perturbatively.

The motivation of this thesis is as follows. So far, GFERG for general scalar field theories has not been well-studied. One of the reasons is that the GFERG equation, the differential equation for the scale dependence of the Wilsonian effective action S_Λ, is defined by the diffusion equation and takes a different form for each scalar field theory. The appropriate diffusion equation heavily depends on details of the theory, such as its symmetry, interactions, and information of target space, and is generally not easy to find. However, because GFERG can manifestly preserve non-linearity or local symmetry of the system, it is important to study GFERG of general scalar theories to get a deeper understanding of them. In particular, fixed points of the GFERG equations and the RG flow structure around them are of interest.

The central part of this thesis is based on my work [1]. We introduce a general GFERG equation based on a diffusion equation of a scalar field theory and study its fixed point structure. This GFERG equation can be applied to a broad class of non-linear sigma models, whose diffusion equation is given by a polynomial of the fields. We find that it has additional non-linear terms compared to the standard Wilson-Polchinski (WP) equation, which is a typical flow equation of the Wilsonian effective action S_Λ. Then, we show that it gives the same UV fixed points as the WP equation. Moreover, we discuss that it has a similar RG flow structure around the fixed point to the WP equation due to the vanishing of the additional non-linear terms. We also find that the relevant operators have the same scaling dimensions as those in the WP equation, while the irrelevant operators are not the case. These results mean that the GFERG gives the correct critical exponents, renormalized trajectories, and the same predictions as the ordinary ERG for the low-energy behavior of the physical systems.

Organization of the thesis

This thesis is organized as follows. In Chapter 2, we review the basics of Exact Renormalization Group. We define an effective action that describes at a focused energy scale and calls it the Wilsonian effective action. Then we study the flow equation for the Wilsonian effective action, called the ERG equation, and give some examples. Then, we search UV fixed points of the ERG equation. Furthermore, we study the flow structure around the fixed points and determine the critical exponents. In Chapter 3, we review Gradient Flow Exact Renormalization Group (GFERG). GFERG was initially proposed to define the Wilsonian effective action for gauge theories in a gauge-invariant manner. First, we introduce its definition within the framework of GFERG and show its gauge invariance. Next, we derive the GFERG equation, the counterpart of the ERG equation in GFERG. We also review recent developments on GFERG.

Chapter 4 is the central part of this thesis. Based on my work [1], we discuss GFERG for scalar field theories in general and investigate its fixed point structure. Then, we explicitly write down the GFERG equation based on an arbitrary polynomial diffusion equation and discuss its fixed points. Furthermore, we calculate the scaling dimensions of operators around the fixed points by solving the GFERG equation to the leading order of the deviations from the fixed point. Chapter 5 is devoted to a summary and conclusion.

Chapter 2

Review of Exact Renormalization Group

We briefly review Exact Renormalization Group (ERG) in this chapter. The following review is based on Refs. [11–13]. (See Refs. [14–17] for more details.)

2.1 What is Exact Renormalization Group

ERG is a method to study the physics of a system under varying the energy scale on which we focus. Let us consider a physical system described by QFT and assume that its physics is determined by the partition function Z in the path-integral formalism:

$$Z = \int D\phi \, e^{S[\phi]}, \tag{2.1.1}$$

where ϕ is the elementary field of the system and S is the action.

In the context of ERG, the physics at the energy scale Λ is effectively described by the Wilsonian effective action. Intuitively, it is obtained by integrating out the modes with momenta p^2 higher than the energy scale Λ^2:

$$e^{S_\Lambda} := \int D\phi_{p^2>\Lambda^2} \, e^{S}. \tag{2.1.2}$$

However, the integration on the right-hand side needs to be clarified due to UV divergences. More specifically, let us consider correlation functions such as

$$\langle 0| \, T(\phi(x_1)\cdots\phi(x_n)) \, |0\rangle \, . \tag{2.1.3}$$

Because correlation functions generally have the UV divergences, we have to introduce a UV regulator to define the Wilsonian effective action. A traditional way to do so is by deforming the bare action as

$$S \to S + \Delta S_\Lambda, \tag{2.1.4}$$

$$\Delta S_\Lambda := -\frac{1}{2}\int_p \phi(-p)R_\Lambda(p)\phi(p), \tag{2.1.5}$$

where $R_\Lambda(p)$ is a UV regulator. This regulator should satisfy the following conditions:

$$R_\Lambda(p) \to \infty \quad \text{as} \quad p \to \infty, \tag{2.1.6a}$$

$$R_\Lambda(p) \to 0 \quad \text{as} \quad p \to 0. \tag{2.1.6b}$$

The first condition ensures that the high-momentum modes of the field do not contribute to the partition function. The second condition means that the UV cutoff function does not change the low-energy physic of the system.

Next let us define the Wilsonian effective action S_Λ at the lower energy scale Λ than the bare scale Λ_0. For example, consider a scalar field theory and its partition function Z at the bare scale Λ_0 given by

$$Z = \int D\phi\, e^{-S-\Delta S_\Lambda}, \tag{2.1.7}$$

$$= \int D\phi\, \exp\left(-\int_p \phi(p)\frac{p^2}{2K(p^2/\Lambda_0^2)}\phi(-p) + S^{\text{int}}_{\Lambda_0}[\phi]\right), \tag{2.1.8}$$

where ϕ is the scalar field, $K(p^2/\Lambda_0^2)$ is the cutoff function, and $S^{\text{int}}_{\Lambda_0}$ gives the interaction terms. The condition Eq. (2.1.6a) for the UV regulator requires this cutoff function $K(p^2/\Lambda)$ to vanish sufficiently fast as $p \to \infty$. The quantities on the right-hand side of this equation are measured at the bare scale Λ_0 and we want to rewrite this relation with the new cutoff Λ like

$$Z = \int D\phi\, \exp\left(-\int_p \phi(p)\frac{p^2}{2K(p^2/\Lambda^2)}\phi(-p) + S^{\text{int}}_{\Lambda}[\phi]\right) \tag{2.1.9}$$

while keeping the partition function Z invariant. In other words, we want to express the partition function by the quantities measured by the new cutoff Λ so that the Wilsonian effective action preserves the partition function (or the thermal free energy) [1]:

$$Z = \int D\phi\, e^{S_{\tau_0}[\phi]} = \int D\phi\, e^{S_\tau[\phi]}. \tag{2.1.10}$$

The following identity [2] helps us find how related $S^{\text{int}}_{\Lambda_0}$ and S^{int}_{Λ} are:

$$\int D\phi_1 D\phi_2 \exp\left(-\int_p \left(\frac{1}{A(p)}\phi_1(-p)\phi_1(p) + \frac{1}{B(p)}\phi_2(-p)\phi_2(p)\right) + S[\phi_1+\phi_2]\right)$$

[1] This identity is the reason why the ERG has the word "exact" in its name: It keeps all the information on quantum fluctuations at any energy scale.

[2] Its proof is given in the following:

$$\int D\phi_1 D\phi_2 \exp\left(-\int_p \left(\frac{1}{A(p)}\phi_1(-p)\phi_1(p) + \frac{1}{B(p)}\phi_2(-p)\phi_2(p)\right) + S[\phi_1+\phi_2]\right)$$

$$\propto \int D\phi_1 \exp\Big(- \int_p \frac{1}{A(p)+B(p)} \phi_1(-p)\phi_1(p) + S[\phi_1] \Big). \quad (2.1.14)$$

We can rewrite the partition function Z with this formula as

$$Z = \int D\phi \, \exp\Big(- \int_p \frac{p^2}{2K(p^2/\Lambda^2)} \phi(-p)\phi(p) \Big)$$
$$\times \int D\phi' \, \exp\Big(- \int_p \frac{p^2}{2(K(p^2/\Lambda_0^2) - K(p^2/\Lambda^2))} \phi'(p)\phi'(-p) + S^{\text{int}}_{\Lambda_0}[\phi+\phi'] \Big). \quad (2.1.15)$$

Comparing this equation and Eq. (2.1.9), we can define the interaction term S^{int}_{Λ} of the Wilsonian effective action at the new energy scale Λ as

$$e^{S^{\text{int}}_{\Lambda}[\phi]} = \int D\phi' \, \exp\Big(- \int_p \frac{p^2}{2(K(p^2/\Lambda_0^2) - K(p^2/\Lambda^2))} \phi'(-p)\phi'(p) + S^{\text{int}}_{\Lambda_0}[\phi+\phi'] \Big). \quad (2.1.16)$$

Note that by shifting $\phi' \to \phi' - \phi$, this can be represented as

$$e^{S^{\text{int}}_{\Lambda}[\phi]} = \int D\phi'$$
$$\times \exp\Big(- \int_p \frac{p^2}{2(K(p^2/\Lambda_0^2) - K(p^2/\Lambda^2))} (\phi'(-p) - \phi(-p))(\phi'(p) - \phi(p)) + S^{\text{int}}_{\Lambda_0}[\phi'] \Big). \quad (2.1.17)$$

It should be noted that the cutoff function decreases monotonically at sufficiently large momentum and that the kinetic term of the integrated field ϕ' has a non-zero value only at $\Lambda^2 \lesssim p^2 \lesssim \Lambda_0^2$. Therefore, the expression Eq. (2.1.16) tells us that the new interaction term S^{int}_{Λ} is given by interacting out the on-shell modes ϕ' with momenta between the old and new cutoffs (Λ_0^2 and Λ^2).

2.2 Flow of Wilsonian effective action

The definition Eq. (2.1.17) of the Wilsonian effective action is too complicated to analyze directly. Instead, we can study the flow of the Wilsonian effective action. As seen in the following, its flow is represented as a differential equation called the "ERG equation".

$$= \int D\phi_1 D\phi_2 \exp\Big(- \int_p \Big(\frac{1}{A(p)} (\phi_1(-p) - \phi_2(-p))(\phi_1(p) - \phi_2(p)) + \frac{1}{B(p)} \phi_2(-p)\phi_2(p) \Big) + S[\phi_1] \Big) \quad (2.1.11)$$

$$= \int D\phi_1 D\phi_2 \exp\Big(- \int_p \Big(\frac{1}{A(p)} - \frac{B(p)}{A(p)(A(p)+B(p))} \Big) \phi_1(-p)\phi_1(p) + \frac{1}{B(p)} \phi_2(-p)\phi_2(p) + S[\phi_1] \Big) \quad (2.1.12)$$

$$\propto \int D\phi_1 \exp\Big(- \int_p \frac{1}{A(p)+B(p)} \phi_1(-p)\phi_1(p) + S[\phi_1] \Big) \quad (2.1.13)$$

2.2.1 Wilson-Polchinski equation

A typical example of ERG equation is the Wilson-Polchinski (WP) equation, which is given by

$$\Lambda\frac{\partial S_\Lambda^{\rm int}}{\partial\Lambda} = -\int_p \Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\frac{1}{2p^2}\left(\frac{\delta^2 S_\Lambda^{\rm int}}{\delta\phi(p)\delta\phi(-p)} + \frac{\delta S_\Lambda^{\rm int}}{\delta\phi(p)}\frac{\delta S_\Lambda^{\rm int}}{\delta\phi(-p)}\right). \tag{2.2.1}$$

This equation gives a flow on theory space and defines a renormalization procedure non-perturbatively. The terms on the right-hand side represents the quantum effect, corresponding to Feynmann diagrams with loops.

Derivation of the WP equation is following. Differentiating the definition Eq. (2.1.17) of $S_\Lambda^{\rm int}$ with respect to Λ, we obtain

$$\begin{aligned}
&\Lambda\frac{\partial}{\partial\Lambda}e^{S_\Lambda^{\rm int}}\\
&= -\int_p \Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\frac{p^2}{2(K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2))^2}\int D\phi'\,(\phi'(-p)-\phi(-p))(\phi'(p)-\phi(p))\\
&\qquad\times\exp\left(-\int_q \phi'(q)\frac{p^2}{2(K(q^2/\Lambda_0^2)-K(q^2/\Lambda^2))}\phi'(-q) + S_{\Lambda_0}^{\rm int}[\phi+\phi']\right).
\end{aligned} \tag{2.2.2}$$

Here, let us consider second functional derivative of Eq. (2.1.17):

$$\begin{aligned}
&\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}e^{S_\Lambda^{\rm int}}\\
&= \int D\phi'\left[\frac{p^4}{(K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2))^2}(\phi'(-p)-\phi(-p))(\phi'(p)-\phi(p)) - \frac{p^2}{K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2)}\delta(0)\right]\\
&\times\exp\left(-\int_p \frac{p^2}{2(K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2))}(\phi'(-p)-\phi(-p))(\phi'(p)-\phi(p)) + S_{\Lambda_0}^{\rm int}[\phi']\right)
\end{aligned} \tag{2.2.3}$$

$$\begin{aligned}
&= -\frac{p^2}{K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2)}\delta(0)e^{S_\Lambda^{\rm int}}\\
&+ \frac{p^4}{(K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2))^2}\int D\phi'\,(\phi'(-p)-\phi(-p))(\phi'(p)-\phi(p))\\
&\times\exp\left(-\int_p \frac{p^2}{2(K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2))}(\phi'(-p)-\phi(-p))(\phi'(p)-\phi(p)) + S_{\Lambda_0}^{\rm int}[\phi']\right)
\end{aligned} \tag{2.2.4}$$

Therefore, we get

$$\Lambda\frac{\partial}{\partial\Lambda}e^{S_\Lambda^{\rm int}} = -\int_p \Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\frac{1}{2p^2}\left(\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} + \frac{p^2}{K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2)}\delta(0)\right)e^{S_\Lambda^{\rm int}}, \tag{2.2.5}$$

equivalently,

$$\Lambda\frac{\partial S_\Lambda^{\rm int}}{\partial\Lambda} = -\int_p \Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\frac{1}{2p^2}\left(\frac{\delta^2 S_\Lambda^{\rm int}}{\delta\phi(p)\delta\phi(-p)} + \frac{\delta S_\Lambda^{\rm int}}{\delta\phi(p)}\frac{\delta S_\Lambda^{\rm int}}{\delta\phi(-p)}\right)$$
$$-\int_p \Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\frac{1}{2(K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2))}\delta(0). \quad (2.2.6)$$

The second term can be eliminated by adding a constant term to $S_\Lambda^{\rm int}$ as

$$S_\Lambda^{\rm int} \to S_\Lambda^{\rm int} + \int^\Lambda d\Lambda' \int_p \frac{\partial K(p^2/\Lambda'^2)}{\partial\Lambda'}\frac{1}{2(K(p^2/\Lambda_0^2)-K(p^2/\Lambda'^2))}\delta(0). \quad (2.2.7)$$

Finally, we get

$$\Lambda\frac{\partial S_\Lambda^{\rm int}}{\partial\Lambda} = -\int_p \Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\frac{1}{2p^2}\left(\frac{\delta^2 S_\Lambda^{\rm int}}{\delta\phi(p)\delta\phi(-p)} + \frac{\delta S_\Lambda^{\rm int}}{\delta\phi(p)}\frac{\delta S_\Lambda^{\rm int}}{\delta\phi(-p)}\right). \quad (2.2.8)$$

In fact, the generating functional $W_k[J]$, defined as

$$e^{W_\Lambda[J]} := \int D\phi\, e^{S-\frac{1}{2}\int_k \phi(-k)R_\Lambda\phi(k)+\int_q J(q)\phi(q)}, \quad (2.2.9)$$

satisfies the WP equation. Let us check that. Differentiating Eq. (2.2.9) with respect to τ, we get

$$\Lambda\frac{\partial}{\partial\Lambda}e^{W_\Lambda[J]} = -\frac{1}{2}\int_q \Lambda\frac{\partial R_\Lambda(q)}{\partial\Lambda}\int D\phi\,\phi(q)\phi(-q)e^{S-\frac{1}{2}\int_k \phi(-k)R_\Lambda\phi(k)+\int_q J(q)\phi(q)} \quad (2.2.10)$$
$$= -\frac{1}{2}\int_q \Lambda\frac{\partial R_\Lambda(q)}{\partial\Lambda}\frac{\delta^2}{\delta J(q)\delta J(-q)}\int D\phi\, e^{S-\frac{1}{2}\int_k \phi(-k)R_\Lambda\phi(k)+\int_J(q)\phi(q)} \quad (2.2.11)$$
$$= -\frac{1}{2}\int_q \Lambda\frac{\partial R_\Lambda(q)}{\partial\Lambda}\frac{\delta^2}{\delta J(q)\delta J(-q)}e^{W_\Lambda[J]} \quad (2.2.12)$$
$$= -e^{W_\Lambda[J]}\frac{1}{2}\int_q \Lambda\frac{\partial R_\Lambda(q)}{\partial\Lambda}\left(\frac{\delta^2 W_\Lambda}{\delta J(q)\delta J(-q)} + \frac{\delta W_\Lambda}{\delta J(q)}\frac{\delta W_\Lambda}{\delta J(-q)}\right), \quad (2.2.13)$$

equivalently,

$$\Lambda\frac{\partial W_\Lambda}{\partial\Lambda} = -\frac{1}{2}\int_q \Lambda\frac{\partial R_\Lambda(q)}{\partial\Lambda}\left(\frac{\delta^2 W_\Lambda}{\delta J(q)\delta J(-q)} + \frac{\delta W_\Lambda}{\delta J(q)}\frac{\delta W_\Lambda}{\delta J(-q)}\right). \quad (2.2.14)$$

Why the generating function W_Λ and the Wilsonian effective action $S_\Lambda^{\rm int}$ satisfy the almost the same equation? This can be seen from their definitions. The definition Eq. (2.1.17) of the Wilsonian effective action $S_\Lambda^{\rm int}$ is rewritten as

$$e^{S_\Lambda^{\rm int}} = \int D\phi'\,\exp\Bigg(-\int_p \frac{p^2}{2(K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2))}\phi'(-p)\phi'(p) + S_{\Lambda_0}^{\rm int}[\phi']$$
$$+\int_p J(-p)\phi(p) - \int_p \frac{1}{2(K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2))}\phi(-p)\phi(p)\Bigg), \quad (2.2.15)$$

where $J(p) := p^2\phi(p)/(K(p^2/\Lambda_0^2) - K(p^2/\Lambda^2))$. Comparing this equation and Eq. (2.2.9), it is easily seen that there is a correspondence between them like

$$\frac{p^2}{K(p^2/\Lambda_0^2) - K(p^2/\Lambda^2)}\phi(p) \leftrightarrow J(p), \tag{2.2.16a}$$

$$\int_p \frac{-p^2}{2(K(p^2/\Lambda_0^2) - K(p^2/\Lambda^2))}\phi'(-p)\phi'(p) + S^{\rm int}_{\Lambda_0}[\phi'] \leftrightarrow S[\phi] - \frac{1}{2}\int_k \phi(-k)R_\Lambda(k)\phi(k), \tag{2.2.16b}$$

$$\int_p \frac{p^2}{2(K(p^2/\Lambda_0^2) - K(p^2/\Lambda^2))}\phi(-p)\phi(p) + S^{\rm int}_{\Lambda}[\phi] \leftrightarrow W_\Lambda[J]. \tag{2.2.16c}$$

Especially, comparing the quadratic terms in the both hand sides of Eq. (2.2.19c), we get

$$\frac{p^2}{K(p^2/\Lambda_0^2) - K(p^2/\Lambda^2)} \leftrightarrow S^{(2)}(p) + R_\Lambda(p), \tag{2.2.17}$$

where

$$S^{(2)}(p)\delta(p+q) := -\left.\frac{\delta^2 S}{\delta\phi(q)\delta\phi(p)}\right|_{\phi=0} \tag{2.2.18}$$

is the quadratic term in S.

Let us concretely check this. The correspondence is given by

$$J(p) = \frac{p^2}{K(p^2/\Lambda_0^2) - K(p^2/\Lambda^2)}\phi(p), \tag{2.2.19a}$$

$$W_\Lambda[J] = \int_p \frac{p^2}{2(K(p^2/\Lambda_0^2) - K(p^2/\Lambda^2))}\phi(-p)\phi(p) + S^{\rm int}_\Lambda[\phi], \tag{2.2.19b}$$

$$R_\Lambda(p) = \frac{p^2}{K(p^2/\Lambda_0^2) - K(p^2/\Lambda^2)} - S^{(2)}(p). \tag{2.2.19c}$$

Then, using Eq. (2.2.19b) and differentiating $S^{\rm int}_\Lambda[\phi]$ in terms of Λ, we get

$$\begin{aligned}
&\Lambda\frac{\partial S^{\rm int}_\Lambda[\phi]}{\partial\Lambda} \\
&= \Lambda\frac{\partial}{\partial\Lambda}W_\Lambda\left[\frac{p^2}{K(p^2/\Lambda_0^2) - K(p^2/\Lambda^2)}\phi(p)\right] - \Lambda\frac{\partial}{\partial\Lambda}\int_p \frac{p^2}{2(K(p^2/\Lambda_0^2) - K(p^2/\Lambda^2))}\phi(-p)\phi(p)
\end{aligned} \tag{2.2.20}$$

$$\begin{aligned}
&= \left[\Lambda\frac{\partial W_\Lambda[J]}{\partial\Lambda} + \int_p\left(\Lambda\frac{\partial}{\partial\Lambda}\frac{p^2}{K(p^2/\Lambda_0^2) - K(p^2/\Lambda^2)}\phi(p)\right)\frac{\delta W_\Lambda[J]}{\delta J(p)}\right]_{J(p)=\frac{p^2}{K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2)}\phi(p)} \\
&\qquad - \int_p\left(\Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\right)\frac{p^2}{2(K(p^2/\Lambda_0^2) - K(p^2/\Lambda^2))^2}\phi(-p)\phi(p)
\end{aligned} \tag{2.2.21}$$

The right hand side of this equation is given by

(R.H.S of Eq. (2.2.21))

$$= -\frac{1}{2}\int_q \Lambda\frac{\partial R_\Lambda(q)}{\partial\Lambda}\left(\frac{\delta^2 W_\Lambda}{\delta J(q)\delta J(-q)} + \frac{\delta W_\Lambda}{\delta J(q)}\frac{\delta W_\Lambda}{\delta J(-q)}\right)\Bigg|_{J(p)=\frac{p^2}{K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2)}\phi(p)}$$

$$+\int_p \left(\Lambda\frac{\partial}{\partial\Lambda}\frac{p^2}{K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2)}\phi(p)\right)\times$$

$$\frac{\delta}{\delta J(p)}\left(\int_p \frac{p^2}{2(K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2))}\phi(-p)\phi(p) + S_\Lambda^{\text{int}}[\phi]\right)\Bigg|_{J(p)=\frac{p^2}{K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2)}\phi(p)}$$

$$-\int_p \left(\Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\right)\frac{p^2}{2(K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2))^2}\phi(-p)\phi(p) \tag{2.2.22}$$

$$\begin{aligned}
= &-\frac{1}{2}\int_q \Lambda\frac{\partial}{\partial\Lambda}\left(\frac{q^2}{K(q^2/\Lambda_0^2)-K(q^2/\Lambda^2)} - S^{(2)}(q)\right)\left(\frac{K(q^2/\Lambda_0^2)-K(q^2/\Lambda^2)}{q^2}\right)^2\\
&\times\Bigg[\frac{\delta^2}{\delta\phi(q)\delta\phi(-q)}\left(\int_p \frac{p^2}{2(K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2))}\phi(-p)\phi(p) + S_\Lambda^{\text{int}}\right)\\
&+\frac{\delta}{\delta\phi(q)}\left(\int_p \frac{p^2}{2(K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2))}\phi(-p)\phi(p) + S_\Lambda^{\text{int}}\right)\\
&\times\frac{\delta}{\delta\phi(-q)}\left(\int_p \frac{p^2}{2(K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2))}\phi(-p)\phi(p) + S_\Lambda^{\text{int}}\right)\Bigg]\\
&+\int_p \left(\Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\right)\frac{1}{K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2)}\phi(p)\times\\
&\left(\frac{p^2}{K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2)}\phi(-p) + \frac{\delta S_\Lambda^{\text{int}}[\phi]}{\delta\phi(p)}\right)\\
&-\int_p \left(\Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\right)\frac{p^2}{2(K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2))^2}\phi(-p)\phi(p)
\end{aligned} \tag{2.2.23}$$

$$\begin{aligned}
= &-\int_q \Lambda\frac{\partial K(q^2/\Lambda^2)}{\partial\Lambda}\frac{1}{2q^2}\Bigg[\frac{q^2}{2(K(q^2/\Lambda_0^2)-K(q^2/\Lambda^2))}\delta(0) + \frac{\delta^2 S_\Lambda^{\text{int}}}{\delta\phi(q)\delta\phi(-q)}\\
&+\left(\int_p \frac{q^2}{K(q^2/\Lambda_0^2)-K(q^2/\Lambda^2)}\phi(-q) + \frac{\delta S_\Lambda^{\text{int}}}{\delta\phi(q)}\right)\left(\int_p \frac{q^2}{K(q^2/\Lambda_0^2)-K(q^2/\Lambda^2)}\phi(q) + \frac{\delta S_\Lambda^{\text{int}}}{\delta\phi(-q)}\right)\Bigg]\\
&+\int_p \left(\Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\right)\frac{1}{K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2)}\phi(p)\left(\frac{p^2}{K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2)}\phi(-p) + \frac{\delta S_\Lambda^{\text{int}}[\phi]}{\delta\phi(p)}\right)\\
&-\int_p \left(\Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\right)\frac{p^2}{2(K(p^2/\Lambda_0^2)-K(p^2/\Lambda^2))^2}\phi(-p)\phi(p)
\end{aligned} \tag{2.2.24}$$

$$\begin{aligned}
= &\int_q \Lambda\frac{\partial K(q^2/\Lambda^2)}{\partial\Lambda}\frac{1}{2q^2}\left[\frac{\delta^2 S_\Lambda^{\text{int}}}{\delta\phi(q)\delta\phi(-q)} + \frac{\delta S_\Lambda^{\text{int}}}{\delta\phi(q)}\frac{\delta S_\Lambda^{\text{int}}}{\delta\phi(-q)}\right]\\
&+\int_q \Lambda\frac{\partial K(q^2/\Lambda^2)}{\partial\Lambda}\frac{1}{2q^2}\frac{q^2}{2(K(q^2/\Lambda_0^2)-K(q^2/\Lambda^2))}\delta(0)
\end{aligned} \tag{2.2.25}$$

Therefore, we get

$$\Lambda \frac{\partial S_\Lambda^{\rm int}[\phi]}{\partial \Lambda} = \int_q \Lambda \frac{\partial K(q^2/\Lambda^2)}{\partial \Lambda} \frac{1}{2q^2} \left[\frac{\delta^2 S_\Lambda^{\rm int}}{\delta\phi(q)\delta\phi(-q)} + \frac{\delta S_\Lambda^{\rm int}}{\delta\phi(q)} \frac{\delta S_\Lambda^{\rm int}}{\delta\phi(-q)} \right] - \int_q \Lambda \frac{\partial K(q^2/\Lambda^2)}{\partial \Lambda} \frac{1}{2q^2} \frac{q^2}{2(K(q^2/\Lambda_0^2) - K(q^2/\Lambda^2))} \delta(0). \quad (2.2.26)$$

Again, the last term in this equation is just a constant[3], and can be remove by adding some constant term to $S_\Lambda^{\rm int}$. Finally, we get

$$\Lambda \frac{\partial S_\Lambda^{\rm int}[\phi]}{\partial \Lambda} = \int_q \Lambda \frac{\partial K(q^2/\Lambda^2)}{\partial \Lambda} \frac{1}{2q^2} \left[\frac{\delta^2 S_\Lambda^{\rm int}}{\delta\phi(q)\delta\phi(-q)} + \frac{\delta S_\Lambda^{\rm int}}{\delta\phi(q)} \frac{\delta S_\Lambda^{\rm int}}{\delta\phi(-q)} \right]. \quad (2.2.27)$$

This is nothing but the WP equation Eq. (2.2.8) for the Wilsonian effective action $S_\Lambda^{\rm int}$.

2.2.2 Wetterich equation

Another example of ERG equations is Wetterich equation [6]. This equation represents a flow of one-particle irreducible (1PI) effective action Γ_Λ. The 1PI action Γ_Λ is defined as the Legendre transformation of $W_k[J]$:

$$\Gamma_\Lambda[\phi] \coloneqq \left(-W_\Lambda[J] + \int_p J(p)\phi(p) \right) \bigg|_{J=J_\Lambda[\phi]} - \frac{1}{2} \int_p \phi(-p) R_\Lambda(p) \phi(p), \quad (2.2.28)$$

where $J_\Lambda[\phi]$ is the solution to

$$\phi(p) = \frac{\delta W_\Lambda[J]}{\delta J(p)} \bigg|_{J=J_\Lambda[\phi]}. \quad (2.2.29)$$

The motivation for considering the 1PI effective action Γ_Λ is that it has a simpler renormalization structure than the generating function W_Λ. It is because Γ_Λ is a generating function 1PI diagrams, while W_Λ is that of connected diagrams including both 1PI and non-1PI diagrams.

The Wetterich equation is given by

$$\Lambda \frac{\partial \Gamma_\Lambda}{\partial \Lambda} = \mathrm{Tr} \left[\Lambda \frac{\partial R_\Lambda}{\partial \Lambda} (\Gamma_\Lambda^{(2)} + R_\Lambda)^{-1} \right], \quad (2.2.30)$$

where Tr means the trace over momentum space and internal degrees of freedom, and $\Gamma^{(2)}$ is defined as

$$\Gamma_\Lambda^{(2)} \coloneqq \frac{\delta^2 \Gamma_\Lambda[\phi]}{\delta\phi(p)\delta\phi(q)}. \quad (2.2.31)$$

We derive this equation in the following. Differentiating Eq. (2.2.28) with respect to Λ, we get

$$\Lambda \frac{\partial \Gamma_\Lambda}{\partial \Lambda}$$

[3] Miraculously, this constant term is the same as that in the WP equation Eq. (2.2.6) for $S_\Lambda^{\rm int}$ before adding the constant to it.

$$= \left(-\Lambda\frac{\partial W_\Lambda[J]}{\partial\Lambda} - \int_p \Lambda\frac{\partial J_\Lambda(p)}{\partial\Lambda}\frac{\delta}{\delta J(p)}\left(W_\Lambda[J] - \int_x J(x)\phi(x)\right)\right)\bigg|_{J=J_\Lambda[\phi]} - \frac{1}{2}\int_p \phi(-p)\Lambda\frac{\partial R_\Lambda(p)}{\partial\Lambda}\phi(p), \tag{2.2.32}$$

$$= \frac{1}{2}\int_q \Lambda\frac{\partial R_\Lambda(q)}{\partial\Lambda}\left(\frac{\delta^2 W_\Lambda}{\delta J(q)\delta J(-q)} + \frac{\delta W_\Lambda}{\delta J(q)}\frac{\delta W_\Lambda}{\delta J(-q)}\right)\bigg|_{J=J_\Lambda[\phi]} - \int_p \Lambda\frac{\partial J_\Lambda(p)}{\partial\Lambda}\left(\frac{\delta W_\Lambda}{\delta J(p)}\bigg|_{J=J_\Lambda[\phi]} - \phi(p)\right) - \frac{1}{2}\int_p \phi(-p)\Lambda\frac{\partial R_\Lambda(p)}{\partial\Lambda}\phi(p) \tag{2.2.33}$$

$$= \frac{1}{2}\int_q \Lambda\frac{\partial R_\Lambda(q)}{\partial\Lambda}\frac{\delta^2 W_\Lambda}{\delta J(q)\delta J(-q)}\bigg|_{J=J_\Lambda[\phi]}. \tag{2.2.34}$$

We used Eq. (2.2.29) in the last equality. Here, taking the second functional derivative over $\phi(p)$ and $\phi(q)$ in Eq. (2.2.28), we get

$$\frac{\delta^2\Gamma[\phi]}{\delta\phi(p)\delta\phi(q)} = \frac{\delta}{\delta\phi(q)}\left[J_\Lambda[\phi](p) - \int_q \frac{\delta J_\Lambda[\phi]}{\delta\phi(q)}\left(\frac{\delta W_\Lambda}{\delta J(q)}\bigg|_{J=J_\Lambda[\phi]} - \phi(q)\right) - R_\Lambda(p)\phi(-p)\right] \tag{2.2.35}$$

$$= \frac{\delta J_\Lambda[\phi](p)}{\delta\phi(q)} - R_\Lambda(p)\delta(p+q). \tag{2.2.36}$$

On the other hand, taking the derivative of Eq. (2.2.29) in terms of $\phi(q)$, we get

$$\delta(p-q) = \int_r \frac{J_\Lambda[\phi](r)}{\phi(q)}\,\frac{\delta^2 W_\Lambda}{\delta J(q)\delta J(r)}\bigg|_{J=J_\Lambda[\phi]} \tag{2.2.37}$$

$$= \int_r \left(\frac{\delta^2\Gamma[\phi]}{\delta\phi(r)\delta\phi(p)} + R_\Lambda(r)\delta(p+r)\right)\frac{\delta^2 W_\Lambda}{\delta J(q)\delta J(r)}\bigg|_{J=J_\Lambda[\phi]}. \tag{2.2.38}$$

This equation yields that regarding the momenta p, q and r as indices of the matrix, we conclude that

$$\frac{\delta^2 W_\Lambda}{\delta J(p)\delta J(q)}\bigg|_{J=J_\Lambda[\phi]} = \left(\Gamma^{(2)} + R_\Lambda\right)^{-1}_{p,q}. \tag{2.2.39}$$

Substituting this relation into Eq. (2.2.34), we get the Wetterich equation Eq. (2.2.30).

2.2.3 Dimensionless Variables

In this subsection, we consider the flow of the full action S_Λ, rather than the interaction part S_Λ^{int} and derive the WP equation for the full action S_Λ. We also rewrite it in terms of dimensionless variables for later use.

First, let us define the full Wilsonian effective action S_Λ as

$$S_\Lambda[\phi] := -\int_p \phi(-p)\frac{1}{2K(p^2/\Lambda^2)}\phi(p) + S_\Lambda^{\mathrm{int}}[\phi]. \tag{2.2.40}$$

Then, we study its flow equation. Substituting $S_\Lambda^{\text{int}} = S_\Lambda + \int_p \phi(-p)\frac{1}{2K(p^2/\Lambda^2)}\phi(p)$ into the WP equation Eq. (2.2.8) of S_Λ^{int}, we get

$$\begin{aligned}
&\Lambda\frac{\partial}{\partial\Lambda}\left(S_\Lambda + \int_p \phi(-p)\frac{1}{2K(p^2/\Lambda^2)}\phi(p)\right) \\
&\qquad = -\int_p \Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\frac{1}{2p^2}\left[\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\left(S_\Lambda + \int_p \phi(-p)\frac{1}{2K(p^2/\Lambda^2)}\phi(p)\right)\right. \\
&\left. + \frac{\delta}{\delta\phi(p)}\left(S_\Lambda + \int_p \phi(-p)\frac{1}{2K(p^2/\Lambda^2)}\phi(p)\right)\frac{\delta}{\delta\phi(-p)}\left(S_\Lambda + \int_p \phi(-p)\frac{1}{2K(p^2/\Lambda^2)}\phi(p)\right)\right].
\end{aligned} \tag{2.2.41}$$

The left hand side of this equation is given by

$$(\text{R.H.S of Eq. (2.2.41)}) = \Lambda\frac{\partial S_\Lambda}{\partial\Lambda} - \int_p \Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\frac{1}{2K^2(p/\Lambda)}\phi(-p)\phi(p). \tag{2.2.42}$$

On the other hand, the right hand side is calculated as

$$\begin{aligned}
&(\text{L.H.S of Eq. (2.2.41)}) \\
&= -\int_p \Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\frac{1}{2p^2}\left[\frac{\delta^2 S_\Lambda}{\delta\phi(p)\delta\phi(-p)}\right. \\
&\quad \left. + \left(\frac{\delta S_\Lambda}{\delta\phi(p)} + \frac{p^2}{K(p^2/\Lambda^2)}\phi(-p)\right)\left(\frac{\delta S_\Lambda}{\delta\phi(-p)} + \frac{p^2}{K(p^2/\Lambda^2)}\phi(p)\right)\right]
\end{aligned} \tag{2.2.43}$$

$$\begin{aligned}
&= -\int_p \Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\frac{1}{2p^2}\left[\frac{\delta^2 S_\Lambda}{\delta\phi(p)\delta\phi(-p)} + \frac{\delta S_\Lambda}{\delta\phi(p)}\frac{\delta S_\Lambda}{\delta\phi(-p)}\right. \\
&\qquad\qquad \left. + 2\frac{p^2}{K(p^2/\Lambda^2)}\phi(p)\frac{\delta S_\Lambda}{\delta\phi(p)} + \frac{p^4}{K^2(p/\Lambda)}\phi(-p)\phi(p)\right]
\end{aligned} \tag{2.2.44}$$

Therefore, combining this equation and Eq. (2.2.42), we get

$$\begin{aligned}
&\Lambda\frac{\partial S_\Lambda}{\partial\Lambda} \\
&= -\int_p \Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\frac{1}{2p^2}\left(2\frac{p^2}{K(p^2/\Lambda^2)}\phi(p)\frac{\delta S_\Lambda}{\delta\phi(p)} + \frac{\delta^2 S_\Lambda}{\delta\phi(p)\delta\phi(-p)} + \frac{\delta S_\Lambda}{\delta\phi(p)}\frac{\delta S_\Lambda}{\delta\phi(-p)}\right)
\end{aligned} \tag{2.2.45}$$

$$= -\int_p \left[\Lambda\frac{\partial \log K(p^2/\Lambda^2)}{\partial\Lambda}\phi(p)\frac{\delta S_\Lambda}{\delta\phi(p)} + \Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\frac{1}{2p^2}\left(\frac{\delta^2 S_\Lambda}{\delta\phi(p)\delta\phi(-p)} + \frac{\delta S_\Lambda}{\delta\phi(p)}\frac{\delta S_\Lambda}{\delta\phi(-p)}\right)\right]. \tag{2.2.46}$$

This equation represents the flow of the full action S_Λ and is also called "Wilson-Polchinski equation".

Let us introduce the following dimensionless quantities[4]:

$$\Lambda := \Lambda_R e^{-\tau}, \tag{2.2.47a}$$

[4] The mass dimension of the functional derivative is determined from the relation that $\delta\phi(p)/\delta\phi(q) = \delta(p-q)$, rather than minus of that of the field $\phi(q)$. This is because the mass dimension of the delta function is not 0 but $-D$ due to the identity $\int d^D p\, \delta(p) = 1$.

$$\tilde{p} \coloneqq p/\Lambda, \tag{2.2.47b}$$

$$\tilde{\phi}(\tilde{p}) \coloneqq \Lambda^{\frac{D+2}{2}} \phi(p), \tag{2.2.47c}$$

$$\frac{\delta}{\delta\tilde{\phi}(\tilde{p})} \coloneqq \Lambda^{\frac{D-2}{2}} \frac{\delta}{\delta\phi(p)}, \tag{2.2.47d}$$

$$\tilde{S}_\tau[\tilde{\phi}] \coloneqq S_\Lambda[\phi], \tag{2.2.47e}$$

where the reference scale Λ_R is an arbitrary mass scale.

Note that

$$\Lambda \frac{\partial K(p^2/\Lambda^2)}{\partial \Lambda} = -p \cdot \partial_p K(p^2/\Lambda^2) = -\tilde{p} \cdot \partial_{\tilde{p}} K(\tilde{p}^2). \tag{2.2.48}$$

Also, because $\phi(p)$ is replace by $\Lambda^{-\frac{D+2}{2}}\tilde{\phi}(\tilde{p}) = \Lambda^{-\frac{D+2}{2}}\tilde{\phi}(p/\Lambda)$, which depends on Λ. Its dependence is calculated as

$$\Lambda \frac{\partial}{\partial\Lambda}\left(\Lambda^{-\frac{D+2}{2}}\tilde{\phi}(p/\Lambda)\right) = -\Lambda^{-\frac{D+2}{2}}\left(\frac{D+2}{2} + \tilde{p}\cdot\partial_{\tilde{p}}\right)\tilde{\phi}(\tilde{p}) \tag{2.2.49}$$

Therefore, the second term is given by

$$\begin{aligned}
&\partial_\tau \tilde{S}_\tau[\tilde{\phi}] \\
&= -\Lambda\frac{\partial}{\partial\Lambda}\tilde{S}_\tau[\tilde{\phi}] &(2.2.50)\\
&= -\Lambda\frac{\partial}{\partial\Lambda}S_\Lambda[\phi]\Bigg|_{\phi\to\Lambda^{-\frac{D+2}{2}}\tilde{\phi}} - \int_p \left(\Lambda\frac{\partial}{\partial\Lambda}\Lambda^{-\frac{D+2}{2}}\tilde{\phi}(p/\Lambda)\right)\Lambda^{-\frac{D-2}{2}}\frac{\delta S_\tau[\tilde{\phi}]}{\delta\tilde{\phi}(p/\Lambda)}, &(2.2.51)\\
&= -\Lambda\frac{\partial}{\partial\Lambda}S_\Lambda[\phi]\Bigg|_{\phi\to\Lambda^{-\frac{D+2}{2}}\tilde{\phi}} + \Lambda^{-D}\int_p \left(\frac{D+2}{2} + p\cdot\partial_p\right)\tilde{\phi}(p/\Lambda)\frac{\delta S_\tau[\tilde{\phi}]}{\delta\tilde{\phi}(p/\Lambda)}, &(2.2.52)\\
&= -\Lambda\frac{\partial}{\partial\Lambda}S_\Lambda[\phi]\Bigg|_{\phi\to\Lambda^{-\frac{D+2}{2}}\tilde{\phi}} + \int_{\tilde{p}} \left(\frac{D+2}{2} + \tilde{p}\cdot\partial_{\tilde{p}}\right)\tilde{\phi}(\tilde{p})\frac{\delta S_\tau[\tilde{\phi}]}{\delta\tilde{\phi}(\tilde{p})}. &(2.2.53)
\end{aligned}$$

Note that p is the integration variable and does not receive the differentiation over Λ. We transformed the integration variable p into $p = \tilde{p}\Lambda$ in the last equality.

The first term is calculated as

$$\begin{aligned}
&-\Lambda\frac{\partial}{\partial\Lambda}S_\Lambda[\phi]\Bigg|_{\phi\to\Lambda^{-\frac{D+2}{2}}\tilde{\phi}} \\
&= \Lambda^{-D}\int_p\Bigg[-\frac{p\cdot\partial_p K(p^2/\Lambda^2)}{K(p^2/\Lambda^2)}\tilde{\phi}(p/\Lambda)\frac{\delta\tilde{S}_\tau}{\delta\tilde{\phi}(p/\Lambda)} \\
&\quad -\frac{p\cdot\partial_p K(p^2/\Lambda^2)}{2p^2/\Lambda^2}\left(\frac{\delta^2\tilde{S}_\tau}{\delta\tilde{\phi}(p/\Lambda)\delta\tilde{\phi}(-p/\Lambda)} + \frac{\delta\tilde{S}_\tau}{\delta\tilde{\phi}(p/\Lambda)}\frac{\delta\tilde{S}_\tau}{\delta\tilde{\phi}(-p/\Lambda)}\right)\Bigg] &(2.2.54)\\
&= \int_{\tilde{p}}\left[\frac{\Delta(\tilde{p}^2)}{K(\tilde{p}^2)}\tilde{\phi}(\tilde{p})\frac{\delta\tilde{S}_\tau}{\delta\tilde{\phi}(\tilde{p})} + \frac{\Delta(\tilde{p}^2)}{2\tilde{p}^2}\left(\frac{\delta^2\tilde{S}_\tau}{\delta\tilde{\phi}(\tilde{p})\delta\tilde{\phi}(-\tilde{p})} + \frac{\delta\tilde{S}_\tau}{\delta\tilde{\phi}(\tilde{p})}\frac{\delta\tilde{S}_\tau}{\delta\tilde{\phi}(-\tilde{p})}\right)\right]. &(2.2.55)
\end{aligned}$$

In the last equality, we changed the integration variable p to $\tilde{p}\Lambda$ and defined $\Delta(\tilde{p}^2)$ as

$$\Delta(\tilde{p}^2) := -\tilde{p} \cdot \partial_{\tilde{p}} K(\tilde{p}^2). \tag{2.2.56}$$

We finally get from Eq. (2.2.53) and Eq. (2.2.55)

$$\partial_\tau \tilde{S}_\tau[\tilde{\phi}] = \int_{\tilde{p}} \left(\frac{\Delta(\tilde{p}^2)}{K(\tilde{p}^2)} + \frac{D+2}{2} + \tilde{p} \cdot \partial_{\tilde{p}} \right) \tilde{\phi}(\tilde{p}) \frac{\delta S_\tau[\tilde{\phi}]}{\delta \tilde{\phi}(\tilde{p})} + \int_{\tilde{p}} \left[\frac{\Delta(\tilde{p}^2)}{2\tilde{p}^2} \left(\frac{\delta^2 \tilde{S}_\tau}{\delta \tilde{\phi}(\tilde{p}) \delta \tilde{\phi}(-\tilde{p})} + \frac{\delta \tilde{S}_\tau}{\delta \tilde{\phi}(\tilde{p})} \frac{\delta \tilde{S}_\tau}{\delta \tilde{\phi}(-\tilde{p})} \right) \right]. \tag{2.2.57}$$

This equation is the dimensionless version of the WP equation. In the following, we remove the tilde to denote dimensionless variables for notational simplicity.

Scaling Theorem and Anomalous Dimension

Here, we introduce the anomalous dimension. It has been shown in Ref. [13] that the Wilsonian effective action S_τ satisfies

$$\left\langle\!\left\langle \phi(p_1 e^{\tau-\tau_0}) \cdots \phi(p_n e^{\tau-\tau_0}) \right\rangle\!\right\rangle_{S_\tau}^{K(1-K)} = e^{-(\tau-\tau_0)n(D+2)/2} \langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_{S_{\tau_0}}^{K(1-K)}. \tag{2.2.58}$$

This equation represents the response of scaling in the correlation functions. We call it "scaling relation" in the following. The modified correlation function $\langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_S^{K(1-K)}$ is defined as

$$\langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_S^{K(1-K)} := \int D\phi \, e^{S[\phi]} \exp\left(-\int_p \frac{K(p^2)(1-K(p^2))}{2p^2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right) \frac{\phi(p_1)}{K(p_1)} \cdots \frac{\phi(p_n)}{K(p_n)}. \tag{2.2.59}$$

We can generalize $K(p^2)(1-K(p^2))$ in this expression to an arbitrary function $k(p^2)$ in the scaling relation by introducing the wavefunction renormalization factor as

$$\left\langle\!\left\langle \phi(p_1 e^{\tau-\tau_0}) \cdots \phi(p_n e^{\tau-\tau_0}) \right\rangle\!\right\rangle_{S_\tau}^{k} = Z_\tau^n e^{-(\tau-\tau_0)n(D+2)/2} \langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_{S_{\tau_0}}^{k}, \tag{2.2.60}$$

where Z_τ is the wavefunction renormalization factor defined as

$$Z_\tau := \exp\left(\int_{\tau_0}^{\tau} \gamma_s ds \right) \tag{2.2.61}$$

with the anomalous dimension γ_s. The generalized modified correlation function with $k(p^2)$ is defined as

$$\langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_S^{k} := \int D\phi \, e^{S[\phi]} \hat{s} \frac{\phi(p_1)}{K(p_1)} \cdots \frac{\phi(p_n)}{K(p_n)}, \tag{2.2.62}$$

where $\hat{s}$ is called "scrambler", defined as

$$\hat{s} := \exp\left(-\int_p \frac{k(p^2)}{2p^2} \frac{\delta}{\delta\phi(p)} \frac{\delta}{\delta\phi(-p)}\right). \tag{2.2.63}$$

To keep the IR physics or more specifically, to avoid unphysical pole at $p^2 = 0$, the scrambler should be analytic around $p^2 = 0$. This requires that $k(p^2)$ should vanish at $p^2 = 0$.

We can derive the WP equation with $k(p^2)$ and the anomalous dimension from Eq. (2.2.60). Eq. (2.2.60) yields that

$$Z_\tau^{-n} e^{(\tau-\tau_0)n(D+2)/2} \left\langle\!\left\langle \phi(p_1 e^{\tau-\tau_0}) \cdots \phi(p_n e^{\tau-\tau_0}) \right\rangle\!\right\rangle_{S_\tau}^k \tag{2.2.64}$$

is τ-independent. Therefore, we get

$$\begin{aligned}
0 &= \partial_\tau \left(Z_\tau^{-n} e^{(\tau-\tau_0)n(D+2)/2} \left\langle\!\left\langle \phi(p_1 e^{\tau-\tau_0}) \cdots \phi(p_n e^{\tau-\tau_0}) \right\rangle\!\right\rangle_{S_\tau}^k \right) && (2.2.65)\\
&= \partial_\tau \int D\phi \, e^{S_\tau} \hat{s} \prod_{i=1}^n Z_\tau^{-1} e^{(\tau-\tau_0)(D+2)/2} \frac{\phi(p_i e^{\tau-\tau_0})}{K(p_i^2 e^{2(\tau-\tau_0)})} && (2.2.66)\\
&= \int D\phi \left[\partial_\tau e^{S_\tau} \hat{s} + e^{S_\tau} \hat{s} \int_p \left(\frac{\Delta(p^2)}{K(p^2)} + \frac{D+2}{2} - \gamma_\tau + p \cdot \partial_p \right) \phi(p) \cdot \frac{\delta}{\delta\phi(p)} \right] \\
&\qquad \times \prod_{i=1}^n Z_\tau^{-1} e^{(\tau-\tau_0)(D+2)/2} \frac{\phi(p_i e^{\tau-\tau_0})}{K(p_i^2 e^{2(\tau-\tau_0)})} && (2.2.67)\\
&= \int D\phi \Bigg[\partial_\tau e^{S_\tau} \\
&\quad + e^{S_\tau} \int_p \left(\frac{\Delta(p^2)}{K(p^2)} + \frac{D+2}{2} - \gamma_\tau + p \cdot \partial_p \right) \left(\phi(p) - \frac{k(p^2)}{p^2} \frac{\delta}{\delta\phi(-p)} \right) \cdot \frac{\delta}{\delta\phi(p)} \Bigg] && (2.2.68)\\
&\qquad \times \hat{s} \prod_{i=1}^n Z_\tau^{-1} e^{(\tau-\tau_0)(D+2)/2} \frac{\phi(p_i e^{\tau-\tau_0})}{K(p_i^2 e^{2(\tau-\tau_0)})}. \\
&= \int D\phi \Bigg[\partial_\tau e^{S_\tau} \\
&\quad - \int_p \frac{\delta}{\delta\phi(p)} \left(\frac{\Delta(p^2)}{K(p^2)} + \frac{D+2}{2} - \gamma_\tau + p \cdot \partial_p \right) \left(\phi(p) + \frac{k(p^2)}{p^2} \frac{\delta}{\delta\phi(-p)} \right) e^{S_\tau} \Bigg] && (2.2.69)\\
&\qquad \times \hat{s} \prod_{i=1}^n Z_\tau^{-1} e^{(\tau-\tau_0)(D+2)/2} \frac{\phi(p_i e^{\tau-\tau_0})}{K(p_i^2 e^{2(\tau-\tau_0)})}.
\end{aligned}$$

We used the following relation:

$$\hat{s}\phi(p) = \left(\phi(p) - \frac{k(p^2)}{p^2} \frac{\delta}{\delta\phi(-p)} \right) \hat{s}. \tag{2.2.70}$$

We used partial integration formula in the last equality. Because Eq. (2.2.69) holds for an arbitrary number n of fields, it should be satisfied that

$$\partial_\tau e^{S_\tau} = \int_p \frac{\delta}{\delta\phi(p)} \left(\frac{\Delta(p^2)}{K(p^2)} + \frac{D+2}{2} - \gamma_\tau + p \cdot \partial_p \right) \left(\phi(p) + \frac{k(p^2)}{p^2} \frac{\delta}{\delta\phi(-p)} \right) e^{S_\tau}. \tag{2.2.71}$$

Note that

$$\begin{aligned}
&\int_p \frac{\delta}{\delta\phi(p)} p\cdot\partial_p \left(\frac{k(p^2)}{p^2}\frac{\delta}{\delta\phi(-p)}\right) \\
&= \int_p \left[\left(p\cdot\partial_p \frac{k(p^2)}{p^2}\right)\frac{\delta}{\delta\phi(p)}\frac{\delta}{\delta\phi(-p)} + \frac{k(p^2)}{p^2}\frac{\delta}{\delta\phi(p)} p\cdot\partial_p \frac{\delta}{\delta\phi(-p)}\right] && (2.2.72)\\
&= \int_p \left[\left(p\cdot\partial_p \frac{k(p^2)}{p^2}\right)\frac{\delta}{\delta\phi(p)}\frac{\delta}{\delta\phi(-p)} + \frac{k(p^2)}{p^2}\frac{1}{2} p\cdot\partial_p \left(\frac{\delta}{\delta\phi(p)}\frac{\delta}{\delta\phi(-p)}\right)\right] && (2.2.73)\\
&= \int_p \left[\left(p\cdot\partial_p \frac{k(p^2)}{p^2} - \frac{1}{2}\partial_p\cdot\left(p\frac{k(p^2)}{p^2}\right)\right)\frac{\delta}{\delta\phi(p)}\frac{\delta}{\delta\phi(-p)}\right] && (2.2.74)\\
&= \int_p \left[-\left(\frac{D}{2}\frac{k(p^2)}{p^2} + p\cdot\partial_p \frac{k(p^2)}{2p^2}\right)\frac{\delta}{\delta\phi(p)}\frac{\delta}{\delta\phi(-p)}\right] && (2.2.75)
\end{aligned}$$

Also, we neglect the constant term, generated by exchanging $\phi(p)$ and $\delta/\delta\phi(p)$. Finally, we get

$$\begin{aligned}
\partial_\tau S_\tau = \int_p \Bigg[&\left(\frac{\Delta(p^2)}{K(p^2)} + \frac{D+2}{2} - \gamma_\tau + p\cdot\partial_p\right)\phi(p)\frac{\delta S_\tau}{\delta\phi(p)} \\
&+ \frac{1}{2p^2}\left(2\frac{\Delta(p^2)}{K(p^2)}k(p^2) + p\cdot\partial_p k(p^2) - 2\gamma_\tau p^2\right)\left(\frac{\delta^2 S_\tau}{\delta\phi(p)\delta\phi(-p)} + \frac{\delta S_\tau}{\delta\phi(p)}\frac{\delta S_\tau}{\delta\phi(-p)}\right)\Bigg]. \qquad (2.2.76)
\end{aligned}$$

2.3 Fixed Point

In this section, we consider the fixed point of the ERG equation. The fixed point of a flow is defined by the condition $\partial_\tau S_\tau = 0$. If the Wilsonian effective action S_τ converges to some finite action S^* in the $\tau \to \infty$ limit, it is obvious that the fixed point action S^* is τ-independent, that is, $\partial_\tau S^* = 0$. Because the energy scale we focus on is given by $\Lambda := \Lambda_0 e^{-\tau}$, Λ vanishes in the $\tau \to \infty$ limit. This means that the fixed point action S^* describes the extremely low-energy physics of the theory. Since all massive degrees of freedom are integrated out, the fixed point action S^* should contain only massless particles. Also, the condition $\partial_\tau S^* = 0$ has another physical meaning that it is invariant under the scale transformation. This yields that it has no dimensionful parameters that characterize IR physics. In statistical physics, a system can be characterized by its correlation length, which measures how far two parts in the system are correlated. If the system undergoes a (second-order) phase transition, this correlation length diverges, and at a critical point, there is believed to be no energy scale that characterizes the system. Therefore, the fixed point of the RG flow is expected to correspond to these critical theories in statistical physics. These fixed point theories should have scale, translation, and rotational invariance.

There is a famous no-go theorem on the symmetry of QFTs proved by Coleman and Mandula [18]. This theorem states that "If a QFT has (1) the Poincáre symmetry, (2) only finite number of particles whose mass are below given M, and (3) some interactions, then its symmetry must be the direct product of the Poincáre group and some Lie group." From this theorem, there seems to be no possibility of any QFTs with the scale invariance apparently. However, there is a loophole in the proof of this theorem: If the theory contains only massless particles, conformal symmetry, which includes the Poincáre symmetry and the scale symmetry, is exceptionally allowed. This loophole is consistent with the above argument; scale-invariant theories can only describe the physics of massless particles.

The generators of Poincáre and scale transformations form a sub-algebra of a bigger Lie algebra. This bigger Lie algebra is the conformal algebra, and the corresponding Lie group is the conformal group. Therefore, we naively expect that the scale- and rotational-invariant theory has conformal symmetry. However, this expectation needs to be clarified and is still to be resolved. As for two-dimensional theories, A.B.Zamoldchikov showed a famous theorem named *C-theorem* [19]. This theorem states that scale-invariant theories have the conformal symmetry and gives the central charge, representing the effective degrees of freedom of the theory. A generalization of the C-theorem to the four-dimensional case called "A-theorem", has been proposed and attempted to prove [20–22]. However, whether the fixed point theories have the conformal symmetry has yet to be revealed.

In the following subsections, we give some examples of the fixed point of the WP equation for scalar fields. The fixed point condition is given by

$$\begin{aligned} 0 = \int_p \Bigg[& \left(\frac{\Delta(p^2)}{K(p^2)} + \frac{D+2}{2} - \gamma_\tau + p \cdot \partial_p \right) \phi_i(p) \frac{\delta S^*}{\delta \phi_i(p)} \\ & + \frac{1}{2p^2} \left(2\frac{\Delta(p^2)}{K(p^2)} k(p^2) + p \cdot \partial_p k(p^2) - 2\gamma_\tau p^2 \right) \left(\frac{\delta^2 S^*}{\delta\phi_i(p)\delta\phi_i(-p)} + \frac{\delta S^*}{\delta\phi_i(p)} \frac{\delta S^*}{\delta\phi_i(-p)} \right) \Bigg], \end{aligned} \tag{2.3.1}$$

where we generalized the WP equation that we derived in the previous subsection to $O(N)$ vector model ($\phi \to \phi_i \, (i = 1, \ldots, N)$), as can be done straightforwardly.

2.3.1 Gaussian fixed point

Let us first consider the Gaussian fixed point, whose action does not have any interactions and only contains a quadratic term of the fields:

$$S^* = -\frac{1}{2} \int_p \phi_i(-p) F(p^2) \phi_i(p). \tag{2.3.2}$$

We also assume $\gamma_\tau = 0$. Then, the fixed point condition is given by

$$\begin{aligned} 0 = \int_p \Bigg[& \left(\frac{\Delta(p^2)}{K(p^2)} + \frac{D+2}{2} + p \cdot \partial_p \right) \phi(p) \frac{\delta S^*}{\delta \phi(p)} \\ & + \frac{1}{2p^2} \left(2\frac{\Delta(p^2)}{K(p^2)} k(p^2) + p \cdot \partial_p k(p^2) \right) \frac{\delta S^*}{\delta\phi(p)} \frac{\delta S^*}{\delta\phi(-p)} \Bigg]. \end{aligned} \tag{2.3.3}$$

We have dropped the second functional derivation of S^* in Eq. (2.3.1) because it gives just a trivial constant term, which can be removed by adding corresponding constant term to S^*. Substituting this ansatz into Eq. (2.3.3), we get

$$0 = \int_p \phi_i(-p)\phi_i(p) \times \left[\left(-\frac{\Delta(p^2)}{K(p^2)} - 1\right) F(p^2) + p^2 F'(p^2) + \frac{F^2(p^2)}{2p^2}\left(2\frac{\Delta(p^2)}{K(p^2)}k(p^2) + p\cdot\partial_p k(p^2)\right)\right]. \quad (2.3.4)$$

Because this equation holds for an arbitrary value of the fields $\phi_i(p)$, it is required for each p that

$$0 = \left(-\frac{\Delta(p^2)}{K(p^2)} - 1\right) F(p^2) + p^2 F'(p^2) + \frac{F^2(p^2)}{2p^2}\left(2\frac{\Delta(p^2)}{K(p^2)}k(p^2) + p\cdot\partial_p k(p^2)\right). \quad (2.3.5)$$

We can solve this equation as follows. Multiplying F^{-2}, we get

$$0 = \left(-\frac{\Delta(p^2)}{K(p^2)} - 1\right) F^{-1} - p^2 (F^{-1})' + \frac{1}{2p^2}\left(2\frac{\Delta}{K}k + p\cdot\partial_p k\right). \quad (2.3.6)$$

Decomposing F as

$$F^{-1} = p^{-2} G(p^2) \quad (2.3.7)$$

and substituting this equation into Eq. (2.3.6), we get

$$0 = -p^{-2}\frac{\Delta}{K}G^{-1} - G' + \frac{1}{2p^2}\left(2\frac{\Delta}{K}k + p\cdot\partial_p k\right) \quad (2.3.8)$$

$$= -((\log K^{-2})' G + G') + \big((\log K^{-2})' k + k'\big), \quad (2.3.9)$$

equivalently,

$$(K^{-2}G)' = (K^{-2}k)', \quad (2.3.10)$$

which is solved by

$$G(p^2) = CK^2(p^2) + k(p^2). \quad (2.3.11)$$

C is an arbitrary constant. This constant can be determined by requiring that ϕ_i should be canonically defined, i.e., the coefficient of its kinetict term is $1/2$:

$$\frac{1}{2} = \frac{1}{2}\left.\frac{\mathrm{d}F}{\mathrm{d}p^2}\right|_{p^2=0} = \frac{1}{2C}, \quad (2.3.12)$$

which yields to $C = 1$. Finally, we get

$$F(p^2) = \frac{p^2}{K^2(p^2) + k(p^2)} \quad (2.3.13)$$

and the Gaussian fixed point action is given by

$$S^* = -\frac{1}{2}\int_p \phi_i(-p)\frac{p^2}{K^2(p^2)+k(p^2)}\phi_i(p). \tag{2.3.14}$$

If we take the original Polchinski convention ($k(p^2) = K(p^2)(1-K(p^2))$), $F(p^2)$ is just given by

$$F(p^2) = \frac{p^2}{K(p^2)}. \tag{2.3.15}$$

2.3.2 Wilson-Fisher fixed point

The Wilson-Fisher fixed point [2, 23] is a non-trivial fixed point in $4-\epsilon$ dimension. In the following, we use ϵ-expansion to derive this fixed point. We adopt the Polchinski's convention here for simplicity. The fixed point action is given by

$$S^* = -\frac{1}{2}\int_p \frac{p^2}{K(p^2)}\phi_i(-p)\phi_i(p) + \int_x V(x), \tag{2.3.16a}$$

$$V(x) = \frac{m_*^2}{2}\phi_i(x)\phi_i(x) + \frac{\lambda_*}{8}(\phi_i(x)\phi_i(x))^2. \tag{2.3.16b}$$

The values of m_*^2 and λ_* are given by

$$m_*^2 := \frac{\epsilon}{4}\frac{N+2}{N+8}, \quad \lambda_* := -\epsilon\frac{8\pi^2}{N+8}. \tag{2.3.17}$$

Let us derive Eq. (2.3.16) in the following. We use the local potential approximation, under which the Wilsonian effective action at the Wilson-Fisher fixed point is expanded as

$$S^* = S^{(0)} + S^I \tag{2.3.18a}$$

$$S^{(0)} = -\frac{1}{2}\int_p G(p^2)\phi_i(-p)\phi_i(p), \tag{2.3.18b}$$

$$S^I = \sum_{n=1}^{\infty}\epsilon^n S^{(n)}, \tag{2.3.18c}$$

$$= \sum_{n=1}^{\infty}\frac{g_{2n}}{n!2^n}(\phi_i(x)^2)^n. \tag{2.3.18d}$$

In addition, we assume that

$$g_2 = \mathcal{O}(\epsilon), \quad g_{2n} = \mathcal{O}\big(\epsilon^{n-1}\big) \quad (n \geq 2) \tag{2.3.19}$$

and

$$\gamma = \mathcal{O}\big(\epsilon^2\big). \tag{2.3.20}$$

This assumption is justified because it does solves the fixed point condition Eq. (4.1.24) within the local potential approximation order by order of ϵ.

Zeroth order

Substituting the ansatz into Eq. (4.1.24) and focusing on the zeroth terms of the ϵ-expansion, we get

$$0 = \int_p \left[\left(\frac{\Delta(p^2)}{K(p^2)} + \frac{4+2}{2} + p \cdot \partial_p \right) \phi_i(p) \frac{\delta S^{(0)}}{\delta \phi_i(p)} + \frac{\Delta(p^2)}{2p^2} \frac{\delta S^{(0)}}{\delta \phi_i(p)} \frac{\delta S^{(0)}}{\delta \phi_i(-p)} \right]. \tag{2.3.21}$$

Remember that $D = 4 - \epsilon$. We dropped the second functional derivative of $S^{(0)}$ since it gives just a constant term. This equation is the same as that Eq. (2.3.3) of the Gaussian fixed point with $k(p^2) = K(p^2)(1 - K(p^2))$. Therefore, $G(p^2)$ and $S^{(0)}$ are given by

$$G(p^2) = \frac{p^2}{K(p^2)}, \tag{2.3.22}$$

$$S^{(0)} = -\frac{1}{2} \int_p \frac{p^2}{K(p^2)} \phi_i(-p)\phi_i(p). \tag{2.3.23}$$

First order

Focusing on the first-order term, we get

$$0 = \epsilon \int_p \Bigg[-\frac{1}{2} \phi_i(p) \frac{\delta S^{(0)}}{\delta \phi_i(p)} + \left(\frac{\Delta(p^2)}{K(p^2)} + \frac{4+2}{2} + p \cdot \partial_p \right) \phi_i(p) \frac{\delta S^{(1)}}{\delta \phi_i(p)} + \frac{\Delta(p^2)}{2p^2} \left(\frac{\delta^2 S^{(1)}}{\delta \phi_i(p) \delta \phi_i(-p)} + 2 \frac{\delta S^{(0)}}{\delta \phi_i(p)} \frac{\delta S^{(1)}}{\delta \phi_i(-p)} \right) \Bigg] \tag{2.3.24}$$

Substituting Eq. (2.3.23) and

$$S^{(1)} = \int_x \left(\frac{g_2}{2} \phi_i(x)^2 + \frac{g_4}{8} \left(\phi_i(x)^2\right)^2 \right) \tag{2.3.25}$$

into this equation and focusing on the mass terms, we get

$$0 = \int_x \phi_i(x)^2 \left(g_2 + \frac{N+2}{2} g_4 c_1 \right), \tag{2.3.26}$$

equivalently,

$$g_2 = -\frac{N+2}{2} c g_4, \tag{2.3.27}$$

where $c := \int_p \Delta(p^2)/2p^2 = -\int_p K'(p^2)$. On the other hand, the fixed point condition for the quartic coupling in automatically saturated with an arbitrary value of g_4.

Second order

Focusing on the first-order term, we get

$$0 = \epsilon^2 \int_p \Bigg[\bigg(\frac{\Delta(p^2)}{K(p^2)} + \frac{4+2}{2} + p \cdot \partial_p \bigg) \phi_i(p) \frac{\delta S^{(2)}}{\delta \phi_i(p)} - \frac{1}{2} \phi_i(p) \frac{\delta S^{(1)}}{\delta \phi_i(p)} + \frac{\Delta(p^2)}{2p^2} \bigg(\frac{\delta^2 S^{(2)}}{\delta \phi_i(p) \delta \phi_i(-p)} + \frac{\delta S^{(1)}}{\delta \phi_i(p)} \frac{\delta S^{(1)}}{\delta \phi_i(-p)} + 2 \frac{\delta S^{(0)}}{\delta \phi_i(p)} \frac{\delta S^{(2)}}{\delta \phi_i(-p)} \bigg) \Bigg]. \tag{2.3.28}$$

Focusing on the sextic and quartic terms, we get

$$0 = \frac{1}{8} \int_x (\phi_i(x)^2)^2 (-2\epsilon g_4 + 4(N+8) c g_6), \tag{2.3.29a}$$

$$0 = \frac{1}{3! 2^3} \int_x (\phi_i(x)^2)^3 \big(-2g_6 - 2K'(0) g_4^2\big). \tag{2.3.29b}$$

Therefore, g_6 is given by

$$g_6 = -K'(0) g_4^2 \tag{2.3.30}$$

and the value of g_4 satisfies

$$0 = -\epsilon g_4 - 2(N+8) K'(0) c g_4^2, \tag{2.3.31}$$

which is solved by

$$g_4 = \epsilon \frac{(2c)^{-1}}{N+8} (-K'(0))^{-1}. \tag{2.3.32}$$

Substituting this equation into Eq. (2.3.27), we get

$$g_2 = -\frac{\epsilon}{(-4K'(0))} \frac{N+2}{N+8}. \tag{2.3.33}$$

If we take take $K(p^2) = e^{-p^2}$, $c = (4\pi)^{-2}$ and $K'(0) = -1$, and we obtain

$$g_2 = -\frac{\epsilon}{4} \frac{N+2}{N+8}, \quad g_4 = \epsilon \frac{8\pi^2}{N+8}, \tag{2.3.34}$$

which reproduces Eq. (4.2.4).

2.4 Flow Structure around Fixed Point, Critical Exponents and Universality

In this section, we focus on a flow structure around a fixed point. Consider the bare action is very close to a fixed point as

$$S_{\tau_0} = S^* + \delta S_{\tau_0} \tag{2.4.1}$$

where S^* is a fixed point of the RG flow. We use the linear analysis to track the flow of the difference $S_\tau - S$ from the fixed point. That is, setting

$$S_\tau = S^* + \delta S_\tau \tag{2.4.2}$$

and taking the first-order terms of δS_τ, we get the linearized ERG equation as

$$\partial_\tau \delta S_\tau = \hat{R}\delta S_\tau + \mathcal{O}\big(\delta S_\tau^2\big). \tag{2.4.3}$$

Then, we expand δS_τ with a complete set of operators $\{\mathcal{O}_i(x)\}$ as

$$\delta S_\tau = \delta g_i(\tau) \int_x \mathcal{O}_i(x), \tag{2.4.4}$$

where each $\delta g_i(\tau)$ is a expansion coefficient and serves as a τ-dependent coupling constant of $\mathcal{O}_i(x)$. The complete set $\{\mathcal{O}_i(x)\}$ includes all kinds of field operators such as $\phi^n(x), (\partial^2)^m\phi(x), (\partial_\mu\phi(x))^2\phi(x)^2$. Substituting this expression to the linearized ERG equation Eq. (2.4.3), we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\delta g_i(\tau) = R_{ij}\delta g_j(\tau), \tag{2.4.5}$$

where R_{ij} is defined as

$$\hat{R}\int_x \mathcal{O}_j(x) =: R_{ij}\int_x \mathcal{O}_i(x). \tag{2.4.6}$$

R_{ij} can be regarded as a infinitely large matrix on the theory space and its indices run on the basis $\{\mathcal{O}_i\}$. Note that $\hat{R}$ and R_{ij} are τ-indepedent, which means so is the flow structure around the fixed point. Eq. (2.4.5) is valid for S_τ is very close to S^*, i.e., $|\delta g_i(\tau)| \ll 1$.

We can solve Eq. (2.4.5) by utilizing eigenvector of R_{ij}. The right-eigenvectors $\{v^{(a)}\}$ of R_{ij} are defined as

$$R_{ij}v_j^{(a)} = \lambda_a v_i^{(a)}. \tag{2.4.7}$$

Note that the indices i and a are not summed in this equation, while j is. Since $\{v^{(a)}\}$ forms a complete set, the coupling $\delta g_j(\tau)$ can be expanded as

$$\delta g_i(\tau) = \delta c^{(a)}(\tau)v_i^{(a)}, \tag{2.4.8}$$

where each $\delta c^{(a)}(\tau)$ is a τ-dependent expansion coefficient. Substituting this equation into Eq. (2.4.5) and focusing on the coefficient of $v^{(a)}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\delta c^{(a)}(\tau) = \lambda_a \delta c^{(a)}(\tau). \tag{2.4.9}$$

Note that the index a is not summed. This equation is solved by

$$\delta c^{(a)}(\tau) = e^{\lambda_a(\tau-\tau_0)}\delta c^{(a)}(\tau_0). \tag{2.4.10}$$

Then, $\delta g_i(\tau)$ and δS_τ is given by

$$\delta g_i(\tau) = \sum_a e^{\lambda_a(\tau-\tau_0)}\delta c^{(a)}(\tau_0)v_i^{(a)}, \tag{2.4.11}$$

$$\delta S_\tau = \sum_{a,i} e^{\lambda_a(\tau-\tau_0)} \delta c^{(a)}(\tau_0) v_i^{(a)} \int_x \mathcal{O}_i(x). \tag{2.4.12}$$

The latter equation means that the operator $\mathcal{O}^a := \int_x v_i^{(a)} \mathcal{O}_i(x)$ is an eigenvector of the linearized ERG equation with the eigenvalue λ_a:

$$\hat{R}\mathcal{O}^a = \lambda_a \mathcal{O}^a, \tag{2.4.13}$$

which can be verified by combining Eq. (2.4.6) and Eq. (2.4.8). These eigenvalues λ_a (called "scaling dimension") are quite important because they determine the flow around the fixed in the direction of $\mathcal{O}^a$ (called "scaling operator") on the theory space. They controls responses of the system to perturbations of the fixed point theory S^* by δS_τ and are observable via experiments. Especially, the sign of λ_a is very important. If λ_a is positive, zero or negative, the corresponding scaling operator $\mathcal{O}^a$ is referred to as relevant, marginal or irrelevant, respectively.

The latter equation Eq. (2.4) means that the operator $\mathcal{O}^a := \int_x v_i^{(a)} \mathcal{O}_i(x)$ is an eigenvector of the linearized ERG equation with the eigenvalue λ_a:

$$\hat{R}\mathcal{O}^a = \lambda_a \mathcal{O}^a, \tag{2.4.14}$$

which can be verified by combining Eq. (2.4.6) and Eq. (2.4.8). These eigenvalues λ_a (called "scaling dimension") are pretty significant because they determine the flow around the fixed in the direction of $\mathcal{O}^a$ (called "scaling operator") on the theory space. They control responses of the system to perturbations of the fixed point theory S^* by δS_τ and are observable via experiments. In particular, the sign of the eigenvalues λ_a is significant. If it is positive, zero, or negative, the corresponding scaling operators are referred to as relevant, marginal, or irrelevant, respectively.

If δS_{τ_0} contains only irrelevant operators, the perturbed action S_τ finally falls into the fixed point S^*. This yields that the same fixed point S^* universally describes the IR behavior of theories only with irrelevant perturbations. We refer to the subspace in the theory space that gets sucked into the same fixed point S^* as the "critical surface" of S^*. The critical surface is spanned by the irrelevant operators, at least around the fixed point S^*, and usually has infinitely large dimensions.

On the other hand, if δS_{τ_0} contains relevant operators, S_τ goes away from S^* along their directions and then S_τ finally reaches a different fixed point from S^*. Therefore, these operators cannot be ignored to study the IR property of S_{τ_0}. The subspace spanned by the relevant operators is called the "renormalized trajectory" of S^*. Usually, a fixed point has a limited number of relevant operators, so the dimension of its renormalized trajectory is finite.

At this point, we can discuss the notion of the "universality of quantum field theory". Combing the above discussion, we see that the coupling of the irrelevant operators in the

Wilsonian effective action S_τ exponentially decreases along the RG flow around the fixed point S^* while those of the relevant operators increase. Therefore, we find that the IR behavior of the perturbed theory S_{τ_0} is described only by the fixed point S^* and its relevant operators. This observation yields that many kinds of UV theories have the same IR behavior, known as "universality". In particular, if we tune the couplings of these UV theories to make them on a critical point, their IR physics should be described by the fixed point S^*. These UV theories belong to the same "universality class". The universality property is quite powerful in studying a physical system because it ensures enough to have a finite number of couplings to describe its IR behavior, even though we do not know its description in the UV region!

Before moving on to examples, we comment on (1) the marginal operators and (2) the renormalization-scheme independence of the scaling dimensions.

(1) The marginal operators seems not to flow by their definition, but remember that Eq. (2.4.3) is just the lowest order approximation in terms of δS_τ. Therefore, marginal operators could turn relevant or irrelevant due to the non-linear effect of higher-order terms of δS_τ, and these operators are marginally relevant or irrelevant, respectively. Of course, there can be those remaining marginal even after taking the non-linear effect into account, and these operators are called "exactly marginal". If there are some exactly marginal operators around a fixed point, the fixed point is not a point but an object with higher dimensions, such as a line or a surface.

(2) The scaling dimensions are defined as the eigenvalue of the linearized ERG equation. We can discuss that various ERG equations have common scaling dimensions. Let us consider two different ERG equations and corresponding Wilsonian effective actions S_τ, S'_τ. We impose that these Wilsonian effective actions agree with each other at a bare scale $\tau = \tau_0$:.

$$S_{\tau_0} = S'_{\tau_0}. \tag{2.4.15}$$

Then, they are expanded at an arbitrary scale τ as

$$S_\tau = g_i(\tau) \int_x \mathcal{O}_i(x), \tag{2.4.16a}$$

$$S'_\tau = g'_i(\tau) \int_x \mathcal{O}_i(x), \tag{2.4.16b}$$

where $g_i(\tau)$ or $g'_i(\tau)$ is the τ-dependent coupling of $\mathcal{O}_i(x)$ in S_τ or S'_τ, respectively. Eq. (2.4.15) requires that $g_i(\tau)$ and $g'_i(\tau)$ should have the same value at $\tau = \tau_0$:

$$g_i(\tau_0) = g'_i(\tau_0). \tag{2.4.17}$$

As for $\tau \neq \tau_0$, these couplings do not necessarily agree since the Wilsonian effective actions S_τ and S'_τ follow different ERG equations. However, when both S_τ and S'_τ converge to some finite actions in $\tau \to \infty$ limit, we expect a diffeomorphism between these couplings. The reason to expect so is given in the following. When the Wilsonian effective action is expanded as Eq. (2.4.16), an ERG equation gives an autonomous system of first-order differential equations for $g_i(\tau)$ or $g'_i(\tau)$. If $g_i(\infty)$ converges to a finite value, it is expected that $g_i(\tau)$ never takes the same value along τ, i.e., it holds that if $\tau_1 \neq \tau_2$,

$$g_i(\tau_1) \neq g_i(\tau_2). \tag{2.4.18}$$

Although this statement holds in the finite-dimensional case due to the uniqueness of the solution, remember that the theory space has infinitely large dimensions. Given this statement, we can construct one-to-one correspondence between $g_i(\tau)$ and $g'_i(\tau)$ at an arbitrary scale parameter τ, which gives the diffeomorphism between them. We denote this diffeomorphism as

$$g_i(\tau) = g_i(g'(\tau)). \tag{2.4.19}$$

With this diffeomorphism, let us study the relation between flow structures around fixed points. Each fixed point, S^* or S'^*, is specified by its finite coupling constants, g_i^* or $g_i'^*$. Note that these values of the coupling constants are related through the diffeomorphism Eq. (2.4.19) as

$$g_i^* = g_i(g'^*). \tag{2.4.20}$$

When perturbing S^* and S'^* to study the flow structures around them as Eq. (2.4.2) and setting $g_i(\tau) = g_i^* + \delta g_i(\tau)$ and $g'_i(\tau) = g_i'^* + \delta g'_i(\tau)$, we get

$$S_\tau = S^* + \sum_i \delta g_i(\tau) \int_x \mathcal{O}_i(x), \tag{2.4.21a}$$

$$S'_\tau = S'^* + \sum_i \delta g'_i(\tau) \int_x \mathcal{O}_i(x). \tag{2.4.21b}$$

Here, thorough the diffeomorphism Eq. (2.4.19), $\delta g_i(\tau)$ and $\delta g'_i(\tau)$ are related as

$$g_i^* + \delta g_i(\tau) = g_i(g'^* + \delta g'(\tau)). \tag{2.4.22}$$

If $|\delta g_i(\tau)|$ and $|\delta g'_i(\tau)|$ are sufficiently small and we expand this equation up to the first-order of them, we obtain

$$\delta g_i(\tau) = J_{ij} \delta g'_j(\tau), \tag{2.4.23}$$

where J_{ij} is the τ-*indepedent* matrix defined as $J_{ij} := \partial g_i / \partial g'_j \big|_{g'=g'^*}$. Then, assuming that the linearized ERG equations for $|\delta g_i(\tau)|$ and $|\delta g'_i(\tau)|$ are given by

$$\partial_\tau \delta g_i(\tau) = R_{ij} \delta g_j(\tau), \tag{2.4.24a}$$

$$\partial_\tau \delta g'_i(\tau) = R'_{ij} \delta g'_j(\tau), \tag{2.4.24b}$$

and using Eq. (2.4.23), we find that R_{ij} and R'_{ij} are related as

$$R'_{il} = J^{-1}_{ij} R_{jk} J_{kl}, \tag{2.4.25}$$

or, in the matrix notation,

$$R' = J^{-1} R J. \tag{2.4.26}$$

Therefore, if we denote the eigenvalues of R_{ij}, R_{ij} as $\lambda^{(a)}, \lambda'^{(a)}$, respectively, they are determined from

$$\det\bigl(R - \lambda^{(a)} I\bigr) = 0, \tag{2.4.27a}$$

$$\det\Bigl(R' - \lambda'^{(a)} I\Bigr) = 0, \tag{2.4.27b}$$

where I is the unit matrix. From these equations and Eq. (2.4.26), we find that $\lambda^{(a)}$ satisfies

$$\begin{aligned} 0 &= \det\bigl(R - \lambda^{(a)} I\bigr) = \det\bigl(J^{-1}(R' - \lambda^{(a)} I)J\bigr) = (\det J)^{-1} \det\bigl(R' - \lambda^{(a)} I\bigr) \det J \\ &= \det\bigl(R' - \lambda^{(a)} I\bigr), \end{aligned} \tag{2.4.28}$$

which is nothing but the eigenvalue equation Eq. (2.4.27b) for $\lambda'^{(a)}$. Therefore, we conclude that $\{\lambda^{(a)}\}$ and $\{\lambda'^{(a)}\}$ agrees as a set. It means that two different (linearized) ERG equations Eq. (2.4.24) have common scaling dimensions; in other words, the scaling dimensions do not depend on the renormalization scheme.

Remember that the above discussion of the renormalization-scheme independence of the critical exponents should *not* be regarded as proof due to some less rigorous arguments. For example, although we have relied on the analogy to the linear algebra on a finite-dimensional space throughout the discussion, the theory space (or the space that is spanned by the coupling $g_i(\tau)$) has infinitely large dimensions, and it is not obvious to take the infinitely-large-dimension limit. Another subtle point is that we have assumed that the Wilsonian effective actions S_τ and S'_τ converge to some finite actions. However, there maybe exist cases where at least one goes infinitely far away or forms a limit cycle and then does not converge in $\tau \to \infty$ limit. Although these possibilities seem unphysical, they have yet to be entirely and rigorously ruled out. It is beyond the scope of this review to clarify these subtle points.

2.4.1 Gaussian Fixed Point

Let us consider the flow structure around the Gaussian fixed point. We adopt the Polchinski's convention ($k(p^2) = K(p^2)(1 - K(p^2))$) here for simplicity. Subsisting Eq. (2.4.2) into the WP equation, we get the linearized WP equation around the fixed point S^* as

$$\partial_\tau \delta S_\tau = \int_p \Bigg[\left(\frac{\Delta(p^2)}{K(p^2)} + \frac{D+2}{2} - \gamma_\tau + p\cdot\partial_p \right)\phi(p) \\ + \frac{1}{p^2}\left(2\frac{\Delta(p^2)}{K(p^2)}k(p^2) + p\cdot\partial_p k(p^2) - 2p^2\gamma_\tau \right)\frac{\delta S^*}{\delta\phi(-p)} \Bigg] \frac{\delta(\delta S_\tau)}{\delta\phi(p)}. \quad (2.4.29)$$

Substituting the concrete form of the Gaussian fixed point action (Eq. (2.3.2) and Eq. (2.3.15)), we can show that

$$\frac{\Delta(p^2)}{K(p^2)}\phi(p) + \frac{1}{p^2}\left(2\frac{\Delta(p^2)}{K(p^2)}k(p^2) + p\cdot\partial_p k(p^2) \right)\frac{\delta S^*}{\delta\phi(-p)} = 0. \quad (2.4.30)$$

Therefore, we obtain

$$\partial_\tau \delta S_\tau = \int_p \left(\frac{D+2}{2} + p\cdot\partial_p \right)\phi(p)\cdot\frac{\delta(\delta S_\tau)}{\delta\phi(p)}. \quad (2.4.31)$$

From this equation $\hat{R}$ in Eq. (2.4.3) is given by

$$\hat{R} = \int_p \left(\frac{D+2}{2} + p\cdot\partial_p \right)\phi(p)\cdot\frac{\delta}{\delta\phi(p)} = \int_x \left(-\frac{D-2}{2} - x\cdot\partial_x \right)\phi(x)\cdot\frac{\delta}{\delta\phi(x)} \quad (2.4.32)$$

The eigenoperators of this $\hat{R}$ is easy to find. This is the generator of the scale transformation and acts on δS_τ as

$$\hat{R}\delta S_\tau[\phi(x)] = -\frac{\partial}{\partial\lambda}\delta S_\tau\left[\lambda^{\frac{D-2}{2}}\phi(\lambda x)\right]\Big|_{\lambda=1}. \quad (2.4.33)$$

The eigenoperators are given just given by a product of the field ϕ with or without derivatives, such as $\phi(x)^n$, $(\partial^2)^m\phi(x)$ and $(\partial_\mu\phi(x))^2$, and the eigenvalue is the mass dimension of their couplings. For example, let us consider $\int_x \phi(x)^n$. Operating $\hat{R}$ on this operator, we get

$$\hat{R}\left[\int_x \phi(x)^n\right] = -\frac{\partial}{\partial\lambda}\left[\int_x \left(\lambda^{\frac{D-2}{2}}\phi(\lambda x)\right)^n\right]\Big|_{\lambda=1} \quad (2.4.34)$$

$$= -\frac{\partial}{\partial\lambda}\lambda^{\frac{D-2}{2}n-D}\left[\int_x (\phi(x))^n\right]\Big|_{\lambda=1} \quad (2.4.35)$$

$$= \left(D - n\frac{D-2}{2}\right)\left[\int_x (\phi(x))^n\right]. \quad (2.4.36)$$

In the second equation, we changed the integration variable x to $\lambda^{-1}x$. This equation shows that the eigenvalue of $\int_x \phi(x)^n$ is $D - n(D-2)/2$. These operators are included in δS_τ as

$$\delta S_\tau = \sum_n c^{(n)}(\tau)\int_x \phi(x)^n \quad (2.4.37)$$

and, in the dimensionful notation, the mass dimension of δS_τ, x and ϕ are 0, -1 and $(D-2)/2$, respectively. Then, the mass dimension of the coupling $c^{(n)}(\tau)$ must be $D -$

$n(D-2)/2$ and agrees with the eigenvalue. It is straightforward to apply the above discussion to operators with any derivatives.

The sign of the mass dimension of its couplings determines whether a scaling operator is relevant or irrelevant. For example, consider the four-dimensional case ($D=4$) and rotational invariant operators. There, relevant operators are given by $\phi(x), \phi(x)^2, \partial^2\phi(x)$, and $\phi(x)^3$. The marginal ones are the kinetic term $(\partial_\mu\phi(x))^2$ and the quartic interaction $\phi(x)^4$. Others are all irrelevant. Although we have givens operators that respect the rotational invariance, there are additional relevant/marginal operators if we do not care about this symmetry.

2.4.2 Wilson-Fisher fixed point

In this section, we study the RG flow structure of the WP equation around the WF fixed point using the local potential approximation.

Within this approximation, S_τ takes the following form:

$$S_\tau = S^* + \delta S(\tau), \tag{2.4.38}$$

where

$$\begin{cases} \delta S(\tau) = \displaystyle\int_x V \\ V = \displaystyle\sum_{n=2}^{N_{\max}} \frac{g_{2n}(\tau)}{2^n n!} (\phi_a(x)^2)^n. \end{cases} \tag{2.4.39}$$

Substituting these equations into the linearized WP equation, we get

$$\begin{aligned} \partial_\tau g_{2n} = (-2n+4+(n-1)\epsilon)g_{2n} + \frac{N+2n}{16\pi^2}\Big(1+\frac{\epsilon}{2}\log 4\pi\Big)g_{2n+2} \\ + 4nm_*^2 g_{2n} + 4n(n-1)\lambda_* g_{2n-2} \end{aligned} \tag{2.4.40}$$

to $\mathcal{O}(\epsilon)$. For simplicity, let us define χ_n as

$$\chi_n(\tau) := \left(\frac{N}{32\pi^2}\right)^{n-1} g_{2n}(\tau), \tag{2.4.41}$$

and Eq. (2.4.40) is rewritten in terms of them as

$$\partial_\tau \chi_n = \hat{R}_{nm}\chi_m, \tag{2.4.42}$$

where

$$\hat{R}_{nm} := A_{nm} + \epsilon B_{nm}, \tag{2.4.43}$$

$$A_{nm} := 2(2-n)\delta_{n,m} + 2\frac{N+2n}{N}\delta_{n+1,m}, \tag{2.4.44}$$

$$B_{nm} := \left(\frac{2(N+5)}{N+8}n-1\right)\delta_{n,m} - \frac{N}{N+8}n(n-1)\delta_{n-1,m} + \log 4\pi\frac{N+2n}{N}\delta_{n+1,m}. \quad (2.4.45)$$

Note that the eigenvalues of $\hat{R}_{nm}$ do not change under the above transformation Eq. (2.4.41).

Let us calculate eigenvalues of $\hat{R}_{nm}$ by the perturbation theory with respect to ϵ. The left- and right-eigenvector of A_{nm} of the eigenvalue 2 is given by

$$v_1^L = (1, 0, \ldots, 0), \quad (2.4.46)$$
$$v_1^R = \big(1, (v_1^R)_n\big), \quad (2.4.47)$$

where

$$(v_1^R)_n = \prod_{m=2}^{n} \frac{1}{m-1}\frac{N+2m-2}{N} \quad (2.4.48)$$

for $n \geq 2$. Note that $v_1^R \cdot v_1^L = 1$ and v_1^L corresponds to the operator $\frac{1}{2}\phi_a^2$. Then, the eigenvalue of $\hat{R}_{nm}$ at $\mathcal{O}(\epsilon)$ is calculated as

$$2 + \epsilon\frac{v_2^R \cdot B \cdot v_2^L}{v_2^R \cdot v_2^L} = 2 - \frac{N+2}{N+8}\epsilon. \quad (2.4.49)$$

Next, let us calculate the correction for the eigenvalue 0. The left- and right-eigenvector of A_{nm} with the eigenvalue 0 is given by

$$v_2^L = \left(-\frac{N+2}{N}, 1, 0\ldots, 0\right), \quad (2.4.50)$$
$$v_2^R = \big(0, 1, (v_2^R)_n\big), \quad (2.4.51)$$

where

$$(v_2^R)_n = \prod_{m=3}^{n} \frac{1}{m-2}\frac{N+2m-2}{N} \quad (2.4.52)$$

for $n \geq 3$. Note that $v_2^R \cdot v_2^L = 1$ and v_2^L corresponds to the operator $\frac{1}{8}(\phi_a^2)^2 - \frac{N+2}{64\pi^2}\phi_a^2$. Then, the correction to the eigenvalue 0 at $\mathcal{O}(\epsilon)$ is given by

$$0 + \epsilon\frac{v_2^R \cdot B \cdot v_2^L}{v_2^R \cdot v_2^L} = -\epsilon. \quad (2.4.53)$$

Finally, let us calculate the correction for the eigenvalue -2. The left- and right-eigenvector of A_{nm} with the eigenvalue -2 is given by

$$v_3^L = \left(\frac{(N+2)(N+4)}{2N^2}, -\frac{N+4}{N}, 1, 0\ldots, 0\right) \quad (2.4.54)$$
$$v_3^R = \big(0, 0, 1, (v_3^R)_n\big), \quad (2.4.55)$$

where

$$(v_3^R)_n = \prod_{m=4}^{n} \frac{1}{m-3} \frac{N+2m-2}{N} \tag{2.4.56}$$

for $n \geq 4$. Note that $v_3^R \cdot v_3^L = 1$ and v_3^L corresponds to the operator $\frac{(N+2)(N+4)}{(64\pi^2)^2}\phi_a^2 - \frac{N+4}{256\pi^2}(\phi_a^2)^2 + \frac{1}{48}(\phi_a^2)^3$. Then, the correction for the eigenvalue -2 at $\mathcal{O}(\epsilon)$ is

$$-2 + \epsilon \frac{x_2^R \cdot B \cdot x_2^L}{x_2^R \cdot x_2^L} = -2 - \frac{N-26}{N+8}\epsilon. \tag{2.4.57}$$

Note that these results hold for any number of the truncation level N_{max}.

Chapter 3

Review of Gradient Flow Exact Renormalization Group

We review Gradient Flow Exact Renormalization Group (GFERG) [7–10, 24–26] in this chapter.

3.1 What is GFERG?

GFERG is a method to define the Wilsonian effective action based on a diffusion equation. This diffusion equation is called "Gradient Flow", initially proposed in the context of lattice simulations of QFT [27–31].

The motivation of GFERG is the problem of treating QFTs with gauge symmetry in ERG. Naively speaking, introducing a UV cutoff conflicts with local symmetry. To see this, let us consider the following infinitesimal gauge transformation of fields as

$$\phi_i(x) \to \phi_i(x) + \lambda(x)\delta\phi_i(x), \tag{3.1.1}$$

where $\lambda(x)$ is the gauge transformation parameter. Then, the gauge transformation of S_k is given by

$$\delta S_k = \delta S_0 - \delta\left(\frac{1}{2}\int_q \phi_i(-q)R_k(q)\phi_i(-q)\right) = -\int_q \phi_i(-q)R_k(q)\delta\phi_i(-q). \tag{3.1.2}$$

We used the gauge-invariance of the undeformed action S_0 ($\delta S_0 = 0$). For a generic cutoff function R_k, this δS_k does not vanish. This means that introducing the UV cutoff breaks the gauge symmetry in general.

The idea of GFERG is that the solution to the Wilson-Polchinski equation is given by

$$e^{S_\tau[\phi]} = \hat{s}^{-1}\int D\phi' \prod_x \delta\left(\phi(x) - e^{\int_{\tau_0}^{\tau} d\tau'((D-2)/2+\gamma_{\tau'})}\phi'(t-t_0, xe^{\tau-\tau_0})\right)\hat{s}e^{S_{\tau_0}[\phi']}, \tag{3.1.3}$$

where $\phi'(t,x)$ is the solution to the following diffusion equation:

$$\partial_t \phi'(t,x) = \partial_x^2 \phi'(t,x) \tag{3.1.4}$$

with $\phi'(0,x) = \phi'(x)$ and $t - t_0 = e^{2(\tau-\tau_0)} - 1$. Recall that $\hat{s}$ is the scrambler, defined as

$$\hat{s} := \exp\left(-\frac{1}{2}\int_p \frac{k(p^2)}{p^2} \frac{\delta}{\delta\phi(p)} \frac{\delta}{\delta\phi(-p)}\right). \tag{3.1.5}$$

This representation implies that the coarse-graining by diffusion can be used to define an RG flow.

This one-parameter deformation of fields via the diffusion equation has been studied in the context of "gradient flow". The gradient flow is a method to construct composite operators without equal-point singularity. It has been shown by Ref. [13] that correlation functions of the flowed field are UV finite with wave function renormalization factor Z_τ:

$$Z_\tau^{-n/2} \langle \phi(t,x_1) \dots \phi(t,x_n) \rangle_\phi < \infty \tag{3.1.6}$$

even for the equal point case (e.g. $x_1 = x_2$).

The gradient flow equation for gauge fields are proposed by Refs. [27–30]. It is given by

$$\partial_t A'_\mu(t,x) = D'_\nu F'_{\nu\mu}(t,x) + \alpha_0 D'_\mu \partial_\nu A'_\nu(t,x), \quad A'_\mu(0,x) = A'_\mu(x), \tag{3.1.7}$$

where

$$D'_\nu F'_{\nu\mu}(t,x) := \partial_\nu F'_{\nu\mu}(t,x) + g_{\tau_0}\left[A'_\nu(t,x), F'_{\nu\mu}(t,x)\right], \tag{3.1.8}$$

$$F'_{\nu\mu}(t,x) := \partial_\nu A'_\mu(t,x) - \partial_\mu A'_\nu(t,x) + g_{\tau_0}\left[A'_\nu(t,x), A'_\mu(t,x)\right]. \tag{3.1.9}$$

$$D'_\mu \partial_\nu A'_\nu(t,x) := \partial_\mu \partial_\nu A'_\nu(t,x) + g_{\tau_0}\left[A'_\mu(t,x), \partial_\nu A'_\nu(t,x)\right] \tag{3.1.10}$$

and g_{τ_0} is the gauge coupling at $\tau = \tau_0$. [1] Because the first term $D'_\nu F'_{\nu\mu}$ on the right-hand side is expanded as

$$D'_\nu F'_{\nu\mu} = \partial_x^2 A'_\mu + \cdots, \tag{3.1.11}$$

the above gradient flow equation Eq. (3.4.1) is regarded as a gauge-covariant diffusion equation. Luscher and Weiss showed that the correlation functions of the flowed field $A_\mu(t,x)$ are UV finite even without additional wavefunction renormalization, once the original ($t = 0$) theory is renormalized.

[1]It should be noted that the variables in the context of gradient flow such as A_μ and x are dimensionful, while they are dimensionless in the context of ERG.

3.2 GFERG for gauge fields

It has been proposed by Ref. [8] to define a Wilsonian effective action via the gradient flow equation Eq. (3.4.1) by

$$e^{S_\tau[A_\mu]} = \hat{s}^{-1} \int DA'_\mu \prod_{x,\mu} \delta\Big(A_\mu(x) - e^{\int_{\tau_0}^{\tau} d\tau'((D-2)/2+\gamma_{\tau'})} A'_\mu(t-t_0, xe^{\tau-\tau_0})\Big) \hat{s} e^{S_{\tau_0}[A'_\mu]}, \tag{3.2.1}$$

where $k(p^2)$ is taken as p^2. They called this framework "Gradient Flow Exact Renormalization Group (GFERG)".

The advantage of GFERG is that it can define an RG flow that preserves the theory's symmetry or non-linearity. For example, S_τ is manifestly gauge-invariant at an arbitrary scale parameter τ. Their discussion is given in the following.

Let us consider the following infinitesimal gauge-transformation:

$$\delta A_\mu(x) = \partial_\mu \chi(x) + g_\tau [A_\mu(x), \chi(x)], \tag{3.2.2}$$

equivalently,

$$\delta A^a_\mu(x) = \partial_\mu \chi^a(x) + g_\tau f_{abc} A^a_\mu(x) \chi^b(x), \tag{3.2.3}$$

where $\chi(x)$ is the gauge-transformation parameter, f_{abc} are structure constants of the gauge Group G[2], and g_τ is the gauge coupling at τ defined as

$$g_\tau \coloneqq g_{\tau_0} e^{-\int_{\tau_0}^{\tau} d\tau'((D-4)/2+\gamma_{\tau'})}. \tag{3.2.4}$$

The Wilsonian effective action transforms under this gauge transformation as

$$e^{S_\tau[A_\mu+\delta A_\mu]} = \hat{s} \int DA'_\mu \prod_{x,\mu} \delta\Big(A_\mu(x) + \delta A_\mu(x) - e^{\int_{\tau_0}^{\tau} d\tau'((D-2)/2+\gamma_{\tau'})} A'_\mu(t-t_0, xe^{\tau-\tau_0})\Big) \hat{s} e^{S_{\tau_0}[A'_\mu]}. \tag{3.2.5}$$

Note that scrambler is invariant under the gauge-transformation. This is because the functional derivative transforms as

$$\frac{\delta}{\delta A^a_\mu(x)} \to \frac{\delta}{\delta\big(A^a_\mu(x) + \partial_\mu \chi^a(x) + g_\tau f_{abc} A^b_\mu(x) \chi^c(x)\big)} = (\delta_{ab} - g_\tau f_{abc} \chi_c(x)) \frac{\delta}{\delta A^b_\mu(x)} + \mathcal{O}\big(\chi^2\big), \tag{3.2.6}$$

and then

[2] We adopt the convention where generators T^a of Lie algebra of G are anti-hermitian and f_{abc} is defined as $[T^a, T^b] = f_{abc} T^c$.

$$\hat{s} = \exp\left(-\frac{1}{2}\int \frac{\delta}{\delta A^a_\mu(x)}\frac{\delta}{\delta A^a_\mu(x)}\right)$$
$$\to \exp\left(-\frac{1}{2}\int (\delta_{ab} - g_\tau f_{abc}\chi_c(x))(\delta_{ad} - g_\tau f_{ade}\chi_e(x))\frac{\delta}{\delta A^b_\mu(x)}\frac{\delta}{\delta A^d_\mu(x)}\right) + \mathcal{O}(\chi^2)$$
$$= \exp\left(-\frac{1}{2}\int \frac{\delta}{\delta A^a_\mu(x)}\frac{\delta}{\delta A^a_\mu(x)}\right) + \mathcal{O}(\chi^2). \quad (3.2.7)$$

On the other hand, the delta function is rewritten as

$$\delta\Big(A_\mu(x) + \delta A_\mu(x) - e^{\int_{\tau_0}^{\tau} d\tau'((D-2)/2+\gamma_{\tau'})} A'_\mu(t-t_0, xe^{\tau-\tau_0})\Big)$$
$$= \delta\left(A_\mu(x) + \partial_\mu\chi(x) + g_\tau[A_\mu(x), \chi(x)] - \frac{g_{\tau_0}}{g_\tau}e^{\tau-\tau_0}A'_\mu(t-t_0, xe^{\tau-\tau_0})\right) \quad (3.2.8)$$
$$= \delta\left(A_\mu(x) + \partial_\mu\chi(x) + g_{\tau_0}e^{\tau-\tau_0}\left[A'_\mu(t-t_0, xe^{\tau-\tau_0}), \chi(x)\right] - \frac{g_{\tau_0}}{g_\tau}e^{\tau-\tau_0}A'_\mu(t-t_0, xe^{\tau-\tau_0})\right) \quad (3.2.9)$$
$$= \delta\left(A_\mu(x) - \frac{g_{\tau_0}}{g_\tau}e^{\tau-\tau_0}\left(A'_\mu(t, xe^{\tau-\tau_0}) - \partial_\mu\hat{\chi}(xe^{\tau-\tau_0}) - g_{\tau_0}\left[A'_\mu(t-t_0, xe^{\tau-\tau_0}), \hat{\chi}(xe^{\tau-\tau_0})\right]\right)\right). \quad (3.2.10)$$

In the second equality, we used the fact that the solution of the delta function in the second line is given by $A_\mu(x) = \frac{g_{\tau_0}}{g_\tau}e^{\tau-\tau_0}A'_\mu(t, xe^{\tau-\tau_0}) + \mathcal{O}(\chi)$ and the difference of the arguments of the delta function of the second and third line is $\mathcal{O}(\chi^2)$. In the last equality, we define $\hat{\chi}(x)$ as

$$\hat{\chi}(xe^{\tau-\tau_0}) := \frac{g_\tau}{g_{\tau_0}}\chi(x). \quad (3.2.11)$$

From Eq. (3.2.10), we see that the gauge transformation for $A_\mu(x)$ with the parameter $\chi(x)$ is equivalent to the following transformation of $A'(t-t_0, xe^{\tau-\tau_0})$:

$$A'(t-t_0, xe^{\tau-\tau_0}) \to A'_\mu(t, xe^{\tau-\tau_0}) - \partial_\mu\hat{\chi}(xe^{\tau-\tau_0}) - g_{\tau_0}\left[A'_\mu(t-t_0, xe^{\tau-\tau_0}), \hat{\chi}(xe^{\tau-\tau_0})\right], \quad (3.2.12)$$

or, in the dimensionful notation,

$$A'(t-t_0, x) \to A'_\mu(t, x) - \partial_\mu\hat{\chi}(x) - g_{\tau_0}\left[A'_\mu(t-t_0, x), \hat{\chi}(x)\right]. \quad (3.2.13)$$

This is nothing but the gauge-transformation of $A'(t-t_0, x)$ with the gauge-transformation paremeter $-\hat{\chi}(x)$.

In fact, this gauge-transformation of $A'(t-t_0, x)$ can be canceled by a gauge-transformation of the initial value $A'(x)$, which is discussed in the following. For notational convenience, we denote the solution to the gradient flow equation Eq. (3.4.1) at the flow time s with an general initial condition $A(0, x) = B_\mu(x)$ as $A_\mu[s, x; B_\mu]$. Let us study the solution to the gradient flow equation when we perturb initial value

$A(x)$ to $A(x) + \partial_\mu \xi(x) + g_{\tau_0}[A_\mu(x), \xi(x)]$. The solution at the flow time s is denoted as $A_\mu[s, x; A_\mu + \partial_\mu \xi + g_{\tau_0}[A_\mu, \xi]]$ with our notation, and is decomposed as,

$$A_\mu[s, x; A_\mu + \partial_\mu \xi + g_{\tau_0}[A_\mu, \xi]] = A_\mu[s, x; A_\mu] + \partial_\mu \xi(s, x) + g_{\tau_0}[A_\mu[s, x; A_\mu], \xi(s, x)]. \tag{3.2.14}$$

Substituting this ansatz into Eq. (3.4.1), we find that $\xi(t, x)$ should satisfy

$$0 = D'_\mu(\partial_s - \alpha_0 D'_\nu \partial_\nu)\xi(s, x), \tag{3.2.15}$$

where D'_μ is the covariant derivative with $A_\mu(s, x)$, defined in Eq. (3.1.10). This equation gives how the gauge-transformation parameter develops along the gradient flow. Especially, if we require that $\xi(s, x)$ should be the solution to

$$\partial_s \xi(s, x) = \alpha_0 D'_\nu \partial_\nu \xi(s, x) \tag{3.2.16}$$

with the boundary condition

$$\xi(t - t_0, x) = \hat{\chi}(x), \tag{3.2.17}$$

we get from Eq. (3.2.14) at $s = t - t_0$

$$\begin{aligned} &A_\mu[t - t_0, x; A_\mu + \partial_\mu \xi + g_{\tau_0}[A_\mu, \xi]] \\ &\qquad = A_\mu[t - t_0, x; A_\mu] + \partial_\mu \hat{\chi}(x) + g_{\tau_0}[A_\mu[t - t_0, x; A_\mu], \hat{\chi}(x)]. \end{aligned} \tag{3.2.18}$$

This equation means that if we perform the gauge-transformation of $A'(x)$ with the parameter $\xi(0, x)$, $A'(t - t_0, x)$ receives that with the parameter $\hat{\chi}(x)$. Finally, we see from Eq. (3.2.13) and Eq. (3.2.18) that the gauge-transformation of A_μ is canceled by that of $A'(x)$ with the parameter $\xi_0(x) := \xi(0, x)$.

Using this fact, we can rewrite Eq. (3.2.5) as

$$\begin{aligned} e^{S_\tau[A_\mu + \delta A_\mu]} = \hat{s} \int DA'_\mu \prod_{x,\mu} \delta\Big(A_\mu(x) - e^{\int_{\tau_0}^{\tau} d\tau' ((D-2)/2 + \gamma_{\tau'})} A'_\mu(t - t_0, x e^{\tau - \tau_0})\Big) \\ \times \hat{s} e^{S_{\tau_0}\left[A'_\mu + \partial_\mu \xi_0 + g_{\tau_0}\left[A'_\mu, \xi_0\right]\right]}. \end{aligned} \tag{3.2.19}$$

We assumed that the path-integral measure DA'_μ is gauge-invariant and used the gauge-invariance of the scrambler. If the bare action $e^{S_{\tau_0}}$ is gauge-invariant, i.e., S_{τ_0} satisfies $S_{\tau_0}\left[A'_\mu\right] = S_{\tau_0}\left[A'_\mu + \partial_\mu \xi_0 + g_{\tau_0}\left[A'_\mu, \xi_0\right]\right]$, then

$$\begin{aligned} &e^{S_\tau[A_\mu + \delta A_\mu]} \\ &= \hat{s}^{-1} \int DA'_\mu \prod_{x,\mu} \delta\Big(A_\mu(x) - e^{\int_{\tau_0}^{\tau} d\tau' ((D-2)/2 + \gamma_{\tau'})} A'_\mu(t - t_0, x e^{\tau - \tau_0})\Big) \hat{s} e^{S_{\tau_0}[A'_\mu]} \qquad (3.2.20) \\ &= e^{S_\tau[A_\mu]}, \qquad (3.2.21) \end{aligned}$$

that is, $S_\tau[A_\mu + \delta A_\mu] = S_\tau[A_\mu]$ holds at an arbitrary scale parameter τ. This shows the manifiest gauge-invariance of the Wilsonian effective action defined via GFERG.

3.3 GFERG equation

We derive the flow equation of the Wilsonian effective action, called "GFERG equation", corresponding to the ERG equation in ERG. For Yang-Mills theory, it is given by

$$\begin{aligned}
&\partial_\tau e^{S_\tau[A_\mu]} \\
&= \int_x \frac{\delta}{\delta A_\mu}\left[-2D_\nu F_{\nu\mu}(x) - 2\alpha_0 D_\mu \partial_\nu A_\nu(x) + \left(-\frac{D-2}{2} - \gamma_\tau + x_\mu \partial_\mu\right) A_\mu(x)\right]\Bigg|_{A_\mu \to A_\mu + \frac{\delta}{\delta A_\mu}} \\
&\qquad \times e^{S_\tau[A_\mu]}. \qquad (3.3.1)
\end{aligned}$$

Let us derive the GFERG equation. Differentiating Eq. (3.2.1) with regard to τ, we get

$$\begin{aligned}
&\partial_\tau e^{S_\tau} \\
&= \hat{s}^{-1} \int DA'_\mu \prod_{y,\rho} \partial_\tau \delta\Big(A_\rho(y) - e^{\int_{\tau_0}^{\tau} d\tau' ((D-2)/2 + \gamma_{\tau'})} A'_\rho(t - t_0, y e^{\tau - \tau_0})\Big) \hat{s} e^{S_{\tau_0}[A'_\mu]} \qquad (3.3.2) \\
&= \hat{s}^{-1} \int DA'_\mu \int_x \frac{\delta}{\delta A_\mu(x)} \Bigg\{ \left[\left(-\frac{D-2}{2} - \gamma_\tau - e^{\tau - \tau_0} x \cdot \partial_x - \frac{\mathrm{d}t}{\mathrm{d}\tau}\partial_t\right) A'_\mu(t - t_0, x e^{\tau - \tau_0})\right] \\
&\qquad \times e^{\int_{\tau_0}^{\tau} d\tau' ((D-2)/2 + \gamma_{\tau'})} \prod_{y,\rho} \delta\Big(A_\rho(y) - e^{\int_{\tau_0}^{\tau} d\tau' ((D-2)/2 + \gamma_{\tau'})} A'_\rho(t - t_0, y e^{\tau - \tau_0})\Big) \hat{s} e^{S_{\tau_0}[A'_\mu]} \Bigg\} \\
&\qquad\qquad (3.3.3) \\
&= \hat{s}^{-1} \int DA'_\mu \int_x \frac{\delta}{\delta A_\mu(x)} \Bigg\{ \Bigg[\left(-\frac{D-2}{2} - \gamma_\tau - e^{\tau - \tau_0} x \cdot \partial_x\right) A'_\mu(t - t_0, x e^{\tau - \tau_0}) \\
&\qquad - 2 e^{2(\tau - \tau_0)} \big(D_\nu F_{\nu\mu}(t - t_0, x e^{\tau - \tau_0}) + \alpha_0 D_\mu \partial_\nu A_\nu(t - t_0, x e^{\tau - \tau_0})\big)\Bigg] \qquad (3.3.4) \\
&\qquad\qquad \times \prod_{y,\rho} \delta\Big(A_\rho(y) - e^{\int_{\tau_0}^{\tau} d\tau' ((D-2)/2 + \gamma_{\tau'})} A'_\rho(t - t_0, y e^{\tau - \tau_0})\Big) \hat{s} e^{S_{\tau_0}[A'_\mu]} \Bigg\} \\
&= \hat{s}^{-1} \int DA'_\mu \\
&\times \int_x \frac{\delta}{\delta A_\mu(x)} \Bigg\{ \left[\left(-\frac{D-2}{2} - \gamma_\tau - x \cdot \partial_x\right) A_\mu(x) - 2D_\nu F_{\nu\mu}(x) - 2\alpha_0 D_\mu \partial_\nu A_\nu(x))\right] \\
&\qquad\qquad \times \prod_{y,\rho} \delta\Big(A_\rho(y) - e^{\int_{\tau_0}^{\tau} d\tau' ((D-2)/2 + \gamma_{\tau'})} A'_\rho(t - t_0, y e^{\tau - \tau_0})\Big) \hat{s} e^{S_{\tau_0}[A'_\mu]} \Bigg\} \\
&\qquad\qquad (3.3.5) \\
&= \hat{s}^{-1} \int_x \frac{\delta}{\delta A_\mu(x)} \left[\left(-\frac{D-2}{2} - \gamma_\tau - x \cdot \partial_x\right) A_\mu(x) - 2D_\nu F_{\nu\mu}(x) - 2\alpha_0 D_\mu \partial_\nu A_\nu(x))\right] \hat{s} \\
&\qquad \times \hat{s}^{-1} \int DA'_\mu \prod_{y,\rho} \delta\Big(A_\rho(y) - e^{\int_{\tau_0}^{\tau} d\tau' ((D-2)/2 + \gamma_{\tau'})} A'_\rho(t - t_0, y e^{\tau - \tau_0})\Big) \hat{s} e^{S_{\tau_0}[A'_\mu]} \\
&\qquad\qquad (3.3.6)
\end{aligned}$$

$$= \hat{s}^{-1} \int_x \frac{\delta}{\delta A_\mu(x)} \left[\left(-\frac{D-2}{2} - \gamma_\tau - x \cdot \partial_x \right) A_\mu(x) - 2D_\nu F_{\nu\mu}(x) - 2\alpha_0 D_\mu \partial_\nu A_\nu(x)) \right] \hat{s} \times e^{S_\tau[A_\mu]}, \tag{3.3.7}$$

where

$$D_\nu F_{\nu\mu}(x) := \partial_\nu F_{\nu\mu}(x) + g_\tau[A_\nu(x), F_{\nu\mu}(x)], \tag{3.3.8}$$

$$F_{\nu\mu}(x) := \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x) + g_\tau[A_\nu(x), A_\mu(x)]. \tag{3.3.9}$$

$$D_\mu \partial_\nu A_\nu(x) := \partial_\mu \partial_\nu A_\nu(x) + g_\tau[A_\mu(x), \partial_\nu A_\nu(x)]. \tag{3.3.10}$$

Note that the covariant derivative D_μ or the field strength $F_{\mu\nu}$ without prime ($'$) are defined with respect to the gauge coupling g_τ at the scale parameter τ, distinguished from those with the prime defined by Eq. (3.1.10).

Furthermore we can easily show that

$$\hat{s}^{-1} A_\mu(x) \hat{s} = A_\mu(x) + \frac{\delta}{\delta A_\mu(x)}, \tag{3.3.11}$$

and with this relation in Eq. (3.3.7), we get the GFERG equation Eq. (3.3.1) for the gauge fields.

3.4 Recent Developments

In this section, we briefly review recent developments on GFERG.

3.4.1 Inclusion of fermion fields

Inclusion of fermionic field in the framework of GFERG is discussed in [9]. The Wilsonian effective action including the fermion field ψ is defined as

$$\begin{aligned} e^{S_\tau[A,\psi,\bar{\psi}]} := \hat{s}^{-1} \int & [DA' D\bar{\psi}' D\psi'] \\ & \times \prod_{x,\mu,a} \delta\Big(A^a_\mu(x) - e^{\int_{\tau_0}^{\tau} d\tau' [(D-2)/2 + \gamma_{\tau'}]} A^{a\prime}_{\ \mu}(t - t_0, e^{\tau-\tau_0} x) \Big) \\ & \times \prod_x \delta\Big(\psi(x) - e^{\int_{\tau_0}^{\tau} d\tau' [(D-1)/2 + \gamma_{F\tau'}]} \psi'(t - t_0, e^{\tau-\tau_0} x) \Big) \\ & \times \prod_x \delta\Big(\bar{\psi}(x) - e^{\int_{\tau_0}^{\tau} d\tau' [(D-1)/2 + \gamma_{F\tau'}]} \bar{\psi}'(t - t_0, e^{\tau-\tau_0} x) \Big) \\ & \times \hat{s} e^{S_{\tau_0}[A',\psi'\bar{\psi}']}, \end{aligned} \tag{3.4.1}$$

where $t - t_0 = e^{2(\tau-\tau_0)} - 1$. The scrambler $\hat{s}$ is defined as

$$\hat{s} := \exp\left[-\int_x \frac{1}{2} \frac{\delta^2}{\delta A_\mu(x)^2} \right] \exp\left[i \int_x \frac{\overrightarrow{\delta}}{\delta \bar{\psi}(x)} \frac{\overrightarrow{\delta}}{\delta \psi(x)} \right], \tag{3.4.2}$$

where $\overrightarrow{\delta}/\delta\psi(x)$ and $\overrightarrow{\delta}/\delta\bar{\psi}(x)$ are the left-functional derivative in terms of them. While the flow equation for the gauge field $A_\mu(x)$ is the same as , that for the fermion field $\psi(x)$ is given by

$$\partial_t\psi'(t,x) = \big(D'_\mu D'_\mu - \alpha_0\partial_\mu A'_\mu(t,x)\big)\psi'(t,x), \quad \psi'(0,x) = \psi'(x). \tag{3.4.3a}$$

The GFERG equation for this Wilsonian effective action $S_\tau[A,\psi]$ is given by

$$\begin{aligned}
&\partial_\tau e^{S_\tau[A_\mu]} = \\
&\quad \mathrm{tr}\int_x \Bigg\{ \frac{\delta}{\delta A_\mu(x)}\left[-2D_\nu F_{\nu\mu}(x) - 2\alpha_0 D_\mu\partial_\nu A_\nu(x) + \left(-\frac{D-2}{2} - \gamma_\tau - x_\mu\partial_\mu\right)A_\mu(x)\right] \\
&\qquad + \frac{\overrightarrow{\delta}}{\delta\psi(x)}\left[2D_\mu D_\mu - 2\alpha_0\partial_\mu A_\mu(x) + \frac{D-1}{2} + \gamma_{F\tau} + x_\mu\partial_\mu\right]\psi(x) \\
&\qquad + \frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\left[2D^*_\mu D^*_\mu + 2\alpha_0\partial_\mu A_\mu(x) + \frac{D-1}{2} + \gamma_{F\tau} + x_\mu\partial_\mu\right]\bar{\psi}(x) \\
&\qquad \Bigg\}\Bigg|_{A_\mu\to A_\mu+\frac{\delta}{\delta A_\mu},\psi\to\psi+i\frac{\overrightarrow{\delta}}{\delta\bar{\psi}},\bar{\psi}\to\bar{\psi}-i\frac{\overrightarrow{\delta}}{\delta\psi}} e^{S_\tau[A_\mu]}.
\end{aligned} \tag{3.4.4}$$

The operations $\psi \to \psi + i\frac{\overrightarrow{\delta}}{\delta\bar{\psi}}$ and $\bar{\psi} \to \bar{\psi} - i\frac{\overrightarrow{\delta}}{\delta\psi}$ come from the following identity:

$$\hat{s}^{-1}\psi(x)\hat{s} = \psi(x) + i\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}, \tag{3.4.5a}$$

$$\hat{s}^{-1}\bar{\psi}(x)\hat{s} = \bar{\psi}(x) - i\frac{\overrightarrow{\delta}}{\delta\psi(x)}. \tag{3.4.5b}$$

They also discuss the chiral symmetry of the Wilsonian effective action defined via GFERG. One of their novel results is that the chiral symmetry generated by

$$\hat{\gamma}_5 := \int_x \left[\gamma_5\psi(x)\frac{\overrightarrow{\delta}}{\delta\psi(x)} + \bar{\psi}(x)\gamma_5\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\right], \tag{3.4.6}$$

is *not* preserved along the GFERG flow. It is because the scrambler $\hat{s}$ does not commute with $\hat{\gamma}_5$, while so do the flow equations. Instead, the Wilsonian effective action preserves the modified chiral transformation generated by

$$\begin{aligned}
\hat{\Gamma}_5 &:= \hat{s}^{-1}\gamma_5\hat{s} \\
&= \int_x \left[\gamma_5\left(\psi(x) + i\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\right)\frac{\overrightarrow{\delta}}{\delta\psi(x)} + \left(\bar{\psi}(x) - i\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right)\gamma_5\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\right].
\end{aligned} \tag{3.4.7}$$

More specifically, if the bare action S_{τ_0} is invariant under the modified chiral transformation ($\hat{\Gamma}_5 e^{S_{\tau_0}} = 0$), so does the Wilsonian effective action S_τ at an arbitrary scale parameter:

$$\hat{\Gamma}_5 e^{S_\tau} = 0. \tag{3.4.8}$$

This condition gives the following constraint on the Wilsonian effective action S_τ:

$$\begin{aligned}
0 &= e^{-S_\tau}\hat{\Gamma}_5 e^{S_\tau} \\
&= \int_x \Bigg[S_\tau \frac{\overleftarrow{\delta}}{\delta\psi(x)}\gamma_5\psi(x) + \bar{\psi}(x)\gamma_5\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S_\tau \\
&\quad + 2iS_\tau\frac{\overleftarrow{\delta}}{\delta\psi(x)}\gamma_5\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S_\tau - 2i\,\mathrm{tr}\left(\gamma_5\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S_\tau\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right)\Bigg].
\end{aligned} \tag{3.4.9}$$

This constraint is non-linear concerning the Wilsonian effective action S_τ; However, when the fermionic part of the Wilsonian effective action S_τ is just given by the bilinear of the fermion fields $\bar{\psi}(x), \psi(x)$ as

$$S_\tau = \int_x \bar{\psi}(x)D\psi(x) + (\text{terms only with } A_\mu), \tag{3.4.10}$$

Eq. (3.4.9) is calculated as

$$\gamma_5 D + D\gamma_5 + 2D\gamma_5 D = 0. \tag{3.4.11}$$

This equation resembles the famous Ginsparg-Wilson relation [32], a condition to keep the following modified chiral transformation on the lattice:

$$\delta\psi(x) = \gamma_5(1-D)\psi(x), \quad \delta\bar{\psi}(x) = \bar{\psi}(x)\gamma_5(1-D). \tag{3.4.12}$$

It is natural for the constraint to be reduced to the GW relation because GFERG effectively introduces a UV cutoff Λ corresponding to the inverse of the lattice spacing a.

To derive Eq. (3.4.11), we neglected the last term in Eq. (3.4.9) because it gives just a constant term. However, this term corresponds to the chiral anomaly [33, 34] in four-dimensional spacetime. The detailed calculation is given in [25].

3.4.2 Perturbative analysis of QED

Perturbative analysis of Quatum Elecetrodynamics is studied in [10]. In this work, they perform the gauge-fixing and introduce the ghost field $c(x)$, the anti-ghost field $\bar{c}(x)$ and the Nakanishi-Lautrap field $B(x)$. Then after integrating out the NL field, they solve the

GFERG equation order by order of the electric coupling e_τ. The Wilson aciton is defined as

$$
\begin{aligned}
e^{S_\tau[A,\bar{c},c,\psi,\bar{\psi}]} := \hat{s}^{-1} \int [DA'D\bar{c}'Dc'D\bar{\psi}'D\psi'] \\
\times \prod_{x,\mu} \delta\Big(A_\mu(x) - e^{\int_{\tau_0}^{\tau} d\tau'[(D-2)/2+\gamma_{\tau'}]} A'_\mu(t-t_0, e^{\tau-\tau_0}x)\Big) \\
\times \prod_{x} \delta\Big(c(x) - e^{\int_{\tau_0}^{\tau} d\tau'[(D-4)/2+\gamma_{\tau'}]} c'(t-t_0, e^{\tau-\tau_0}x)\Big) \\
\times \prod_{x} \delta\Big(\bar{c}(x) - e^{\int_{\tau_0}^{\tau} d\tau'[D/2-\gamma_{\tau'}]} \bar{c}'(t-t_0, e^{\tau-\tau_0}x)\Big) \\
\times \prod_{x} \delta\Big(\psi(x) - e^{\int_{\tau_0}^{\tau} d\tau'[(D-1)/2+\gamma_{F\tau'}]} \psi'(t-t_0, e^{\tau-\tau_0}x)\Big) \\
\times \prod_{x} \delta\Big(\bar{\psi}(x) - e^{\int_{\tau_0}^{\tau} d\tau'[(D-1)/2+\gamma_{F\tau'}]} \bar{\psi}'(t-t_0, e^{\tau-\tau_0}x)\Big) \\
\times \hat{s} e^{S_{\tau_0}[A',\bar{c}',c',\psi',\bar{\psi}']}.
\end{aligned}
\tag{3.4.13}
$$

The scrambler $\hat{s}$ here is defined as

$$
\hat{s} := \exp\left[-\int_x \frac{1}{2}\frac{\delta^2}{\delta A_\mu(x)^2}\right] \exp\left[\int_x \frac{\delta}{\delta c(x)}\frac{\delta}{\delta \bar{c}(x)}\right] \exp\left[i\int_x \frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\frac{\overrightarrow{\delta}}{\delta\psi(x)}\right], \tag{3.4.14}
$$

Note that the gauge group is not $SU(N)$ but $U(1)$ and so $A_\mu(x)$ has no indices for internal degrees of freedom. The gradient flow equations for the fields are given by

$$\partial_t A'_\mu(t,x) = \partial^2 A'_\mu(t,x), \tag{3.4.15a}$$
$$\partial_t c'(t,x) = \partial^2 c'(t,x), \tag{3.4.15b}$$
$$\partial_t \bar{c}'(t,x) = \partial^2 \bar{c}'(t,x), \tag{3.4.15c}$$
$$\partial_t \psi'(t,x) = \big(D'_\mu D'_\mu + ie_0\partial_\mu A'_\mu(t,x)\big)\psi'(t,x), \tag{3.4.15d}$$

where e_0 is an arbitrary real number, regarded as the electric charge of the fermion field $\psi'(x)$, and $D'_\mu := \partial_\mu - ie_0 A'_\mu(t,x)$. These flow equation are consistent (i.e., $[\partial_t, \delta_B] = 0$) with the following on-shell BRST transformation:

$$\theta\delta_B A'_\mu(t,x) = \theta\partial_\mu c'(t,x), \tag{3.4.16a}$$
$$\theta\delta_B c'(t,x) = \theta\frac{1}{\xi_0}\partial_\mu A'_\mu(t,x), \tag{3.4.16b}$$
$$\theta\delta_B \bar{c}'(t,x) = 0, \tag{3.4.16c}$$
$$\theta\delta_B \psi'(t,x) = \theta ie_0 c'(t,x), \tag{3.4.16d}$$

where θ is an arbitrary Grassmann-odd number and ξ_0 is an arbitary number, regarded as the gauge-parameter in the R_ξ gauge.

The GFERG equation is given by

$$\partial_\tau e^{S_\tau[A_\mu]} = \mathrm{tr} \int_x \Bigg\{ \frac{\delta}{\delta A_\mu(x)} \Bigg[-2\partial^2 A_\mu(x) + \Bigg(-\frac{D-2}{2} - \gamma_\tau - x_\mu \partial_\mu \Bigg) A_\mu(x) \Bigg]$$
$$+ \frac{\delta}{\delta c(x)} \Bigg(2\partial^2 + \frac{D+4}{2} + \gamma_\tau + x_\mu \partial_\mu \Bigg) c(x) + \frac{\delta}{\delta \bar{c}(x)} \Bigg(2\partial^2 + \frac{D}{2} - \gamma_\tau + x_\mu \partial_\mu \Bigg) \bar{c}(x)$$
$$+ \frac{\overrightarrow{\delta}}{\delta \psi(x)} \Bigg(2D_\mu D_\mu + 2ie_0 \partial_\mu A_\mu(x) + \frac{D-1}{2} + \gamma_{F\tau} + x_\mu \partial_\mu \Bigg) \psi(x)$$
$$+ \frac{\overrightarrow{\delta}}{\delta \bar{\psi}(x)} \Bigg(2D^*_\mu D^*_\mu - 2ie_0 \partial_\mu A_\mu(x) + \frac{D-1}{2} + \gamma_{F\tau} + x_\mu \partial_\mu \Bigg) \bar{\psi}(x)$$
$$\Bigg\}\Bigg|_{A_\mu \to A_\mu + \frac{\delta}{\delta A_\mu}, \psi \to \psi + i\frac{\overrightarrow{\delta}}{\delta \bar{\psi}}, \bar{\psi} \to \bar{\psi} - i\frac{\overrightarrow{\delta}}{\delta \psi}, c \to c + \frac{\delta}{\delta \bar{c}}, \bar{c} \to \bar{c} - \frac{\delta}{\delta c}} e^{S_\tau[A_\mu]}, \quad (3.4.17)$$

where tr means the trace over the spinor indices, $D_\mu := \partial_\mu - ie_\tau A_\mu(x)$ and e_τ is the electric charge at the scale parameter τ, defined as

$$e_\tau := e_{\tau_0} \exp\Bigg(-\int_{\tau_0}^{\tau} d\tau' \Bigg(\frac{D-4}{2} + \gamma_{\tau'} \Bigg) \Bigg). \quad (3.4.18)$$

To derive this GFERG equation, we used the fact that

$$\hat{s}^{-1} c(x) \hat{s} = c(x) + \frac{\delta}{\delta \bar{c}(x)}, \quad (3.4.19a)$$
$$\hat{s}^{-1} \bar{c}(x) \hat{s} = \bar{c}(x) - \frac{\delta}{\delta c(x)}. \quad (3.4.19b)$$

This GFERG equation dose not preserve the BRST transformation, generated by

$$\hat{\delta} := \mathrm{tr} \int_x \Bigg[\partial_\mu c(x) \frac{\delta}{\delta A_\mu(x)} + \frac{1}{\xi_\tau} \partial_\mu A_\mu(x) \frac{\delta}{\delta c(x)}$$
$$+ ie_\tau c(x) \psi(x) \frac{\overrightarrow{\delta}}{\delta \psi(x)} - ie_\tau c(x) \bar{\psi}(x) \frac{\overrightarrow{\delta}}{\delta \bar{\psi}(x)} \Bigg], \quad (3.4.20)$$

but the modified one, generated by

$$\hat{\hat{\delta}} := \hat{s}^{-1} \hat{\delta} \hat{s} \quad (3.4.21)$$
$$= \mathrm{tr} \int_x \Bigg[\partial_\mu \Bigg(c(x) + \frac{\delta}{\delta \bar{c}(x)} \Bigg) \frac{\delta}{\delta A_\mu(x)} + \frac{1}{\xi_\tau} \partial_\mu \Bigg(A_\mu(x) + \frac{\delta}{\delta A_\mu(x)} \Bigg) \frac{\delta}{\delta c(x)}$$
$$+ ie_\tau \Bigg(c(x) + \frac{\delta}{\delta \bar{c}(x)} \Bigg) \psi(x) \frac{\overrightarrow{\delta}}{\delta \psi(x)} - ie_\tau \Bigg(c(x) + \frac{\delta}{\delta \bar{c}(x)} \Bigg) \bar{\psi}(x) \frac{\overrightarrow{\delta}}{\delta \bar{\psi}(x)} \Bigg], \quad (3.4.22)$$

where ξ_τ is the gauge parameter at the scale parameter τ, defined as

$$\xi_\tau := \xi_{\tau_0} \exp\Bigg(2\int_{\tau_0}^{\tau} d\tau' \Bigg(\frac{D-4}{2} + \gamma_{\tau'} \Bigg) \Bigg). \quad (3.4.23)$$

This modified BRST invariance results in the Ward-Takahashi (WT) identity for S_τ, given by

$$
\begin{aligned}
0 &= e^{-S_\tau}\hat{\hat{\delta}} e^{S_\tau} \qquad (3.4.24)\\
&= \mathrm{tr}\int_x \Bigg[\partial_\mu\left(c(x)+\frac{\delta S_\tau}{\delta\bar{c}(x)}\right)\frac{\delta S_\tau}{\delta A_\mu(x)}+\frac{1}{\xi_\tau}\partial_\mu\left(A_\mu(x)+\frac{\delta S_\tau}{\delta A_\mu(x)}\right)\frac{\delta S_\tau}{\delta c(x)}\\
&\quad + ie_\tau\left(c(x)+\frac{\delta S_\tau}{\delta\bar{c}(x)}\right)\psi(x)\frac{\overrightarrow{\delta}}{\delta\psi(x)}S_\tau - ie_\tau\left(c(x)+\frac{\delta S_\tau}{\delta\bar{c}(x)}\right)\bar{\psi}(x)\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S_\tau\\
&\quad \partial_\mu\frac{\delta^2 S_\tau}{\delta\bar{c}(x)\delta A_\mu(x)}+\frac{1}{\xi_\tau}\partial_\mu\frac{\delta}{\delta A_\mu(x)}\frac{\delta S_\tau}{\delta c(x)}+ie_\tau\frac{\delta}{\delta\bar{c}(x)}\left(\psi(x)\frac{\overrightarrow{\delta}}{\delta\psi(x)}S_\tau-\bar{\psi}(x)\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S_\tau\right)\Bigg].
\end{aligned}
$$

(3.4.25)

Although this constraint is non-linear and complicated to analyze, we can solve the GFERG equation and confirm that S_τ does saturate the WT identity order by order of e_τ.

Let us perturbatively solve the GFERG equation up to the first order of the electric charge e_τ and confirm that the WT identity is saturated. The GFERG equation is explicitly written down as

$$
\begin{aligned}
&\partial_\tau e^{S_\tau}\\
&= \int_k\left[\left(2k^2+\frac{D+2}{2}-\gamma_\tau+k\cdot\partial_k\right)A_\mu(k)\cdot\frac{\delta}{\delta A_\mu(k)}e^{S_\tau}+\left(2k^2+1-\gamma_\tau\right)\frac{\delta^2}{\delta A_\mu(k)\delta A_\mu(-k)}e^{S_\tau}\right]\\
&\quad+\int_k\Bigg[\left(2k^2+\frac{D}{2}+k\cdot\partial_k\right)\bar{c}(-k)\cdot\frac{\overrightarrow{\delta}}{\delta\bar{c}(-k)}e^{S_\tau}+e^{S_\tau}\frac{\overleftarrow{\delta}}{\delta c(k)}\left(2k^2+\frac{D+4}{2}+k\cdot\partial_k\right)c(k)\\
&\qquad -2(2k^2+1)\frac{\overrightarrow{\delta}}{\delta\bar{c}(-k)}e^{S_\tau}\frac{\overleftarrow{\delta}}{\delta c(k)}\Bigg]\\
&\quad+\int_p\Bigg[e^{S_\tau}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\left(2p^2+\frac{D+1}{2}-\gamma_{F\tau}+p\cdot\partial_p\right)\psi(p)+\left(2p^2+\frac{D+1}{2}-\gamma_{F\tau}+p\cdot\partial_p\right)\bar{\psi}(p)\cdot\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(p)}e^{S_\tau}\\
&\qquad -i\left(4p^2+1-2\gamma_{F\tau}\right)\mathrm{tr}\,\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}e^{S_\tau}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\Bigg]\\
&\quad+\mathrm{tr}\int d^Dx\,\frac{\delta}{\delta\bar{\psi}(x)}\left\{4ie_\tau\left[A_\mu(x)+\frac{\delta}{\delta A_\mu(x)}\right]\partial_\mu-2e_\tau^2\left[A_\mu(x)+\frac{\delta}{\delta A_\mu(x)}\right]\left[A_\mu(x')+\frac{\delta}{\delta A_\mu(x')}\right]\right\}\\
&\qquad\times e^{S_\tau}\left[\bar{\psi}(x)+i\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right]\\
&\quad+\mathrm{tr}\int d^Dx\left\{-4ie_\tau\left[A_\mu(x)+\frac{\delta}{\delta A_\mu(x)}\right]\partial_\mu-2e_\tau^2\left[A_\mu(x)+\frac{\delta}{\delta A_\mu(x)}\right]\left[A_\mu(x')+\frac{\delta}{\delta A_\mu(x')}\right]\right\}\\
&\qquad\times\left[\psi(x)+i\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\right]e^{S_\tau}\frac{\overleftarrow{\delta}}{\delta\psi(x)} \qquad (3.4.26)
\end{aligned}
$$

Here, we consider the continuum limit, i.e., setting the cutoff Λ_0 infinity. This limit is not evident because we must tune the couplings appropriately to get finite correlation functions in the limit. We do not give the details here[3]; instead we just assume that the Wilsonian effective action S_τ is a function of e_τ and ξ_τ and does not depend on τ explicitly after taking this limit. In addition, we consider the massless QED for simplicity and assume that the Wilsonian effective action S_τ does not depend on the fermion mass. Then, S_τ is expanded as

$$S_\tau = \sum_{n=0}^{\infty} e_\tau^n S_\tau^{(n)}. \tag{3.4.27}$$

Under this assumption, $\partial_\tau S_\tau$ is given by

$$\partial_\tau S_\tau = \left(\partial_\tau e_\tau \frac{\partial}{\partial e_\tau} + \partial_\tau \xi_\tau \frac{\partial}{\partial \xi_\tau}\right) S_\tau \tag{3.4.28}$$

$$= \left(\left[\frac{4-D}{2} - \gamma(e_\tau^2)\right] e_\tau \frac{\partial}{\partial e_\tau} + 2\gamma(e_\tau^2)]\xi_\tau \frac{\partial}{\partial \xi_\tau}\right) S_\tau. \tag{3.4.29}$$

Therefore, the GFERG equation is given by

$$\left(\left[\frac{4-D}{2} - \gamma(e_\tau^2)\right] e_\tau \frac{\partial}{\partial e_\tau} + 2\gamma(e_\tau^2)\xi_\tau \frac{\partial}{\partial \xi_\tau}\right) S_\tau = (\text{R.H.S of Eq.(3.4.26)}) \tag{3.4.30}$$

and we solve this equation to determine $S_\tau^{(n)}$ order by order.

Tree level

Let us study the Gaussian fixed point $S_\tau^{(0)}$ in the GFERG equation with $e_\tau = \gamma_\tau = \gamma_{F\tau} = 0$. It is given by

$$S_\tau^{(0)} = -\frac{1}{2}\int_k A_\mu(k)A_\nu(-k)\left[\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{k^2}{e^{-2k^2}+k^2} + \frac{k_\mu k_\nu}{k^2}\frac{k^2}{\xi_\tau e^{-2k^2}+k^2}\right]$$
$$- \int_k \bar{c}(-k)c(k)\frac{k^2}{e^{-2k^2}+k^2} - \int_p \bar{\psi}(p)\frac{\not{p}}{e^{-2p^2}+\not{p}}\phi(p). \tag{3.4.31}$$

It can be easily seen that $S^{(0)}$ satisfies the WT identity at $\mathcal{O}((e_\tau)^0)$.

In the following analysis, we assume that the ghost terms in S are just given by those in $S^{(0)}$, equivalently, $S^{(n)}$ $(n \geq 1)$ does not contain any ghost terms. This is justified because the ghosts decouple from the gauge and matter fields in R_ξ gauge. Then, the WT identity is reduced to

$$e_\tau \int_p \left[\bar{\psi}(-p-k)\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S_\tau - S_\tau\frac{\overleftarrow{\delta}}{\delta\psi(p+k)}\psi(p)\right] = \frac{\xi_\tau e^{-2k^2}+k^2}{\xi_\tau e^{-2k^2}}k_\mu\frac{\delta S_I}{\delta A_\mu(k)}, \tag{3.4.32}$$

where $S_I := S_\tau - S_\tau^{(0)}$.

[3]See Refs. [11,14–17] for details.

First order

Let us study the solution to the GFERG equation at order of e_τ. Here, we assume that

$$S_\tau^{(1)} = \int_{p,k} \widetilde{\bar{\Psi}}(-p-k) V_\mu(p,k) \widetilde{\mathcal{A}}_\mu(k) \widetilde{\Psi}(p), \tag{3.4.33a}$$

$$\gamma_\tau = \mathcal{O}\left(e_\tau^2\right), \tag{3.4.33b}$$

$$\gamma_{F\tau} = \mathcal{O}\left(e_\tau^2\right), \tag{3.4.33c}$$

where $\widetilde{\mathcal{A}}_\mu$, $\widetilde{\Psi}$ and $\widetilde{\bar{\Psi}}$ are defined as

$$\widetilde{\mathcal{A}}_\mu(k) \coloneqq e^{k^2} \mathcal{A}_\mu(k), \tag{3.4.34a}$$

$$\widetilde{\Psi}(p) \coloneqq e^{p^2} \Psi(p), \tag{3.4.34b}$$

$$\widetilde{\bar{\Psi}}(-p) \coloneqq e^{p^2} \bar{\Psi}(-p), \tag{3.4.34c}$$

where

$$\mathcal{A}_\mu(k) \coloneqq A_\mu(k) + \frac{\delta S_\tau^{(0)}}{\delta A_\mu(-k)} = e^{-2k^2} h_{\mu\nu}(k) A_\nu(k), \tag{3.4.35a}$$

$$\Psi(p) \coloneqq \psi(p) + i \frac{\overrightarrow{\delta}}{\delta \bar{\psi}(p)} S_\tau^{(0)} = e^{-2p^2} \frac{1}{i} h_F(p) \psi(p), \tag{3.4.35b}$$

$$\bar{\Psi}(-p) \coloneqq \bar{\psi}(-p) + i S_\tau^{(0)} \frac{\overleftarrow{\delta}}{\delta \psi(p)} = \bar{\psi}(-p) e^{-2p^2} \frac{1}{i} h_F(p), \tag{3.4.35c}$$

and

$$h_{\mu\nu}(k) \coloneqq \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \frac{1}{e^{-2k^2} + k^2} + \frac{k_\mu k_\nu}{k^2} \frac{\xi_\tau}{\xi_\tau e^{-2k^2} + k^2}, \tag{3.4.36a}$$

$$h_F(p) \coloneqq \frac{1}{e^{-2p^2} + i \not{p}}. \tag{3.4.36b}$$

$h_{\mu\nu}$ or h_F is the high-momentum propagator of the gauge field A_μ or the fermion field ψ. For later use, it should be noted here that these variables satisfy

$$\begin{aligned} &\left(k \cdot \partial_k + \frac{D+2}{2}\right) \widetilde{\mathcal{A}}_\mu(k) \cdot \frac{\delta}{\delta \widetilde{\mathcal{A}}_\mu(k)} \\ &= \left[\left(2k^2 + \frac{D+2}{2} + k \cdot \partial_k\right) A_\mu(k) + 2\left(2k^2 + 1\right) \frac{\delta S_\tau^{(0)}}{\delta A_\mu(-k)}\right] \frac{\delta}{\delta A_\mu(k)}, \end{aligned} \tag{3.4.37a}$$

$$\begin{aligned} &\frac{\overleftarrow{\delta}}{\delta \widetilde{\Psi}(p)} \left(p \cdot \partial_p + \frac{D+1}{2}\right) \widetilde{\Psi}(p) \\ &= \frac{\overleftarrow{\delta}}{\delta \psi(p)} \left[\left(2p^2 + \frac{D+1}{2} + p \cdot \partial_p\right) \psi(p) + i\left(4p^2 + 1\right) \frac{\overrightarrow{\delta}}{\delta \bar{\psi}(-p)} S_\tau^{(0)}\right], \end{aligned} \tag{3.4.37b}$$

$$\left(p\cdot\partial_p+\frac{D+1}{2}\right)\tilde{\bar{\Psi}}(-p)\cdot\frac{\overrightarrow{\delta}}{\delta\tilde{\bar{\Psi}}(-p)}$$
$$=\left[\left(2p^2+\frac{D+1}{2}+p\cdot\partial_p\right)\bar{\psi}(-p)+i\left(4p^2+1\right)S_\tau^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\right]\cdot\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}. \tag{3.4.37c}$$

Substituting the perturbative expansion Eq. (3.4.27) of $S^{(1)}$ into the GFERG equation and focusing on $\mathcal{O}(e_\tau)$ terms, we get

$$\begin{aligned}
-\frac{D-4}{2}S_\tau^{(1)} &= \int_k\left(\left(2k^2+\frac{D+2}{2}+k\cdot\partial_k\right)A_\mu(k)+2\left(2k^2+1\right)\frac{\delta S^{(0)}}{\delta A_\mu(-k)}\right)\frac{\delta S^{(1)}}{\delta A_\mu(k)}\\
&+\int_p\Bigg[\left(\left(2p^2+\frac{D+1}{2}+p\cdot\partial_p\right)\bar{\psi}(-p)+i\left(4p^2+1\right)S^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\right)\cdot\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S_\tau^{(0)}.\\
&\quad+S_\tau^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\left(\left(2p^2+\frac{D+1}{2}+p\cdot\partial_p\right)\psi(p)+i\left(4p^2+1\right)\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S_\tau^{(0)}\right),\\
&\qquad-i\left(4p^2+1\right)\mathrm{tr}\,\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S_\tau^{(1)}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\Bigg]\\
&+4i\,\mathrm{tr}\int_x\left(\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S^{(0)}\right)\left(A_\mu(x)+\frac{\delta S^{(0)}}{\delta A_\mu(x)}\right)\partial_\mu\left(\bar{\psi}(x)+iS^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right)\\
&-4i\,\mathrm{tr}\int_x\left(A_\mu(x)+\frac{\delta S^{(0)}}{\delta A_\mu(x)}\right)\partial_\mu\left(\psi(x)+\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S^{(0)}\right)\left(S^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(x)}\right).
\end{aligned} \tag{3.4.38}$$

In terms of the variables $\widetilde{\mathcal{A}}_\mu$, $\widetilde{\Psi}$ and $\tilde{\bar{\Psi}}$, this equation is rewritten as

$$\begin{aligned}
-\frac{D-4}{2}S_\tau^{(1)} &= \int_k\left(k\cdot\partial_k+\frac{D+2}{2}\right)\widetilde{\mathcal{A}}_\mu(k)\cdot\frac{\delta S_\tau^{(1)}}{\delta\widetilde{\mathcal{A}}_\mu(k)}\\
&+\int_p\Bigg[S_\tau^{(1)}\frac{\overleftarrow{\delta}}{\delta\widetilde{\Psi}(p)}\left(p\cdot\partial_p+\frac{D+1}{2}\right)\widetilde{\Psi}(p)+\left(p\cdot\partial_p+\frac{D+1}{2}\right)\tilde{\bar{\Phi}}(-p)\cdot\frac{\overrightarrow{\delta}}{\delta\tilde{\bar{\Psi}}(-p)}S_\tau^{(1)}\\
&\qquad-i\left(4p^2+1\right)\mathrm{tr}\,\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S_\tau^{(1)}\frac{\overleftarrow{\delta}}{\delta\psi(p)}\Bigg]\\
&+4\int_{p,k}\left(e^{(p+k)^2-p^2-k^2}(\not{p}+\not{k})p_\mu+e^{p^2-(p+k)^2-k^2}\not{p}(p+k)_\mu\right)\tilde{\bar{\Phi}}(-p-k)\widetilde{\mathcal{A}}_\mu(k)\widetilde{\Psi}(p).
\end{aligned}$$
(3.4.39)

Substituting the ansatz Eq. (3.4.33a) of $S^{(1)}$ into this equation, we get the following inhomogeneous differential equation for $V_\mu(p,k)$:

$$(p\cdot\partial_p+k\cdot\partial_k)V_\mu(p,k)=4\left(e^{(p+k)^2-p^2-k^2}(\not{p}+\not{k})p_\mu+e^{p^2-(p+k)^2-k^2}\not{p}(p+k)_\mu\right). \tag{3.4.40}$$

We neglected the term $\mathrm{tr}\,\overrightarrow{\delta}/\delta\bar{\psi}(-p)S^{(1)}\overleftarrow{\delta}/\delta\psi(p)$ to derive this equation. This is justified by $\mathrm{tr}\,V_\mu(p,-p)=0$, which will be shown later.

Using the formula in Appendix A.2, we obtain the solution to this equation as

$$V_\mu(p,k) \\ = a_\mu + 2F((p+k)^2 - p^2 - k^2)(\not{p}+\not{k})p_\mu + 2F(p^2-(p+k)^2-k^2)\not{p}(p+k)_\mu, \quad (3.4.41)$$

where c_μ is an arbitrary constant vector, and $F(x) := (e^x - 1)/x$. c_μ can be fixed from the WT identity. Substituting the perturbative expansion Eq. (3.4.27) of $S^{(1)}$ into the WT identity Eq. (3.4.32) and focusing on $\mathcal{O}(e_\tau)$ terms, we obtain

$$\int_p \left[\bar{\psi}(-p-k)\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(-p)}S_\tau^{(0)} - S_\tau^{(0)}\frac{\overleftarrow{\delta}}{\delta\psi(p+k)}\psi(p)\right] = \frac{\xi_\tau e^{-2k^2}+k^2}{\xi_\tau e^{-2k^2}}k_\mu\frac{\delta S_\tau^{(1)}}{\delta A_\mu(k)}, \quad (3.4.42)$$

which requires V_μ to satisfy

$$e^{(p+k)^2-p^2-k^2}h_F^{-1}(p+k) - e^{p^2-(p+k)^2-k^2}h_F^{-1}(p) = k_\mu V_\mu(p,k). \quad (3.4.43)$$

From this equation, we find that c_μ is determined as $c_\mu = \gamma_\mu$. Finally, $V_\mu(p,k)$ is given by

$$V_\mu(p,k) \\ = \gamma_\mu + 2F((p+k)^2 - p^2 - k^2)(\not{p}+\not{k})p_\mu + 2F(p^2-(p+k)^2-k^2)\not{p}(p+k)_\mu. \quad (3.4.44)$$

Note that $\operatorname{tr} V_\mu(p,-p) = 0$, which was claimed in the above.

We emphasize that the WT identity is saturated up to the first order of the electric coupling e_τ. Although we can determine the second-order contribution $S_\tau^{(2)}$ to the Wilsonian effective action S_τ by straightforward calculations, we here stop to calculate more. See the original paper [10] for the detailed calculations of the second-order analysis.

3.4.3 GFERG equation for 1PI effective action

The flow equation for the 1PI effective action Γ_Λ is discussed in [26]. The 1PI effective action is defined in terms of dimensionless variables as

$$\Gamma_\tau[\mathcal{A}_\mu,\Psi,\bar{\Psi}] \\ := S_\tau[A_\mu,\psi,\bar{\psi}] + \int_x \left[\frac{1}{2}(\mathcal{A}_\mu(x)-A_\mu(x))^2 - i\big(\bar{\Psi}(x)-\bar{\psi}(x)\big)(\Psi(x)-\psi(x))\right], \quad (3.4.45)$$

where $A_\mu(x), \psi(x)$ and $\bar{\psi}(x)$ are substituted by the solution to the following equations:

$$\mathcal{A}_\mu(x) = A_\mu(x) + \frac{\delta S}{\delta A_\mu(x)}, \quad (3.4.46a)$$

$$\Psi(x) = \psi(x) + i\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S, \quad (3.4.46b)$$

$$\bar{\Psi}(x) = \bar{\psi}(x) + iS\frac{\overleftarrow{\delta}}{\delta\psi(x)}. \quad (3.4.46c)$$

These composite operators are fundamental variables with respect to the modified correlation functions. Note that $\mathcal{A}_\mu(x)$, $\Psi(x)$ and $\bar{\Psi}(x)$ are rewritten as

$$\mathcal{A}_\mu(x) = e^{-S_\tau}\hat{s}A_\mu(x)\hat{s}^{-1}e^{S_\tau}, \tag{3.4.47a}$$

$$\Psi(x) = e^{-S_\tau}\hat{s}\psi(x)\hat{s}^{-1}e^{S_\tau}, \tag{3.4.47b}$$

$$\bar{\Psi}(x) = e^{-S_\tau}\hat{s}\bar{\psi}(x)\hat{s}^{-1}e^{S_\tau}. \tag{3.4.47c}$$

Let us consider inserting $\mathcal{A}_\mu(x)$ to $\langle\!\langle \mathcal{A}_{\mu_1}(x_1)\cdots\mathcal{A}_{\mu_n}(x_n)\rangle\!\rangle^k_{S_\tau}$ as

$$\int DA_\mu\, e^{S_\tau}\mathcal{A}_\mu(x)\hat{s}^{-1}(\mathcal{A}_{\mu_1}(x_1)\cdots\mathcal{A}_{\mu_n}(x_n)) \tag{3.4.48}$$

The flow equation for the 1PI effective action Γ_τ is given by

$$\partial_\tau\Gamma_\tau = \partial_\tau S, \tag{3.4.49}$$

where the right-hand side is the gauge and fermion part of the GFERG equation, expressed in terms of $\mathcal{A}_\mu(x), \Psi(x)$, and $\bar{\Psi}(x)$, rather than $A_\mu(x), \psi(x)$, and $\bar{\psi}(x)$. Recall that because the WP equation for the scalar field contains only the first and second functional derivatives, the Wetterich equation is given in a simple form. However, the GFERG equation for QED includes up to the fourth functional derivative, and then the corresponding flow equation for the 1PI effective action Γ_τ becomes quite complicated. S ee the original paper [26] for details.

Chapter 4

Fixed Point Structure of GFERG for Scalar Field Theories

In this section, we discuss GFERG for general gradient flow equation of scalar field theories and show that it has the same fixed point structure as that of the WP equation. We also compare scaling dimensions of operators between the GFERG equation and the WP equation around the fixed points. The following discussions are based on my work [1].

The rest of this chapter is organized as follows. In Section 4.1, we introduce a general gradient flow equation for a scalar field theory and derive the GFERG equation based on this flow equation following Ref. [8]. Then, we study the fixed points of the GFERG equation. We also investigate the RG flow structure around the fixed points. In Section 4.2, we discuss fixed points of the GFERG equation of the $O(N)$ non-linear sigma model in $4-\epsilon$ dimensions, as an example. Then, we illustrate the result in the previous section, focusing on the WF fixed point. Note that throughout this chapter, we work on the dimensionless framework.

4.1 GFERG for General Gradient Flow

4.1.1 GFERG Equation

Let us consider a general gradient flow equation defined by the following differential equation:

$$\partial_t \varphi_a(t,x) = F_a[\varphi](t,x), \quad \varphi_a(0,x) = \phi_a(x), \tag{4.1.1}$$

where ϕ_a is a real scalar field, and a labels all kinds of fields in the theory. $F_a[\varphi](t,x)$ is an arbitrary functional of φ_a's. The variables t and x denote a fictitious time called the flow time and the D-dimensional (Euclidean space) coordinate, respectively. The gradient flow continuously deforms the fields ϕ_a defined on the D-dimensional Euclidean space along the flow time t.

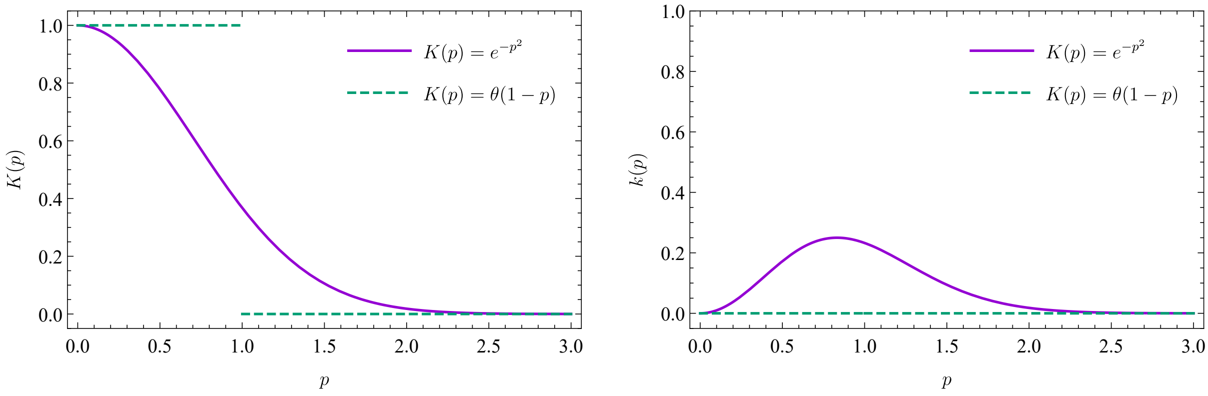

Figure 4.1: Plots of the cutoff functions $(K(p), k(p))$ [1]. The left figure depicts the momentum dependence of some examples of $K(p)$. The right figure shows that of $k(p) = K(p)(1-K(p))$, corresponding to each example of $K(p)$. Note that the momentum p here is a dimensionless quantity, normalized by the cutoff Λ.

In this paper, we assume that $F_a[\varphi]$ is expanded as a polynomial of $\vec{\varphi}$ like

$$F_a[\varphi](t,x) = \partial_\mu^2 \varphi_a(t,x) + \sum_{n=n_{\min}}^{\infty} \int_{x_1,\ldots,x_n} f_a^{a_1,\ldots,a_n}(x; x_1,\ldots,x_n; \partial_{x_1},\ldots,\partial_{x_n}) \prod_{j=1}^{n} \varphi_{a_j}(t,x_j) \quad (4.1.2)$$

where the expansion coefficient $f_a^{a_1,\ldots,a_n}$ depends on x_i's and contains partial derivatives with respect to them, and $n_{\min}$ is a positive integer larger than one.[1] For example, gradient flow equations for the non-linear sigma model in two-dimensions are proposed as $F_a = \partial_\mu^2\varphi_a - (\varphi_b\partial_\mu^2\varphi_b)\varphi_a$ for $a,b = 1,\ldots,N$ in Ref. [36], or as $F_a = \partial_\mu^2\varphi_a + \varphi_a\partial_\mu\varphi_b\partial_\mu\varphi_b + \varphi_a(\varphi_b\partial_\mu\varphi_b)^2/(1-(\varphi_c)^2)$ for $a,b,c = 1,\ldots,N-1$ in Ref. [37]. In the $O(N)$ linear sigma model, the gradient flow equation is just given by the diffusion equation: $F_a = \partial_\mu^2\varphi_a$ [38].

Following Ref. [8], it is straightforward to define the Wilson action associated with Eq. (4.1.1) as

$$e^{S_\tau[\phi_a]} = \exp\left[\int_{x,y} \frac{1}{2}\mathcal{D}(x-y)\frac{\delta^2}{\delta\phi_a(x)\delta\phi_a(y)}\right] \times \int [D\phi'_a] \prod_{x',a} \delta(\phi_a(x) - e^{\tau(D-2)/2} Z_\tau^{1/2} \varphi'_a(t, x'e^\tau)) \times \exp\left[-\int_{x'',y''} \frac{1}{2}\mathcal{D}(x-y)\frac{\delta^2}{\delta\phi'_a(x'')\delta\phi'_a(y'')}\right] e^{S_{\tau=0}[\phi'_a]}, \quad (4.1.3)$$

where φ'_a is the solution to the general flow equation Eq. (4.1.1) with the initial condition $\varphi'_a(0,x) = \phi'_a(x)$ and Z_τ is the wave function renormalization factor depending on τ. The

[1] Note that the linear term is the same as the diffusion equation while the non-linear terms take general forms in the above gradient flow equation. In principle, a more general linear term can be considered in the GFERG framework as well. See Ref. [35] for such cases.

relation between the flow time t of the gradient flow and τ in the GFERG equation is given by

$$t := e^{2\tau} - 1. \tag{4.1.4}$$

$\mathcal{D}(x-y)$ is defined by

$$\mathcal{D}(x-y) := \int_p e^{ip(x-y)} \frac{k(p)}{p^2}. \tag{4.1.5}$$

$K(p)$ and $k(p)$ are the cutoff function satisfying

$$K(0) = 1, \quad K(\infty) = 0, \quad k(0) = 0, \tag{4.1.6}$$

and we set $K(p) = e^{-p^2}$ and $k(p) = K(p)(1 - K(p))$ in this paper. See Fig. 4.1 for plots of their p-dependence.

Note that Eq. (4.1.3) is not invariant under target space diffeomorphism of the fields and seems inapplicable for non-linear sigma models with a curved target space metric. Instead, this expression should be interpreted as one for non-linear sigma models embedded in higher-dimensional Euclidean space. This prescription is ensured by Nash's embedding theorem, which states that we can embed an arbitrary Riemann manifold into a Euclidean space $\mathbb{R}^m$ of sufficiently large dimensions m with some constraints for the fields (coordinates of the target space). Then the consistency under the diffeomorphism does not matter, and the gradient flow equation is required to preserve the constraint instead.

By differentiating S_τ with respect to τ, we get the GFERG equation for S_τ. It is given by

$$\begin{aligned}\frac{\partial}{\partial\tau} e^{S_\tau[\phi_a]} = \exp&\left[\lambda(\tau)^2 \int_{x,y} \frac{1}{2}\mathcal{D}(x-y) \frac{\delta^2}{\delta\widetilde{\phi}_a(x)\delta\widetilde{\phi}_a(y)}\right] \\ &\times \int d^D x' \frac{\delta}{\delta\widetilde{\phi}_a(x')} \left[-2F_a[\widetilde{\phi}](x) - \left(\frac{D-2}{2} + \gamma_\tau + x'_\rho \partial'_\rho\right)\widetilde{\phi}_a(x')\right] \\ &\qquad \times \exp\left[\lambda(\tau)^2 \int_{x'',y''} \frac{1}{2}\mathcal{D}(x''-y'') \frac{\delta^2}{\delta\widetilde{\phi}_a(x'')\delta\widetilde{\phi}_a(y'')}\right] e^{S_\tau[\phi'_a]},\end{aligned} \tag{4.1.7}$$

where $\widetilde{\phi}$ is the rescaled field defined as

$$\widetilde{\phi}_a := \lambda(\tau)\phi_a \tag{4.1.8}$$

and $\lambda(\tau)$ is given by

$$\lambda(\tau) := e^{-\tau(D-2)/2} Z_\tau^{-1/2}. \tag{4.1.9}$$

Furthermore, using the relation

$$\widehat{\widetilde{\phi}}_a(x) \coloneqq \widetilde{\phi}_a(x) + \lambda^2 \int_y \mathcal{D}(x-y) \frac{\delta}{\delta\widetilde{\phi}_a(y)} \tag{4.1.10}$$

$$= \exp\left[\lambda^2 \int_{x,y} \frac{1}{2}\mathcal{D}(x-y)\frac{\delta^2}{\delta\widetilde{\phi}_a(x)\delta\widetilde{\phi}_a(y)}\right]\widetilde{\phi}_a(x) \exp\left[-\lambda^2 \int_{x,y} \frac{1}{2}\mathcal{D}(x-y)\frac{\delta^2}{\delta\widetilde{\phi}_a(x)\delta\widetilde{\phi}_a(y)}\right], \tag{4.1.11}$$

Eq. (4.1.7) can be written in a compact form:

$$\frac{\partial}{\partial\tau}e^{S_\tau[\phi_a]} = \int d^Dx \frac{\delta}{\delta\widetilde{\phi}_a(x)}\left[-2F[\widehat{\widetilde{\phi}}](x) - \left(\frac{D-2}{2} + \gamma_\tau + x_\nu\partial_\nu\right)\widehat{\widetilde{\phi}}_a(x)\right]e^{S_\tau[\phi_a]}, \tag{4.1.12}$$

where the anomalous dimension γ_τ is defined as

$$\gamma_\tau \coloneqq \frac{1}{2}\frac{d\log Z_\tau}{d\tau}. \tag{4.1.13}$$

More specifically, Eq. (4.1.12) can be written in the following form:

$$\begin{aligned}
\frac{\partial}{\partial t}e^{S_\tau[\phi_a]} = &\int_p \Bigg\{\left[\left(2p^2 + \frac{D+2}{2} - \gamma_\tau\right)\phi_a(p) + p_\mu\frac{\partial}{\partial p_\mu}\phi_a(p)\right]\frac{\delta}{\delta\phi_a(p)} \\
&+ \frac{1}{p^2}\left[4p^2k(p) + 2p^2\frac{dk(p)}{dp^2} - 2\gamma_\tau k(p)\right]\frac{1}{2}\frac{\delta^2}{\delta\phi_a(p)\delta\phi_a(-p)}\Bigg\}e^{S_\tau[\phi_a]} \\
&- 2\sum_{n=n_{\min}}^{\infty}\lambda(\tau)^{n-1}\int_{x,x_1,\dots,x_n}\frac{\delta}{\delta\phi_a(x)}\Bigg\{f_a^{a_1,\dots,a_n}\left(\phi_{a_1}(x_1) + \int_{y_1}\mathcal{D}(x_1-y_1)\frac{\delta}{\delta\phi_{a_1}(y_1)}\right)\times\cdots \\
&\times\left(\phi_{a_n}(x_n) + \int_{y_n}\mathcal{D}(x_n-y_n)\frac{\delta}{\delta\phi_{a_n}(y_n)}\right)\Bigg\}e^{S_\tau[\phi_a]}.
\end{aligned} \tag{4.1.14}$$

We have ignored ordering of $\delta/\delta\phi_a$ and ϕ_a in the first and second lines because it only changes the Wilson action S_τ by a field-independent constant.

In Ref. [8], the sigma model with a single scalar field is considered, where $F[\varphi]$ is just given by $\partial_\mu^2\varphi$. There, Z_τ is necessary from the renormalizability of correlation functions of the flowed field $\varphi(t,x)$, that is, to keep the quantity

$$Z_\tau^{n/2}\left\langle \exp\left[-\int_p \frac{k(p)}{p^2}\frac{1}{2}\frac{\delta^2}{\delta\phi(p)\phi(-p)}\right]\varphi(t,p_1)\cdots\varphi(t,p_n)\right\rangle_{S_{\tau=0}} \tag{4.1.15}$$

UV-finite after performing the renormalization of the original theory with identification of $t = e^{2\tau} - 1$. The GFERG equation of this model becomes

$$\begin{aligned}
\frac{\partial}{\partial\tau}e^{S_\tau[\phi_a]} = &\int_p \Bigg\{\left[\left(2p^2 + \frac{D+2}{2} - \gamma_\tau\right)\phi_a(p) + p_\mu\frac{\partial}{\partial p_\mu}\phi_a(p)\right]\frac{\delta}{\delta\phi_a(p)} \\
&+ \frac{1}{p^2}\left[4p^2k(p) + 2p^2\frac{dk(p)}{dp^2} - 2\gamma_\tau k(p)\right]\frac{1}{2}\frac{\delta^2}{\delta\phi_a(p)\delta\phi_a(-p)}\Bigg\}e^{S_\tau[\phi_a]},
\end{aligned} \tag{4.1.16}$$

which is nothing but the WP equation. Comparing the general GFERG equation Eq. (4.1.14) with this equation, we readily notice that the former has the extra non-linear terms accompanied with one or more factors of $\lambda(\tau)$. We study their effect on the fixed points and RG flow structure around the fixed points in the following sections.

4.1.2 Fixed Points

In this section, we study fixed points of the GFERG equation Eq. (4.1.14) for the general gradient flow equation. Then, we show that the fixed points of the WP equation [5] appear in the $\tau \to \infty$ limit along the GFERG flow.

Let us consider the solution S_τ to the GFERG equation and the limiting value of S_τ as $\tau \to \infty$. If S_τ converges in this limit to some finite action S^*, it becomes τ-independent, i.e., $\partial_\tau S^* = 0$. Therefore, S^* satisfies

$$\begin{aligned}
0 = &\int_p \Bigg\{ \left[\left(2p^2 + \frac{D+2}{2} - \gamma \right) \phi_a(p) + p_\mu \frac{\partial}{\partial p_\mu} \phi_a(p) \right] \frac{\delta}{\delta \phi_a(p)} \\
&\qquad + \frac{1}{p^2} \left[4p^2 k(p) + 2p^2 \frac{dk(p)}{dp^2} - 2\gamma k(p) \right] \frac{1}{2} \frac{\delta^2}{\delta \phi_a(p) \delta \phi_a(-p)} \Bigg\} e^{S^*} \\
&- 2 \sum_{n=n_{\min}}^{\infty} \lambda(\infty)^{n-1} \int_{x,x_1,\ldots,x_n} \frac{\delta}{\delta \phi_a(x)} \left\{ f_a^{a_1,\ldots,a_n} \left(\phi_{a_1}(x_1) + \int_{y_1} \mathcal{D}(x_1 - y_1) \frac{\delta}{\delta \phi_{a_1}(y_1)} \right) \times \ldots \right. \\
&\qquad \left. \times \left(\phi_{a_n}(x_n) + \int_{y_n} \mathcal{D}(x_n - y_n) \frac{\delta}{\delta \phi_{a_n}(y_n)} \right) \right\} e^{S^*}, \qquad (4.1.17)
\end{aligned}$$

where γ is the anomalous dimension at the fixed point theory S^* defined as

$$\gamma \coloneqq \lim_{\tau \to \infty} \gamma_\tau. \qquad (4.1.18)$$

We should determine the value of $\lambda(\infty)$ to solve this equation concretely. Note that the asymptotic behavior of Z_τ is given by

$$Z_\tau \sim e^{2\gamma\tau} \qquad (4.1.19)$$

as $\tau \to \infty$. Then the asymptotic behavior of $\lambda(\tau)$ is given by

$$\lambda(\tau) \sim e^{-\tau(D-2+2\gamma)/2} \qquad (4.1.20)$$

from the definition of $\lambda(\tau)$ (Eq. (4.1.9)). From this equation, we readily find that the signature of $D - 2 + 2\gamma$ controls the convergence of $\lambda(\infty)$. In particular, $\lambda(\infty)$ vanishes when

$$D - 2 + 2\gamma > 0. \qquad (4.1.21)$$

It is indeed found that $D - 2 + 2\gamma$ should be positive from physical viewpoints. Because the fixed point action S^* is invariant under the GFERG flow, it should have the conformal symmetry. There, the connected two-point function scales as

$$\langle \phi(x) \phi(0) \rangle_{\text{conneted}} \propto \frac{1}{x^{D-2+2\gamma}}. \qquad (4.1.22)$$

According to the cluster decomposition principle, the two-point function factorizes into the product of one-point functions when $|x|$ goes to infinity. Therefore we get

$$\langle\phi(x)\phi(0)\rangle_{\text{conneted}} = \langle\phi(x)\phi(0)\rangle - \langle\phi(x)\rangle\langle\phi(0)\rangle \to 0 \quad (|x| \to \infty). \tag{4.1.23}$$

This fact requires that γ should satisfy Eq. (4.1.21).

Then, we conclude that $\lambda(\infty) = 0$ and S^* satisfies

$$0 = \int_p \left\{ \left[\left(2p^2 + \frac{D+2}{2} - \gamma \right) \phi_a(p) + p_\mu \frac{\partial}{\partial p_\mu} \phi_a(p) \right] \frac{\delta}{\delta\phi_a(p)} + \frac{1}{p^2} \left[4p^2 k(p) + 2p^2 \frac{dk(p)}{dp^2} - 2\gamma k(p) \right] \frac{1}{2} \frac{\delta^2}{\delta\phi_a(p)\delta\phi_a(-p)} \right\} e^{S^*}, \tag{4.1.24}$$

which is nothing but the fixed point condition of the WP equation. Therefore, we find the fixed points of the WP equation appear along the general GFERG flow as $\tau \to \infty$.

Here we give a comment on those fixed points. Because the GFERG flow depends on the RG flow time, S_τ cannot stay at S^* at finite flow time. In other words, even if the Wilson action S_τ equals to S^* at some finite time $\tau = \tau_1$, $\partial_\tau S_\tau\big|_{\tau=\tau_1}$ is not zero because of the extra non-linear terms proportional to powers of $\lambda(\tau_1)$ in the GFERG equation. Therefore, the GFERG equation does not have the same fixed points as those of the WP equation *at a finite flow time*, and they appear *in the large flow time limit*. Note that the GFERG equation has no fixed point at the finite flow time. See Sec. **??** for a detailed argument.

4.1.3 Flow Structure around Fixed Points

In the previous subsection, we have found that the fixed point action S^* of the WP equation appears in the $\tau \to \infty$ limit along the GFERG flow. This means that there can be a solution to the GFERG equation that flows into S^* as $\tau \to \infty$.

In this subsection, we study the RG flow structure around a fixed point after a long time. We investigate the time evolution of the GFERG equation after a long time $\tau = \tau_0 \gg 1$ so that $\exp(-\tau_0(D - 2 + 2\gamma)/2) \ll 1$. Let us consider perturbing S_τ from a fixed point of the GFERG equation at $\tau = \tau_0$ as

$$S_{\tau=\tau_0} = S^* + \sum_A \delta c^A \mathcal{O}_A, \tag{4.1.25}$$

where δc^A is a small fluctuation around the fixed point ($\left|\delta c^A\right| \ll 1$) and $\mathcal{O}_A$'s form a complete set of operators (defined later). If we set $\tau = \tau' + \tau_0$, the GFERG equation is given by

$$\frac{\partial}{\partial\tau'} e^{S_\tau[\phi_a]} = \int_p \left\{ \left[\left(2p^2 + \frac{D+2}{2} - \gamma_\tau \right) \phi_a(p) + p_\mu \frac{\partial}{\partial p_\mu} \phi_a(p) \right] \frac{\delta}{\delta\phi_a(p)} \right.$$

$$
+\frac{1}{p^2}\left[4p^2k(p)+2p^2\frac{dk(p)}{dp^2}-2\gamma_\tau k(p)\right]\frac{1}{2}\frac{\delta^2}{\delta\phi_a(p)\delta\phi_a(-p)}\Bigg\}e^{S_\tau[\phi_a]}
$$

$$
-2\sum_{n=n_{\min}}^{\infty}(\lambda(\tau'+\tau_0))^{n-1}\int_{x,x_1,\dots,x_n}\frac{\delta}{\delta\phi_a(x)}\left\{f_a^{a_1,\dots,a_n}\left(\phi_{a_1}(x_1)+\int_{y_1}\mathcal{D}(x_1-y_1)\frac{\delta}{\delta\phi_{a_1}(y_1)}\right)\times\dots\right.
$$

$$
\left.\times\left(\phi_{a_n}(x_n)+\int_{y_n}\mathcal{D}(x_n-y_n)\frac{\delta}{\delta\phi_{a_n}(y_n)}\right)\right\}e^{S_\tau[\phi_a]}. \quad (4.1.26)
$$

Note that the asymptotic behavior of $\lambda(\tau'+\tau_0)$ as $\tau_0\to\infty$ is given by $e^{-\tau'(D-2+2\gamma)/2}\lambda_0$, where λ_0 is defined as $\lambda_0 := \exp(-\tau_0(D-2+2\gamma)/2)$. Because both of λ_0 and δc^A are sufficiently small, the solution S_τ can be expanded in terms of λ_0 and δc^A as

$$
S_\tau = S^* + \sum_A(\delta c^A\xi^A(\tau')+\lambda_0^{n_{\min}-1}\zeta^A(\tau'))\mathcal{O}_A + (\text{higher-order terms}). \quad (4.1.27)
$$

Note that the leading contribution from λ_0 in the expansion should be proportional to $\lambda_0^{n_{\min}-1}$ since λ_0 appears in Eq. (4.1.26) as $\lambda_0^{n_{\min}-1}$ at the leading order. Substituting this equation into the GFERG equation and focusing on the terms up to the linear order of δc^A and $\lambda_0^{n_{\min}-1}$, we get

$$
\partial_{\tau'}\sum_A(\delta c^A\xi^A(\tau')+\lambda_0^{n_{\min}-1}\zeta^A(\tau'))\mathcal{O}_A = \hat{R}\sum_A(\delta c^A\xi^A(\tau')+\lambda_0^{n_{\min}-1}\zeta^A(\tau'))\mathcal{O}_A
$$

$$
+\lambda_0^{n_{\min}-1}e^{-\tau'(n_{\min}-1)(D-2+2\gamma)/2}H(S^*), \quad (4.1.28)
$$

where

$$
\hat{R} := \int_p\Bigg\{\left[\left(2p^2+\frac{D+2}{2}-\gamma\right)\phi_a(p)+p_\mu\frac{\partial}{\partial p_\mu}\phi_a(p)\right]\frac{\delta}{\delta\phi_a(p)}
$$

$$
+\frac{1}{p^2}\left[4p^2k(p)+2p^2\frac{dk(p)}{dp^2}-2\gamma k(p)\right]\frac{\delta S^*}{\delta\phi_a(p)}\frac{\delta}{\delta\phi_a(-p)}\Bigg\} \quad (4.1.29)
$$

and

$$
H(S^*) := -2e^{-S^*}\int_{x,x_1,\dots,x_{n_{\min}}}\frac{\delta}{\delta\phi_a(x)}\left\{f_a^{a_1,\dots,a_{n_{\min}}}\left(\phi_{a_1}(x_1)+\int_{y_1}\mathcal{D}(x_1-y_1)\frac{\delta}{\delta\phi_{a_1}(y_1)}\right)\times\dots\right.
$$

$$
\left.\times\left(\phi_{a_{n_{\min}}}(x_{n_{\min}})+\int_{y_{n_{\min}}}\mathcal{D}(x_{n_{\min}}-y_{n_{\min}})\frac{\delta}{\delta\phi_{a_n}(y_{n_{\min}})}\right)\right\}e^{S^*} \quad (4.1.30)
$$

Comparing each term of $\mathcal{O}(\lambda_0^{n_{\min}-1})$ and $\mathcal{O}(\delta c^A)$ in the left and right hand sides of Eq. (4.1.28), we get

$$
\sum_A\delta c^A\partial_{\tau'}\xi^A(\tau')\mathcal{O}_A = \sum_A\delta c^A\xi^A(\tau')\hat{R}\mathcal{O}_A \quad (4.1.31)
$$

$$
\sum_A\partial_{\tau'}\zeta^A(\tau')\mathcal{O}_A = \sum_A\zeta^A(\tau')\hat{R}\mathcal{O}_A + e^{-\tau'(n_{\min}-1)(D-2+2\gamma)/2}H(S^*). \quad (4.1.32)
$$

Because $\hat{R}$ is a time-independent operator on the functional space, $\mathcal{O}_A$ can be taken as its eigenoperator satisfying

$$\hat{R}\mathcal{O}_A = x_A \mathcal{O}_A \quad (A = 1, 2, \ldots) \tag{4.1.33}$$

with its eigenvalue x_A. Note that the index A is not summed in this equation. Since $\{\mathcal{O}_A\}$ forms a complete set on the functional space, $H(S^*)$ can be expanded as

$$H(S^*) = h^A \mathcal{O}_A, \tag{4.1.34}$$

where h^A is an expansion coefficient. Substituting this expression into Eq. (4.1.31) and Eq. (4.1.32), and focusing on the each coefficient of $\mathcal{O}_A$, we get

$$\partial_{\tau'}\xi^A(\tau') = x_A \xi^A(\tau'), \tag{4.1.35}$$

$$\partial_{\tau'}\zeta^A(\tau') = x_A \zeta^A(\tau') + e^{-\tau'(n_{\min}-1)(D-2+2\gamma)/2} h^A. \tag{4.1.36}$$

for each $A = 1, 2, \cdots$. The solutions to these equations are given by

$$\xi^A(\tau') = e^{x_A \tau'}, \tag{4.1.37}$$

$$\zeta^A(\tau') = \frac{e^{x_A\tau'} - e^{-(n_{\min}-1)(D-2+2\gamma)\tau'/2}}{x_A + (n_{\min}-1)(D-2+2\gamma)/2} h^A, \tag{4.1.38}$$

where we have used the initial conditions $\xi^A(0) = 1$ and $\zeta^A(0) = 0$. Finally, we get

$$S_\tau = S^* + \sum_A \left(\delta c^A e^{x_A\tau'} + \lambda_0^{n_{\min}-1} \frac{e^{x_A\tau'} - e^{-(n_{\min}-1)(D-2+2\gamma)\tau'/2}}{x_A + (n_{\min}-1)(D-2+2\gamma)/2} h^A \right) \mathcal{O}_A \tag{4.1.39}$$

to the order of $\lambda_0^{n_{\min}-1}$ and δc^A. Note that if $x_A + (n_{\min}-1)(D-2+2\gamma)/2 = 0$, we have

$$\left. \frac{e^{x_A\tau'} - e^{-(n_{\min}-1)(D-2+2\gamma)\tau'/2}}{x_A + (n_{\min}-1)(D-2+2\gamma)/2} \right|_{x_A + (n_{\min}-1)(D-2+2\gamma)/2 = 0} = \tau' e^{x_A\tau'}. \tag{4.1.40}$$

Let us discuss scaling dimensions of operators at the fixed point. In the conventional ERG formalism such as the WP equation, the scaling dimension d_A of an operator $\mathcal{O}_A$ can be determined from the time-evolution of S_τ in the direction of $\mathcal{O}_A$. For example, let us consider the WP equation, which corresponds to the case $H(S^*) = 0$, i.e., $h^A = 0$. There S_τ is given by

$$S_\tau = S^* + \sum_A \delta c^A e^{x_A\tau'} \mathcal{O}_A. \tag{4.1.41}$$

Because the time-dependence has a simple form of $e^{x_A\tau'}$, the scaling dimension d_A is defined as $d_A = x_A$.

On the other hand, in the case of the GFERG equation, the time-dependence of S_τ in $\mathcal{O}_A$ is a linear combination of $e^{x_A\tau'}$ and $e^{-(n_{\min}-1)(D-2+2\gamma)\tau'/2}$ (see Eq. (4.1.39)).

Therefore we should be careful about defining relevant or irrelevant operators and their scaling dimensions in this case. Recall that whether an operator is relevant or irrelevant corresponds to whether the amplitude of its coupling increases or not (i.e., its linearized flow departs from/converges to the fixed point) as τ' increases. Therefore, we find that an operator with positive (negative) x_A should be called relevant (irrelevant) in GFERG like the conventional ERG formalism.

Let us discuss the scaling dimensions of relevant operators from the viewpoint of observable quantities in experiments. They can be measured by tuning parameters so that the system undergoes a phase transition. There, the observable quantities are determined by the infrared (IR) behavior of the system, which is described by the renormalized trajectory of the fixed point. Since the renormalized trajectory is defined by taking the IR limit ($\tau_0 \to \infty$) with tuning the relevant (bare) couplings, one is led to consider the $\lambda_0 \to 0$ limit to define the scaling dimensions of relevant operators. Because the time-dependence of S_τ in the direction of $\mathcal{O}_A$ in this limit is the same as the WP equation, we should define their scaling dimensions d_A as $d_A = x_A$ like the conventional ERG formalism.

As for irrelevant operators, their scaling dimensions should be determined as the convergence speed to the fixed point when the theory sits on a critical surface. From Eq. (4.1.39), we see that for a sufficiently large time $\tau' \gg 1$, the coefficient of the (irrelevant) operator $\mathcal{O}_A$ is given by

$$\delta c^A e^{x_A \tau'} + \lambda_0^{n_{\min}-1} \frac{e^{x_A \tau'} - e^{-(n_{\min}-1)(D-2+2\gamma)\tau'/2}}{x_A + (n_{\min}-1)(D-2+2\gamma)/2} h^A \propto e^{-\tau' \min(|x_A|,(n_{\min}-1)(D-2+2\gamma)/2)} \quad (\tau' \gg 1). \quad (4.1.42)$$

Therefore, from the above argument, the scaling dimension d_A of the irrelevant operator $\mathcal{O}_A$ should be defined as $d_A = -\min(|x_A|, n_{\min}(D-2+2\gamma))$.

Here we comment on the case where the expansion coefficient h^A becomes zero. In this case, time-dependence of S_τ in the direction of the corresponding operator is the same as in the WP equation, i.e., $\delta c^A e^{x_A \tau'}$. Therefore, the scaling dimension of this operator is given by x_A regardless of whether they are relevant or irrelevant. We will encounter this case in the next section.

4.2 Example : Non-linear Sigma Model in $4-\epsilon$ Dimensions

In this section, we illustrate our results in Section 4.1 with the $O(N)$ non-linear sigma model in $4-\epsilon$ dimensions and the WF fixed point.

4.2.1 GFERG Equation

Lagrangian of the non-linear sigma model is given by

$$\mathcal{L} = \frac{1}{2g^2}\partial_\mu\phi_a\partial^\mu\phi_a, \tag{4.2.1}$$

where ϕ_a $(i = 1, \ldots, N)$ is a real scalar field constrained by

$$\phi_a\phi_a = 1. \tag{4.2.2}$$

Note that this constraint requires the physical degree of freedom to be $N-1$. g^2 is a bare coupling constant.

It is well-known that this model is defined non-perturbatively on the renormalized trajectory of the WF fixed point [23]. The $O(N)$ liner sigma model with the quartic interaction also belongs to this WF universality class after the $O(N)$ symmetry breaks spontaneously down to $O(N-1)$ with a negative mass term. By setting $D = 4-\epsilon$ and solving the fixed point condition Eq. (4.1.24) of the WP equation with the ϵ expansion [39], we get the action S^*_{WF} at the WF fixed point up to $\mathcal{O}(\epsilon)$ as

$$S^*_{\mathrm{WF}} = -\int_p \frac{p^2}{2K(p)}\phi_a(p)\phi_a(-p) + \int_x \left(\frac{m_*^2}{2}\phi_a(x)^2 - \frac{\lambda_*}{8}(\phi_a(x)^2)^2\right), \tag{4.2.3}$$

where the (dimensionless) couplings m_*^2 and λ_* are defined as [2]

$$m_*^2 := \frac{\epsilon}{4}\frac{N+2}{N+8}, \quad \lambda_* := -\epsilon\frac{8\pi^2}{N+8}. \tag{4.2.4}$$

If the $O(N)$ symmetry is spontaneously broken, the theory contains one massive mode and $N-1$ Nambu-Goldstone (NG) modes. When we focus on a much lower energy scale compared to the mass of the former, the massive particle becomes sufficiently heavy to decouples from the NG modes. These remaining NG bosons correspond to the $N-1$ physical degrees of freedom in the $O(N)$ non-linear sigma model in the IR region.

The gradient flow equation for this model is given in Ref. [36] as

$$\partial_t\varphi_a = \partial_\mu^2\varphi_a - (\varphi_b\partial_\mu^2\varphi_b)\varphi_a \tag{4.2.5}$$

with the initial condition $\varphi_a(0,x) = \phi_a(x)$ for $a, b = 1, \ldots, N$. An advantage of adopting this flow equation is that in two-dimensions, correlation functions of the flowed field $\varphi_a(t,x)$ are UV-finite *without* additional wave function renormalization, i.e., Z_τ can be set to unity. On the other hand, Z_τ cannot be omitted in the present case, which is obvious from the following results in order for the WF fixed point to exist.

The Wilson action of this model can be defined in the same way as Eq. (4.1.3) via the solution $\varphi'_a(t,x)$ to Eq. (4.2.5) with the initial condition $\varphi'_a(0,x) = \phi_a(x)$. The GFERG equation associated with this Wilson action is given by

[2] λ^* in Eq. (4.2.4) is different from that in Ref. [39] by a factor 2. There seems to be a typo in Ref. [39].

$$
\begin{aligned}
\frac{\partial}{\partial\tau}e^{S_\tau[\phi_a]} = \int_p \Bigg\{ &\left[\left(2p^2 + \frac{D+2}{2} - \gamma_\tau\right)\phi_a(p) + p_\mu \frac{\partial}{\partial p_\mu}\phi_a(p)\right]\frac{\delta}{\delta\phi_a(p)} \\
&+ \frac{1}{p^2}\left[4p^2 k(p) + 2p^2\frac{dk(p)}{dp^2} - 2\gamma_\tau k(p)\right]\frac{1}{2}\frac{\delta^2}{\delta\phi_a(p)\delta\phi_a(-p)}\Bigg\} e^{S_\tau[\phi_a]} \\
+ 2\lambda(\tau)^2 \int_x \frac{\delta}{\delta\phi_a(x)} &\left(\phi_b(x) + \int_y \mathcal{D}(x-y)\frac{\delta}{\delta\phi_b(y)}\right)\partial_\mu^2\left(\phi_b(x) + \int_y \mathcal{D}(x-y)\frac{\delta}{\delta\phi_b(y)}\right) \\
&\times \left(\phi_a(x) + \int_y \mathcal{D}(x-y)\frac{\delta}{\delta\phi_a(y)}\right)e^{S_\tau[\phi_a]}. \quad (4.2.6)
\end{aligned}
$$

The terms in the third and the fourth lines of this equation are peculiar to GFERG, compared to the WP equation. Note that this GFERG equation is invariant under the global $O(N)$ symmetry and expected to preserve the constraint $\phi_a^2 = \text{const.}$ in the correlation functions.

4.2.2 Wilson-Fisher Fixed Point

Let us confirm that the GFERG equation Eq. (4.2.6) has the WF fixed point in the $\tau \to \infty$ limit. Since S^*_{WF} satisfies $\partial_\tau S^*_{\text{WF}} = 0$ and the fixed point condition of the WP equation Eq. (4.1.24), all we have to confirm is the vanishing of $\lambda(\infty)$. As was seen in Section 4.1.2, the asymptotic behavior of $\lambda(\tau)$ at the large flow time is controlled by the signature of the quantity $D - 2 + 2\gamma$. The anomalous dimension γ can be explicitly calculated with the ϵ expansion at this fixed point [39] and is given to $\mathcal{O}(\epsilon^2)$ by

$$
\gamma = \frac{N+2}{(N+8)^2}\frac{\epsilon^2}{2}. \quad (4.2.7)
$$

Then, we get

$$
D - 2 + 2\gamma = 2 - \epsilon + \frac{N+2}{(N+8)^2}\epsilon^2 \quad (4.2.8)
$$

to $\mathcal{O}(\epsilon^2)$ in $D = 4 - \epsilon$. Since ϵ is small within the ϵ expansion, this quantity is positive. Recalling that $\lambda(\tau)$ behaves asymptotically as $\tau \to \infty$ like

$$
\lambda(\tau) \sim \exp\left(-\frac{\tau}{2}(D - 2 + 2\gamma)\right), \quad (4.2.9)
$$

we conclude that $\lambda(\tau)$ vanishes at $\tau = \infty$. Therefore, we readily find that the action S^*_{WF} at the WF fixed point satisfies the GFERG equation in the $\tau \to \infty$ limit.

Let us see the relationship between the signature of $D - 2 + 2\gamma$ and the cluster decomposition principle concretely from the two-point function of the WF fixed point action. According to [39], the connected two-point function of ϕ_a is given to $\mathcal{O}(\epsilon^2)$ by

$$
\langle \phi_a(p)\phi_b(-p)\rangle_{\text{connected}} = \frac{\delta_{ab}}{(p^2)^{1-\gamma}} \quad (4.2.10)
$$

in the momentum space. By performing the inverse Fourier transformation, we get

$$\langle \phi_a(x)\phi_b(0)\rangle_{\text{connected}} \propto \frac{\delta_{ab}}{x^{D-2+2\gamma}} \tag{4.2.11}$$

in the position space. From this equation, we can explicitly confirm that $D-2+2\gamma>0$ follows from the cluster decomposition principle.

4.2.3 Perturbative Solution around WF Fixed Point

In this subsection, we solve Eq. (4.2.6) to $\mathcal{O}(\epsilon)$ around the WF fixed point and study the flow structure around it. The solution to the general GFERG equation is already given in Eq. (4.1.39). In the present case, $D=4-\epsilon$ and $n_{\text{min}}=3$, and then S_τ is given to the linear order in $\epsilon, \delta c^A$ and λ_0^2 by

$$S_\tau = S^* + \sum_A \left(\delta c^A e^{x_A\tau'} + \lambda_0^2 \frac{e^{x_A\tau'} - e^{-(2-\epsilon+2\gamma)\tau'}}{x_A+2-\epsilon+2\gamma} h^A\right)\mathcal{O}_A. \tag{4.2.12}$$

h^A is defined in the same way as Eq. (4.1.34), where $H(S^*)$ is given by

$$\begin{aligned}H(S^*) \coloneqq -2e^{-S^*}\int_x \frac{\delta}{\delta\phi_a(x)}\left(\phi_b(x) + \int_y \mathcal{D}(x-y)\frac{\delta}{\delta\phi_b(y)}\right)\\ \times\, \partial_\mu^2\left(\phi_b(x) + \int_y \mathcal{D}(x-y)\frac{\delta}{\delta\phi_b(y)}\right)\left(\phi_a(x) + \int_y \mathcal{D}(x-y)\frac{\delta}{\delta\phi_a(y)}\right)e^{S^*}.\end{aligned} \tag{4.2.13}$$

Here we study contributions of some eigenoperators to S_τ concretely around the WF fixed point. To this end, we must specify the eigenoperators $\{\mathcal{O}_A\}$ of the linearized WP equation around it. The Wilson action is decomposed around the fixed point as $S_\tau = S^* + \delta S(\tau)$, and we use the local potential approximation (LPA), in which the fluctuation $\delta S(\tau)$ takes the following form:

$$\begin{cases} \delta S(\tau) = \displaystyle\int_x V \\ \quad V = \displaystyle\sum_{n=2}^{N_{\max}} \frac{g_{2n}(\tau)}{2^n n!}(\phi_a(x)^2)^n, \end{cases} \tag{4.2.14}$$

where $g_{2n}(\tau)$ is the τ-dependent coupling of the $2n$-point vertex and $N_{\max}$ is the truncation level of the LPA larger than 2.

By substituting Eq. (4.2.14) into the linearized WP equation with respect to δS, we can write down the time evolution equation for $g_n(\tau)$ and calculate $\hat{R}$ explicitly. Then we get a set of its eigenoperators $\mathcal{O}_A$ by diagonalizing it. The detailed calculations are shown in Sec. 2.4.2 and we just cite its results here; $\hat{R}$ has only one relevant operator

$$\mathcal{O}_1 = \phi_a(x)^2 + \mathcal{O}(\epsilon) \quad \text{with} \quad x_1 = 2 - \epsilon\frac{N+2}{N+8} + \mathcal{O}(\epsilon^2), \tag{4.2.15}$$

and the other local operators are all irrelevant. An example of the irrelevant operators is

$$\mathcal{O}_2 = (\phi_a(x)^2)^2 - \frac{N+2}{8\pi^2}\phi_a(x)^2 + \mathcal{O}(\epsilon) \quad \text{with} \quad x_2 = -\epsilon + \mathcal{O}(\epsilon^2). \tag{4.2.16}$$

Note that this result does not depend on the truncation level $N_{\max}$.

Then we can calculate the expansion coefficient h^A for these operators and their contributions to S_τ. Because the right hand side of Eq. (4.2.13) has one factor of the Laplacian ∂_μ^2, $H(S^*)$ is expanded with field operators with two or more derivatives. Thus as far as $\mathcal{O}_A$ is a linear combination of operators without derivatives like $(\phi_a(x)^2)^n$, its expansion coefficient h^A of $H(S^*)$ is zero. Therefore, we find that their contributions to S_τ within the LPA are given by

$$S_\tau = S^* + \int_x \Big[\delta c^1 e^{(2-\epsilon(N+2)/(N+8))\tau'}\mathcal{O}_1 + \delta c^2 e^{-\epsilon\tau'}\mathcal{O}_2\Big] + \sum_{A\neq 1,2}\left(\delta c^A e^{x_A\tau'} + \lambda_0^2 \frac{e^{x_A\tau'} - e^{-(D-2+2\gamma)\tau'}}{x_A + 2 - \epsilon} h^A\right)\mathcal{O}_A. \tag{4.2.17}$$

Finally, let us discuss the scaling dimensions of the eigenoperators $\mathcal{O}_A$ around the fixed point. As we have seen in the previous paragraph, the expansion coefficient h^A for a linear combination of the field operators without derivatives like $(\phi_a(x)^2)^n$ is zero in the present case of the gradient flow equation Eq. (4.2.5). Thus, the time dependence of S_τ in the direction of such operators is just given by $e^{x_A t}$ as seen from Eq. (4.2.17). Because this time dependence agrees with the WP equation, we conclude that such operators have the same scaling dimensions as the WP equation. For operators with derivatives, which appear when one goes beyond the LPA, their expansion coefficients h^A do not vanish in general. In such a case, the non-linear terms in the gradient flow equation give a difference between the GFERG equation and the WP equation. Therefore the scaling dimensions of those operators are different from those of the WP equation if they are irrelevant. It should be noted that this result highly depends on the form of the non-linear terms in the gradient flow equation

4.3 Miscellaneous Comments

In this section, we give some comments on our result as follows.

Is GFERG a kind of ERG?

As stated above, irrelevant operators in the GFERG equation have different scaling dimensions from those in the WP equation in general. On the other hand, it is believed that different schemes provide the same scaling dimensions within ERG by a field redefinition. Thus it seems that *GFERG is not a kind of ERG* but an alternative framework to study the low-energy physics in the Wilsonian sense. We, however, emphasize that GFERG gives the same prediction on the low-energy renormalized theory as ERG. This is because they have the same renormalized trajectories and critical exponents around the fixed points. This point will be confirmed by further studies elsewhere.

Gaussian fixed point for $O(N)$ non-linear sigma model

The Gaussian fixed point can arise in the GFERG equation of the $O(N)$ non-linear sigma model in addition to the WF one. This fact seems mysterious because the $O(N)$ non-linear sigma model belongs to the universality class characterized by the WF fixed point rather than the Gaussian one. The existence of the Gaussian fixed point seems extra. However, we should note that the action at the Gaussian fixed point does not satisfy the constraint $\phi_a^2 = \text{const.}$, which is always respected by the GFERG flow. Thus this Gaussian fixed point is only apparent and that the corresponding flow will never converge to it whatever the initial condition of the GFERG equation is. In other words, any initial points satisfying the condition $\phi_a^2 = 1$ at $\tau = 0$ do not flow into the Gaussian fixed point for $\tau \to \infty$.

Loophole

We also comment on our discussion to obtain the fixed points of the GFERG equation in Section 4.1.2. Although the vanishing of $\lambda(\tau)$ is essential there, we have an exceptional case in which $\lambda(\tau)$ does not depend on the RG flow time ($d\lambda(\tau)/d\tau = 0$), i.e., $2\gamma_\tau = 2 - D$ holds for an arbitrary flow time τ. In particular, this equation requires γ_τ to be zero in two-dimensions. This means that Z_τ is also time-independent constant, and we do not need to perform the additional wave function renormalization for the fields φ_a to keep their correlation functions UV finite. An example is the $O(N)$ non-linear sigma model in two-dimensions [36], with which the GFERG equation associated can have fixed points that are not covered by our argument.

Fixed Points of GFERG Equation at Finite Time

Here, we comment on fixed points of the general GFERG equation at the finite flow time. Because $\partial_\tau S^* = 0$ is required at an arbitrary time, S^* should satisfy

$$\left(\phi_a(x) + \int_y \mathcal{D}(x-y)\frac{\delta}{\delta\phi_a(y)}\right)e^{S^*} = 0 \tag{4.3.1}$$

in addition to the fixed point condition of the WP equation. The solution to this equation can be found easily and given by

$$S^* = -\frac{1}{2}\int_{x,y}\mathcal{D}(x-y)\phi_a(x)\phi_a(y) = -\frac{1}{2}\int_p \frac{k(p)}{p^2}\phi_a(p)\phi_a(-p). \tag{4.3.2}$$

The fixed point condition of the WP equation requires $k(p)$ to satisfy

$$\left(4p^2k(p) + 2p^2\frac{dk(p)}{dp^2} - 2\gamma_\tau k(p)\right)\left(\left(\frac{k(p)}{p^2}\right)^2 - 1\right) = 0 \tag{4.3.3}$$

Because $k(p)$ is given by $K(p)(1-K(p)) = e^{-p^2}(1-e^{-p^2})$, it does not satisfy Eq. (4.3.3) and therefore we have no fixed point at finite time.

Preserving Constraint along GFERG Flow

As is related with the GFERG flow for $O(N)$ non-linear sigma model, we can define an RG flow that preservers the constranit for the field by GFERG. In other word, if the gradient flow equation preserves the constraint of fields, the GFERG flow also preserves it. The discussion is given as follows.

Let us explain our setup in more details. Consider some constrained fields (such as the non-linear sigma model) by $G(\phi_i(x)) = 0$, and the gradient flow equation $\partial_t\varphi_i(t,x) = F_i(\varphi(t,x))$ which satisfies

$$0 = \partial_t G(\varphi_i(t,x)) = \partial_t\varphi_i(t,x)\frac{\partial G}{\partial\varphi_i(t,x)} = F_i(\varphi(t,x))\frac{\partial G}{\partial\varphi_i(t,x)}. \tag{4.3.4}$$

This condition guarantees that the constraint is preserved under the gradient flow, i.e.,

$$G(\varphi_i(t,x)) = 0 \tag{4.3.5}$$

holds for an arbitrary flow time t and spacetime point x.

The Wilson action S_τ is defined as

$$e^{S_\tau[\phi]} = \hat{s}_\phi^{-1}\int[D\phi']\prod_{x,i}\delta(\phi_i(x) - \varphi'_i(t, e^\tau x))\hat{s}_{\phi'}e^{S_{\tau=0}[\phi']}, \tag{4.3.6}$$

where $\varphi'_i(t,x)$ is the solution to the gradient flow equation with the initial condition of $\varphi'_i(0,x) = \phi'_i(x)$ and $t = e^{2\tau} - 1$. Here we emphasize that we do *not* consider the rescaling factor ($e^{\tau(D-2)/2}Z_\tau^{1/2}$) of the fields in the above definition. This normalization seems natural in the models such as the two-dimensional non-linear sigma model.

We claim that the Wilson action satisfies

$$\langle\!\langle G(\phi)\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)\rangle\!\rangle_S^k = 0 \tag{4.3.7}$$

for an arbitrary scale parameter τ. Recall that the modified correlation function of the product of operators $\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)$ is defined as

$$\langle\!\langle\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)\rangle\!\rangle_{S_\tau}^k := \int[D\phi]\hat{s}(\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n))e^{S_\tau}. \tag{4.3.8}$$

Eq. (4.3.7) means that the constraint $G(\phi) = 0$ always holds in the modified correlation functions, and the GFERG flow keeps the information of the target space.

Here is the proof of Eq. (4.3.7). The left hand side of Eq. (4.3.7) is

$$\langle\!\langle G(\phi)\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)\rangle\!\rangle_S^k \tag{4.3.9}$$

$$= \int [D\phi]\, \hat{s}(G(\phi)\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n))e^{S_\tau} \tag{4.3.10}$$

$$= \int [D\phi]\, G(\phi)\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)\hat{s}e^{S_\tau} \tag{4.3.11}$$

$$= \int [D\phi' D\phi]\, G(\phi)\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)\prod_{x,i}\delta(\phi_i(x) - \varphi'_i(t, e^\tau x))\hat{s}e^{S_{\tau=0}[\phi'])} \tag{4.3.12}$$

$$= \int [D\phi']\, G(\varphi'(t, e^\tau x))\varphi'_{i_1}(t, e^\tau x_1)\cdots\varphi'_{i_n}(t, e^\tau x_n)\hat{s}e^{S_{\tau=0}[\phi']}. \tag{4.3.13}$$

Because Eq. (4.3.5) holds for an arbitrary flow time and spacetime point, $G(\varphi'(t, e^\tau x))$ also vanishes. Therefore we have obtained Eq. (4.3.7).

Chapter 5

Conclusion

In the first half of this thesis (Chapters 2 and 3), we have reviewed Exact Renormalization Group (ERG) and Gradient Flow Exact Renormalization Group (GFERG), a new framework to define the Wilsonian effective action via the diffusion equation. In the latter half (Chapter 4), based on my work [1], we applied GFERG to general scalar field theories and discussed their fixed point structures.

In Chapter 2, we have reviewed the basics of Exact Renormalization Group. We have defined an effective action $S_\Lambda[\phi]$ that describes the physics at a focused energy scale Λ. Then we studied the flow equation for $S_\Lambda[\phi]$ and gave some examples. We also have considered an IR limit to find fixed points of the ERG equation. The renormalized trajectory from the fixed point describes the IR physics. At this point, we were led to the notion of "universality", which states that the IR behavior of various UV theories can be described universally by the same renormalized trajectory. Furthermore, we have studied the flow structure around the fixed points and have determined the critical exponents. We have illustrated these notions by the Gaussian and Wilson-Fisher fixed points.

In Chapter 3, we have reviewed Gradient Flow Exact Renormalization Group. GFERG was initially proposed to define the Wilsonian effective action for gauge theories in a gauge-invariant manner. Because this framework utilizes coarse-graining via the diffusion equation to define the Wilsonian effective action, we can define an RG flow that respects local symmetries or non-linearity of the system with it. Then we derived the GFERG equation, the counterpart of the ERG equation in GFERG. We also have reviewed recent developments on GFERG: Inclusion of fermion fields, the perturbative analysis of QED, and the flow equation for the one-particle irreducible effective action.

Chapter 4 is the main part of this thesis. We have discussed GFERG for scalar field theories in general and investigated its fixed point structure. We have explicitly written down the GFERG equation based on an arbitrary polynomial diffusion equation and then discussed its fixed point action. Remarkably, the fixed points appear for a large flow time limit and are precisely the same as those of the Wilson-Polchinski (WP) equation. It is because the non-linear terms in the GFERG equation involves $\lambda(\tau) := Z_\tau^{-1/2} \exp(-\tau(D-2)/2)$, where Z_τ is the wavefunction renormalization factor, and $\lambda(\tau)$ vanishes in the IR limit. These non-linear terms originates from those in the diffusion equation. Furthermore, we have calculated the scaling dimensions of operators around the fixed points by solving the GFERG equation to the leading order of the deviations from the fixed points and $\lambda(\tau_0)$, where τ_0 is the scaler parameter corresponding to the bare scale Λ_0. We find that the relevant operators around the fixed points of the GFERG equation have the same scaling dimensions as those of the WP equation, while the irrelevant operators have different ones generally. Therefore, the critical exponents around the fixed points of the GFERG equation are the same as those of the corresponding fixed points of the WP equation, resulting in the same prediction for its low-energy physics.

As a possible future direction, it is interesting to consider gauge theories within GFERG. Since the most plausible point of GFERG is its manifest gauge invariance, it would help us to investigate their fixed point structures in a gauge invariant way. However, the situation there is expected to be different from the case of scalar field theories. The key point of our analysis is the vanishing of $\lambda(\tau)$, and this quantity should not vanish for gauge theories or gravity at large flow times. Indeed, as was discussed in the original paper [8], the counterpart of $\lambda(\tau)$ in the pure Yang-Mills theory is given by $Z_\tau^{-1/2} \exp(-\tau(D-4)/2)$, and becomes τ-independent constant because Z_τ can be set to unity in $D = 4$. Thus our present argument is not applied to them straightforwardly and we need more detailed arguments for GFERG in these theories, which is left as future work.

Another future direction is the asymptotically safe gravity. It is known that the Einstein-Hilbert action is not renormalizable around the Gaussian fixed point. It is because the Ricci scalar is expanded as an infinite-order polynomial of the fluctuation of the gravitational field around the flat metric and contains many irrelevant interactions. It is called the "continuum limit" to remove the cutoff and define the quantum field theory, and we can determine the continuum limit of a theory on the renormalized trajectory of a fixed point. The attempt to define quantum gravity around any non-trivial fixed point is called the "asymptotic safety" program. It is known that there can be a non-physical fixed point along RG flows without manifest diffeomorphism invariance [40–44]. By contrast, GFERG can define an RG flow that manifestly preserves local symmetries, and we expect that GFERG is helpful to define one with manifest diffeomorphism invariance and to contribute to the asymptotic safety program.

It is also interesting to study non-linear sigma models in two dimensions by the GFERG method, a loophole of our discussion in Chapter 4 . Some non-linear sigma models (e.g., $O(N)$ non-linear sigma model or $\mathbb{CP}^{N-1}$ model) have similar properties to Yang-Mills theory, such as gauge redundancy and field configurations with non-trivial topology or asymptotic freedom. Studying these models with the GFERG method may give clues to the understanding of the non-perturbative aspects of the gauge theories.

As seen in this thesis, GFERG is a promising approach to studying non-linear systems within the framework of ERG. In particular, it may provide new insights into the theories with non-trivial target space, such as gauge theories or non-linear sigma models. We believe that GFERG will become one of the standard methods to analyze QFTs non-perturbatively in the future and hope that this thesis will contribute to it.

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Appendix A

Notation

A.1 Notation

In this paper, we use the following compact notation for integrals:

$$\int_x := \int d^D x, \qquad \int_p := \int \frac{d^D p}{(2\pi)^D}, \tag{A.1.1}$$

where p denotes a momentum. The Dirac's delta function in real space is given by

$$\delta^D(x) = \int_p e^{ip\cdot x}. \tag{A.1.2}$$

The Fourier transformation of $\phi(x)$ is

$$\phi(x) = \int_p \phi(p) e^{ip\cdot x}, \qquad \phi(p) = \int_x \phi(x) e^{-ip\cdot x}. \tag{A.1.3}$$

The functional derivative with respect to the field in the momentum space $\phi(p)$ is defined by the Fourier transformation as

$$\frac{\delta}{\delta\phi(p)} := \int_x e^{ip\cdot x} \frac{\delta}{\delta\phi(x)}, \tag{A.1.4}$$

which satisfies the following normalization:

$$\frac{\delta}{\delta\phi(q)}\phi(p) = \int_x e^{i(q-p)\cdot x} = (2\pi)^D \delta^D(p-q). \tag{A.1.5}$$

The Laplacian ∂_μ^2 is defined as

$$\partial_\mu^2 := \sum_{\mu=1}^{D} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\mu}. \tag{A.1.6}$$

$p \cdot \partial_p$ and $x \cdot \partial_x$ are defined as

$$p \cdot \partial_p \coloneqq p_\mu \frac{\partial}{\partial p_\mu}, \tag{A.1.7a}$$

$$x \cdot \partial_x \coloneqq x^\mu \frac{\partial}{\partial x^\mu}. \tag{A.1.7b}$$

The Feynman slash notation such as $\not{p}$ is defined as

$$\not{p} \coloneqq p_\mu \gamma^\mu, \tag{A.1.8}$$

where γ^μ is the gamma matrices.

A.2 Useful Formula

Consider the following partial differential equation:

$$\left(x_i \frac{\partial}{\partial x_i} + \zeta \right) F(x) = f(x). \tag{A.2.1}$$

A particular solution to this equation is given by

$$F_p(x) = \int_0^1 \frac{d\alpha}{\alpha} \alpha^\zeta f(\alpha x), \tag{A.2.2}$$

which satisfies

$$\lim_{\alpha \to 0} F_p(\alpha x) = 0. \tag{A.2.3}$$

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