Application of quasiconformal surgery to some transcendental meromorphic functions

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Abstract

Rational functions are meromorphic in the Riemann sphere. Many results on dynamics of rational functions are known. In particular, for rational functions, the Fatou-Shishikura inequality holds and is best possible in some sense. In addition, irrationally indifferent periodic points are Cremer points if their multipliers satisfy some condition, and all bounded type Siegel disks are bounded by quasicircles containing critical points. Transcendental meromorphic functions are not rational and have an essential singularity at ∞. Transcendental functions and rational functions have quite different properties. However, transcendental functions with finitely many singular values share important dynamical properties with rational functions. Therefore, we can expect the generalization of the results for rational functions to such transcendental functions.

Quasiconformal surgery is an important technique. Roughly speaking, it modifies given meromorphic functions to new meromorphic functions with given dynamical properties. In this thesis, we apply quasiconformal surgery technique to some transcendental meromorphic functions with finitely many singular values in order to obtain various kinds of results:

(1) Eremenko and Lyubich showed the Fatou-Shishikura inequality for transcendental entire functions in the Speiser class in [EL]. We show that the inequality is best possible on the analogy of the Fatou-Shishikura inequality for rational functions in [Shi].

(2) Let $\mathscr{S}$ be the set of all transcendental entire functions of the form

$$P(z) \exp (Q(z)),$$

where P and Q are polynomials. By using the theory of polynomial-like mappings, we tell that irrationally indifferent fixed points of some functions in $\mathscr{S}$ are Cremer points by their multipliers. We also use the theory and obtain some transcendental entire function in $\mathscr{S}$ with bounded type Siegel disks centered at points other than the origin bounded by quasicircles containing critical points, with a Siegel disk bounded by a Jordan curve containing critical points, which is not a quasicircle, and with a Siegel disk bounded by a quasicircle without critical points. Moreover, we construct some functions in $\mathscr{S}$ with the Julia sets of Lebesgue measure zero and with the Julia sets of positive Lebesgue measure. Those are some extensions of Geyer's result in [Gey], Cremer's result in [Cr1], Zakeri's result in [Za2], and Keen and Zhang's result in [KeZ].

(3) We discuss a one parameter family of some transcendental meromorphic functions with one pole, two critical points, and a bounded type Siegel disk centered at the origin. We show that if two critical values coincide, then the boundary of the Siegel disk is a quasicircle containing exactly one critical point, and the set of parameters for which two critical values coincide is countably infinite. We also show that for any parameter in some uncountable set, the boundary of the Siegel disk is a quasicircle containing exactly one critical point. In addition, we can choose a parameter so that the boundary of the Siegel disk is a quasicircle containing exactly two critical points. This is an extension of the result in [Za2] or [KeZ].

Contents

Chapter 1

Introduction and the results

Iteration of a meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ for an initial point $z_0 \in \mathbb{C}$ yields the following sequence:

$$z_0, z_1 = f(z_0), z_2 = f(z_1) = f(f(z_0)) = f^2(z_0), \cdots, z_n = f^n(z_0),$$

where $f^n = \overbrace{f \circ f \circ \cdots \circ f}^{n}$, whenever z_0 is not a preimage of ∞ by $f, f^2, \cdots, f^{n-1}$. The main purpose of studying complex dynamics is to understand the behavior of such sequence $\{z_0, z_1, \cdots\}$ as $n \to \infty$. The Riemann sphere $\widehat{\mathbb{C}}$ is divided into the *Fatou set* $F(f)$ and the *Julia set* $J(f)$. (Their names come from two pioneers Fatou and Julia of complex dynamics.) The set $F(f)$ is open, and hence the set $J(f)$ is closed. Roughly speaking, if $z_0 \in F(f)$, then $\sigma(f^n(z_0'), f^n(z_0))$ is small enough for any z_0' in small enough neighborhood of z_0 and any $n \in \mathbb{N}$, where σ denotes the spherical metric (see Proposition 2.3). We call this property the *stability* of dynamics on $F(f)$. On the other hand, if $z_0 \in J(f)$, then there can exist a point z_0' in any small neighborhood of z_0 such that $\sigma(f^N(z_0'), f^N(z_0))$ is large for some $N \in \mathbb{N}$ (see Proposition 2.5). This property is often called the *sensitive dependence on initial conditions* of the dynamics on $J(f)$. Those two sets are fundamental objects to study in complex dynamics.

A *polynomial* $P : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 2$ has the form

$$P(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0,$$

where $a_j \in \mathbb{C}$ $(j = 0, \cdots, d)$ and $a_d \neq 0$. A *rational function* (*or map*) $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$ has the form

$$R(z) = \frac{P(z)}{Q(z)},$$

where P and Q are mutually prime polynomials with $\max\{\deg P, \deg Q\} = d$. By definition, rational functions include polynomials. There are many results on dynamics of rational functions or polynomials. A meromorphic function on $\mathbb{C}$ which is not rational is called *transcendental*. Since transcendental meromorphic functions have an essential singularity at ∞, we cannot define the value at ∞ naturally in contrast to rational functions. For a rational function R of degree d and any $w \in \widehat{\mathbb{C}}$, the equation $R(z) = w$ of z has d solutions counted with multiplicity. On the other hand, for transcendental functions, such equation can have infinitely many solutions. These are big differences between rational case and transcendental case. Therefore, we cannot extend results for rational functions to transcendental meromorphic functions in general. For a transcendental meromorphic or rational function f defined on $\mathbb{C}$ or $\widehat{\mathbb{C}}$, a *singular value* $v \in \widehat{\mathbb{C}}$ is a point such that a branch of f^{-1} cannot be defined naturally in any neighborhood of it. It is known that transcendental functions with finitely many singular values share important dynamical properties with rational functions (see Section 2.2). Hence it is natural to expect some extensions of results for rational functions to such transcendental functions. In particular, we give such extensions of the results for rational functions on the *Fatou-Shishikura inequality*, *Cremer points*, and *Siegel disks* bounded by *quasicircles*, which are special Jordan curves and defined in Definition 2.9 (see Theorem 1.1, Proposition 2.24, Theorem 1.3, and Theorem 1.4). *Quasiconformal surgery* is an important technique which makes a given meromorphic function into a new meromorphic function with given dynamical properties (see Section 2.5). We use this technique and obtain the following results of some transcendental meromorphic functions with finitely many singular values:

(1) For rational functions, Shishikura constructed a theory of quasiconformal surgery and showed the *Fatou-Shishikura inequality* in [Shi]. His surgery technique also showed that the inequality is *best possible* in some sense (see Theorem 1.1). The *Speiser class* S is the set of all entire functions with finitely many singular values. As an extension of the result, Eremenko and Lyubich showed the Fatou-Shishikura inequality for transcendental entire functions in the Speiser class in [EL] (see Theorem 1.2). However, it has not been proved that the inequality is best possible. On the analogy of the rational case, we show the inequality is best possible. (See Section 1.1.)

(2) Denote by $\mathscr{S}$ the set of all transcendental entire functions of the form

$$P(z)\exp(Q(z)),$$

where P and Q are polynomials. They belong to the Speiser class S and are *structurally finite* (see Definition 2.3). Thus it is natural to expect results for $f \in \mathscr{S}$ similar to those for polynomials. By using the theory of polynomial-like mappings (see Section 2.5), we construct some functions in $\mathscr{S}$ with *Cremer points*, with *Siegel disks* bounded by *quasicircles* or Jordan curves, with the Julia sets of Lebesgue measure zero, and with the Julia sets of positive Lebesgue measure. These are some extensions of Geyer's result in [Gey], Cremer's result in [Cr1], Zakeri's result in [Za2], and Keen and Zhang's result in [KeZ]. (See Section 1.2.)

(3) We consider a one parameter family of some transcendental meromorphic functions with one pole, at most four singular values, and a *bounded type* Siegel disk centered at the origin. We show that the Siegel disks are bounded by quasicircles containing *critical points* for uncountably many parameters. This is an extension of Zakeri's result in [Za2] or Keen and Zhang's result in [KeZ]. (See Section 1.3.)

1.1 Best possibility of the Fatou-Shishikura inequality for transcendental entire functions in the Speiser class (Theorem A)

For a transcendental meromorphic or rational function f defined on $\mathbb{C}$ or $\widehat{\mathbb{C}}$, a point z is called *periodic* if $f^p(z) = z$ for some minimum $p \in \mathbb{N}$. This p is called the *period* of z. In particular, it is called *fixed* if $p = 1$. Periodic points are important initial points, which often become a key to understand dynamics. The set $\{z, f(z), \cdots, f^{p-1}(z)\}$ is called the *cycle* containing z. Periodic points (resp. cycles containing them) are classified into *repelling*, *attracting* (*or super-attracting*), *rationally indifferent*, or *irrationally indifferent* periodic points (resp. cycles). (See Section 2.1.) Furthermore, we call an irrationally indifferent periodic point z a *Siegel point* if $z \in F(f)$, or a *Cremer point* if $z \in J(f)$. The cycle containing a Siegel point is called a *Siegel cycle*. Similarly, we define *Cremer cycles*.

A connected component of $F(f)$ is called a *Fatou component*. Let U be a Fatou component. Then $f^n(U)$ is contained in some Fatou component U_n for $n = 1, 2, \cdots$. The domain U is called a *wandering domain* if $U_m \neq U_n$ for any distinct $m, n \in \mathbb{N}$ (see [Bak, p.567, Example 5.1]). Otherwise, U is called an *eventually periodic component*. The domain U is called a *periodic component* if there exists the minimum number $n \in \mathbb{N}$ such that $U_n = U$. This n is called the *period* of U. In addition, $\{U, U_1, \cdots, U_{n-1}\}$ is called the *cycle* (*of period* n) containing U. Periodic components are classified into *attractive basins*, *parabolic basins*, *Siegel disks*, *Herman rings*, and *Baker domains* (see Proposition 2.11). Cycles containing attractive basins are called AB-cycles. Similarly, we define PB-cycles, SD-cycles, HR-cycles, and BD-cycles. AB-cycles, PB-cycles, and SD-cycles have close relations with attracting cycles, rationally indifferent cycles, and Siegel cycles respectively (see Proposition 2.9, Proposition 2.10, and Proposition 2.11). Note that entire functions have no HR-cycles and rational functions have no BD-cycles (see Section 2.1).

Points c and $f(c)$ are called a *critical point* and a *critical value* respectively if f fails to be univalent in any neighborhood of c. A point $a \in \widehat{\mathbb{C}}$ is called an *asymptotic value* if f is transcendental and there exists a continuous curve $\gamma(t)$ $(0 \leq t < 1)$ with $\lim_{t\to 1} \gamma(t) = \infty$ and $\lim_{t\to 1} f(\gamma(t)) = a$. Singular values are classified into critical values, asymptotic values, and their accumulation points, which have important relations with periodic components and Cremer cycles (see Proposition 2.12 and Proposition 2.14). Let $S_q \subset S$ be the set of all **transcendental entire** functions which have exactly q distinct **finite** singular values. Let Pol_d be the set of all polynomials of degree $d \geq 2$. Any $f \in Pol_d$ has $d-1$ *critical points* in $\mathbb{C}$ counted with multiplicity and at most $d-1$ singular values in $\mathbb{C}$, which are critical values.

Now we introduce the Fatou-Shishikura inequality for $f \in Pol_d$ and that for $f \in S_q$. When $f \in Pol_d \cup S_q$, we define $n_{\mathrm{rat}}(f)$ as the number of rationally indifferent cycles of f. Similarly we define

$$n_{\mathrm{Si}}(f),\ n_{\mathrm{Cr}}(f),\ n_{\mathrm{PB}}(f),\ n_{\mathrm{SD}}(f)$$

as the number of Siegel cycles, Cremer cycles, PB-cycles, and SD-cycles respectively. In addition,

$$n_{\mathrm{rat}}(f) \leq n_{\mathrm{PB}}(f), \quad n_{\mathrm{Si}}(f) = n_{\mathrm{SD}}(f)$$

hold among these notations (see Proposition 2.10 and Proposition 2.11). It is known that $n_{\mathrm{PB}}(f)$ is a multiple of $n_{\mathrm{rat}}(f)$ (see Proposition 2.10). We define $n_{\mathrm{att}}(f)$ as the number of attracting cycles of f **in** $\mathbb{C}$. Let $n_{\mathrm{AB}}(f)$ be the number of attractive basins of f if $f \in S_q$ and the number of **bounded** attractive basins if $f \in Pol_d$. It follows that

$$n_{\mathrm{att}}(f) = n_{\mathrm{AB}}(f).$$

(See Proposition 2.9.) Note that we adopt the definitions of $n_{\mathrm{att}}(f)$ and $n_{\mathrm{AB}}(f)$ for the inequality in Theorem 1.1.[1)] The following is the Fatou-Shishikura inequality for $f \in Pol_d$, which is some modification of [Shi, p.5, Corollary 2, p.6, Theorem 4]):

Theorem 1.1 ([Shi, p.5, Corollary 2, p.6, Theorem 4])**.** *Let* $f \in Pol_d$. *Then*

$$n_{\mathrm{AB}}(f) + n_{\mathrm{PB}}(f) + n_{\mathrm{SD}}(f) + n_{\mathrm{Cr}}(f) \leq d - 1.$$

Moreover, the inequality is best possible in the following sense: If non-negative integers $m_{\mathrm{AB}}, m_{\mathrm{PB}}$, m_{SD}, *and* m_{Cr} *satisfy*

$$m_{\mathrm{AB}} + m_{\mathrm{PB}} + m_{\mathrm{SD}} + m_{\mathrm{Cr}} \leq d - 1,$$

then there exists a polynomial $P \in Pol_d$ *with*

$$(n_{\mathrm{AB}}(P), n_{\mathrm{PB}}(P), n_{\mathrm{SD}}(P), n_{\mathrm{Cr}}(P)) = (m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}}, m_{\mathrm{Cr}}).$$

[1)] When we regard $f \in Pol_d$ as a rational function defined on the Riemann sphere $\widehat{\mathbb{C}}$, f always has a super-attracting fixed point ∞ and an attractive basin containing ∞. ∞ is a critical point with multiplicity $d-1$. By the Fatou-Shishikura inequality for rational functions of degree d (see [Shi, p.5, Corollary 2]), the sum of the numbers of AB-cycles, PB-cycles, SD-cycles, and Cremer cycles is less than or equal to the number $2d-2$ of critical points in $\widehat{\mathbb{C}}$ counted with multiplicity. We adopt the definition of $n_{\mathrm{att}}(f)(= n_{\mathrm{AB}}(f))$ so that the right-hand side of the modified inequality in Theorem 1.1 becomes the number $d-1$ of critical points in $\mathbb{C}$.

The Fatou-Shishikura inequality for $f \in S_q$ is as follows:

Theorem 1.2 ([EL, p.1005, Theorem 5])**.** *Let $f \in S_q$. Then*

$$n_{\mathrm{AB}}(f) + n_{\mathrm{PB}}(f) + n_{\mathrm{SD}}(f) + n_{\mathrm{Cr}}(f) \leq q.$$

Both functions in Pol_d and those in S_q share some important dynamical properties as follows:

(1) They have finitely many singular values;

(2) They have no Herman rings, no Baker domains, and no wandering domains (see Section 2.1, Proposition 2.15, and Proposition 2.16);

(3) They satisfy the Fatou-Shishikura inequalities (Theorem 1.1 and Theorem 1.2).

According to Theorem 1.1, the Fatou-Shishikura inequality for $f \in Pol_d$ is best possible. From these dynamical properties of $f \in S_q$ similar to those of $f \in Pol_d$, we can expect best possibility of the Fatou-Shishikura inequality for $f \in S_q$ analogous to that for $f \in Pol_d$. We show that this is actually true by constructing structurally finite transcendental entire functions, which belong to S and have the explicit representation (see Definition 2.3 and [T]). Let SF be the set of all structurally finite transcendental entire functions.

Theorem A ([KiN1, p.168, Main Theorem])**.** *The Fatou-Shishikura inequality for $f \in S_q$ is best possible in the following sense: If non-negative integers $m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}}$, and m_{Cr} satisfy*

$$m_{\mathrm{AB}} + m_{\mathrm{PB}} + m_{\mathrm{SD}} + m_{\mathrm{Cr}} \leq q,$$

then there exists a $T \in S_q$ with

$$(n_{\mathrm{AB}}(T), n_{\mathrm{PB}}(T), n_{\mathrm{SD}}(T), n_{\mathrm{Cr}}(T)) = (m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}}, m_{\mathrm{Cr}}).$$

More precisely, T satisfies $n_{\mathrm{PB}}(T) = n_{\mathrm{rat}}(T)$ and $T \in SF$. In addition, every non-repelling periodic point of T has the same period relatively prime with q.

The proof of Theorem A is based on an analogy of [Shi]. Cremer's result only for rational functions in [Cr1] can tell that irrationally indifferent cycles are Cremer cycles if they satisfy some condition which we will name [Cremer (d)] (see Theorem 1.3). Shishikura used the result to prove best possibility of the Fatou-Shishikura inequality for rational functions (see [Shi]). On the other hand, we cannot use the result for transcendental case in general. (Note that such condition can be applicable in some transcendental cases. See Theorem C.) For the proof of Theorem A, we construct Cremer cycles without using his result. This is the main difference between our proof and [Shi]. Moreover, our construction can be also used for the rational case, which leads to a slightly different proof of [Shi, p.6, Theorem 4]. We also give another way how to construct Cremer cycles, where we use another Cremer's result in [Cr2] (see Remark 3.4).

1.2 Some transcendental entire functions with irrationally indifferent fixed points (Theorems B–H)

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental meromorphic function or a polynomial of degree $d \geq 2$. A periodic point $z_0 \in \mathbb{C}$ of period p is called *irrationally indifferent* if $\lambda := (f^p)'(z_0) = e^{2\pi i\theta} (\theta \in \mathbb{R}\backslash\mathbb{Q})$. We call λ the *multiplier* of z_0. If z_0 is a Siegel point, then there exists a maximal f^p-invariant simply connected domain $D \subset F(f)$ containing z_0 on which f^p is conformally conjugate to the rotation $z \mapsto \lambda z$ (see Proposition 2.11 (SD)). In fact, the domain D is a Siegel disk (*centered at* z_0). In addition, we call D *fixed* if $p = 1$. We say that $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is a *Brjuno number* if

$$\sum_n \frac{\log q_{n+1}}{q_n} < \infty,$$

where p_n/q_n is the nth convergent $[a_0; a_1, a_2, \ldots, a_{n-1}]$ to θ coming from the continued fraction expansion

$$\theta = [a_0; a_1, a_2, \ldots, a_{n-1}, \ldots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \cfrac{1}{a_{n-1} + \cdots}}}.$$

The condition is called the *Brjuno condition.* Denote by $\mathcal{B}$ the set of all Brjuno numbers. The set $\mathcal{B}$ is uncountable and dense in $\mathbb{R}$. According to Brjuno and Rüssmann, if $\theta \in \mathcal{B}$, then z_0 is a Siegel point (see [Brj] and [Ru] or [Mil, p.132, Theorem 11.10]). In general, we cannot tell whether z_0 is a Siegel point or a Cremer point if $\theta \notin \mathcal{B}$. This causes a difficulty in constructing Cremer points of transcendental functions. Some solutions are in the proof of Theorem A. For the quadratic polynomial of the form $e^{2\pi i\theta}z + z^2$ ($\theta \in \mathbb{R} \setminus \mathbb{Q}$), Yoccoz showed that the Brjuno condition is optimal for the origin to be a Siegel point (see [Y]). This result is generalized by Okuyama and Geyer (see [Ok1] and [Gey]). In particular, Geyer extended the Yoccoz's result to the transcendental function of the form $e^{2\pi i\theta}ze^z \in \mathscr{S}$ ($\theta \in \mathbb{R} \setminus \mathbb{Q}$) (see [Gey]). We extend Geyer's result as follows:

Theorem B ([KiN2, p.372, Theorem 3])**.** *Let*

$$F_{\theta,c}(z) := e^{2\pi i\theta}z(1 + cz)^{d-1}e^z \in \mathscr{S},$$

where $\theta \in \mathbb{R} \setminus \mathbb{Q}$, an integer $d \geq 2$, and $c \in \mathbb{C}$. Then for c with $|c|$ large enough, $F_{\theta,c}$ has a Siegel point at the origin if and only if $\theta \in \mathcal{B}$.

Remark 1.1. If $|c| > 6\sqrt[d-1]{4e^{3/2}} + 2 =: M(d)$, then the statement of Theorem B holds (see Lemma 4.1 for $M(d)$).

The following is a sufficient condition on multipliers for irrationally indifferent periodic points of rational functions to be Cremer points:

Theorem 1.3 ([Cr1])**.** *Let f be a rational function of degree $d \geq 2$. If f has an irrationally indifferent fixed point z_0 whose multiplier λ satisfies the following condition:*

$$[\text{Cremer } (d)] \qquad \left\{ \sqrt[d^n]{1/|\lambda^n - 1|} \right\}_{n=1}^{\infty} \text{ is unbounded,}$$

then z_0 is a Cremer fixed point. In addition, the set of all λ satisfying [Cremer (d)] *is uncountable and dense in $\{\lambda \mid |\lambda| = 1\}$.*

(See Chapter 2 for the definition of the multiplier when $z_0 = \infty$.) For transcendental entire functions, another Cremer's result says that irrationally indifferent periodic points are Cremer points if their multipliers satisfy some condition different from [Cremer (d)] (see Proposition 2.23). On the other hand, we cannot tell that irrationally indifferent periodic points of transcendental entire functions are Cremer points even if their multipliers satisfy [Cremer (d)] in general. We obtain functions in $\mathscr{S}$ with several Cremer points with multipliers satisfying [Cremer (d)] for some integer d as follows:

Theorem C ([KiN2, p.371, Theorem 2])**.** *Let q be an arbitrary positive integer. Then there exist a $g \in \mathscr{S}$ and a $d \in \mathbb{N}$ such that g has q Cremer fixed points whose multipliers satisfy* [Cremer (d)].

We say that $\theta \in \mathbb{R}$ is of *bounded type* if $\{a_n\}_{n=0}^{\infty}$ is bounded, where $\theta = [a_0; a_1, a_2, \ldots, a_n, \ldots]$ is the continued fraction expansion. (Such numbers are also called *Diophantine numbers of order* 2.) Denote by $\mathcal{D}(2)$ the set of all irrational numbers of bounded type. The set $\mathcal{D}(2)$ is uncountable and dense in $\mathbb{R}$. Note that $\mathcal{D}(2) \subset \mathcal{B}$. Thus if $\theta \in \mathcal{D}(2)$, then a periodic point z_0 with multiplier $\lambda = e^{2\pi i\theta}$ is a Siegel point. In this case, we call z_0 or the Siegel disk D centered at z_0 *bounded type*. The following is from Shishikura's unpublished work on the boundaries of bounded type Siegel disks of polynomials:

Theorem 1.4. *All bounded type Siegel disks of polynomials of degree $d \geq 2$ are bounded by quasicircles containing at least one critical point.*[2)]

Zakeri gave an extension of Theorem 1.4 to functions in $\mathscr{S}$ as follows:

Theorem 1.5 ([Za2])**.** *Let $f \in \mathscr{S}$. Suppose that f has a bounded type fixed Siegel disk centered at the origin. Then the fixed Siegel disk is bounded by a quasicircle containing at least one critical point.*[3)]

As he mentioned in his paper, the space $\mathscr{S}$ is not invariant under linear conjugations which move the origin (see Section 2.5 for conjugation). Thus his result does not say anything about bounded type fixed Siegel disks of $f \in \mathscr{S}$ centered at points other than the origin. As an extension of Theorem 1.5, we construct functions in $\mathscr{S}$ with such Siegel disks bounded by quasicircles containing critical points as follows:

Theorem D ([KiN2, p.371, Theorem 1])**.** *Let q be an arbitrary positive integer. Then there exists a $g \in \mathscr{S}$ such that g has q bounded type fixed Siegel disks each of which is bounded by a quasicircle containing at least one critical point. In addition, those fixed Siegel disks are centered at points other than the origin. We can also construct g so that each boundary of such Siegel disks contains exactly one critical point.*

Let $\mathscr{S}_{2,1} \subset \mathscr{S}$ be the set of all functions in $\mathscr{S}$ of the form

$$(\lambda z + \alpha z^2)e^z,$$

where $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{C} \setminus \{0\}$. Functions in $\mathscr{S}_{2,1}$ have two critical points and two asymptotic values 0 and ∞. Keen and Zhang proved a special case of Theorem D for $f \in \mathscr{S}_{2,1}$ by a different approach from Zakeri's proof. More precisely, they showed the following result:

Theorem 1.6 ([KeZ, p.138, Main Theorem])**.** *Let $\theta \in \mathcal{D}(2)$. Then for the function*

$$g_\alpha(z) := (e^{2\pi i\theta} z + \alpha z^2)e^z \in \mathscr{S}_{2,1},$$

the boundary of the bounded type fixed Siegel disk centered at the origin is a quasicircle containing at least one critical point.

Let $\mathcal{E}$ be the set of all irrational numbers θ satisfying the arithmetic condition:

$$\log a_n = \mathcal{O}(\sqrt{n}) \quad \text{as } n \to \infty,$$

where $\theta = [a_0; a_1, a_2, \ldots, a_n, \ldots]$ is the continued fraction expansion (see [PeZ] for the set $\mathcal{E}$). By definition, one can check that $\mathcal{D}(2) \subset \mathcal{E} \subset \mathcal{B}$. Moreover, it is known that $\mathcal{E} \cap [0,1]$ has full measure (see [PeZ, p.9, Corollary 2.2]). On the other hand, the set $\mathcal{D}(2) \cap [0,1]$ has Lebesgue measure zero. Therefore, the set of all $\theta \in (\mathcal{E} \setminus \mathcal{D}(2)) \cap [0,1]$ has full measure. As an extension of Theorem 1.6, we obtain the following result for g_α with $\theta \in \mathcal{E}$:

2) Zhang extended this result to all bounded type Siegel disks of rational functions in [Zh].

3) Zakeri's analysis also gives another proof of Theorem 1.4. His original statement includes that of Theorem 1.4.

Theorem E ([KiN2, p.373, Theorem 5])**.** *Let $\theta \in \mathcal{E}$, let*

$$g_\alpha(z) := (e^{2\pi i\theta} z + \alpha z^2)e^z,$$

where $\alpha \in \mathbb{C} \setminus \{0\}$, and let $\triangle_\alpha$ be the fixed Siegel disk of g_α centered at the origin. Then there exists a constant $M > 0$ independent of θ such that if $|\alpha| > M$, then $\partial\triangle_\alpha$ is a Jordan curve containing exactly one critical point. Moreover, if $\theta \in \mathcal{E} \setminus \mathcal{D}(2)$ and $|\alpha| > M$, then $\partial\triangle_\alpha$ is a Jordan curve but is not a quasicircle.

Remark 1.2. Let $M(d)$ be as in Remark 1.1. Note that we can take $M := M(2) = 24e^{3/2} + 2$ (see the proof of Theorem E). Our proof of Theorem E for $\theta \in \mathcal{D}(2)$ may be similar to the proof noted in [KeZ, p.146, Remark 2.4] without giving the details, which shows a special case of Theorem 1.6. However, our main target is g_α for $\theta \in \mathcal{E} \setminus \mathcal{D}(2)$. Moreover, we provides the detailed argument.

In contrast to Theorem E, we also have:

Theorem F. *There exists an $f \in \mathscr{S}_{2,1}$ such that it has a Siegel disk centered at the origin whose boundary is a quasicircle without critical points.*

The Lebesgue measure of the Julia set is an important information. Keen and Zhang also constructed an $f \in \mathscr{S}_{2,1}$ with $J(f)$ of Lebesgue measure zero and an attracting fixed point of multiplier λ at the origin (see [KeZ, p.147, Lemma 3.4]). However, they did not say anything on the Lebesgue measure of $J(g_\alpha)$. We have:

Theorem G ([KiN2, p.373, Theorem 6])**.** *For every $\theta \in \mathcal{E}$, there exists a domain $A \subset \{\alpha \mid |\alpha| > M\}$ such that if $\alpha \in A$, then $J(g_\alpha)$ has Lebesgue measure zero, where M is as in Theorem E.*

In contrast to Theorem G, we obtain the following result:

Theorem H ([KiN2, p.372, Theorem 4])**.** *There exist functions f_1 and f_2 in $\mathscr{S}_{2,1}$ such that $J(f_j)$ $(j = 1, 2)$ have positive Lebesgue measure and f_1 (resp. f_2) has a Cremer fixed point (resp. a Siegel fixed point) at the origin.*

The proofs of the results from Theorem B to Theorem H are based on the theory of *polynomial-like mappings*, which allows us to use the results for polynomials even in some transcendental cases. (For polynomial-like mappings, see Section 2.5.)

1.3 The boundaries of bounded type fixed Siegel disks of some transcendental meromorphic functions (Theorem I)

Let $\tilde{\mathscr{S}}$ be the set of all transcendental meromorphic functions of the form

$$R(z) \exp{(Q(z))},$$

where $R(z)$ and $Q(z)$ are a rational function which has at least one pole and a polynomial respectively. Functions in $\tilde{\mathscr{S}}$ and functions in $\mathscr{S}$ share many important properties. For example, they have finitely many critical points, two asymptotic values 0 and ∞, and finitely many zeros. Thus we can expect the result for functions in $\tilde{\mathscr{S}}$ similar to Theorem 1.5 for functions in $\mathscr{S}$.

Question 1. *Let $f \in \tilde{\mathscr{S}}$. Suppose that f has a bounded type fixed Siegel disk centered at the origin. Is the fixed Siegel disk bounded by a quasicircle containing at least one critical point?*

We consider the easiest case as follows: Henceforth fix any irrational number θ of bounded type. Suppose that $f \in \tilde{S}$, the degrees of R and Q are 1, and f has a bounded type Siegel fixed point at the origin with multiplier $\lambda = e^{2\pi i\theta}$. The function f is conformally conjugate to

$$h_\alpha(z) := e^{2\pi i\theta} \frac{z}{1 - \frac{\alpha+1}{\alpha} z} e^{\alpha z}$$

for some $\alpha \in \mathbb{C} \setminus \{0, -1\}$ (see Proposition 5.1). The one parameter family $\{h_\alpha\}_{\alpha \in \mathbb{C}\setminus\{0,-1\}}$ has the following properties:

(1) h_α has two critical points 1 and

$$c_\alpha := \frac{-1}{\alpha+1},$$

two asymptotic values 0 and ∞, and one pole

$$t_\alpha := \frac{\alpha}{\alpha+1}$$

(see Proposition 5.1).

(2) $h_{\alpha'} \neq h_\alpha$ is conformally conjugate to h_α if and only if $\alpha' = 1/(\alpha+1) - 1$ (see Proposition 5.2).

We show the following theorem:

Theorem I ([Nab])**.** *Let $\triangle_\alpha$ be the bounded type fixed Siegel disk of h_α centered at the origin. Then we have the following assertions:*

(i) *If two critical values $h_\alpha(1)$ and $h_\alpha(c_\alpha)$ coincide, then $\triangle_\alpha$ is bounded by a quasicircle containing exactly one critical point. Moreover, the set $\Omega_1 := \{\alpha \mid h_\alpha(1) = h_\alpha(c_\alpha)\}$ is countably infinite.*

(ii) *There exists an uncountable set Ω_2 such that if $\alpha \in \Omega_2$, then $\triangle_\alpha$ is bounded by a quasicircle containing exactly one critical point. Moreover, the quasicircle constant can be taken so that it is independent of $\alpha \in \Omega_2$.*

(iii) *There exists an α such that $\triangle_\alpha$ is bounded by a quasicircle containing exactly two critical points.*

Recall that Keen and Zhang studied the one parameter family

$$\{g_\alpha(z) := (e^{2\pi i\theta} z + \alpha z^2)e^z\}_{\alpha \in \mathbb{C}\setminus\{0\}},$$

where θ is of bounded type. Like h_α, g_α has two critical points, two asymptotic values 0 and ∞, and a bounded type fixed Siegel disk centered at the origin. By Theorem 1.6, the bounded type Siegel disk of g_α centered at the origin is bounded by a quasicircle containing critical points. They also showed that for some α, the boundary of the Siegel disk contains exactly two critical points (see [KeZ]). Therefore, it is natural to expect that Keen and Zhang's method also shows Theorem I. Unfortunately, since h_α has one pole t_α, we cannot use their method as in [KeZ] (and cannot use the method as in [Za2]). Hence in order to show Theorem I, we have to modify Keen and Zhang's argument. We use the result of [CheE] in order to prove Theorem I (i). The proofs of Theorem I (ii) and (iii) are inspired by quasiconformal surgery methods of [Za1], [KeZ], and [CheE].

The thesis is organized as follows: We devote Chapter 2 to preliminaries. In Chapter 3, 4, and 5, we show Theorem A, Theorems B–H, and Theorem I respectively. We make some concluding remarks in Chapter 6. Note that the contents on Theorem A are based on [KiN1], that the contents of the results in Section 1.2 other than Theorem F are based on [KiN2], and that the contents on Theorem I are based on [Nab].

Acknowledgments

The author would like to thank Masashi Kisaka for his valuable discussions and his help in improving the thesis. The author would also like to thank Hiroki Sumi for his valuable comments in improving the thesis.

Chapter 2

Preliminaries

In this chapter, we review basic definitions and facts and introduce *quasiconformal surgery technique*. Let f be a transcendental meromorphic function defined on $\mathbb{C}$ or a rational function of degree $d \geq 2$ defined on $\widehat{\mathbb{C}}$. Recall that if f is transcendental and has a pole, then $f^n(z)$ is defined for all points in $\mathbb{C}$ except for the preimages of ∞ by $f, f^2, \cdots, f^{n-1}$.

2.1 Properties of $F(f)$ and $J(f)$

We say that a family $\mathcal{F}$ of meromorphic functions in a domain $D \subset \widehat{\mathbb{C}}$ is *normal* if for any sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$, there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ which converges to some limit function f_∞ locally uniformly on D with respect to the spherical metric in $\widehat{\mathbb{C}}$. Note that the limit function f_∞ is meromorphic in D. The following is a useful tool to tell that a family is normal (see [Bea, p.57, Theorem 3.3.4]):

Montel's theorem. *Let $\mathcal{F}$ be a set of meromorphic functions in a domain $D \subset \widehat{\mathbb{C}}$. If any function in $\mathcal{F}$ does not take three distinct fixed values in $\widehat{\mathbb{C}}$, then $\mathcal{F}$ is normal.*

We define the *Fatou set* and the *Julia* set as follows:

Definition 2.1.

$$F(f) := \{z \in \widehat{\mathbb{C}} \mid \ \{f^n\}_{n=1}^{\infty} \textit{ is defined and normal in some neighborhood of } z\},$$

$$J(f) := \widehat{\mathbb{C}} \setminus F(f).$$

By definition, $F(f)$ is open, and hence $J(f)$ is closed. For example, we have $J(P) = \mathbb{S}^1 := \{z \mid |z| = 1\}$ for $P(z) = z^2$. Indeed, since $P : \mathbb{D} \to \mathbb{D}$ and $P : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, where $\mathbb{D} := \{z \mid |z| < 1\}$, Montel's theorem shows that $J(P) \subseteq \mathbb{S}^1$. In addition, since $P^n(z) \to 0$ or $P^n(z) \to \infty$ as $n \to \infty$ if $z \notin \mathbb{S}^1$ and $P(\mathbb{S}^1) = \mathbb{S}^1$, $J(P)$ is exactly $\mathbb{S}^1$. Note that if f is transcendental, then ∞ is always in $J(f)$. The two sets have the following important properties:

Proposition 2.1 ([Ber, p.155, Lemma 2])**.** *Both $F(f)$ and $J(f)$ are completely invariant sets in the following sense:*

$$f(F(f)) \subseteq F(f), \quad f^{-1}(F(f)) \subseteq F(f), \quad f(J(f)) \subseteq J(f), \quad f^{-1}(J(f)) \subseteq J(f).$$

Hence we can divide the dynamics of f into $f|_{F(f)} : F(f) \to F(f)$ and $f|_{J(f)} : J(f) \to J(f)$. If f is transcendental and has poles, then f^n is not meromorphic in $\mathbb{C}$ for $n \geq 2$. However, we can define $F(f^n)$ and $J(f^n)$ naturally even in this case (see [Ber, p.155]). Then:

Proposition 2.2 ([Ber, p.155, Lemma 1]). *For every $n \geq 2$,*

$$F(f^n) = F(f), \quad J(f^n) = J(f).$$

The following comes from the Ascoli-Arzelà theorem (see [Bea, p.56, Theorem 3.3.2]):

Proposition 2.3. *Let $z_0 \in F(f)$. Then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\sigma(f^n(z), f^n(z_0)) < \varepsilon$$

for any $n \in \mathbb{N}$ and any z with $\sigma(z, z_0) < \delta$, where σ denotes the spherical metric.

This property is called the *stability* of the dynamics on $F(f)$. If f is rational and U is a Fatou component of f, then $f(U)$ is also a Fatou component. However, for some transcendental f, the set $f(U)$ is not a Fatou component (see examples in [Herr]). The following is known:

Proposition 2.4 ([Herr, p.264, Theorem 1 and Theorem 2]). *Let U and V be a Fatou component of f and the Fatou component containing $f(U)$ respectively. Then the set $V \setminus f(U)$ contains at most two points. Moreover, any point in $V \setminus f(U)$ is an asymptotic value.*

Let

$$\mathcal{O}^+(z) := \bigcup_{n \geq 0} \{f^n(z)\},$$

where this union is for all $n \geq 0$ such that $f^n(z)$ is defined. For any set $X \subset \widehat{\mathbb{C}}$, let

$$\mathcal{O}^+(X) := \bigcup_{z \in X} \mathcal{O}^+(z).$$

It follows from Montel's theorem that:

Proposition 2.5 ([Ber, p.156]). *Let $z_0 \in J(f)$ and let U be a neighborhood of z_0. Then $\widehat{\mathbb{C}} \setminus \mathcal{O}^+(U)$ contains at most two points.*

This yields the complicated behavior of the dynamics on $J(f)$.

Let $\kappa(z) := 1/z$. For a periodic point z_0 of period p, we define the *multiplier* λ by

$$\lambda := \begin{cases} (f^p)'(z_0) & (\text{if } z_0 \in \mathbb{C}) \\ (\kappa \circ f^p \circ \kappa)'(0) & (\text{if } f \text{ is rational and } z_0 = \infty). \end{cases}$$

We say that z_0 (or the cycle containing z_0) is *repelling*, *attracting*, *rationally indifferent*, or *irrationally indifferent* if $|\lambda| > 1, |\lambda| < 1$, $\lambda = e^{2\pi i\theta}$ ($\theta \in \mathbb{Q}$), or $\lambda = e^{2\pi i\theta}$ ($\theta \in \mathbb{R} \setminus \mathbb{Q}$), respectively. The last three cases are called *non-repelling*. In particular, attracting cycles or periodic points are called *super-attracting* if $\lambda = 0$. Note that the points $z_j = f^j(z_0)$ ($j = 0, \cdots, p-1$) in the cycle containing z_0 have the same multiplier λ, where f^0 is the identity.

The following proposition characterizes $J(f)$:

Proposition 2.6 ([Ber, p.160, Theorem 4]). *$J(f)$ is the closure of the set of all repelling periodic points of f.*

Note that f has infinitely many repelling periodic points (see [Ber, p.161, Theorem 5]), and hence $J(f)$ is not empty. Proposition 2.5 shows that repelling periodic points are not isolated points in $J(f)$. Therefore, $J(f)$ has no isolated points.

Proposition 2.7 ([Ber, p.159, Theorem 3]). *$J(f)$ is perfect in the following sense: $J(f)$ is closed and is not empty, and has no isolated points.*

Proposition 2.8 ([Ber, p.155, Lemma 3, p.156], [Bea, p.271, Section 11.9], [Mis]). *$J(f)$ has empty interior or $J(f) = \widehat{\mathbb{C}}$. There exist a rational function and a transcendental meromorphic function such that the Julia sets are $\widehat{\mathbb{C}}$. Let $f_1(z) := (z-2)^2/z^2$, let $f_2(z) := e^z$, and let $f_{3,\lambda}(z) := \lambda \tan z$ ($\lambda \in \mathbb{C} \setminus \{0\}$). Then $J(f_1) = J(f_2) = J(f_{3,\lambda}) = \widehat{\mathbb{C}}$ for suitable values λ.*

Attracting or rationally indifferent periodic points have important relations with periodic components as follows:

Proposition 2.9 ([Bea, p.104, Theorem 6.3.1]). *If z_0 is an attracting periodic point of period p, then $z_0 \in F(f)$. Moreover, there exists a periodic component A containing z_0 of period p such that $f^{np}(z) \to z_0$ as $n \to \infty$ for any $z \in A$.*

We call the periodic component A in Proposition 2.9 an *attractive basin*.

Proposition 2.10 ([Bea, p.110, Theorem 6.5.1, p.131, Theorem 6.5.10]). *Let z_0 be a rationally indifferent periodic point of period p with multiplier $\lambda = \exp(2\pi i r/q)$, where q and r are mutually prime integers. Then $z_0 \in J(f)$. Moreover, the following Taylor expansion holds:*

$$\tau \circ f^{pq} \circ \tau^{-1}(z) = z + az^{kq+1} + O(z^{kq+2}) \quad \text{as } z \to 0,$$

where $\tau(z) := z - z_0$ if $z_0 \in \mathbb{C}$, $\tau(z) := 1/z$ if f is rational and $z_0 = \infty$, $k \in \mathbb{N}$, and $a \neq 0$, and there exist pkq periodic components $B_1, \cdots B_{pkq}$ such that for any point in B_j ($j = 1, \cdots, pkq$), $f^{npq}(z) \to z_0 \in \partial B_j$ as $n \to \infty$ and $B_1, \cdots, B_{pkq}$ form k cycles containing pq periodic components.

We call the periodic components B_j ($j = 1, \cdots, pkq$) in Proposition 2.10 *parabolic basins*.

Remark 2.1. Although [Bea, p.104, Theorem 6.3.1] and [Bea, p.110, Theorem 6.5.1, p.131, Theorem 6.5.10] are for rational functions, their proofs are applicable to transcendental meromorphic functions in $\mathbb{C}$.

We can classify periodic components as follows:

Proposition 2.11 ([Ber, p.163, Theorem 6]). *Let $D \subset F(f)$ be a periodic component of period p. Then there are the following five possibilities:*

(AB) *D is an attractive basin.*

(PB) *D is a parabolic basin.*

(SD) *There exists a Siegel point $z_0 \in D$ of period p with multiplier λ and $f^p|_D$ is conformally conjugate to an irrational rotation $z \mapsto \lambda z$ of the unit disk $\mathbb{D} := \{z \mid |z| < 1\}$. To be more precise, there exists a conformal map $\varphi : D \to \mathbb{D}$ such that*

$$\varphi(z_0) = 0, \quad \varphi \circ f^p \circ \varphi^{-1}(z) = \lambda z$$

hold for $z \in \mathbb{D}$.

(HR) *$f^p|_D$ is conformally conjugate to an irrational rotation $z \mapsto \lambda z$ of the annulus $A_r := \{z \mid r < |z| < 1\}$ ($0 < r < 1$). To be more precise, there exists a conformal map $\varphi : D \to A_r$ such that*

$$\varphi \circ f^p \circ \varphi^{-1}(z) = \lambda z$$

holds for $z \in A_r$.

(BD) *There exists a point $z_0 \in \partial U$ such that $f^p(z_0)$ is not defined and every point $z \in U$ satisfies $f^{pk}(z) \to z_0$ as $k \to \infty$.*

In the case (SD), (HR), or (BD) as in Proposition 2.11, we call D a *Siegel disk* (*centered at* z_0), a *Herman ring*, or a *Baker domain* respectively. Recall that a cycle containing an attractive basin is called an AB-cycle. Similarly, we defined PB-cycles, SD-cycles, HR-cycles, and BD-cycles. Note that entire functions cannot have Herman rings (see [Ber, p.164]).

Let $\operatorname{sing}(f^{-1})$ be the set of all singular values of f in $\mathbb{C}$ (resp. in $\widehat{\mathbb{C}}$) when f is transcendental (resp. when f is rational). Singular values and periodic components (or Cremer cycles) have the following important relations:

Proposition 2.12 ([Ber, p.164, Theorem 7, p.172, Theorem 16])**.** *Let C be a cycle containing a periodic component f. Then the following assertions hold:*

(i) *If C is an AB-cycle or a PB-cycle, then some periodic component in C contains a singular value of f.*

(ii) *If C is a SD-cycle or a HR-cycle, then the union of all boundaries of periodic components in C is contained in $\overline{\mathcal{O}^+(\operatorname{sing}(f^{-1}))}$.*

(iii) *If C is a BD-cycle of period p, then ∞ is an accumulation point of the set*

$$\bigcup_{j=0}^{p-1} f^j(\operatorname{sing}(f^{-1})).$$

Proposition 2.13 ([Bar, p.294, Theorem 4])**.** *If f is transcendental entire and has a cycle of period 1 containing a Baker domain, then there exist constants $K > 1$ and $r_0 > 0$ such that for any $r \geq r_0$, f has a singular value in*

$$\{z \mid r/K < |z| < Kr\}.$$

Proposition 2.14 ([MoNTU, p.76, Theorem 2.4.7])**.** *If f is rational or transcendental entire and $a \in \widehat{\mathbb{C}}$ is a Cremer point of f, then a is an accumulation point of the set $\mathcal{O}^+(\operatorname{sing}(f^{-1}))$.*

2.2 Meromorphic functions with finitely many singular values

In this section, we review properties of some families of meromorphic functions with finitely many singular values. For a critical point c of f, we define the *multiplicity* $m(c)$ of c as follows:

$$m(c) := \begin{cases} k_1 - 1 & (c \in \mathbb{C},\ f(c) \in \mathbb{C}, \text{ and } \lim_{z\to c}(f(z) - f(c))/(z-c)^{k_1} = K) \\ k_2 - 1 & (c \in \mathbb{C},\ f(c) = \infty, \text{ and } \lim_{z\to c}((\kappa \circ f)(z) - (\kappa \circ f)(c))/(z-c)^{k_2} = K) \\ k_3 - 1 & (c = \infty,\ f(c) = \infty, \text{ and } \lim_{z\to 0}((\kappa \circ f \circ \kappa)(z) - (\kappa \circ f \circ \kappa)(0))/z^{k_3} = K), \end{cases}$$

where K is non-zero and finite, $\kappa(z) := 1/z$, and f is rational in the last case. Note that rational functions of degree $d \geq 2$ have $2d-2$ critical points counted with multiplicity and no asymptotic values, and hence they have at most $2d-2$ singular values, which are critical values (see [Bea, p.43, Theorem 2.7.1]).

Sullivan showed that:

Proposition 2.15 ([Su, p.404, Theorem 1])**.** *Rational functions have no wandering domains.*

Therefore, all Fatou components of rational functions are eventually periodic components. By Proposition 2.11, we can understand the dynamics on the Fatou sets of rational functions. In addition, by definition, rational functions have no Baker domains. Let S' be the set of all **transcendental meromorphic** functions with finitely many critical points and asymptotic values. Note that there exists a function f in the Speiser class S such that $f \notin S'$. For example, $f(z) = \sin z$ has infinitely many critical points, two critical values ± 1, and no asymptotic values, and hence $f \in S$ but $f \notin S'$ in this case. As in the case of rational functions, the following is known:

Proposition 2.16 ([EL, p.990, Theorem 1, p.1004, Theorem 3], [BakKL2, p.652, Theorem], [Ber, p.172, Corollary 4])**.** *Let $f \in S \cup S'$. Then f has no wandering domains and Baker domains.*

Remark 2.2. Proposition 2.12 (iii) shows that $f \in S \cup S'$ has no Baker domains. Eremenko and Lyubich showed that if $\mathrm{sing}(f^{-1})$ is bounded for a transcendental entire function f, then f has no Baker domains (see [EL, p.990, Theorem 1]). This is also shown by Proposition 2.13.

We call a periodic component U of f *completely invariant* if $f(U) \subseteq U$ and $f^{-1}(U) \subseteq U$.

Proposition 2.17 ([BakKL1, p.609, Theorem 4.5])**.** *Let $f \in S'$. Then $F(f)$ contains at most two completely invariant components.*

In contrast to rational functions, transcendental functions can have asymptotic values.

Definition 2.2. *Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be transcendental, let $a \in \mathbb{C}$, and let $A(r) \subset \mathbb{C}$ be a connected component of $f^{-1}(\{z \mid |z - a| < r\})$ for any $r > 0$ such that $A(r_1) \subset A(r_2)$ if $r_1 < r_2$. $\{A(r)\}_{r>0}$ is called a transcendental singularity of f^{-1} if*

$$\bigcap_{r>0} \overline{A(r)} \cup \{\infty\} = \{\infty\}.$$

Remark 2.3. By definition, the number of finite asymptotic values of f is less than or equal to that of transcendental singularities of f^{-1}. Therefore, if f has only finitely many critical points and transcendental singularities, then $f \in S'$.

We define *structurally finite* transcendental entire functions as follows:

Definition 2.3. *Let*

$$SF_{k,l} := \left\{ f(z) = \int_0^z (c_k t^k + \cdots + c_0) e^{a_l t^l + \cdots + a_1 t} dt + b \right. \\ \left. \mid b, c_i, a_j \in \mathbb{C}\ (i = 0, \cdots, k,\ j = 1, \cdots, l),\ c_k a_l \neq 0 \right\}$$

for $k \geq 0$ and $l \geq 1$, and let

$$SF := \bigcup_{k \geq 0,\, l \geq 1} SF_{k,l}.$$

A transcendental entire function is called structurally finite if $f \in SF$.

Remark 2.4. This definition of structurally finite transcendental entire functions is based on [T, p.68, Theorem 1].

Structurally finite transcendental entire functions have the following good properties:

Proposition 2.18 ([Ok2, p.347, Theorem 1.1], [T2], [T, p.68, Theorem 1])**.** *If $f \in SF_{k,l}$, then f has exactly k critical points counted with multiplicity and l transcendental singularities of f^{-1}. Conversely, every transcendental entire function f with exactly k critical points and l transcendental singularities of f^{-1} satisfies $f \in SF_{k,l}$.*

By Proposition 2.18 and Remark 2.3, it follows that $SF \subset S \cap S'$.

Definition 2.4. *Entire functions f and g are called topologically equivalent if there exist homeomorphisms $\varphi, \Psi : \mathbb{C} \to \mathbb{C}$ such that*

$$\Psi \circ f = g \circ \varphi.$$

Denote by M_f the set of all entire functions topologically equivalent to $f \in S_q$. By definition, we have $M_f \subset S_q$. We can take M_f as a $(q+2)$-dimensional complex analytic manifold whose topology is locally equivalent to the topology of uniform convergence on compact subsets of $\mathbb{C}$. Also, we can take a local coordinate on any small enough open set $U \subset M_f$

$$\Phi : U \to \mathbb{C}^{q+2}, \qquad \Phi(g) = (\Phi_1(g), \cdots, \Phi_{q+2}(g))$$

such that

$$\{\Phi_1(g), \cdots, \Phi_q(g)\} = \operatorname{sing}(g^{-1});$$

the mapping

$$\Phi(U) \times \mathbb{C} \to \mathbb{C}, \qquad (\Phi(g), z) \mapsto g(z)$$

is analytic. (See [EL, Section 3].)

Proposition 2.19 ([T, p.69, Proposition 2])**.** *Let $f \in SF_{k,l}$. Then every $g \in M_f$ satisfies $g \in SF_{k,l}$.*

We introduce the definition of analytic sets as follows:

Definition 2.5 (Analytic sets)**.** *Let $D \subset \mathbb{C}^n$ $(n \in \mathbb{N})$ be a domain. $A \subset D$ is an analytic set in D if for any $a \in D$, there exist a neighborhood U of a and holomorphic functions $f_1, \cdots, f_p$ in U such that*

$$A \cap U = \{z \in U \mid f_1(z) = \cdots = f_p(z) = 0\}.$$

Remark 2.5. Let A and D be as in Definition 2.5. By definition, A is closed in D. (See [Chi] and [Nar] for the properties of analytic sets.) Note that we can define analytic sets naturally on the topologically equivalent space.

The following is from [Nar, p.54, Proposition 10 (Maximum Principle)]:

Proposition 2.20. *Let A be an analytic set in a domain $D \subset \mathbb{C}^n$, let $f : D \to \mathbb{C}$ be a holomorphic function, and let $a \in A$. If f is not constant in $A \cap U$ for any neighborhood U of a, then $f(A \cap V)$ is a neighborhood of $f(a)$ for any small enough neighborhood V of a.*

2.3 Siegel points and Cremer points

In this section, we review basic facts on Siegel or Cremer points. Siegel showed the existence of Siegel points as follows:

Proposition 2.21 ([Si])**.** *Let z be an irrationally indifferent periodic point with multiplier λ. If there exist positive constants M and k such that λ satisfies the following condition:*

$$\lceil\text{Siegel}\rfloor \qquad \frac{1}{|\lambda^n - 1|} \leq Mn^k \quad (n = 1, 2, \cdots),$$

then z is a Siegel point. In addition, $\lceil$Siegel$\rfloor$ is satisfied by almost every λ in the unit circle $\mathbb{S}^1$.

Note that if $\lambda = e^{2\pi i\theta}$ $(\theta \in \mathbb{R} \setminus \mathbb{Q})$ satisfies $\lceil$Siegel$\rfloor$, then $\theta \in \mathcal{B}$ and θ is called a *Diophantine number*. The set $\mathcal{D}$ of all Diophantine numbers satisfies $\mathcal{D}(2) \subset \mathcal{E} \subset \mathcal{D}$ (see [PeZ, p.8, p.9]).

Remark 2.6. Let g be a holomorphic function in a domain $D \subset \mathbb{C}$ and let $\lambda \in \mathbb{S}^1$ satisfy $\lceil$Siegel$\rfloor$. Suppose that the following Taylor expansion holds:

$$g(z) = z_0 + \lambda(z - z_0) + \Sigma_{n=2}^{\infty} a_n (z - z_0)^n \quad (z \to z_0),$$

where $z_0 \in D$, $a_n \in \mathbb{C}$, and $|a_n| < A^{n-1}$ for any $n \geq 2$ and some $A > 0$. The argument in [Si] and Koebe's theorem in [CaG, p.3, Theorem 1.4] show that there exists an $r > 0$ depending only on λ such that the function equation

$$\varphi^{-1} \circ g \circ \varphi(\zeta) = \lambda\zeta$$

has the conformal solution $\varphi(\zeta) = z_0 + \zeta + \Sigma_{n=2}^{\infty} c_n \zeta^n$ ($c_n \in \mathbb{C}$) from $\{\zeta \mid |\zeta| < r\}$ onto some neighborhood U of z_0, which contains $D \cap \{z = \varphi(\zeta) \mid |z - z_0| < r/(4A)\}$.

There are the following facts which tell that irrationally indifferent periodic points of transcendental functions are Cremer points:

Proposition 2.22. *Let z be an irrationally indifferent periodic point. If there exist periodic points in any punctured neighborhood of z, then z is a Cremer point.*

Proposition 2.23 ([Cr2, p.299])**.** *Let g be a non-linear entire function such that:*

(1) *The origin is an irrationally indifferent fixed point with multiplier λ;*

(2) *It satisfies*

$$\max_{|z| \leq r} |g(z)| \leq F(r)$$

for all large enough $r > 0$ and a positive function F defined for all positive real numbers.

If λ satisfies the following condition for every large enough $r > 0$:

$$\lceil \mathrm{Cremer}(F) \rfloor \qquad \liminf_{n \to \infty} \sqrt[\log F^n(r)]{\lambda^n - 1} = 0,$$

then the origin is a Cremer fixed point. Moreover, the set $\Lambda(F)$ of all λ satisfying $\lceil$Cremer$(F)\rfloor$ is uncountable and dense in the unit circle $\mathbb{S}^1$.

Yoccoz's result on the Brjuno condition for quadratic polynomials is generalized as follows:

Proposition 2.24 ([Y], [Gey, p.3665, Theorem 3.2], and [Ok1, p.849, Theorem 1, p.872, Example 2])**.** *Let*

$$P_{\theta,d}(z) := e^{2\pi i\theta} z(1+z)^{d-1},$$

where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and an integer $d \geq 2$. Then $P_{\theta,d}$ has a Siegel point at the origin if and only if $\theta \in \mathcal{B}$.

2.4 Quasiconformal mappings

In this section, we introduce definitions and basic properties of quasiconformal mappings.

Definition 2.6. *Let $f : [a,b] \to \mathbb{R}$ be continuous. The function f is called absolutely continuous if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\Sigma_k |f(y_k) - f(x_k)| < \varepsilon$ holds for any finitely many mutually disjoint closed intervals $I_k := [x_k, y_k] \subset [a,b]$ satisfying $\Sigma_k (y_k - x_k) < \delta$.*

Proposition 2.25. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous. Then f is differentiable almost everywhere in $[a,b]$.*

Definition 2.7. *Let $D \subset \mathbb{C}$ be a domain and let $\phi : D \to \mathbb{C}$ be expressed as $\phi(z) = u(x,y)+iv(x,y)$, where $z = x+iy (x, y \in \mathbb{R})$, $u(x,y) \in \mathbb{R}$, and $v(x,y) \in \mathbb{R}$. We call ϕ absolutely continuous on lines if $u(x,y)$ and $v(x,y)$ are absolutely continuous on almost all lines parallel to the real-axis and almost all lines parallel to the imaginary-axis in any closed rectangle $\{x+iy \mid a \leq x \leq b,\, c \leq y \leq d\} \subset D$.*

Let ϕ be as above. By Proposition 2.25, we can define ϕ_z and $\phi_{\overline{z}}$ almost everywhere by

$$\phi_z := \frac{1}{2}(\phi_x - i\phi_y), \quad \phi_{\overline{z}} := \frac{1}{2}(\phi_x + i\phi_y),$$

where $\phi_x = u_x + iv_x$ and $\phi_y = u_y + iv_y$. Note that if $\phi(z)$ is differentiable at a point $z_0 \in D$, then it follows from the Cauchy-Riemann equations $\phi_{\overline{z}} = 0$ that $\phi_z(z_0) = \phi'(z_0)$ and $\phi_{\overline{z}}(z_0) = 0$.

Definition 2.8. *Let $1 \leq K < \infty$ and let D be a domain of $\mathbb{C}$. A homeomorphism $\phi : D \to \phi(D)$ is a K-quasiconformal mapping if ϕ satisfies the following conditions:*

(1) *ϕ is absolutely continuous on lines;*

(2) *$|\phi_{\overline{z}}| \leq k|\phi_z|$ holds almost everywhere, where $k := (K-1)/(K+1)$.*

In addition, the K and $\phi_{\overline{z}}/\phi_z$ are called a quasiconformal constant and the complex dilatation of ϕ respectively.

This is one of the equivalent definitions of quasiconformal mappings (see [Ah2] for the other definitions). Quasiconformal mappings between Riemann surfaces are defined by their local coordinates. Note that if $K = 1$ in the definition above, then $\phi_{\overline{z}} = 0$ almost everywhere.

Weyl's lemma ([BraF, p.32, Theorem 1.14])**.** *Quasiconformal mappings are conformal if and only if they are 1-quasiconformal.*

Definition 2.9 (Quasicircles)**.** *A Jordan curve $\gamma \subset \widehat{\mathbb{C}}$ is called a K-quasicircle if there exists a K-quasiconformal mapping $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\gamma = \phi(\mathbb{S}^1)$, where $\mathbb{S}^1 := \{z \mid |z| = 1\}$. This K is called a quasicircle constant of γ.*

By definition, the unit circle $\mathbb{S}^1$ is a 1-quasicircle.

Lemma 2.1 ([Za2, p.488, Lemma 2.2])**.** *Let $\gamma \subset \widehat{\mathbb{C}}$ be a K-quasicircle, let U be a component of $\widehat{\mathbb{C}} \setminus \gamma$, and let $g : \mathbb{D} \to U$ be a conformal mapping. Then g extends to a K^2-quasiconformal mapping of $\widehat{\mathbb{C}}$.*

We can tell whether a Jordan curve is a quasicircle or not by the following lemma:

Lemma 2.2 ([Ah1], [GehH, p.23, Theorem 2.2.5])**.** *Let $\gamma \subset \widehat{\mathbb{C}}$ be a Jordan curve and let $\mathrm{Diam}(X)$ be the Euclidean diameter of a set $X \subset \mathbb{C}$. Then γ is a K-quasicircle for some $K \geq 1$ if and only if there exists a constant $A \geq 1$ such that for every pair of two distinct points $z_1, z_2 \in \gamma \setminus \{\infty\}$,*

$$\min_{j=1,2} \mathrm{Diam}(\gamma_j) \leq A|z_1 - z_2|,$$

where γ_1 and γ_2 are the components of $\gamma \setminus \{z_1, z_2\}$. Moreover, K and A depend only on each other.

The following is useful to extend domains of quasiconformal mappings:

Rickman's lemma ([Ri, p.6, Theorem 1], [CheE, p.2145, Theorem 3.4], [BraF, p.34, Lemma 1.20])**.** *Let U be a domain in $\widehat{\mathbb{C}}$, let $C \subset U$ be closed in U, and let ϕ and Φ be homeomorphisms on U. Assume that ϕ is quasiconformal, that Φ is quasiconformal on $U \setminus C$, and that $\phi = \Phi$ on C. Then Φ is quasiconformal and $\phi_{\overline{z}} = \Phi_{\overline{z}}$ almost everywhere on C.*

Quasiconformal mappings share the following properties:

Proposition 2.26 ([BraF, p.31–p.33, Section 1.3.6], [As, p.37, Theorem 1.1]). *Let $\phi : D \to \phi(D) =: D'$ be a K-quasiconformal mapping for a domain $D \subset \mathbb{C}$. Then the following assertions hold:*

(i) *ϕ^{-1} is also K-quasiconformal.*

(ii) *If $\varphi_1 : D'' \to D$ and $\varphi_2 : D' \to \varphi_2(D')$ are K'-quasiconformal, then $\phi \circ \varphi_1$ and $\varphi_2 \circ \phi$ are KK'-quasiconformal.*

(iii) *For any compact set E in D, there exists an $M > 0$ such that*

$$|\phi(z_1) - \phi(z_2)| \leq M|z_1 - z_2|^{1/K}$$

for any z_1 and z_2 in E.

(iv) *If $D = D' = \mathbb{D}$ and $\phi(0) = 0$, then for any Borel measurable set $U \subset \mathbb{D}$, there exists a constant $C > 0$ depending only on K such that*

$$\text{Area}(\phi(U)) \leq C(\text{Area}(U))^{1/K},$$

where $\text{Area}(X)$ denotes the Lebesgue measure for any measurable set $X \subset \mathbb{C}$.

(v) *ϕ maps a set of Lebesgue measure zero (resp. a set of positive Lebesgue measure) to a set of Lebesgue measure zero (resp. a set of positive Lebesgue measure).*

We defined a normal family of meromorphic functions in a domain. Similarly, we define a normal family of K-quasiconformal mappings in a domain.

Proposition 2.27 ([LV, p.73, Theorem 5.1]). *Let $\mathcal{F}$ be a family of K-quasiconformal mappings in a domain $D \subset \widehat{\mathbb{C}}$. Then the following assertions hold:*

(i) *Suppose that there exists a $C > 0$ such that every $\phi \in \mathcal{F}$ does not take two values whose spherical distance is greater than C. Then $\mathcal{F}$ is normal.*

(ii) *Suppose that there exist a $C > 0$ and three fixed points z_1, z_2, and z_3 in D such that for every $\phi \in \mathcal{F}$,*

$$\sigma(\phi(z_k), \phi(z_l)) > C,$$

where σ denotes the spherical metric, and $1 \leq k \leq 3$, $1 \leq l \leq 3$, and $k \neq l$. Then $\mathcal{F}$ is normal.

The following is from Proposition 2.27 and [LV, p.74, Theorem 5.3]:

Proposition 2.28. *Let $\mathcal{F}$ be a family of K-quasiconformal mappings in a domain $D \subset \widehat{\mathbb{C}}$ such that every $\phi \in \mathcal{F}$ satisfies $\phi(a) = a$ and $\phi(b) = b$ for some two fixed a and b in D. Then $\mathcal{F}' := \{\phi|_{D \setminus \{a,b\}}\}_{\phi \in \mathcal{F}}$ is normal. Moreover, if the set of the all limit functions of $\mathcal{F}'$ does not contain the constants a and b, then $\mathcal{F}$ is normal, and any limit function of $\mathcal{F}$ is a K-quasiconformal mapping in D.*

Proposition 2.29 ([LV, p.70, Theorem 4.3]). *Let $\mathcal{F}$ be a normal family of K-quasiconformal mappings in a domain $D \subset \widehat{\mathbb{C}}$. Then for every compact set $E \subset D$, there exists a constant $C > 0$ such that for every $\phi \in \mathcal{F}$, every $z \in D$, and every $z_0 \in E$,*

$$\sigma(\phi(z), \phi(z_0)) \leq C(\sigma(z, z_0))^{1/K},$$

where σ denotes the spherical metric.

Since conformal mappings have the complex dilatation 0, the limit functions of normal families of them also have the complex dilatation 0. However, the uniform convergence of quasiconformal mappings does not mean the convergence of the complex dilatations in general (see [LV, p.186]). In contrast to this, it is known that:

Lemma 2.3 ([L, p.29, Theorem 4.6]). *Let ϕ_n $(n = 1, 2 \cdots)$ and ϕ be quasiconformal mappings of $\widehat{\mathbb{C}}$ fixing 0, 1, and ∞, and let μ_n and μ be the complex dilatations of ϕ_n and ϕ respectively. If $||\mu_n||_\infty \leq k < 1$ and $\mu_n \to \mu$ almost everywhere as $n \to \infty$, where*

$$||\mu_n||_\infty := \inf\{M \mid |\mu_n| \leq M \text{ almost everywhere}\},$$

then $\phi_n \to \phi$ locally uniformly as $n \to \infty$.

We define *quasiregular mappings* as follows:

Definition 2.10 (Quasiregular mappings). *Let D be an open set of $\mathbb{C}$. A continuous mapping $F : D \to \mathbb{C}$ is a K-quasiregular mapping if F is locally K-quasiconformal except at a discrete set of points in D for some $K \geq 1$. The constant K is called a quasiregular constant.*

Note that Weyl's lemma shows that K-quasiregular mappings are holomorphic if and only if $K = 1$. A mapping $g : D \to \mathbb{C}$ is quasiregular for a domain $D \subset \mathbb{C}$ if F can be expressed as

$$F = f \circ \phi,$$

where $\phi : D \to \phi(D)$ is a quasiconformal mapping and $f : \phi(D) \to F(D)$ is a holomorphic function. (In fact, this is one of the equivalent definitions of quasiregular mappings in a domain. See [BraF, p.55, Definition 1.33, p.56, Definition 1.34].) As in the case of quasiconformal mappings, quasiregular mappings between Riemann surfaces are defined by their local coordinates.

Finally, we introduce the definition of *quasisymmetric mappings* on $\mathbb{S}^1$ and the way of constructing quasiconformal mappings on $\mathbb{D}$ by extending them.

Definition 2.11 (Quasisymmetric mappings). *Let $s : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation preserving homeomorphism and let $\tilde{s} : \mathbb{R} \to \mathbb{R}$ be the continuous mapping satisfying*

$$s \circ \Gamma = \Gamma \circ \tilde{s},$$

where

$$\Gamma : \mathbb{R} \to \mathbb{S}^1, \quad x \mapsto e^{2\pi i x}.$$

s is called k-quasisymmetric if it satisfies

$$\frac{1}{k} \leq \frac{\tilde{s}(x+t) - \tilde{s}(x)}{\tilde{s}(x) - \tilde{s}(x-t)} \leq k$$

for some $k \geq 1$ and any $x \in \mathbb{R}$ and any $t > 0$.

The Beurling-Ahlfors extension ([BeuA], [BraF, p.83–p.84]). *Let s and $\tilde{s}$ be as in Definition 2.11, and let $\mathbb{H} \subset \mathbb{C}$ be the upper half plane. Define $\hat{s}_1$ on $\mathbb{H} \cup \mathbb{R}$ by*

$$\hat{s}_1(x+iy) := \frac{1}{2}\int_0^1 \{\tilde{s}(x+ty) + \tilde{s}(x-ty)\}dt + i\int_0^1 \{\tilde{s}(x+ty) - \tilde{s}(x-ty)\}dt,$$

where $x \in \mathbb{R}$ and $y \geq 0$. (Note that $\hat{s}_1|_{\mathbb{H}} : \mathbb{H} \to \mathbb{H}$ and $\hat{s}_1|_{\mathbb{R}} = \tilde{s}$.) Then there exists a constant $K \geq 1$ depending only on k such that $\hat{s}_1|_{\mathbb{H}} : \mathbb{H} \to \mathbb{H}$ is K-quasiconformal. In addition, there exists an orientation preserving homeomorphism $\hat{s}_2 : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that:

(a) $\hat{s}_2|_{\mathbb{S}^1} = s$ *and* $\hat{s}_2|_{\mathbb{D}}$ *is a* K*-quasiconformal mapping fixing the origin;*

(b) *It satisfies*

$$\hat{s}_2 \circ \hat{\Gamma} = \hat{\Gamma} \circ \hat{s}_1,$$

where

$$\hat{\Gamma} : \mathbb{H} \cup \mathbb{R} \to \overline{\mathbb{D}}, \quad z \mapsto e^{2\pi i z}.$$

Definition 2.12 (Rotation numbers)**.** *Let* $f : \mathbb{S}^1 \to \mathbb{S}^1$ *be an orientation preserving homeomorphism and let* Γ *be as in Definition 2.11. Define the rotation number* $\mathrm{rot}(f)$ *of* f *by the fractional part of*

$$\lim_{n\to\infty} \frac{F^n(x) - x}{n},$$

where $x \in \mathbb{R}$ *and* $F : \mathbb{R} \to \mathbb{R}$ *satisfies*

$$\Gamma \circ F = f \circ \Gamma.$$

Remark 2.7. Let $\mathrm{rot}(f)$ and F be as in Definition 2.12. According to [Po], the rotation number $\mathrm{rot}(f)$ is independent of $x \in \mathbb{R}$ and F.

Next, we introduce the following version of the Herman-Świątek theorem: We call $f : \mathbb{S}^1 \to \mathbb{S}^1$ a *critical circle map* if f is an orientation preserving homeomorphism which has at least one critical point.

Lemma 2.4 ([CheE, p.2147, Theorem 3.8], [Herm2], [Herm3], and [Św])**.** *Let* $\mathcal{F}$ *be a family of holomorphic maps defined in a neighborhood of* $\mathbb{S}^1$ *with the following properties:*

(a) *There exists an open annulus* A *containing* $\mathbb{S}^1$ *such that every* $f \in \mathcal{F}$ *is defined in* A*;*

(b) $f(\mathbb{S}^1) = \mathbb{S}^1$ *and* $f|_{\mathbb{S}^1}$ *is a critical circle map;*

(c) *There exists an* $R > 0$ *such that for every* $f \in \mathcal{F}$ *and every* $n \geq 1$*, the rotation number* $\mathrm{rot}(f|_{\mathbb{S}^1})$ *satisfies* $a_n \leq R$*, where* $\mathrm{rot}(f|_{\mathbb{S}^1}) = [a_0; a_1, \ldots, a_n, \ldots]$ *is the continued fraction expansion;*

(d) $\mathcal{F}$ *is precompact on* A *for the Euclidean metric.*

Then there exists a $k > 1$ *such that for every* $f \in \mathcal{F}$*,* $f|_{\mathbb{S}^1}$ *is* k*-quasisymmetrically conjugate to a rotation. To be more precise, there exists a* k*-quasisymmetric map* $s : \mathbb{S}^1 \to \mathbb{S}^1$ *such that for any* $z \in \mathbb{S}^1$*,*

$$s \circ f|_{\mathbb{S}^1} \circ s^{-1}(z) = e^{2\pi i\theta} z =: R_\theta(z),$$

where $\theta = \mathrm{rot}(f|_{\mathbb{S}^1})$*.*

Let f and s be as in Lemma 2.4, and let $\hat{s} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be a homeomorphism obtained by the Beurling-Ahlfors extension of s, which is K-quasiconformal in $\mathbb{D}$, where K depends only on k. Then we can extend $f|_{\mathbb{S}^1}$ to

$$\hat{f} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}, \quad \hat{f} := \hat{s}^{-1} \circ R_\theta \circ \hat{s},$$

which is K^2-quasiconformal in $\mathbb{D}$ (see Proposition 2.26 (i) and (ii)).

2.5 Quasiconformal surgery

For a given meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$, quasiconformal surgery consists of the following steps:

(1) We construct a quasiregular mapping $F : \mathbb{C} \to \widehat{\mathbb{C}}$ from f with some appropriate properties. (For example, we choose an appropriate quasiconfomal mapping $\phi : \mathbb{C} \to \mathbb{C}$ so that $F = f \circ \phi$.)

(2) We show the existence of a quasiconformal mapping $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\varphi(\infty) = \infty$ and

$$G := \varphi \circ F \circ \varphi^{-1} : \mathbb{C} \to \widehat{\mathbb{C}}$$

is meromorphic.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \widehat{\mathbb{C}} \\ \Big\downarrow \varphi & & \Big\downarrow \varphi \\ \mathbb{C} & \xrightarrow{G} & \widehat{\mathbb{C}} \end{array}$$

The maps F and G are called (quasiconformally) *conjugate* by φ. Since φ is a homeomorphism, the two sequences $\{z, F(z), F^2(z), \cdots, F^n(z)\}$ and $\{\varphi(z), G(\varphi(z)), G^2(\varphi(z)), \cdots, G^n(\varphi(z))\}$ have one-to-one correspondence whenever $F^n(z)$ is defined. Therefore, G preserves important dynamical properties of F. For example, if z is a periodic point of period p, a critical point, or a singular value of F, then $\varphi(z)$ is a periodic point of period p, a critical point, or a singular value of G respectively. Suppose that z is a periodic point of F and its multiplier is defined as in the case of meromorphic functions. If φ is conformal in a neighborhood of z, an easy calculation shows that the periodic points z of F and $\varphi(z)$ of G have the same multiplier. Hence the construction of F is important for the new meromorphic function G obtained to have appropriate dynamical properties.

Let $V \subset \mathbb{C}$ be open. A Lebesgue measurable function $\mu : V \to \mathbb{D}$ is called a *Beltrami coefficient* in V.

Lemma 2.5. *Let μ be as above and let $f : U \to V = f(U)$ be a quasiregular mapping for an open set $U \subset \mathbb{C}$. Then*

$$f^*\mu(u) := \frac{f_{\overline{z}}(u) + \mu(f(u))\overline{f_z(u)}}{f_z(u) + \mu(f(u))\overline{f_{\overline{z}}(u)}}$$

is well-defined for almost all $u \in U$. In addition, for an open set $D \subset \mathbb{C}$ and a quasiregular mapping $g : D \to U = g(D)$,

$$(f \circ g)^*\mu(a) = g^*(f^*\mu)(a)$$

holds for almost all $a \in D$.

Remark 2.8. Note that if f as in Lemma 2.5 is holomorphic, then

$$||f^*\mu||_\infty = ||\mu||_\infty,$$

since $f_{\overline{z}} = 0$ yields

$$f^*\mu(u) = \frac{\mu(f(u))\overline{f_z(u)}}{f_z(u)}$$

for almost all u.

$f^*\mu$ is called a *pullback* of μ by f. Pullbacks of quasiregular mappings have the following property:

Lemma 2.6. *Let $\phi : D \to \phi(D)$ be a quasiregular mapping in a domain $D \subset \mathbb{C}$. Let $\mu_0 := 0$ be a Beltrami coefficient in the domain $\phi(D)$. Then $\phi^*\mu_0 = \phi_{\overline{z}}/\phi_z$ holds almost everywhere in D. In addition, $(\phi^{-1})^*\mu_\phi = \mu_0$ holds almost everywhere in $\phi(D)$, where μ_ϕ is a Beltrami coefficient in D which equals the complex dilatation $\phi_{\overline{z}}/\phi_z$ almost everywhere.*

Definition 2.13. *Let $F : \mathbb{C} \to \widehat{\mathbb{C}}$ be a quasiregular mapping and let $\mu : \mathbb{C} \to \mathbb{D}$ be a Beltrami coefficient in $\mathbb{C}$. We call μ F-invariant (or invariant by F) if $F^*\mu = \mu$ holds almost everywhere in $\mathbb{C}$.*

The most important gear for quasiconformal surgery is the following lemma, which is some modification of [BraF, p.60, Lemma 1.39]:

Lemma 2.7. *Let $F : \mathbb{C} \to \widehat{\mathbb{C}}$ be a quasiregular mapping and let μ be a Beltrami coefficient in $\mathbb{C}$. If $||\mu||_\infty \leq k < 1$ and μ is F-invariant, then there exists an exactly one quasiconformal mapping $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with the following properties:*

(i) *$\varphi_{\overline{z}}/\varphi_z = \mu$ holds almost everywhere in $\mathbb{C}$;*

(ii) *φ fixes 0, 1, and ∞;*

(iii) *$\varphi \circ F \circ \varphi^{-1} : \mathbb{C} \to \widehat{\mathbb{C}}$ is meromorphic.*

Proof. The existence of φ with properties (i) and (ii) follows directly from the Integrability Theorem (see [BraF, p.40, Theorem 1.27 and Theorem 1.28]). The property (iii) follows from properties (i) and (ii) as follows: By Proposition 2.26 (ii), $\varphi \circ F \circ \varphi^{-1}$ is quasiregular. By Lemma 2.5, Lemma 2.6, and the construction,

$$(\varphi \circ F \circ \varphi^{-1})^* \mu_0 = (\varphi^{-1})^*(F^*(\varphi^* \mu_0)) = (\varphi^{-1})^*(F^* \mu) = (\varphi^{-1})^* \mu = \mu_0$$

holds almost everywhere in $\mathbb{C}$. Thus μ_0 is $(\varphi \circ F \circ \varphi^{-1})$-invariant, and hence $(\varphi \circ F \circ \varphi^{-1})_{\overline{z}} = 0$ holds almost everywhere in $\mathbb{C}$. Since $\varphi \circ F \circ \varphi^{-1}$ is 1-quasiregular, Weyl's lemma shows the property (iii). ☐

The following is some modification of [Shi, p.7, Lemma 1]:

Lemma 2.8 ([Shi, p.7, Lemma 1])**.** *Let $F : \mathbb{C} \to \widehat{\mathbb{C}}$ be a K-quasiregular mapping with the following properties:*

(a) *$F(U) \subset U$ for some domain $U \subset \mathbb{C}$;*

(b) *$F|_U = \phi^{-1} \circ f \circ \phi$, where $\phi : U \to U$ is K'-quasiconformal and $f : U \to U$ is holomorphic;*

(c) *$F_{\overline{z}} = 0$ almost everywhere in $\mathbb{C} \setminus F^{-1}(U)$.*

Then there exists an M-quasiconformal mapping $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that:

(i) *The quasiconformal constant M depends only on K;*

(ii) *φ fixes 0, 1, and ∞, and*

$$G := \varphi \circ F \circ \varphi^{-1} : \mathbb{C} \to \widehat{\mathbb{C}}$$

is meromorphic;

(iii) *$\varphi \circ \phi^{-1}$ is conformal in U and φ is conformal on the interior of $O := \mathbb{C} \setminus \bigcup_{n=1}^{\infty} F^{-n}(U)$.*

Proof. Let μ_ϕ be a Beltrami coefficient in U which corresponds to $\phi_{\overline{z}}/\phi_z$ almost everywhere. We construct an F-invariant Beltrami coefficient μ_F in $\mathbb{C}$ as follows: We define $\mu_F(u) := (F^n)^* \mu_\phi(u)$ for almost all $u \in \bigcup_{n=0}^{\infty} F^{-n}(U)$, where $F^0 = \mathrm{Id}$. Otherwise, set $\mu_F := \mu_0 = 0$. The property (c) and Lemma 2.5 show that $F^* \mu_F = \mu_F$ almost everywhere in O, where $\mu_F = \mu_0$. By Lemma 2.5, Lemma 2.6, and the property (b), we have

$$(F|_U)^* \mu_\phi = (\phi^{-1} \circ f \circ \phi)^* \mu_\phi = \phi^*(f^*((\phi^{-1})^* \mu_\phi)) = \phi^*(f^* \mu_0) = \phi^* \mu_0 = \mu_\phi,$$

since f is holomorphic, and hence $f^* \mu_0 = \mu_0$ in U. From this and the definition of μ_F for almost all points in $\bigcup_{n=0}^{\infty} F^{-n}(U)$, we have $F^* \mu_F = \mu_F$ there. Moreover, since the property (c) implies that F is holomorphic $\mathbb{C} \setminus F^{-1}(U)$, it follows from Remark 2.8 that there exists a constant $0 < k < 1$, depending only on K, such that $||\mu_F||_\infty \leq k$. Therefore, μ_F is an F-invariant Beltrami coefficient with $||\mu_F||_\infty \leq k < 1$, $\mu_F = \mu_\phi$ in U, and $\mu_F = \mu_0$ in O, and hence Lemma 2.7 shows the existence of φ. ☐

Remark 2.9. Let F be as in Lemma 2.8. From the construction, there exists a periodic component of G which contains $\varphi(U)$. In particular, when $U = \mathbb{D}$ and f is the rotation R_θ, G has a Siegel disk containing $\varphi(\mathbb{D})$. In addition, if $\mathbb{S}^1$ contains a critical point of F, then $\partial\varphi(\mathbb{D})$ also contains a critical point of G. Therefore, $\varphi(\mathbb{D})$ is exactly a Siegel disk of G. Ghys generalized this surgery technique (see [Gh] and [BraF, p.226, Theorem 7.10]).

Douady and Hubbard are pioneers of quasiconformal surgery. They introduced *polynomial-like mappings* as follows:

Definition 2.14 (Polynomial-like mappings ([DH, p.294]))**.** *Let U and V be bounded simply connected subdomains of $\mathbb{C}$ with $\overline{U} \subset V$. The triple $(f; U, V)$ is called a polynomial-like mapping of degree d if $f : U \to V$ is a proper and holomorphic mapping of degree d.*

Let P and $(f; U, V)$ be a polynomial of degree $d \geq 2$ and a polynomial-like mapping of degree d respectively. Define *the filled Julia set K_P of P* by

$$K_P := \{z \in \mathbb{C} \mid \{P^n(z)\} \text{ is bounded}\}.$$

Note that $\partial K_P = J(P)$. On the other hand, *the filled Julia set K_f of* $(f; U, V)$ is defined by

$$K_f := \{z \in U \mid f^n(z) \in U \quad \text{for every integer } n \geq 1\}.$$

Moreover, we define the Julia set $\tilde{J}_f$ of $(f; U, V)$ in the sense of [DH] by

$$\tilde{J}_f := \partial K_f.$$

Note that if $f = F|_U$, where F is a transcendental meromorphic function, then $\tilde{J}_f$ is a proper subset of $J(F)$. We say that $(f; U, V)$ and P are *hybrid equivalent* (*by* ϕ) if there is a quasiconformal mapping ϕ from a neighborhood of K_f onto a neighborhood of K_P such that:

$$\phi \circ f(z) = P \circ \phi(z) \quad (\text{near } K_f);$$

$$\phi_{\overline{z}}(z) = 0 \quad (\text{almost everywhere on } K_f).$$

The straightening theorem ([DH, p.296, Theorem 1])**.** *Let $(f; U, V)$ be a polynomial-like mapping of degree $d \geq 2$. Then there exists a polynomial P of degree d such that $(f; U, V)$ and P are hybrid equivalent by some ϕ. Furthermore, ϕ can be extended as a quasiconformal mapping of $\widehat{\mathbb{C}}$ fixing ∞.*

Sketch of the proof. We can choose simply connected domains $U' \subset U$ and $V' \subset V$ such that $\partial U'$ and $\partial V'$ are analytic Jordan curves and $(f; U', V')$ is a polynomial-like mapping of degree d (see [BraF, p.220, Remark 7.2]). From the smoothness of boundaries $\partial U'$ and $\partial V'$, one can extend f to a quasiregular mapping $F : \mathbb{C} \to \mathbb{C}$ which is of degree d and conformally conjugate to $z \mapsto z^d$ in a neighborhood of ∞, and can construct an F-invariant Beltrami coefficient μ_F on $\mathbb{C}$ which equals 0 in K_f and satisfies $||\mu_F||_\infty \leq k < 1$. Therefore, Lemma 2.7 shows that the straightening theorem. □

Transcendental meromorphic functions defined on $\mathbb{C}$ and polynomials have many different properties. However, if a transcendental meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ restricted to some bounded simply-connected domain U is a polynomial-like mapping, then the straightening theorem shows that $(f|_U; U, V)$ and some polynomial P are hybrid equivalent by some ϕ.

$$\begin{array}{ccc} K_{f|_U} & \xrightarrow{f} & K_{f|_U} \\ \big\downarrow \phi & & \big\downarrow \phi \\ K_P & \xrightarrow{P} & K_P \end{array}$$

In this case, we can understand the dynamics of $f|_U$ by using results which is applicable only to polynomials.

We introduce the Riemann-Hurwitz formula for domains, which is useful to find the structure of polynomial-like mappings.

The Riemann-Hurwitz formula for domains ([MoNTU, p.10, Lemma 1.1.5]). *Let U and V be domains in $\widehat{\mathbb{C}}$ bounded by finitely many mutually disjoint Jordan curves and let $f : U \to V$ be a proper, onto, and holomorphic mapping of degree d with N critical points counted with multiplicity. Then the following formula holds:*

$$(2 - n) + N = d(2 - m),$$

where n and m are the number of boundary components of the boundary ∂U and the number of ∂V respectively.

Chapter 3

Best possibility of the Fatou-Shishikura inequality for transcendental entire functions in the Speiser class —Proof of Theorem A—

Shishikura showed best possibility of the Fatou-Shishikura inequality for rational functions by his quasiconformal surgery which converts one Siegel cycle into one repelling, attracting, rationally indifferent, or Cremer cycle (see [Shi]). To be more precise, he constructed a rational function with several Siegel cycles. From this function, he obtained a new rational function with one more repelling, attracting, rationally indifferent, or Cremer cycle and one less Siegel cycle by his quasiconformal surgery. This process is repeated until the numbers of cycles are as desired. Basically, we follow his method for the rational case. However, there are differences between rational functions and transcendental entire functions as we noted in Chapter 1. Thus we have to modify his proof at each step. The differences are as follows: We modify his quasiconformal surgery technique, since the value at ∞ cannot be defined naturally in contrast to the rational case. Shishikura obtained functions in some analytic sets in the topological space defined by the coefficients of rational functions. On the other hand, we use the theory of the topologically equivalent space in Section 2.2 and construct functions in some analytic sets defined on the space. The critical difference is in our construction of Cremer cycles. Shishikura constructed Cremer cycles one by one. More precisely, he made one Siege cycle into one Cremer cycle with multiplier satisfying [Cremer (d)] in Theorem 1.3, which is specific to rational functions. We cannot use this for our case. Thus we have to make Cremer cycles of T in a different way. We do not construct Cremer cycles one by one because our construction does not guarantee that one Cremer cycle constructed is kept unchanged while we construct another Cremer cycle. Instead, we construct all Cremer cycles of T in the final step.

Here we give the sketch of the proof. If $(m_{\rm AB}, m_{\rm PB}, m_{\rm SD}, m_{\rm Cr}) \neq (0,0,0,0)$, we construct T by the following procedure: First of all, we construct a $T_0 \in S_q \cap SF_{0,q}$ which has q Siegel cycles (Lemma 3.1). Next, we take T_0 as T if $(m_{\rm AB}, m_{\rm PB}, m_{\rm SD}, m_{\rm Cr}) = (0,0,q,0)$. Let $(m_{\rm AB}, m_{\rm PB}, m_{\rm SD}, m_{\rm Cr}) \neq (0,0,q,0)$. If $m_{\rm Cr} = 0$, we convert T_0 into $T \in M_{T_0}$ by making one Siegel cycle repelling, attracting, or rationally indifferent repeatedly. This step by step procedure is done by quasiconformal surgery (Lemma 3.4) or some argument on analytic sets (Lemma 3.5). If $m_{\rm Cr} \neq 0$, we convert T_0 into a $\tilde{T}$ with

$$(n_{\rm AB}(\tilde{T}), n_{\rm PB}(\tilde{T}), n_{\rm SD}(\tilde{T}), n_{\rm Cr}(\tilde{T})) = (m_{\rm AB}, m_{\rm PB}, m_{\rm SD} + m_{\rm Cr}, 0)$$

in the manner above. Then we convert $\tilde{T}$ into T by making $m_{\rm Cr}$ Siegel cycles of $\tilde{T}$ into $m_{\rm Cr}$ Cremer cycles of T at a time (Lemma 3.6).

Let p be any positive integer relatively prime with q. Put $\lambda = \exp(2\pi i/p)$. Set

$$f_\alpha(z) := (1+\alpha)\lambda \int_0^z e^{t^q}\, dt \in SF_{0,q} \quad \text{for } \alpha \in \mathbb{C} \setminus \{-1\}.$$

Since f_α has q distinct finite asymptotic values and no critical values (see [Ne, p.168, 2.3]), we have $f_\alpha \in S_q$.

Lemma 3.1. *There is an uncountable set $A \subset \mathbb{C} \setminus \{-1\}$ such that f_α ($\alpha \in A$) has q Siegel cycles of period p whose multipliers satisfy the condition* $\lceil$Siegel$\rfloor$ *of Proposition 2.21.*

Proof. An easy calculation shows that

$$f_\alpha(z) = (1+\alpha)\lambda z \left(1 + \frac{z^q}{q+1} + \frac{z^{2q}}{2(2q+1)} + \cdots\right).$$

From this and Proposition 2.10, we get

$$f_0^p(z) = z\{1 + c_0 z^{pkq} + O(|z|^{(pk+1)q})\} \quad \text{as } z \to 0,$$

where $c_0 \neq 0$, $k \geq 1$. In addition, there are kq PB-cycles of period p. By Theorem 1.2, we have $kq \leq q$. Thus we obtain $k = 1$. It follows that

$$f_\alpha^p(z) = z\{(1+\alpha)^p + c(\alpha) z^{pq} + O(|z|^{(p+1)q})\} \quad \text{as } \alpha, z \to 0,$$

where $c(\alpha)$ is a holomorphic function of α, with $c(0) = c_0$. Set $X = z^q$. Let $f_\alpha^p(z) = zF(X, \alpha)$. Thus we have

$$F(X,\alpha) = (1+\alpha)^p \left(1 + \frac{c(\alpha)}{(1+\alpha)^p} X^p + O(|X|^{p+1})\right) \quad \text{as } \alpha, X \to 0.$$

By the construction and Rouché's theorem, if $\alpha \neq 0$ is small enough, $F(X,\alpha) = 1$ has p different solutions $X = \zeta_1(\alpha), \cdots, \zeta_p(\alpha)$ with $\zeta_j(\alpha) \neq 0$ and $\zeta_j(\alpha) \to 0$ ($\alpha \to 0$) for $j = 1, \cdots, p$. Therefore, f_α has q cycles $C_1(\alpha), \cdots, C_q(\alpha)$ of period p for small enough $\alpha \neq 0$. They consist of pq q-th roots of $\zeta_j(\alpha)$ ($j = 1, \cdots, p$). Moreover, it follows that

$$\frac{\partial F}{\partial X}|_{(\zeta_j(\alpha_0),\alpha_0)} \neq 0 \quad (j = 1, \cdots, p)$$

for every small enough $\alpha_0 \neq 0$. By the implicit function theorem, $\zeta_j(\alpha)$ ($j = 1, \cdots, p$) are holomorphic functions of α on some neighborhood of α_0. It follows that

$$\Sigma(\alpha) := \Sigma_{j=1}^p \zeta_j(\alpha)$$

is a holomorphic function of α on some punctured neighborhood of 0. By the construction, $C_j(\alpha)$ ($j = 1, \cdots, q$) have the same multiplier $\sigma(\alpha)$. An easy calculation shows that

$$\sigma(\alpha) = (1+\alpha)^p e^{\Sigma(\alpha)}.$$

Thus we have $\sigma(\alpha) \to 1$ as $\alpha \to 0$ and $\sigma(\alpha)$ is holomorphic on some punctured neighborhood of 0. Set $\sigma(0) = 1$. By the Riemann removable singularity theorem, $\sigma(\alpha)$ is holomorphic on some neighborhood U of 0. It follows that $\sigma(U)$ is a neighborhood of 1. Hence there is an uncountable set $A \subset U$ such that $\sigma(\alpha)$ ($\alpha \in A$) satisfies the condition $\lceil$Siegel$\rfloor$ of Proposition 2.21. Thus f_α ($\alpha \in A$) has q Siegel cycles of period p with multiplier $\sigma(\alpha)$. □

Lemma 3.1 shows the existence of $T_0 \in S_q \cap SF_{0,q}$ with q Siegel cycles of period p. We convert T_0 into T with non-repelling cycles of period p. Henceforth, we construct T with $p = 1$ for simplicity. The case $p \geq 2$ is shown exactly in the same way.

We call $a \in \operatorname{sing}(f^{-1})$ *eventually repelling* if $f^n(a)$ is a repelling periodic point for some $n \geq 0$. Let $n_{\mathrm{ER}}(f)$ be the number of eventually repelling singular values of $f \in S_q$. We define $C(f)$ for $f \in S_q$ by

$$C(f) := (n_{\mathrm{AB}}(f), n_{\mathrm{PB}}(f), n_{\mathrm{SD}}(f), n_{\mathrm{Cr}}(f), n_{\mathrm{ER}}(f)).$$

The combination $C(f)$ always satisfies

$$n_{\mathrm{rat}}(f) \leq n_{\mathrm{PB}}(f), \quad n_{\mathrm{AB}}(f) + n_{\mathrm{PB}}(f) + n_{\mathrm{SD}}(f) + n_{\mathrm{Cr}}(f) + n_{\mathrm{ER}}(f) \leq q.$$

(See Proposition 2.10 for the former. The latter follows from Shishikura's idea used in the proof of Theorem 1.2. See [Shi, p.25] for details.) They yield the following lemma:

Lemma 3.2. *Suppose that non-negative integers $n_{\mathrm{AB}}, n_{\mathrm{rat}}, n_{\mathrm{SD}}, n_{\mathrm{Cr}}$, and n_{ER}, and $f \in S_q$ satisfy*

$$n_{\mathrm{AB}} + n_{\mathrm{rat}} + n_{\mathrm{SD}} + n_{\mathrm{Cr}} + n_{\mathrm{ER}} = q,$$

$$n_{\mathrm{AB}}(f) \geq n_{\mathrm{AB}}, \cdots, n_{\mathrm{ER}}(f) \geq n_{\mathrm{ER}}.$$

Then $n_{\mathrm{PB}}(f) = n_{\mathrm{rat}}(f)$ and

$$C(f) = (n_{\mathrm{AB}}, n_{\mathrm{rat}}, n_{\mathrm{SD}}, n_{\mathrm{Cr}}, n_{\mathrm{ER}}).$$

The following is the fundamental lemma for our quasiconformal surgery, which is some modification of Lemma 2.8 and [Shi, p.7, Lemma 1, p.9, Lemma 3]):

Lemma 3.3 ([Shi, p.7, Lemma 1, p.9, Lemma 3])**.** *For $\varepsilon \in \mathbb{C}$ in a neighborhood of 0, set a quasiregular mapping*

$$g_\varepsilon = f \circ \Psi_\varepsilon,$$

where f is an entire function and $\Psi_\varepsilon : \mathbb{C} \to \mathbb{C}$ is a quasiconformal mapping. Suppose that g_ε satisfies the following conditions:

(1) $||(\Psi_\varepsilon)_{\overline{z}}/(\Psi_\varepsilon)_z||_\infty \to 0$ $(\varepsilon \to 0)$ *and* $\Psi_\varepsilon \to \mathrm{Id}_{\mathbb{C}}$ $(\varepsilon \to 0)$ *locally uniformly on* $\mathbb{C}$*;*

(2) *There exists an open set E_ε such that $g_\varepsilon(E_\varepsilon) \subset E_\varepsilon$ and $(g_\varepsilon)_{\overline{z}} = 0$ almost everywhere on $E_\varepsilon \cup (\mathbb{C} \setminus (g_\varepsilon)^{-1}(E_\varepsilon))$.*

Then there exists a quasiconformal mapping φ_ε with the following properties:

(a) $\tilde{g}_\varepsilon = \varphi_\varepsilon \circ g_\varepsilon \circ \varphi_\varepsilon^{-1}$ *is an entire function;*

(b) $\varphi_\varepsilon \to \mathrm{Id}_{\mathbb{C}}$ *and* $\tilde{g}_\varepsilon \to f$ *locally uniformly on* $\mathbb{C}$ *as* $\varepsilon \to 0$*;*

(c) φ_ε *is conformal on the interior of* $E_\varepsilon \cup (\mathbb{C} \setminus \bigcup_{n=1}^{\infty} (g_\varepsilon)^{-n}(E_\varepsilon))$.

By quasiconformal surgery, we will reduce the number of SD-cycles by one and increase that of AB-cycles (or PB-cycles, eventually repelling singular values) by one. We use the following lemma which is applicable to general structurally finite transcendental entire functions:

Lemma 3.4. *Let $f \in S_q \cap SF_{k,l}$. We assume the following conditions:*

(1) *Every non-repelling periodic point of f is a fixed point;*

(2) *Every Siegel point of f has the multiplier satisfying the condition* $\lceil$Siegel$\rfloor$ *of Proposition 2.21;*

(3) *f satisfies* $n_{\mathrm{SD}}(f) \geq 2$, $n_{\mathrm{PB}}(f) = n_{\mathrm{rat}}(f)$, $n_{\mathrm{Cr}}(f) = 0$, *and*

$$n_{\mathrm{AB}}(f) + n_{\mathrm{PB}}(f) + n_{\mathrm{SD}}(f) + n_{\mathrm{ER}}(f) = q.$$

Then for every neighborhood $N \subset M_f$ *of* f*, there exist* $g_j \in N (j = 1, 2, 3)$ *with the following properties:*

(a) *Every non-repelling periodic point of* g_j *is a fixed point whose multiplier is not* 1*;*

(b) *Every Siegel point of* g_j *has the multiplier satisfying the condition* $\lceil$Siegel$\rfloor$ *of Proposition 2.21;*

(c) g_j *has a Siegel point with a preimage other than itself;*

(d)

$$g_j \in SF_{k,l}, \quad n_{\mathrm{PB}}(g_j) = n_{\mathrm{rat}}(g_j),$$

and

$$\begin{aligned} C(g_1) &= (n_{\mathrm{AB}}(f) + 1, n_{\mathrm{PB}}(f), n_{\mathrm{SD}}(f) - 1, 0, n_{\mathrm{ER}}(f)), \\ C(g_2) &= (n_{\mathrm{AB}}(f), n_{\mathrm{PB}}(f) + 1, n_{\mathrm{SD}}(f) - 1, 0, n_{\mathrm{ER}}(f)), \\ C(g_3) &= (n_{\mathrm{AB}}(f), n_{\mathrm{PB}}(f), n_{\mathrm{SD}}(f) - 1, 0, n_{\mathrm{ER}}(f) + 1). \end{aligned}$$

Proof. First of all, we show the existence of g_1 and g_2. There are at least two Siegel points of f, say z_0 and z_1. Hence we can assume that z_0 is not a Picard exceptional value and has a preimage $z^* \neq z_0$. By using the Lagrange interpolating polynomial, one can construct a polynomial P such that

$$P\left(\frac{1}{z_1 - z^*}\right) = 0, \qquad P'\left(\frac{1}{z_1 - z^*}\right) = -(z_1 - z^*)^2;$$
$$P\left(\frac{1}{a - z^*}\right) = 0, \qquad P'\left(\frac{1}{a - z^*}\right) = 0$$

if $a \neq z_1$ is a non-repelling fixed point or $a = f^n(b)$, where $n \geq 0$ and b is an eventually repelling singular value. Let ρ be an increasing C^∞ function on $[0, \infty)$ satisfying $\rho = 0$ on $[0, 1]$ and $\rho = 1$ on $[2, \infty)$. Let d be the degree of P. We define $H_\varepsilon : \mathbb{C} \to \mathbb{C}$ for $\varepsilon \in \mathbb{C} \setminus \{0\}$ by

$$H_\varepsilon(z) := \begin{cases} z + \varepsilon\rho\left(|\varepsilon|^{-1/(3d)}|z - z^*|\right) P\left(1/(z - z^*)\right) & (z \neq z^*) \\ z^* & (z = z^*). \end{cases}$$

Let $H_0 : \mathbb{C} \to \mathbb{C}$ be the identity. An easy calculation shows that $H_\varepsilon : \mathbb{C} \to \mathbb{C}$ is a quasiconformal mapping for ε small enough. Set a quasiregular mapping

$$F_\varepsilon := f \circ H_\varepsilon$$

for ε small enough. The mappings H_ε and F_ε have the following properties:

(i) $||(H_\varepsilon)_{\bar{z}}/(H_\varepsilon)_z||_\infty \to 0$ $(\varepsilon \to 0)$ and $H_\varepsilon \to H_0$ $(\varepsilon \to 0)$ locally uniformly on $\mathbb{C}$;

(ii) H_ε is conformal on

$$V_\varepsilon := \{z \in \mathbb{C} \mid |z - z^*| > 2|\varepsilon|^{1/(3d)}\},$$

and hence F_ε is holomorphic there and we can define the multipliers of periodic points of F_ε there as in the case of entire functions;

(iii) There is a neighborhood U_ε of z_0 such that $F_\varepsilon(U_\varepsilon) = U_\varepsilon$ and $\mathbb{C} \setminus V_\varepsilon \subset F_\varepsilon^{-1}(U_\varepsilon)$;

(iv) z_1 is a fixed point of F_ε with multiplier $(1 + \varepsilon)f'(z_1)$;

(v) If $z \neq z_1$ is a non-repelling fixed point of f, then z is a fixed point of F_ε with multiplier $f'(z)$;

(vi) If z is an eventually repelling singular value of f, then

$$F_\varepsilon^n(z) = f^n(z) \quad (n = 1, 2, \cdots)$$

and

$$F_\varepsilon'(\tilde{z}) = f'(\tilde{z}) \quad \text{for any } \tilde{z} \in \{f^n(z) \mid n = 0, 1, \cdots\}.$$

By the construction, $F_\varepsilon'(z_0) = f'(z_0)$ satisfies the condition $\lceil$Siegel$\rfloor$ of Proposition 2.21. This yields the property (iii) (see Remark 2.6 or [Si] and [Shi, p.26, STEP 2]). The other properties follow directly from the construction. From (i), (ii), and (iii), we can apply Lemma 3.3 to $g_\varepsilon = F_\varepsilon$ and $E_\varepsilon = U_\varepsilon$ for ε small enough. It follows from this, (iv), (v), and (vi) that there exists a quasiconformal mapping $\phi_\varepsilon : \mathbb{C} \to \mathbb{C}$ with the following properties:

(i)'
$$G_\varepsilon := \phi_\varepsilon \circ F_\varepsilon \circ \phi_\varepsilon^{-1}$$
is an entire function;

(ii)' $\phi_\varepsilon \to \mathrm{Id}_{\mathbb{C}}$ and $G_\varepsilon \to f$ locally uniformly on $\mathbb{C}$ as $\varepsilon \to 0$;

(iii)' $\phi_\varepsilon(z_1)$ is a fixed point of G_ε with multiplier $(1+\varepsilon)f'(z_1)$;

(iv)' If $z \neq z_1$ is a non-repelling fixed point of f, then $\phi_\varepsilon(z)$ is a fixed point of G_ε with multiplier $f'(z)$;

(v)' If z is an eventually repelling singular value of f, then $\phi_\varepsilon(z)$ is an eventually repelling singular value of G_ε.

From (i)', we have $G_\varepsilon = \phi_\varepsilon \circ f \circ (H_\varepsilon \circ \phi_\varepsilon^{-1})$, where ϕ_ε and $H_\varepsilon \circ \phi_\varepsilon^{-1}$ are quasiconformal mappings (see Proposition 2.26 (i) and (ii)). Therefore, G_ε and f are topologically equivalent. By Proposition 2.19, we have $G_\varepsilon \in SF_{k,l}$. In addition, we obtain $G_\varepsilon \in N$ from (ii)'. By the construction, G_ε has a Siegel point $\phi_\varepsilon(z_0)$ with a preimage $\phi_\varepsilon(z^*) \neq \phi_\varepsilon(z_0)$. It follows from (iii)' that some G_{ε_1}(resp. G_{ε_2}) $\in N$ has an attracting fixed point $\phi_{\varepsilon_1}(z_1)$ (resp. a rationally indifferent fixed point $\phi_{\varepsilon_2}(z_1)$ whose multiplier is not 1). From (iv)' and (v)', we see that

$$n_{\mathrm{SD}}(G_\varepsilon) \geq n_{\mathrm{SD}}(f) - 1, \qquad n_{\mathrm{AB}}(G_\varepsilon) \geq n_{\mathrm{AB}}(f),$$

$$n_{\mathrm{rat}}(G_\varepsilon) \geq n_{\mathrm{rat}}(f), \qquad n_{\mathrm{ER}}(G_\varepsilon) \geq n_{\mathrm{ER}}(f).$$

It follows from the construction and Lemma 3.2 that G_{ε_1}(resp. G_{ε_2}) satisfies the properties (a)$\sim$(d) of g_1 (resp. g_2).

Finally, we show the existence of g_3. Suppose that $n_{\mathrm{ER}}(G_\varepsilon) = n_{\mathrm{ER}}(f)$ for any ε small enough. Let $\tilde{J}(F_\varepsilon) \subset \widehat{\mathbb{C}}$ be the closure of the set of all repelling periodic points of F_ε. (Note that all periodic points of F_ε are in V_ε for ε small enough.) By following Shishikura's idea in [Shi, p.27, STEP 3], one can show that there exist a neighborhood N_0 of 0 and a continuous mapping

$$\Gamma : N_0 \times J(f) \to \tilde{J}(F_\varepsilon),$$

where $\varepsilon \in N_0$. Hence we have $z_1 \notin \tilde{J}(F_\varepsilon)$ for $\varepsilon \in \mathbb{C}$ small enough. (Recall that z_1 is a Siegel point of f.) However, from (iv), we can vary the multiplier of z_1 and make z_1 into a repelling fixed point of F_ε. This is a contradiction. Therefore, we have $n_{\mathrm{ER}}(G_{\varepsilon_3}) \geq n_{\mathrm{ER}}(f) + 1$ for some ε_3. It follows from the construction and Lemma 3.2 that G_{ε_3} satisfies the properties of g_3. □

Remark 3.1. In the proof of Lemma 3.4, we constructed an attracting fixed point $\phi_{\varepsilon_1}(z_1)$ of G_{ε_1} near z_1 with multiplier $(1+\varepsilon_1)f'(z_1)$ ($\varepsilon_1 \in \mathbb{C}$). On the other hand, there exists a similar way to make attracting cycles. More precisely, suppose that f has an irrationally (or a rationally) indifferent fixed point z_1' with a preimage other than itself. As in [Shi, Section 4], some modification of our surgery enables us to perturb f so that the fixed point near z_1' has multiplier $(1-\varepsilon)f'(z_1')$ ($\varepsilon > 0$). (If z_1' is rationally indifferent, there is no problem without the condition that z_1' has a preimage other than itself.)

The assumption of Lemma 3.4 requires $n_{\mathrm{SD}}(f) \geq 2$. On the other hand, when $n_{\mathrm{SD}}(f) \geq 1$, we can reduce the number of SD-cycles by one and increase that of AB-cycles (or PB-cycles) by one as follows:

Lemma 3.5. *Let $f \in S_q \cap SF_{k,l}$. We assume the following conditions:*

(1) *Every non-repelling periodic point of f is a fixed point whose multiplier is not* 1*;*

(2) *Every Siegel point of f has the multiplier satisfying the condition* ⌈Siegel⌋ *of Proposition 2.21;*

(3) *f has a Siegel point with a preimage other than itself;*

(4) *f satisfies $n_{\mathrm{PB}}(f) = n_{\mathrm{rat}}(f)$, $n_{\mathrm{Cr}}(f) = 0$, and*

$$n_{\mathrm{AB}}(f) + n_{\mathrm{PB}}(f) + n_{\mathrm{SD}}(f) + n_{\mathrm{ER}}(f) = q.$$

Then for every neighborhood $N \subset M_f$ of f, there exist $g_j \in N$ ($j = 1, 2$) with the following properties:

(a) *Every non-repelling periodic point of g_j is a fixed point whose multiplier is not* 1*;*

(b) *Every Siegel point of g_j has the multiplier satisfying the condition* ⌈Siegel⌋ *of Proposition 2.21;*

(c)
$$g_j \in SF_{k,l}, \quad n_{\mathrm{PB}}(g_j) = n_{\mathrm{rat}}(g_j),$$
and
$$C(g_1) = (n_{\mathrm{AB}}(f)+1, n_{\mathrm{PB}}(f), n_{\mathrm{SD}}(f)-1, 0, n_{\mathrm{ER}}(f)),$$
$$C(g_2) = (n_{\mathrm{AB}}(f), n_{\mathrm{PB}}(f)+1, n_{\mathrm{SD}}(f)-1, 0, n_{\mathrm{ER}}(f)).$$

Proof. Let z_0 be a Siegel point of f with a preimage other than itself. Let $\{\zeta_1, \cdots, \zeta_n\}$ be the set of all non-repelling fixed points of f other than z_0, if any. By the assumption, we have $f'(\zeta_j) \neq 1$ for $j = 1, \cdots, n$. By the implicit function theorem, there exist a neighborhood $W \subset N$ of f and neighborhoods U_{ζ_j} of ζ_j ($j = 1, \cdots, n$) such that every $g \in W$ has a unique fixed point $\alpha_j(g)$ in U_{ζ_j} and $\alpha_j(g)$ is a holomorphic function on W. Thus

$$A_{\zeta_j} := \{g \in W \mid g'(\alpha_j(g)) = f'(\zeta_j)\}$$

is an analytic set in W when it is expressed by a local coordinate on W. Let $\{\eta_1, \cdots, \eta_{n_{\mathrm{ER}}(f)}\}$ be the set of all eventually repelling singular values, when $n_{\mathrm{ER}}(f) \geq 1$. Then there exist some integers $n_t \geq 0$ and $m_t \geq 1$ such that $f^{n_t}(\eta_t)$ is a repelling periodic point of f with period m_t for $t = 1, \cdots, n_{\mathrm{ER}}(f)$. If we take small enough W, there exist neighborhoods U_{η_t} of η_t ($t = 1, \cdots, n_{\mathrm{ER}}(f)$) such that every $g \in W$ has a unique singular value $\beta_t(g)$ in U_{η_t}. Moreover, there exists some $1 \leq t' \leq q$ such that $\beta_t(g) = \Phi_{t'}(g)$, where $\Phi(g) = (\Phi_1(g), \cdots, \Phi_{q+2}(g))$ is a local coordinate on W. (See Section 2.2 for local coordinates on M_f.) Hence $\beta_t(g)$ is holomorphic on W. Thus

$$A_{\eta_t} := \{g \in W \mid g^{m_t+n_t}(\beta_t(g)) = g^{n_t}(\beta_t(g))\}$$

is an analytic set in W. If we take small enough W, every $g \in A_{\eta_t}$ has an eventually repelling singular value $\beta_t(g)$. We define Z by

$$Z := \begin{cases} (\bigcap_{j=1}^{n} A_{\zeta_j}) \cap (\bigcap_{t=1}^{n_{\mathrm{ER}}(f)} A_{\eta_t}) & (n \geq 1, n_{\mathrm{ER}}(f) \geq 1) \\ \bigcap_{j=1}^{n} A_{\zeta_j} & (n \geq 1, n_{\mathrm{ER}}(f) = 0) \\ \bigcap_{t=1}^{n_{\mathrm{ER}}(f)} A_{\eta_t} & (n = 0, n_{\mathrm{ER}}(f) \geq 1) \\ W & (n = 0, n_{\mathrm{ER}}(f) = 0). \end{cases}$$

By definition, Z is an analytic set in W and every $g \in Z$ satisfies

$$n_{\mathrm{SD}}(g) \geq n_{\mathrm{SD}}(f) - 1, \qquad n_{\mathrm{AB}}(g) \geq n_{\mathrm{AB}}(f),$$

$$n_{\mathrm{rat}}(g) \geq n_{\mathrm{rat}}(f), \qquad n_{\mathrm{ER}}(g) \geq n_{\mathrm{ER}}(f).$$

In addition, by Proposition 2.19, every $g \in Z$ satisfies $g \in SF_{k,l}$.

If we take small enough W, the implicit function theorem shows that there exists a holomorphic function $x(g)$ on W such that

$$g(x(g)) = x(g), \qquad x(f) = z_0.$$

(Recall that z_0 is a Siegel point of f with a preimage other than itself.) Consider a holomorphic function

$$\lambda(g) := g'(x(g))$$

on W. As in Remark 3.1, some modification of the proof of Lemma 3.4 enables us to convert f into some $g_0 \in Z$ with $|\lambda(g_0)| < 1$. Thus λ is not constant on Z. Since we can choose small enough W, Proposition 2.20 shows that $\lambda(Z)$ is a neighborhood of $f'(z_0)$. Hence $x(\tilde{g}_1)$ and $x(\tilde{g}_2)$ are an attracting fixed point of $\tilde{g}_1$ and a rationally indifferent fixed point of $\tilde{g}_2$ whose multiplier is not 1 respectively, for some $\tilde{g}_1, \tilde{g}_2 \in Z$. It follows from the construction and Lemma 3.2 that $\tilde{g}_1$ and $\tilde{g}_2$ satisfy the properties of g_1 and those of g_2 respectively. □

Remark 3.2. From the proof of Lemma 3.5, we can convert f into some $g \in Z$ so that the multiplier of the fixed point $x(g)$ near z_0 becomes any value in some open set containing $f'(z_0)$. From Remark 3.1, if f satisfies the assumption of Lemma 3.5 other than (3) and has one rationally indifferent fixed point z_0', similar argument goes well. More precisely, we can perturb f so that the multiplier of the fixed point near z_0' becomes any value in some open set containing $f'(z_0')$.

Lemma 3.5 does not require $n_{\mathrm{SD}}(f) \geq 2$. Therefore, one may think that Lemma 3.4 is not needed. However, by Lemma 3.4, we can convert a Siegel cycle without preimages other than itself or increase the number of eventually repelling singular values. This is an advantage of Lemma 3.4.

When $n_{\mathrm{SD}}(f) \geq 1$, we can convert some Siegel cycles into Cremer cycles at a time as follows:

Lemma 3.6. *Suppose that $f \in S_q \cap SF_{k,l}$ satisfies the assumption of Lemma 3.5. Then for every neighborhood $N \subset M_f$ of f and every m with $1 \leq m \leq n_{\mathrm{SD}}(f)$, there exists a $g_* \in N$ with the following properties:*

(a) *Every non-repelling periodic point of g_* is a fixed point;*

(b)

$$g_* \in SF_{k,l}, \qquad n_{\mathrm{PB}}(g_*) = n_{\mathrm{rat}}(g_*),$$

and

$$C(g_*) = (n_{\mathrm{AB}}(f), n_{\mathrm{PB}}(f), n_{\mathrm{SD}}(f) - m, m, n_{\mathrm{ER}}(f)).$$

Proof. Let $z_1, \cdots, z_m$ be m Siegel points containing a point with a preimage other than itself. By the implicit function theorem, there exist a neighborhood $W' \subset M_f$ of f and holomorphic functions $x_j(g)$ $(j = 1, \cdots, m)$ on W' satisfying

$$g(x_j(g)) = x_j(g), \qquad x_j(f) = z_j.$$

As in the proof of Lemma 3.5, we can construct an analytic set Z' in W' such that every $g \in Z'$ satisfies $g \in SF_{k,l}$ and

$$n_{\mathrm{SD}}(g) \geq n_{\mathrm{SD}}(f) - m, \qquad n_{\mathrm{AB}}(g) \geq n_{\mathrm{AB}}(f),$$

$$n_{\mathrm{rat}}(g) \geq n_{\mathrm{rat}}(f), \qquad n_{\mathrm{ER}}(g) \geq n_{\mathrm{ER}}(f).$$

Now we define $R \subset Z'$ by

$$R := \{g \in Z' \mid x_j(g)\ (j = 1, \cdots, m) \text{ are rationally indifferent fixed points of } g\}.$$

We construct g_* as a limit of some sequence of functions in R.

First of all, we convert f into some $g_1 \in R$ by applying Lemma 3.4 or Lemma 3.5 repeatedly. Set

$$A_{a,b}(x) := \{z \in \mathbb{C} \mid a < |z - x| < b\},$$

where $0 < a < b$ and $x \in \mathbb{C}$. In addition, we define $0 \leq \theta_j(g) < 1$ for $g \in W'$ and every j $(1 \leq j \leq m)$ by

$$g'(x_j(g)) = |g'(x_j(g))| e^{2\pi i \theta_j(g)}.$$

Recall that rationally indifferent periodic points are in the Julia set, which is the closure of the set of all repelling periodic points. Hence g_1 has periodic points in any punctured neighborhood of each of $x_j(g_1)$ $(j = 1, \cdots, m)$. Thus there exist $r_1 > 0$ and $0 < r_2 < r_1/2$ such that g_1 has some p_j-periodic point in each of annuli $A_{r_2,r_1}(x_j(g_1))$ $(j = 1, \cdots, m)$. By applying Rouché's theorem to $g_1^{p_j}(z) - z$ and $g^{p_j}(z) - g_1^{p_j}(z)$, there exists a closed neighborhood $U_1 \subset W'$ of g_1 such that every $g \in U_1$ has some (p_j-)periodic point in each of annuli $A_{r_2,r_1}(x_j(g))$ $(j = 1, \cdots, m)$. In addition, if we take U_1 small enough, it follows from the continuity of $\theta_j(g)$ at g_1 that every $g \in U_1$ satisfies

$$|\theta_j(g_1) - \theta_j(g)| = \left| \frac{p_{1,j}}{q_{1,j}} - \theta_j(g) \right| < \frac{1}{2(q_{1,j})^2} \qquad (j = 1, \cdots, m),$$

where $p_{1,j}/q_{1,j} = \theta_j(g_1)$ and $p_{1,j}, q_{1,j} \in \mathbb{N}$ are mutually prime. From Remark 3.2, we can convert g_1 into some $g \in Z'$ so that the multiplier of each of $x_j(g)$ $(j = 1, \cdots, m)$ becomes any value in some neighborhood of $g_1'(x_j(g_1))$. Thus we can get some $g_2 \in (U_1 \setminus \{g_1\}) \cap R$ such that

$$g_1'(x_j(g_1)) \neq g_2'(x_j(g_2)) \qquad (j = 1, \cdots, m).$$

From the construction similar to that of U_1, there exist a closed neighborhood $U_2 \subset U_1$ of g_2 and $0 < r_3 < r_2/2$ such that:

(1) Every $g \in U_2$ has some periodic point in each of annuli $A_{r_3,r_2}(x_j(g))$ $(j = 1, \cdots, m)$;

(2) Every $g \in U_2$ satisfies

$$|\theta_j(g_2) - \theta_j(g)| = \left| \frac{p_{2,j}}{q_{2,j}} - \theta_j(g) \right| < \frac{1}{2(q_{2,j})^2} \qquad (j = 1, \cdots, m),$$

where $p_{2,j}/q_{2,j} = \theta_j(g_2)$ and $p_{2,j}, q_{2,j} \in \mathbb{N}$ are mutually prime.

By repeating this procedure, we get functions $g_n \in R$, closed neighborhoods $U_n \subset W'$ of g_n, and $r_n > 0$, for $n = 1, 2, \cdots$, such that:

(i) $g'_{n_1}(x_j(g_{n_1})) \neq g'_{n_2}(x_j(g_{n_2}))$ $(j = 1, \cdots, m)$ if $n_1 \neq n_2$;

(ii) $U_n \supset U_{n+1}$;

(iii) $r_{n+1} < r_1/2^n$;

(iv) Every $g \in U_n$ has some periodic point in each of annuli $A_{r_{n+1}, r_n}(x_j(g))$ $(j = 1, \cdots, m)$;

(v) Every $g \in U_n$ satisfies

$$|\theta_j(g_n) - \theta_j(g)| = \left| \frac{p_{n,j}}{q_{n,j}} - \theta_j(g) \right| < \frac{1}{2(q_{n,j})^2} \qquad (j = 1, \cdots, m),$$

where $p_{n,j}/q_{n,j} = \theta_j(g_n)$ and $p_{n,j}, q_{n,j} \in \mathbb{N}$ are mutually prime.

Set

$$K := \{g \in Z' \mid |g'(x_j(g))| = 1 \ (j = 1, \cdots, m)\}.$$

From (ii) and a standard argument, we get some $g_\infty \in (\bigcap_{n=1}^{\infty} U_n) \cap K$. It follows from this, (iii), and (iv) that g_∞ has some periodic points in any punctured neighborhood of each of $x_j(g_\infty)$ $(j = 1, \cdots, m)$. Next, we show that $x_j(g_\infty)$ $(j = 1, \cdots, m)$ are irrationally indifferent fixed points of g_∞. It follows from $g_\infty \in (\bigcap_{n=1}^{\infty} U_n) \cap K$ and (v) that

$$|\theta_j(g_n) - \theta_j(g_\infty)| = \left| \frac{p_{n,j}}{q_{n,j}} - \theta_j(g_\infty) \right| < \frac{1}{2(q_{n,j})^2}$$

for every $n \geq 1$ and every j $(1 \leq j \leq m)$. Then an easy calculation shows that rational numbers $\theta_j(g_n) (n = 1, 2, \cdots)$ are best approximations (of the second kind) of $\theta_j(g_\infty)$ in the sense of Khinchin (see [Kh, Section 6] and [Sho, p.130]). In addition, it follows from (i) that $\theta_j(g_n)$ $(n = 1, 2, \cdots)$ are different from each other. Thus $\theta_j(g_\infty)$ is an irrational number, since any rational number has at most a finite number of such approximations (see [Kh] and [Ol] for basic facts of continued fractions). Therefore, $x_j(g_\infty)$ $(j = 1, \cdots, m)$ are irrationally indifferent fixed points. By Proposition 2.22, they are m Cremer fixed points. It follows from the construction and Lemma 3.2 that g_∞ satisfies the properties of g_*. □

Remark 3.3. Let f_α be as in Lemma 3.1. Any function $f \in M_{f_\alpha}$ with a Siegel fixed point always satisfies the assumption (3) of Lemma 3.5 and Lemma 3.6 that the point has a preimage other than itself. (Hence in Lemma 3.4, if $f \in M_{f_\alpha}$, then the property (c) of g_j is obvious.) This is due to the following reason: The function f_α does not have any exceptional point with only one preimage. Indeed, if f_α has such a point b, then it needs to have the form $(z - \beta)e^{h(z)} + b$, where β is the preimage and $h(z)$ is an entire function. Since f_α is of finite order, $h(z)$ must be a polynomial. This is a contradiction. In addition, any function $f \in M_{f_\alpha}$ also satisfies the property, since covering properties are preserved in M_{f_α}.

Here, we are ready to prove Theorem A.

Proof of Theorem A. By Lemma 3.1, there exists a $T_0 := f_\alpha \in S_q \cap SF_{0,q}$ with q Siegel points of period 1. Hence we have already shown the Theorem A when $(m_{\rm AB}, m_{\rm PB}, m_{\rm SD}, m_{\rm Cr}) = (0, 0, q, 0)$. Thus we show Theorem A when $(m_{\rm AB}, m_{\rm PB}, m_{\rm SD}, m_{\rm Cr}) \neq (0, 0, q, 0)$. We construct T whose non-repelling cycles have the same period 1.

First of all, suppose that $q = 1$ and $(m_{\rm AB}, m_{\rm PB}, m_{\rm SD}, m_{\rm Cr}) \neq (0, 0, 0, 0)$. From Remark 3.3, $T_0 = f_\alpha$ satisfies the assumptions of Lemma 3.5 and Lemma 3.6. We get T by applying Lemma

3.5 or Lemma 3.6 to T_0. More precisely, we convert one Siegel cycle of T_0 into one attracting (or rationally indifferent, Cremer) cycle of T.

Next, suppose that $q \geq 2$ and $(m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}}, m_{\mathrm{Cr}}) \neq (0,0,0,0)$.

(i) When $m_{\mathrm{Cr}} = 0$, we convert T_0 into T by decreasing the number of Siegel cycles and increasing that of attracting cycles (or rationally indifferent cycles, eventually repelling singular values). To be more precise, we apply Lemma 3.4 repeatedly until we get a function with only one SD-cycle. Since Lemma 3.4 ensures that the function satisfies the assumption of Lemma 3.5, we can apply Lemma 3.5 to it for the last step.

(ii) When $m_{\mathrm{Cr}} \neq 0$, we construct T by the following steps:

(STEP 1) As in the case (i), we construct a $\tilde{T} \in SF_{0,q}$ with $n_{\mathrm{PB}}(\tilde{T}) = n_{\mathrm{rat}}(\tilde{T})$ and

$$C(\tilde{T}) = (m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}} + m_{\mathrm{Cr}}, 0, q - \Sigma),$$

where $\Sigma = m_{\mathrm{AB}} + m_{\mathrm{PB}} + m_{\mathrm{SD}} + m_{\mathrm{Cr}}$. Note that $\tilde{T}$ has a Siegel point with a preimage other than itself.

(STEP 2) By the construction, $\tilde{T}$ satisfies the assumption of Lemma 3.6. We get T by applying Lemma 3.6 to $\tilde{T}$ so that m_{Cr} Siegel cycles of $\tilde{T}$ become m_{Cr} Cremer cycles of T.

Finally, suppose that $(m_{\mathrm{AB}}, m_{\mathrm{PB}}, m_{\mathrm{SD}}, m_{\mathrm{Cr}}) = (0,0,0,0)$. If $q \geq 2$, set

$$h_\varepsilon(z) := \varepsilon z e^{z^{q-1}} \in S_q \cap SF_{q-1,q-1} \qquad \text{for } \varepsilon \in \mathbb{C} \setminus \{0\}.$$

An easy calculation shows that h_ε has an asymptotic value 0 and $q-1$ critical values $z_j(\varepsilon)$ ($j = 1, \cdots, q-1$) expressed as

$$z_j(\varepsilon) = \varepsilon \sqrt[q-1]{\frac{1}{(q-1)e}}\, e^{i\theta_j},$$

where

$$\theta_j = \frac{(2j-1)\pi}{q-1}.$$

Consider the equation on ε

$$h_\varepsilon(z_1(\varepsilon)) = z_1(\varepsilon).$$

This yields

$$F(\varepsilon) := h_\varepsilon(z_1(\varepsilon))/z_1(\varepsilon) = \varepsilon e^{z_1(\varepsilon)^{q-1}} = 1.$$

Obviously, $F(\varepsilon)$ is a holomorphic function of ε. It is easy to see that some ε_0 satisfies $F(\varepsilon_0) = 1$. Thus $z_1(\varepsilon_0)$ is a fixed point of h_{ε_0}. In addition, the other critical values $z_j(\varepsilon_0)$ ($j = 2, \cdots, q-1$) are fixed points because

$$h_{\varepsilon_0}(z_j(\varepsilon_0)) = e^{i(\theta_j - \theta_1)} h_{\varepsilon_0}(z_1(\varepsilon_0)) = e^{i(\theta_j - \theta_1)} z_1(\varepsilon_0) = z_j(\varepsilon_0).$$

Thus all singular values of h_{ε_0} are fixed. If a transcendental entire function f has a non-repelling cycle, f has a singular value a such that $\{f^n(a)\}_{n \in \mathbb{N}}$ is an infinite set (see Proposition 2.12 and Proposition 2.14). It follows from this fact that we can take h_{ε_0} as T. Also, if $q = 1$,

$$w(z) := 2\pi i e^z \in S_1 \cap SF_{0,1}$$

can be taken as T. Indeed, w has an asymptotic value 0 with

$$w(0) = 2\pi i, \quad w(2\pi i) = 2\pi i. \qquad \square$$

Remark 3.4. Here we note the proof of Theorem A by constructing Cremer cycles one by one as in [Shi]. Let $\Lambda(F)$ be as in Proposition 2.23. By definition, $\Lambda(F)$ depends only on F. For entire functions of finite order, we may take $E(r) := e^{e^r}$ as $F(r)$. Thus for such functions (containing structurally finite transcendental entire functions), Proposition 2.23 implies that irrationally indifferent fixed points with multipliers in $\Lambda(E)$ are Cremer fixed points. It follows from this and the proofs of Lemma 3.4 and Lemma 3.5 that we can convert one Siegel cycle into one Cremer cycle with multiplier in $\Lambda(E)$. Moreover, our construction can keep the multiplier unchanged. Hence we can also construct Cremer cycles of T one by one as in [Shi]. Even when $p \geq 2$, we can also construct Cremer cycles of T with period p by a similar argument. In this case, we can construct Cremer cycles with multipliers in $\Lambda(E^p)$.

Chapter 4

Some transcendental entire functions with irrationally indifferent fixed points

Our proofs of the results are based on the following Main Lemma:

Main Lemma. *Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. Suppose that there exist bounded simply connected domains U and V with the following properties:*

(a) *$(f|_U; U, V)$ is a polynomial-like mapping of degree $d \geq 2$;*

(b) *f has an irrationally indifferent fixed point β in U with multiplier $\lambda = e^{2\pi i\theta}$ $(\theta \in \mathbb{R} \setminus \mathbb{Q})$.*

Then there exists a polynomial P of degree $d \geq 2$ such that $(f|_U; U, V)$ and P are hybrid equivalent by some quasiconformal mapping ϕ, and the following assertions hold:

(1) *Suppose that $\beta = 0$, $\phi(0) = 0$, and*

$$P(z) = P_{\theta,d}(z) := e^{2\pi i\theta} z\,(1+z)^{d-1}\,.$$

If f has a Siegel point at the origin, then $\theta \in \mathcal{B}$.

(2) *If $\lambda = e^{2\pi i\theta}$ satisfies* [Cremer (d)], *then β is a Cremer fixed point.*

(3) *If $\theta \in \mathcal{D}(2)$, then the bounded type fixed Siegel disk centered at β is bounded by a quasicircle containing at least one critical point.*

(4) *Let*

$$I(f) := \{z \mid f^n(z) \to \infty \text{ as } n \to \infty\}.$$

Suppose that $f \in S$, $\mathrm{sing}(f^{-1}) \cap J(f) \subset \tilde{J}_{f|_U}$, and $I(f)$ and $J(P)$ have Lebesgue measure zero. Then $J(f)$ has Lebesgue measure zero.

(5) *If $J(P)$ has positive Lebesgue measure, then $J(f)$ has positive Lebesgue measure.*

Proof of the Main Lemma. By the straightening theorem, there exists a polynomial P of degree d such that $(f|_U; U, V)$ and P are hybrid equivalent by some ϕ. Moreover, ϕ can be extended as a quasiconformal mapping of $\widehat{\mathbb{C}}$ which fixes ∞.

(1) By Proposition 2.24, $P_{\theta,d}$ has a Siegel point at the origin if and only if $\theta \in \mathcal{B}$. Then the assertion (1) follows from this and the assumption.

(2) Suppose that β is a Siegel fixed point of f. By the assumption, $\phi(\beta)$ is a Siegel fixed point of P. On the other hand, since β is in the interior of $K_{f|_U}$ and ϕ is conformal there, we have $P'(\phi(\beta)) = f'(\beta) = \lambda$. Then by Theorem 1.3, $\phi(\beta)$ is a Cremer fixed point of P. This is a contradiction. Hence β is a Cremer fixed point of f.

(3) By the assumption, β is in the interior of $K_{f|_U}$. Since ϕ is conformal there, we obtain $P'(\phi(\beta)) = f'(\beta) = \lambda$. By Theorem 1.4, P has the bounded type fixed Siegel disk D centered at $\phi(\beta)$ bounded by a quasicircle containing at least one critical point. Thus there exists a quasiconformal mapping $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\varphi(\mathbb{S}^1) = \partial D$. Therefore, $\phi^{-1}(\partial D) = \phi^{-1} \circ \varphi(\mathbb{S}^1)$ is a quasicircle, since $\phi^{-1} \circ \varphi$ is quasiconformal in $\widehat{\mathbb{C}}$ (see Proposition 2.26 (i) and (ii)). Evidently we see that $D \subset K_P$. Hence f has the bounded type fixed Siegel disk $\phi^{-1}(D)$ centered at β bounded by the quasicircle $\phi^{-1}(\partial D)$. In addition, since ∂D contains a critical point of P, $\phi^{-1}(\partial D)$ also contains a critical point of f.

(4) From the assumption, we have only to consider the Lebesgue measure of $J(f) \setminus I(f)$. We can see that $J(f) \setminus I(f) = A \cup B$, where

$$A := \{z \in J(f) \mid \text{ for some sequence } \{m_k\}_{k=1}^{\infty} \text{ and some } b \in J(f) \setminus \tilde{J}_{f|_U}, \\ f^{m_k}(z) \to b \text{ as } k \to \infty \text{ or } f^{m_k}(z) = b\};$$

$$B := \bigcup_{n \geq 0} f^{-n}(\tilde{J}_{f|_U}).$$

(By definition, we have $A \cap B = \emptyset$.) We can show that A has Lebesgue measure zero as follows: Fix any point $a \in A$. From the assumption, for some fixed $\delta > 0$ and all large enough k, there is a neighborhood D_k of a which f^{m_k} maps univalently onto

$$B_\delta(b) := \{z \mid |z - b| < \delta\}.$$

Let $m(X)$ be the Lebesgue measure of a measurable set $X \subset \mathbb{C}$. Since $J(f)$ is nowhere dense, by applying the generalized distortion theorem [Con, p.68, Theorem 7.16] to the inverse branch of f^{m_k} which maps $B_\delta(b)$ onto D_k, we see that there exists a constant $L_1 > 0$ independent of k such that

$$\frac{m(D_k \setminus J(f))}{m(D_k)} \geq L_1 \frac{m(B_\delta(b) \setminus J(f))}{m(B_\delta(b))} =: C > 0,$$

where $0 < C < 1$ is a constant independent of k. Since $f^{m_k}(a) \to b$ as $k \to \infty$ or $f^{m_k}(a) = b$, the distortion theorem again shows that

$$\frac{\sup\{|z - a| \mid z \in \partial D_k\}}{\inf\{|z - a| \mid z \in \partial D_k\}}$$

is less than some constant independent of k. Therefore, there exist $r_k > 0$ and $0 < L_2 < 1$ independent of k such that

$$B_{r_k}(a) \supset D_k, \quad \frac{m(D_k)}{m(B_{r_k}(a))} > L_2.$$

By [MoNTU, p.75, Proposition 2.4.5], we obtain $\mathrm{diam}(D_k) \to 0$, and hence $r_k \to 0$ as $k \to \infty$. Since

$$\frac{m(J(f) \cap B_{r_k}(a))}{m(B_{r_k}(a))} < 1 - L_2 C$$

for all large k, Lebesgue's density theorem shows that A has Lebesgue measure zero. Since $J(P)$ has Lebesgue measure zero, $\tilde{J}_{f|_U}$ has Lebesgue measure zero. In addition, since $f \in S$, for every $z \in \mathbb{C}$ except for finitely many singular values, there exists a neighborhood of z on which all branches of f^{-1} are univalent. Thus $f^{-1}(\tilde{J}_{f|_U})$ has Lebesgue measure zero. Similar argument inductively shows that $f^{-2}(\tilde{J}_{f|_U}), f^{-3}(\tilde{J}_{f|_U}), \cdots$ have Lebesgue measure zero, and hence B also has Lebesgue measure zero. Therefore, we have the desired result.

(5) By Proposition 2.26 (v), quasiconformal mappings map sets of positive Lebesgue measure to sets of positive Lebesgue measure. Since $J(P)$ has positive Lebesgue measure, $\tilde{J}_{f|_U} = \phi^{-1}(J(P))$ has positive Lebesgue measure. It follows from $\tilde{J}_{f|_U} \subset J(f)$ that $J(f)$ also has positive Lebesgue measure.

Thus we have completed the proof of the Main Lemma. □

Remark 4.1. In the Main Lemma, the assumption (b) is not necessary for the assertion (4) and (5).

4.1 Proof of Theorem B

Lemma 4.1. *For any integer $d \geq 2$ and*

$$M(d) := 6\sqrt[d-1]{4e^{3/2}} + 2,$$

set

$$F_{\theta,c}(z) := e^{2\pi i\theta} z(1+cz)^{d-1} e^z,$$

where $\theta \in \mathbb{R}$ and $c \in \{c \mid |c| > M(d)\}$. Let

$$V_c := \{z \mid |z| < R\},$$

where

$$R := \frac{1}{4}\left(\frac{|c|}{2} - 1\right)^{d-1} e^{-1/2},$$

and let U_c be a component of $F_{\theta,c}^{-1}(V_c)$ containing 0. (Note that $F_{\theta,c}(0) = 0$.) Then $(F_{\theta,c}|_{U_c}; U_c, V_c)$ is a polynomial-like mapping of degree d which is hybrid equivalent to

$$P_{\theta,d}(z) := e^{2\pi i\theta} z(1+z)^{d-1}$$

by some quasiconformal mapping ϕ satisfying $\phi(0) = 0$.

Proof. Since $|c| > M(d)$, we have

$$D := \{z \mid |z| < 1/2\} \subset V_c.$$

It follows that for $z \in \partial D$,

$$|F_{\theta,c}(z)| \geq \frac{1}{2}\left(\frac{|c|}{2} - 1\right)^{d-1} e^{-1/2} > R,$$

and hence $\overline{U_c} \subset D \subset V_c$. Let

$$D_c := \{z \mid |z| < 2/|c|\}.$$

We see that for $z \in \partial D_c$,

$$|F_{\theta,c}(z)| < (1+2)^{d-1} e^{2/|c|} < 3^{d-1} e < R,$$

and hence $\overline{D_c} \subset U_c$. From this and the definition of $F_{\theta,c}$, $F_{\theta,c}$ has all d zeros 0 and $-1/c$ (with multiplicity $d-1$) in U_c. It follows from this and the construction that $F_{\theta,c}|_{U_c} : U_c \to V_c$ is a proper and holomorphic mapping of degree d with $\overline{U_c} \subset V_c$. We can show that U_c is simply connected as follows: We have

$$F'_{\theta,c}(z) = e^{2\pi i\theta}(1+cz)^{d-2}(cz^2 + (cd+1)z + 1)e^z.$$

Hence $F_{\theta,c}$ has critical points $-1/c$ (with multiplicity $d-2$) and

$$\beta_{\pm} := \frac{-(cd+1) \pm \sqrt{(cd+1)^2 - 4c}}{2c},$$

where double sign corresponds and the real part of $\sqrt{(cd+1)^2 - 4c}/(cd+1)$ is positive. An easy calculation shows that

$$\begin{aligned}
|\beta_-| &= \frac{|cd+1| \cdot |1 + \sqrt{1 - 4c/(cd+1)^2}|}{2|c|} \\
&= \frac{1}{2} \cdot \left| d + \frac{1}{c} \right| \cdot |1 + \sqrt{1 - 4c/(cd+1)^2}| \\
&> \frac{1}{2} \cdot (d-1) \cdot (1 + 1/\sqrt{2}) \\
&> \frac{1}{2}.
\end{aligned}$$

It follows from this and $\overline{U_c} \subset D$ that $\beta_- \notin U_c$. In addition, we also have

$$\begin{aligned}
|\beta_+||c| &= \frac{2|c|}{|cd+1| \cdot |1 + \sqrt{1 - 4c/(cd+1)^2}|} \\
&= \frac{2}{|d + 1/c| \cdot |1 + \sqrt{1 - 4c/(cd+1)^2}|} \\
&< \frac{2}{(d-1) \cdot (1 + 1/\sqrt{2})} \\
&< 2,
\end{aligned}$$

and hence $\beta_+ \in D_c$. From the construction and $\overline{D_c} \subset U_c$, $F_{\theta,c}$ has $d-1$ critical points $-1/c$ (with multiplicity $d-2$) and β_+ in U_c. By the construction, ∂U_c consists of finitely many mutually disjoint Jordan curves. The Riemann-Hurwitz formula for the map $F_{\theta,c}|_{U_c} : U_c \to V_c$ shows that

$$(2-n) + (d-1) = d \cdot (2-1),$$

where n is the number of components of ∂U_c. Thus $n = 1$ and U_c is simply connected. Therefore, $(F_{\theta,c}|_{U_c}; U_c, V_c)$ is a polynomial-like mapping of degree d.

Henceforth we fix c and d with $|c| > M(d)$ so that $(F_{\theta,c}|_{U_c}; U_c, V_c)$ is a polynomial-like mapping of degree d for any $\theta \in \mathbb{R}$. For these fixed c and d, denote $F_{\theta,c}$, U_c, and V_c by F_θ, U, and V respectively. Consider the family $\{(F_\theta|_U; U, V)\}_{\theta \in \mathbb{R}}$. By the straightening theorem, there exists a polynomial p_θ of degree d such that $(F_\theta|_U; U, V)$ and p_θ are hybrid equivalent by some ϕ_θ. From the proof of the straightening theorem (see [DH] or [BraF, p.221, p.222]), we can normalize ϕ_θ so that

$\phi_\theta(0) = 0$ and $\phi_\theta(-1/c) = -1$. By the construction, p_θ has zeros $\phi_\theta(0) = 0$ and $\phi_\theta(-1/c) = -1$ (with multiplicity $d-1$). In addition, p_θ is of degree d. Hence we have

$$p_\theta(z) = P_{\lambda(\theta)}(z) := \lambda(\theta) z(1+z)^{d-1} \quad \text{for some } \lambda(\theta) \in \mathbb{C} \setminus \{0\}.$$

We must show that $\lambda(\theta) = e^{2\pi i\theta}$ for any $\theta \in \mathbb{R}$. Recall that we fixed c and d. Since V is an open disk independent of θ, the preimage U by F_θ is independent of θ. By the construction, ∂U and ∂V are analytic Jordan curves. In addition, F_θ depends continuously on θ. From those properties of $(F_\theta|_U; U, V)$ and some modification of the proof of the straightening theorem, one can show that the map

$$\lambda : \mathbb{R} \to \mathbb{C} \setminus \{0\}, \quad \theta \mapsto \lambda(\theta)$$

is continuous (see Zakeri's argument [Za1, p.218–p.221, 11] and the proof of the straightening theorem [BraF, p.221, p.222]). If $\theta \in \mathcal{B}$, then F_θ has a Siegel fixed point at the origin in the interior of $K_{F_\theta|_U}$. Since ϕ_θ is conformal there, we have $P'_{\lambda(\theta)}(0) = F'_\theta(0)$, and hence $\lambda(\theta) = e^{2\pi i\theta}$ for any $\theta \in \mathcal{B}$. Since $\mathcal{B}$ is dense in $\mathbb{R}$ and λ is continuous in $\mathbb{R}$, we obtain $\lambda(\theta) = e^{2\pi i\theta}$ for any $\theta \in \mathbb{R}$, as required. Our argument above goes well for any fixed c and d with $|c| > M(d)$. Therefore, we have the desired result. □

Remark 4.2. Let $\beta_\pm$ be as in the proof of Lemma 4.1. Note that $|\beta_+||c| \to 1/d$ and $\beta_- \to -d$ as $c \to \infty$.

We prove Theorem B by the Main Lemma (1) and Lemma 4.1 as follows:

Proof of Theorem B. Let $(F_{\theta,c}|_{U_c}; U_c, V_c)$ and $M(d) > 0$ be as in Lemma 4.1. By Lemma 4.1, we can apply the Main Lemma (1) to $(f|_U; U, V) = (F_{\theta,c}|_{U_c}; U_c, V_c)$. Then for $|c| > M(d)$, $F_{\theta,c}$ has a Siegel fixed point at the origin if and only if $\theta \in \mathcal{B}$. □

4.2 Proofs of Theorem C and Theorem D

In order to prove Theorem C and Theorem D, we construct the following function in $\mathscr{S}$:

Lemma 4.2. *Fix any integers $q \geq 1$ and $m \geq 5$. Let*

$$f_\varepsilon(z) := (1+\varepsilon) z(1+z^{mq}) e^{z^q},$$

where ε is a complex number with $|\varepsilon| < 1/2$, let

$$V := \{z \mid |z| < R\},$$

where

$$R := \frac{1}{2}(4^5 - 1)e^{-4},$$

and let U_ε be a component of $f_\varepsilon^{-1}(V)$ containing 0. *(Note that $f_\varepsilon(0) = 0$.) Then $(f_\varepsilon|_{U_\varepsilon}; U_\varepsilon, V)$ is a polynomial-like mapping of degree $mq+1$. Moreover, there exist uncountable sets $\Lambda_j \subset \{\varepsilon \mid |\varepsilon| < 1/2\}$ $(j = 1, 2)$ such that:*

(1) *If $\varepsilon \in \Lambda_1$, then f_ε has q irrationally indifferent fixed points in $U_\varepsilon \setminus \{0\}$ with multipliers satisfying* [Cremer $(mq+1)$]*;*

(2) *If $\varepsilon \in \Lambda_2$, then f_ε has q bounded type Siegel fixed points in $U_\varepsilon \setminus \{0\}$.*

Proof. Define D_q by

$$D_q := \{z \mid |z| < \sqrt[q]{4}\}.$$

Then since $\sqrt[q]{4} \leq 4 < 3e < R$, we have $D_q \subset V$. For $z \in \partial D_q$, we have

$$|f_\varepsilon(z)| \geq (1 - |\varepsilon|) \cdot \sqrt[q]{4} \cdot (4^5 - 1)\, e^{-4} > \left(1 - \frac{1}{2}\right) \cdot 2R = R,$$

and hence

$$\overline{U_\varepsilon} \subset D_q \subset V.$$

It follows that for $z \in \mathbb{S}^1$,

$$|f_\varepsilon(z)| < \left(1 + \frac{1}{2}\right) \cdot 2e = 3e < R,$$

and hence $\overline{\mathbb{D}} \subset U_\varepsilon$. Moreover, f_ε has all $mq + 1$ zeros 0 and mqth roots of -1 in $\overline{\mathbb{D}} \subset U_\varepsilon$. We deduce from this and the construction that $f_\varepsilon|_{U_\varepsilon} : U_\varepsilon \to V$ is a proper and holomorphic mapping of degree $mq+1$ with $\overline{U_\varepsilon} \subset V$. We prove that U_ε is simply connected as follows: An easy calculation shows that

$$f'_\varepsilon(z) = (1 + \varepsilon)e^{z^q} \cdot (F(z) + G(z)),$$

where

$$F(z) := (mq + 1)z^{mq}, \quad G(z) := 1 + qz^q + qz^{(m+1)q}.$$

For $z \in \mathbb{S}^1$, we also have

$$|G(z)| \leq 1 + 2q < |F(z)| = mq + 1,$$

since $m \geq 5$. By applying Rouché's theorem to $F(z)$ and $F(z) + G(z)$, f_ε has mq critical points counted with multiplicity in $\mathbb{D}$. (Henceforth we count the number of critical points with multiplicity.) For $z \in \partial D_q$, it follows that

$$|G(z)| \leq 1 + (4 + 4^{m+1})q < |F(z)| = 4^m + 4^m \cdot mq,$$

since from $m \geq 5$ we obtain

$$4^m \cdot mq \geq 4^m \cdot 5q = (4^{m+1} + 4^m)q.$$

Rouché's theorem again shows that f_ε has mq critical points in D_q. Thus from $\overline{\mathbb{D}} \subset U_\varepsilon \subset D_q$, f_ε has no critical points in $U_\varepsilon \setminus \mathbb{D}$, and hence it has mq critical points in U_ε. By the construction, there are no critical points in ∂U_ε. In addition, we have $\partial V = \{z \mid |z| = R\} = f_\varepsilon(\partial U_\varepsilon)$. It follows that ∂U_ε consists of finitely many mutually disjoint Jordan curves. By applying the Riemann-Hurwitz formula to the map $f_\varepsilon|_{U_\varepsilon} : U_\varepsilon \to V$, we have

$$(2 - n) + mq = (mq + 1) \cdot (2 - 1),$$

where n is the number of components of ∂U_ε. It follows that $n = 1$, and hence U_ε is simply connected. Therefore, $(f_\varepsilon|_{U_\varepsilon}; U_\varepsilon, V)$ is a polynomial-like mapping of degree $mq + 1$.

Suppose that $f_\varepsilon(z) = z$ and $z \neq 0$. Then

$$H(X, \varepsilon) := (1 + X^m)e^X - \frac{1}{1 + \varepsilon} = 0,$$

where $X = z^q$. We have

$$H(0, 0) = 0, \qquad \frac{\partial H}{\partial X}|_{(0,0)} = 1 \neq 0.$$

By the implicit function theorem, there exists a holomorphic function $X(\varepsilon)$ on some neighborhood $D \subset \{\varepsilon \mid |\varepsilon| < 1/2\}$ of $\varepsilon = 0$ such that

$$H(X(\varepsilon), \varepsilon) = 0, \qquad X(0) = 0.$$

We take D small enough so that for $\varepsilon \in D \setminus \{0\}$, f_ε has q different non-zero fixed points

$$z_1(\varepsilon), \cdots, z_q(\varepsilon) \in \mathbb{D} \subset U_\varepsilon,$$

where $(z_j(\varepsilon))^q = X(\varepsilon)$ $(j = 1, \cdots, q)$. In addition, $z_1(\varepsilon), \cdots, z_q(\varepsilon)$ have the same multiplier

$$\lambda(\varepsilon) := (1+\varepsilon)e^{X(\varepsilon)}(1 + qX(\varepsilon) + q(X(\varepsilon))^{m+1} + (mq+1)(X(\varepsilon))^m).$$

It follows that $\lambda(\varepsilon)$ is an open map with $\lambda(0) = 1$. Thus $\lambda(D)$ is a neighborhood of 1. This fact implies that $\lambda(D)$ contains some arc segment I in $\mathbb{S}^1$ containing 1. It follows that there exists an $\varepsilon \in D$ such that $\lambda(\varepsilon) = t$ for any $t \in I$. From this and Theorem 1.3, there exist uncountable sets Λ_j $(j = 1, 2)$ in $D \setminus \{0\}$ such that:

(1) If $\varepsilon \in \Lambda_1$, then $\lambda(\varepsilon)$ satisfies [Cremer $(mq+1)$] in Theorem 1.3;

(2) If $\varepsilon \in \Lambda_2$, then $\lambda(\varepsilon) = e^{2\pi i\theta}$, where θ is of bounded type. □

Instead of Lemma 4.2, we can also use the following lemma:

Lemma 4.3. *Fix any integer $q \geq 1$ and any constant $c \in \mathbb{C}$ satisfying $|c| > \max\{6e^2 + 1, q + 1\}$. Let*

$$f_\varepsilon(z) := (1+\varepsilon)z(1 + cz^q)e^{z^q},$$

where ε is a complex number with $|\varepsilon| < 1/2$, let

$$V := \{z \mid |z| < R\},$$

where

$$R := \frac{1}{2}(|c| - 1)e^{-1},$$

and let U_ε be a component of $f_\varepsilon^{-1}(V)$ containing 0. (Note that $f_\varepsilon(0) = 0$.) Then $(f_\varepsilon|_{U_\varepsilon}; U_\varepsilon, V)$ is a polynomial-like mapping of degree $q+1$ with q critical points. Moreover, there exist uncountable sets $\Lambda_j \subset \{\varepsilon \mid |\varepsilon| < 1/2\}$ $(j = 1, 2)$ such that:

(1) *If $\varepsilon \in \Lambda_1$, then f_ε has q irrationally indifferent fixed points in $U_\varepsilon \setminus \{0\}$ with multipliers satisfying* [Cremer $(q+1)$]*;*

(2) *If $\varepsilon \in \Lambda_2$, then f_ε has q bounded type Siegel fixed points in $U_\varepsilon \setminus \{0\}$.*

Proof. Since $|c| > 6e^2 + 1$, we have $\mathbb{D} \subset V$. It follows that for $z \in \mathbb{S}^1$,

$$|f_\varepsilon(z)| \geq (1 - |\varepsilon|) \cdot (|c| - 1)e^{-1} > \left(1 - \frac{1}{2}\right) \cdot 2R = R,$$

and hence

$$\overline{U_\varepsilon} \subset \mathbb{D} \subset V.$$

Henceforth we define

$$D_q := \{z \mid |z| < \sqrt[q]{1/|c|}\}.$$

It follows that for $z \in \partial D_q$,

$$|f_\varepsilon(z)| < \left(1 + \frac{1}{2}\right) \cdot \sqrt[q]{1/|c|} \cdot \left(1 + |c|\frac{1}{|c|}\right) e^{1/|c|} < 3e < R,$$

and hence

$$\overline{D_q} \subset U_\varepsilon.$$

By definition, f_ε has all $q+1$ zeros 0 and qth roots of $-1/c$ in $\overline{D_q} \subset U_\varepsilon$. Thus from the construction, $f_\varepsilon|_{U_\varepsilon} : U_\varepsilon \to V$ is a proper and holomorphic mapping of degree $q+1$ with $\overline{U_\varepsilon} \subset V$. Next, we show that U_ε is simply connected as follows: It follows that

$$f'_\varepsilon(z) = (1+\varepsilon)e^{z^q} \cdot (F(z) + G(z)),$$

where

$$F(z) := (c + cq + q)z^q, \quad G(z) := 1 + cqz^{2q}.$$

For $z \in \partial D_q$, we also have

$$|F(z)| = \frac{|c + cq + q|}{|c|} \geq \frac{(1+q)|c| - q}{|c|} = \frac{|c| + (|c|-1)q}{|c|},$$

$$|G(z)| \leq 1 + |c|q\left(\frac{1}{|c|}\right)^2 = \frac{|c| + q}{|c|} < \frac{|c| + (|c|-1)q}{|c|},$$

and hence $|F(z)| > |G(z)|$ for $z \in \partial D_q$. We can apply Rouché′s theorem to $F(z)$ and $F(z)+G(z)$. Then f_ε has q critical points counted with multiplicity in D_q. For $z \in \mathbb{S}^1$, it follows from $|c| > q+1$ that

$$|F(z)| = |c + cq + q| \geq (1+q)|c| - q = |c| - q + |c|q > 1 + |c|q \geq |G(z)|.$$

By Rouché′s theorem again, f_ε has q critical points in $\mathbb{D}$. Since $\overline{D_q} \subset U_\varepsilon \subset \mathbb{D}$, f_ε has no critical points in $U_\varepsilon \setminus D_q$, and hence it has q critical points in U_ε. As in the proof of Lemma 4.2, we see that ∂U_ε consists of finitely many mutually disjoint Jordan curves. We can apply the Riemann-Hurwitz formula to the map $f_\varepsilon|_{U_\varepsilon} : U_\varepsilon \to V$. Then we obtain

$$(2 - n) + q = (q+1) \cdot (2-1),$$

where n is the number of components of ∂U_ε. We have $n = 1$, and hence U_ε is simply connected. This shows that $(f_\varepsilon|_{U_\varepsilon}; U_\varepsilon, V)$ is a polynomial-like mapping of degree $q+1$.

From the argument similar to that of Lemma 4.2, we can find Λ_1 and Λ_2. We omit the details. □

We give here the proofs of Theorem C and Theorem D:

Proof of Theorem C. Let $(f_\varepsilon|_{U_\varepsilon}; U_\varepsilon, V)$ and Λ_1 be as in Lemma 4.2 or Lemma 4.3. By Lemma 4.2 or Lemma 4.3, if $\varepsilon \in \Lambda_1$, then f_ε has q irrationally indifferent fixed points $z_1(\varepsilon), \cdots, z_q(\varepsilon)$ in $U_\varepsilon \setminus \{0\}$, whose multipliers satisfy [Cremer $(mq+1)$] or [Cremer $(q+1)$]. When $\varepsilon \in \Lambda_1$, we can apply the Main Lemma (2) to $(f|_U; U, V) = (f_\varepsilon|_{U_\varepsilon}; U_\varepsilon, V)$ and $\beta = z_j(\varepsilon)$ $(j = 1, \cdots, q)$. Then we can take $g := f_\varepsilon$ $(\varepsilon \in \Lambda_1)$. □

Proof of Theorem D. Let $(f_\varepsilon|_{U_\varepsilon}; U_\varepsilon, V)$ and Λ_2 be as in Lemma 4.2 (or Lemma 4.3). From the argument similar to the proof of Theorem C, by the Main Lemma (3), we can take $g := f_\varepsilon$ $(\varepsilon \in \Lambda_2)$. In addition, since f_ε in Lemma 4.3 has only q critical points in U_ε, we can choose g so that each boundary of such q Siegel disks contains exactly one critical point. □

4.3 Proofs of Theorem E and Theorem F

In order to show Theorem E, we prepare the following lemmas:

Lemma 4.4 ([PeZ, p.2, Theorem A]). *Let*

$$P_\theta(z) := e^{2\pi i\theta}z + z^2,$$

where $\theta \in \mathbb{R}\setminus\mathbb{Q}$. If $\theta \in \mathcal{E}$, then the Siegel disk of P_θ centered at the origin is bounded by a Jordan curve containing exactly one critical point. Furthermore, $J(P_\theta)$ has Lebesgue measure zero.

The following is from unpublished Herman's work in 1989 quoted in [Pe, p.1739, Theorem 1.2]:

Lemma 4.5. *Let f be an entire function with a bounded fixed Siegel disk $\triangle$. If $\partial\triangle$ is a quasicircle containing a critical point of f, then $\triangle$ is bounded type.*

Proof of Theorem E. Let $F_{\theta,c}$ be as in Lemma 4.1. It follows that $g_\alpha = F_{\theta,c}$ for $d=2$, $\theta\in\mathcal{E}$, and $c = \alpha/e^{2\pi i\theta}$. Note that P_θ and $P_{\theta,d}$ are conjugate by a linear transformation. Then by Lemma 4.1, for α with $|\alpha| > M(2)$, there exist simply connected domains U_α and V_α containing 0 such that $(g_\alpha|_{U_\alpha}; U_\alpha, V_\alpha)$ and P_θ are hybrid equivalent by φ satisfying $\varphi(0)=0$. From Lemma 4.4 and the argument similar to that in the proof of the Main Lemma (3), $\partial\triangle_\alpha \subset U_\alpha$ is a Jordan curve which contains exactly one critical point. Therefore, we can take $M := M(2)$. In addition, suppose that $\theta \in \mathcal{E}\setminus\mathcal{D}(2)$. Then by Lemma 4.5, $\partial\triangle_\alpha$ is not a quasicircle. □

Unpublished Herman's work [Herm1] and [Gh] show that:

Lemma 4.6 ([BraF, p.228, Theorem 7.13]). *There exists a $\theta\in\mathbb{R}\setminus\mathbb{Q}$ such that the quadratic polynomial $P_\theta(z) = e^{2\pi i\theta}z + z^2$ has a Siegel disk centered at the origin, whose boundary is a quasicircle without critical points.*

Proof of Theorem F. Theorem F follows from Lemma 4.6 and the argument similar to that in the proof of Theorem E. We omit the details. □

4.4 Proofs of Theorem G and Theorem H

Let $I(f)$ be as in the Main Lemma. The Eremenko-Lyubich class $\mathscr{B}$ is the set of all entire functions such that the sets $\operatorname{sing}(f^{-1})\cap\mathbb{C}$ are bounded. By definition, we have $\mathscr{S}\subset S\subset\mathscr{B}$. For the proof of Theorem G, we introduce the following lemmas:

Lemma 4.7 ([Cu, p.91, Theorem 1.3]). *Let $f\in\mathscr{B}$ be a transcendental entire function of finite order and satisfy*

$$\operatorname{sing}(f^{-1}) \subset \{z \mid |z| < r_0\}$$

for some $r_0 > 0$. Set

$$\tilde{\theta}(r) := \operatorname{meas}\{t\in[0,2\pi] \mid |f(re^{it})| < r_0\},$$

where $r>0$ and meas *denotes the one-dimensional Lebesgue measure. Let*

$$E(x) := e^x.$$

Suppose that $\tilde{\theta}(r)\geq\theta_0(r)$ for large $r>0$, where $\theta_0(r)$ is continuous and non-increasing and satisfies

$$\sum_{k=1}^{\infty}\theta_0(E^k(0)) = \infty.$$

Then $I(f)$ has Lebesgue measure zero.

Lemma 4.8. *Let $f \in \mathscr{S}$. Then $I(f)$ has Lebesgue measure zero.*

Proof. Let

$$f(z) = P(z)\exp(Q(z)),$$

where P and Q are polynomials, and $Q(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_1 z$ for $d \in \mathbb{N}$ and $a_j \in \mathbb{C}\,(j = 1, \cdots, d)$ with $a_d \neq 0$. For any $z \in \mathbb{C} \setminus \{0\}$, we define $0 \leq \arg(z) < 2\pi$ by

$$z = |z|\exp(i \cdot \arg(z)).$$

Set

$$\mathcal{R} := \left\{ z \in \mathbb{C} \setminus \{0\} \mid |\arg(a_d z^d) - \pi| < \frac{\pi}{5} \right\}.$$

Note that $Q(z)/(a_d z^d) \to 1$ as $z \to \infty$. Let r_0 and $\tilde{\theta}(r)$ be as in Lemma 4.7. If $z \in \mathcal{R}$ and $|z|$ is large enough, then $|f(z)| < r_0$, and hence there exists a constant $T > 0$ such that for large $r > 0$,

$$\tilde{\theta}(r) \geq T =: \theta_0(r).$$

By Lemma 4.7, $I(f)$ has Lebesgue measure zero. □

Let β_+ and β_- be critical points of g_α as in the proof of Lemma 4.1, where $g_\alpha = F_{\theta,c}$ for $d = 2$, $\theta \in \mathcal{E}$, and $c = \alpha/e^{2\pi i\theta}$.

Lemma 4.9. *Fix any $\theta \in \mathcal{E}$ and any $0 < \varepsilon < \pi/2$. Suppose that $0 \leq |\arg(\alpha) - \pi| < \pi/2 - \varepsilon$ and $|\alpha|$ is large enough. Then $g_\alpha^2(\beta_-) \in \triangle_\alpha$.*

Proof. Since $\beta_- \to -2$ as $|\alpha| \to \infty$ (see Remark 4.2), it follows from the assumption that there exists a constant $K > 0$ independent of α such that $|g_\alpha^2(\beta_-)| < e^{-K|\alpha|}$ for all large enough $|\alpha|$. We can write

$$g_\alpha(z) = e^{2\pi i\theta}z + \sum_{k=2}^{\infty} a_k z^k,$$

where $a_k = \alpha/(k-2)! + e^{2\pi i\theta}/(k-1)!$. Then we obtain $|a_k| \leq (2|\alpha|)^{k-1}$ for every $k \geq 2$. Recall that θ is a Diophantine number (see Section 2.3 or [PeZ, p.8, p.9]). It follows from [Si] that there exists a constant $L > 0$ independent of α such that $\{z \mid |z| < L/|\alpha|\} \subset \triangle_\alpha$ (see Remark 2.6). Since $|g_\alpha^2(\beta_-)| < e^{-K|\alpha|} < L/|\alpha|$ holds for all large enough $|\alpha|$, we deduce that $g_\alpha^2(\beta_-) \in \triangle_\alpha$. □

We are ready to prove Theorem G.

Proof of Theorem G. By Lemma 4.9, there exists a domain $A \subset \{\alpha \mid |\alpha| > M\}$ such that if $\alpha \in A$, then $g_\alpha^2(\beta_-) \in \triangle_\alpha$. Fix any $\alpha \in A$ and let $g := g_\alpha$. Since $\beta_+ \in \partial\triangle_\alpha$ and $\beta_-, 0 \in F(g)$, we have $\mathrm{sing}(g^{-1}) \cap J(g) = \{g(\beta_+)\} \subset \tilde{J}_{g|_U}$. Since $g \in \mathscr{S}$, Lemma 4.8 shows that $I(g)$ has Lebesgue measure zero. In addition, by Lemma 4.4, $J(P_\theta)$ has Lebesgue measure zero. By the Main Lemma (4), $J(g)$ has Lebesgue measure zero. □

We introduce the following result for quadratic polynomials:

Lemma 4.10 ([BuC, p.674, Theorem 1, Theorem 2])**.** *Let*

$$P_\theta(z) := e^{2\pi i\theta}z + z^2,$$

where $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then there exist irrational numbers $\theta_1 \notin \mathcal{B}$ and $\theta_2 \in \mathcal{B}$ such that $J(P_{\theta_j})\,(j = 1, 2)$ have positive Lebesgue measure and the origin is a Cremer fixed point of P_{θ_1} and a Siegel fixed point of P_{θ_2}.

We show Theorem H by the Main Lemma (5), Lemma 4.1, and Lemma 4.10 as follows:

Proof of Theorem H. Let $F_{\theta,c}$ be as in Lemma 4.1 for $d = 2$ and let $\theta_1 \notin \mathcal{B}$ and $\theta_2 \in \mathcal{B}$ be as in Lemma 4.10. Set

$$f_1 := F_{\theta_1,c}, \qquad f_2 := F_{\theta_2,c}.$$

As in the proof of Theorem E, for any c with $|c| > M(2)$, there exist simply connected domains U_c and V_c containing 0 such that $(f_j|_{U_c}; U_c, V_c)$ $(j = 1, 2)$ and P_{θ_j} are hybrid equivalent by φ satisfying $\varphi(0) = 0$. By Lemma 4.10, $J(P_{\theta_j})$ $(j = 1, 2)$ have positive Lebesgue measure. Hence by applying the Main Lemma (5) to $(f_j|_{U_c}; U_c, V_c)$ and $P = P_{\theta_j}$, we deduce that $J(f_j)$ $(j = 1, 2)$ have positive Lebesgue measure. By the construction, f_1 (resp. f_2) has a Cremer fixed point (resp. a Siegel fixed point) at the origin. $\square$

Chapter 5

The boundaries of bounded type fixed Siegel disks of some transcendental meromorphic functions

5.1 Characterization of the family $\{h_\alpha\}_{\alpha\in\mathbb{C}\setminus\{0,-1\}}$

In this section, we characterize the one parameter family $\{h_\alpha\}_{\alpha\in\mathbb{C}\setminus\{0,-1\}}$ defined in Section 1.3 by the following propositions:

Proposition 5.1. *Let $f \in \tilde{\mathscr{S}}$ have the following properties:*

(a) *f can be written by*

$$f(z) = \frac{az+b}{cz+d}e^{tz},$$

where $ad-bc$, c, and t are non-zero;

(b) *f has a bounded type Siegel fixed point at the origin with multiplier $\lambda = e^{2\pi i\theta}$.*

Then f is conformally conjugate to

$$h_\alpha(z) = e^{2\pi i\theta}\frac{z}{1-\frac{\alpha+1}{\alpha}z}e^{\alpha z}$$

for some $\alpha \in \mathbb{C}\setminus\{0,-1\}$. Moreover, h_α has two critical points 1 and $c_\alpha = -1/(\alpha+1)$, two asymptotic values 0 and ∞, and one pole $t_\alpha = \alpha/(\alpha+1)$.

Proof. Since f has a fixed point at the origin, we have $b=0$, and hence $ad \neq 0$. In addition, it follows from the assumption (b) that $f'(0) = a/d = e^{2\pi i\theta}$. Set

$$s := -c/d \neq 0.$$

Then we can write

$$f(z) = e^{2\pi i\theta}\frac{z}{1-sz}e^{tz}.$$

An easy calculation shows that

$$f'(z) = e^{2\pi i\theta+tz}\frac{-stz^2+tz+1}{(1-sz)^2}.$$

Hence f has two non-zero critical points u and v which are roots of $-stz^2+tz+1=0$. Let

$$L(z) := uz.$$

It follows that $L^{-1} \circ f \circ L$ has two critical points 1 and v/u. Moreover, we obtain

$$\tilde{f}(z) := L^{-1} \circ f \circ L(z) = e^{2\pi i\theta} \frac{z}{1 - \tilde{s}z} e^{\tilde{t}z},$$

where $\tilde{s} = su \neq 0$ and $\tilde{t} = tu \neq 0$. Since $\tilde{f}'(1) = 0$, we have

$$-\tilde{s}\tilde{t} \cdot 1^2 + \tilde{t} \cdot 1 + 1 = 0,$$

and hence $\tilde{s} = (\tilde{t} + 1)/\tilde{t}$. It follows from this, $\tilde{s} \neq 0$ and $\tilde{t} \neq 0$ that $\tilde{t} \in \mathbb{C} \setminus \{0, -1\}$, and hence $\tilde{f}(z) = h_\alpha(z)$, where $\alpha = \tilde{t}$. By the construction, h_α has two critical points 1 and c_α, and one pole t_α. Since the map $z \mapsto e^{\alpha z}$ has two asymptotic values 0 and ∞, and

$$e^{2\pi i\theta} \frac{z}{1 - \frac{\alpha+1}{\alpha} z} \to -e^{2\pi i\theta} \frac{\alpha}{\alpha + 1} \quad (z \to \infty),$$

h_α has two asymptotic values 0 and ∞. □

Proposition 5.2. *Let α and α' be two distinct points in $\mathbb{C} \setminus \{0, -1\}$. Then h_α and $h_{\alpha'}$ are conformally conjugate if and only if $\alpha' = 1/(\alpha + 1) - 1$.*

Proof. Suppose that $\alpha' = 1/(\alpha + 1) - 1$ and

$$l(z) := -(\alpha + 1)z.$$

An easy calculation shows that $l^{-1} \circ h_{\alpha'} \circ l = h_\alpha$.

Suppose that there exists a conformal map $\tilde{l} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\tilde{l}^{-1} \circ h_{\alpha'} \circ \tilde{l} = h_\alpha$. Since both $h_{\alpha'}$ and h_α have an essential singularity at ∞ and only two asymptotic values 0 and ∞, $\tilde{l}$ fixes 0 and ∞. It follows that $\tilde{l}(z) = kz$ for some $k \neq 0$. Moreover, since $\tilde{l}(1) = k$ is a critical point of $h_{\alpha'}$, we have $k = 1$ or $k = -1/(\alpha' + 1)$. Since $h_{\alpha'} \neq h_\alpha$, we have $k \neq 1$, and hence $k = -1/(\alpha' + 1)$ and $\alpha' \neq -2$. Since $h_{\alpha'}$ has another critical point $\tilde{l}(-1/(\alpha + 1)) = 1/\{(\alpha' + 1)(\alpha + 1)\} = 1$, we obtain $\alpha' = 1/(\alpha + 1) - 1$. □

5.2 Proof of Theorem I (i)

We use the following result of [CheE] to prove Theorem I (i):

Lemma 5.1 ([CheE, p.2140, Theorem 1.5.]). *Let $U \subset \widehat{\mathbb{C}}$ be an open set and let a meromorphic function $f : U \to \widehat{\mathbb{C}}$ have the following properties:*

(a) *The set of all singular values of f is contained in $\{a, b, c\}$ for some $a, b, c \in \widehat{\mathbb{C}}$;*

(b) *$a \in U$ and a is a bounded type Siegel fixed point;*

(c) *$c \in \widehat{\mathbb{C}} \setminus U$ or $f(c) = c$.*

Moreover, let γ' be an injective path which goes from a to b while avoiding $\{a, b, c\}$ in between and let γ be a preimage of γ' by f which has an endpoint a. Then one and only one of the following three cases occurs:

(1) *γ ends on a non-critical point in U. In addition, $U = \widehat{\mathbb{C}}$ and f is a Möbius transformation.*

(2) *γ ends on a critical point.* (*We call the critical point the main critical point.*) *In addition, the Siegel disk $\triangle$ centered at a is bounded by a quasicircle which contains the main critical point and does not contain other critical points.*

(3) *γ leaves every compact subset of U. In addition, $\triangle$ does not compactly contained in U.*

Proof of Theorem I (i). By the assumption, h_α has exactly one critical value $h_\alpha(1) = h_\alpha(c_\alpha)$ and two asymptotic value 0 and ∞. Hence we can apply Lemma 5.1 to h_α by putting $U = \mathbb{C}$, $f = h_\alpha$, $a = 0$, $b = h_\alpha(1)$, and $c = \infty$. Since h_α is transcendental, either of the cases (2) and (3) holds. Since $b = h_\alpha(1)$ is not an asymptotic value, the case (3) does not occur. Therefore, the case (2) occurs.

Next, we show the existence of Ω_1. Put $h_\alpha(1) = h_\alpha(c_\alpha)$. Then it follows that

$$F(\alpha) := \frac{1}{(\alpha+1)^2} e^{-\alpha/(\alpha+1)} - e^\alpha = 0.$$

$F(\alpha)$ has an essential singularity at $\alpha = -1$ and does not have an asymptotic value 0 at $\alpha = -1$. By Picard's theorem and Iversen's theorem, the set $\Omega_1 := \{\alpha \mid h_\alpha(1) = h_\alpha(c_\alpha)\}$ is countably infinite (see [I] or [ColL, p.8, Theorem 1.6] for Iversen's theorem). □

Remark 5.1. Two critical points 1 and $c_\alpha = -1/(\alpha+1)$ of h_α coincides only when $\alpha = -2$. By Theorem I (i), $\triangle_{-2}$ is bounded by a quasicircle containing the critical point 1 of h_{-2}.

5.3 Proof of Theorem I (ii)

For $\beta \in \mathbb{C} \setminus \{0\}$, we define

$$f_\beta(z) := \begin{cases} \frac{z}{1-(\beta+1)z/\beta} e^{\beta z} & (\beta \in \mathbb{C} \setminus \{0, -1\}) \\ z e^{-z} & (\beta = -1). \end{cases}$$

Note that if $\beta \to -1$, then $f_\beta \to f_{-1}$ locally uniformly. By the argument in Section 5.1, when $\beta \in \mathbb{C} \setminus \{0, -1\}$, f_β has two critical points 1 and $c_\beta = -1/(\beta+1)$, two asymptotic values 0 and ∞, and one pole $t_\beta = \beta/(\beta+1)$. We have $c_\beta, t_\beta \to \infty$ as $\beta \to -1$. For any $r > 0$, we define

$$B_r := (-1, -1 + r].$$

Henceforth we restrict β to B_r (or $\overline{B_r} = B_r \cup \{-1\}$). We prove Theorem I (ii) by going through the following three steps:

Step 1. By choosing a small enough $r > 0$ and using f_β, we construct an M-quasiregular mapping $F_\beta : \mathbb{C} \to \widehat{\mathbb{C}}$ for every $\beta \in B_r$ with the following properties:
(1) $F_\beta(0) = 0$, $F_\beta(\mathbb{D}) = (\mathbb{D})$, and $F_\beta|_{\mathbb{S}^1}$ is a critical circle map;
(2) F_β and

$$R_\theta(z) := e^{2\pi i\theta} z$$

are quasiconformally conjugate on $\mathbb{D}$;
(3) F_β depends continuously on $\beta \in B_r$;
(4) The constant M is independent of $\beta \in B_r$.

Step 2. We show that there exists an M_1-quasiconformal mapping $\varphi_\beta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which fixes 0, 1, and ∞, and has the following properties:

(1) For some $\alpha \in \mathbb{C} \setminus \{0, -1\}$,

$$G_\beta(z) := \varphi_\beta \circ F_\beta \circ \varphi_\beta^{-1}(z) = e^{2\pi i\theta} \frac{z}{1 - \frac{\alpha+1}{\alpha} z} e^{\alpha z} = h_\alpha,$$

where h_α is as in Section 1.3.

(2) $h_\alpha(= G_\beta)$ has the Siegel disk $\triangle_\alpha$ centered at the origin whose boundary $\partial\triangle_\alpha$ is an M_1-quasicircle containing exactly one critical point 1;

(3) The constant M_1 is independent of $\beta \in B_r$.

Step 3. From Step 2, we define the surgery map

$$\mathcal{S} : B_r \to \mathbb{C} \setminus \{0, -1\}, \quad \beta \mapsto \alpha,$$

where $G_\beta = h_\alpha$. We show that the surgery map $\mathcal{S}$ is continuous and $\mathcal{S}(\beta) \to -1$ as $\beta \to -1$. Since the set $\mathcal{S}(B_r)$ is uncountable and $\partial\triangle_\alpha$ is an M_1-quasicircle containing exactly one critical point 1 for any $\alpha \in \mathcal{S}(B_r)$, we obtain Theorem I (ii) by taking

$$\Omega_2 := \mathcal{S}(B_r).$$

We prepare the following lemmas for the steps above:

Lemma 5.2. *Let $\beta \in \overline{B_r}$, let*

$$D_\beta := \{z \mid |z| < |f_\beta(1)|\},$$

and let U_β be the connected component of $f_\beta^{-1}(D_\beta)$ which contains the origin. (*Note that $f_\beta(0) = 0$.*) *If $r > 0$ is small enough, then $f_\beta|_{U_\beta} : U_\beta \to D_\beta$ is univalent and U_β is simply connected. Moreover, U_β has the following properties:*

(1) *∂U_β is a piecewise smooth Jordan curve containing exactly one critical point* 1*:*

(2) *$U_\beta \subset \overline{\mathbb{D}}$.*

Proof. Suppose that $\beta \in B_r$. f_β has two critical values $f_\beta(1)$ and $f_\beta(c_\beta)$. We have

$$f_\beta(1) = -\beta e^\beta, \quad f_\beta(c_\beta) = -\frac{\beta}{(1+\beta)^2} e^{-\beta/(1+\beta)}.$$

Since $f_\beta(1) \to e^{-1}$ and $f_\beta(c_\beta) \to \infty$ as $\beta \to -1$, we have $f_\beta(c_\beta) \notin D_\beta$ for $r > 0$ small enough. By [CheE, p.2155, Lemma 5.3], $f_\beta|_{U_\beta} : U_\beta \to D_\beta$ is univalent and U_β is simply connected. Obviously, ∂D_β does not contain the asymptotic values 0 and ∞ of f_β. It follows from this that ∂U_β is a Jordan curve (see [CheE, p.2155, Lemma 5.4]). Since ∂U_β is a preimage of ∂D_β by f_β, ∂U_β is piecewise smooth. By the construction, we have $f_\beta([0,1)) \in \mathbb{R}$, $f'_\beta(z) \neq 0$ for any $z \in [0,1)$, $f_\beta(1) > 0$, and $f_\beta(0) = 0$. It follows that $f'_\beta(z) > 0$ for any $z \in [0,1)$, and hence $[0,1) \subset U_\beta$. This implies that ∂U_β contains the critical point 1. An easy calculation shows that $|f_\beta(z)| > f_\beta(1)$ for any $z \in \mathbb{S}^1 \setminus \{1\}$, and hence $U_\beta \subset \overline{\mathbb{D}}$. By the construction, another critical point c_β is not in ∂U_β for $r > 0$ small enough.

Similarly, we can show the case $\beta = -1$. We omit the details. □

Lemma 5.3. *If $r > 0$ is small enough, then there exists a constant $K \geq 1$ such that ∂U_β is a K-quasicircle for all $\beta \in B_r$.*

Proof. The proof is similar to that of [KeZ, p.142, Lemma 2.4]. We have to pay attention to the existence of the pole t_β of f_β for $\beta \in B_r$ and modify the argument.

Suppose that $r > 0$ is small enough so that the statement of Lemma 5.2 holds. We take two distinct points x and y in ∂U_β so that they divide ∂U_β into two Jordan arcs I and I'. For any piecewise smooth arc segment J, let $|J|$ be the Euclidean length of J. We can assume that $|f_\beta(I)| \leq |f_\beta(I')|$ without loss of generality. Let $\mathrm{Diam}(X)$ be as in Lemma 2.2. By Lemma 2.2, we have only to show that there exists a constant $A > 0$ independent of $\beta \in B_r$, x, and y such that

$$Q(\beta, x, y) := \frac{\mathrm{Diam}(I)}{|x - y|} < A. \tag{5.1}$$

Since $f_\beta(I) \subset \partial D_\beta$ and $\partial D_\beta = \{z \mid |z| = f_\beta(1)\}$ is a circle, we have

$$|f_\beta(I)| \le (\pi/2)|f_\beta(x) - f_\beta(y)|. \tag{5.2}$$

Henceforth let L be the straight line segment joining x and y. It follows from (5.2) and $|f_\beta(x) - f_\beta(y)| \le |f_\beta(L)|$ that

$$|f_\beta(I)| \le (\pi/2)|f_\beta(L)|. \tag{5.3}$$

By Lemma 5.2, we have $L \subset \overline{\mathbb{D}}$. In addition, recall that f_β has the pole t_β with $t_\beta \to \infty$ as $\beta \to -1$. Thus if $r > 0$ is small enough, then $t_\beta \notin \overline{\mathbb{D}}$, and hence $t_\beta \notin L$. Therefore, there exists a $q \in L$ such that $|f'_\beta(q)| = \max_{z\in L} |f'_\beta(z)| > 0$. It follows that

$$|f_\beta(L)| \le |f'_\beta(q)||L|. \tag{5.4}$$

By the definition of diameter, there exist $b_1, b_2 \in \overline{I}$ such that $|b_1 - b_2| = \mathrm{Diam}(I)$. Moreover, there also exists $j = 1$ or 2 such that:

$$1 \notin \{z \mid |z - b_j| \le \mathrm{Diam}(I)/5\}.$$

Define

$$\tilde{I} := \{z \mid |z - b_j| \le \mathrm{Diam}(I)/10\} \cap I.$$

By definition, it follows that:

$$|\tilde{I}| \ge \mathrm{Diam}(I)/10; \tag{5.5}$$

$$|z - 1| \ge \mathrm{Diam}(I)/10 \quad \text{for any } z \in \tilde{I}. \tag{5.6}$$

Since $\tilde{I}$ does not contain critical points 1 and c_β of f_β, there exists a $p \in \tilde{I}$ such that $|f'_\beta(p)| = \min_{z\in\tilde{I}} |f'_\beta(z)| > 0$. It follows that

$$|f_\beta(\tilde{I})| \ge |f'_\beta(p)||\tilde{I}|. \tag{5.7}$$

From (5.4), (5.5), (5.7), the definition of $Q(\beta, x, y)$, and $\tilde{I} \subset I$, we see that

$$\begin{aligned}
\frac{|f'_\beta(q)|}{|f'_\beta(p)|} &\ge \frac{|f_\beta(L)|}{|L|} \cdot \frac{|\tilde{I}|}{|f_\beta(\tilde{I})|} \\
&= \frac{|f_\beta(L)|}{|f_\beta(\tilde{I})|} \cdot \frac{|\tilde{I}|}{\mathrm{Diam}(I)} \cdot \frac{\mathrm{Diam}(I)}{|L|} \\
&\ge \frac{1}{10}\frac{|f_\beta(L)|}{|f_\beta(I)|} \cdot Q(\beta, x, y).
\end{aligned} \tag{5.8}$$

It follows from (5.3) that

$$\frac{|f_\beta(I)|}{|f_\beta(L)|} \le \frac{\pi}{2}. \tag{5.9}$$

The inequalities (5.8) and (5.9) yield

$$Q(\beta, x, y) \le 5\pi \frac{|f'_\beta(q)|}{|f'_\beta(p)|}. \tag{5.10}$$

An easy calculation shows that

$$f'_\beta(z) = -\beta^2 \frac{(z-1)(z + 1/(\beta+1))}{(\beta+1)(z - \beta/(\beta+1))^2} e^{\beta z}.$$

Thus we have

$$\frac{|f'_\beta(q)|}{|f'_\beta(p)|} = \frac{|p-\beta/(\beta+1)|^2}{|q-\beta/(\beta+1)|^2}\cdot\frac{|q-1|}{|p-1|}\cdot\frac{|q+1/(\beta+1)|}{|p+1/(\beta+1)|}\cdot|e^{\beta(q-p)}|. \tag{5.11}$$

Since $L \subset \overline{\mathbb{D}}$ and $\tilde{I} \subset I \subset \overline{\mathbb{D}}$, we have $|p| \leq 1$ and $|q| \leq 1$. Thus we obtain for every $\beta \in B_r$,

$$|e^{\beta(q-p)}| < e^{2(1+r)}. \tag{5.12}$$

Moreover, when $r > 0$ is small enough, it follows that for every $\beta \in B_r$,

$$\frac{|p-\beta/(\beta+1)|^2}{|q-\beta/(\beta+1)|^2} < 2; \tag{5.13}$$

$$\frac{|q+1/(\beta+1)|}{|p+1/(\beta+1)|} < 2. \tag{5.14}$$

(This is because the left-hand sides of (5.13) and (5.14) converge to 1 as $\beta \to -1$.) From the triangle inequality, $q \in L$, and the definition of diameter, we see that

$$\begin{aligned}|q-1| &\leq |q-p| + |p-1| \\ &\leq |q-x| + |x-p| + |p-1| \\ &\leq |x-y| + |x-p| + |p-1| \\ &\leq 2\mathrm{Diam}(I) + |p-1|. \end{aligned} \tag{5.15}$$

The inequalities (5.6) and (5.15) show that

$$\begin{aligned}\frac{|q-1|}{|p-1|} &\leq \frac{2\mathrm{Diam}(I)+|p-1|}{|p-1|} \\ &= \frac{2\mathrm{Diam}(I)}{|p-1|}+1 \\ &\leq \frac{2\mathrm{Diam}(I)}{\mathrm{Diam}(I)/10}+1 \\ &= 21. \end{aligned} \tag{5.16}$$

It follows from (5.10)–(5.16) that if $r > 0$ is small enough, then for any $\beta \in B_r$ and any pair of x and y in U_β,

$$Q(\beta, x, y) < 420\pi e^4 =: A,$$

as required. □

Henceforth we suppose that $r > 0$ is small enough so that the statements of Lemma 5.2 and Lemma 5.3 hold.

Lemma 5.4. *Let $\{\beta_n\}_{n\in\mathbb{N}} \subset B_r$ be a sequence with $\beta_n \to \beta_\infty \in \overline{B_r}$ as $n \to \infty$. Then $\partial U_{\beta_n} \to \partial U_{\beta_\infty}$ as $n \to \infty$ with respect to the Hausdorff metric.*

Proof. Suppose that there exists a subsequence $\{\beta'_n\}_{n\in\mathbb{N}} \subset \{\beta_n\}_{n\in\mathbb{N}}$ and a $\delta > 0$ such that the Hausdorff metric between $\partial U_{\beta'_n}$ and $\partial U_{\beta_\infty}$ is greater than δ for any $n \geq 1$. By the Riemann mapping theorem and Carathéodory's theorem, we can take a homeomorphism $\tilde{\omega}_{\beta_n} : \overline{\mathbb{D}} \to \overline{U_{\beta_n}}$ which is conformal in $\mathbb{D}$, and fixes 0 and 1. By Lemma 2.1, we can extend $\tilde{\omega}_{\beta_n}$ into a K^2-quasiconformal mapping ω_{β_n} of $\widehat{\mathbb{C}}$ fixing 0 and 1, where K is as in Lemma 5.3. By Proposition

2.28, there exists a subsequence $\{\beta''_n\}_{n\in\mathbb{N}} \subset \{\beta'_n\}_{n\in\mathbb{N}}$ such that $\omega_{\beta''_n} \to \omega$ locally uniformly on $\mathbb{C}$, where ω is a K^2-quasiconformal mapping of $\widehat{\mathbb{C}}$ fixing 0 and 1. Let

$$\gamma := \omega(\mathbb{S}^1) \subset \mathbb{C}.$$

By the construction, γ is a K^2-quasicircle with $\partial U_{\beta''_n} \to \gamma$ (as $n \to \infty$) with respect to the Hausdorff metric. By Lemma 5.2, we have $U_{\beta''_n} \subset \overline{\mathbb{D}}$ for any $n \geq 1$, and hence $\gamma \subset \overline{\mathbb{D}}$. In addition, from the fact that $f_{\beta''_n} \to f_{\beta_\infty}$ uniformly on $\overline{\mathbb{D}}$ and the definition of D_β, it follows that

$$f_{\beta''_n}(\partial U_{\beta''_n}) \to f_{\beta_\infty}(\gamma), \quad \partial D_{\beta''_n} \to \partial D_{\beta_\infty}$$

with respect to the Hausdorff metric. Since $\partial D_{\beta''_n} = f_{\beta''_n}(\partial U_{\beta''_n})$, we obtain $f_{\beta_\infty}(\gamma) = \partial D_{\beta_\infty}$. By Hurwitz's theorem, f_{β_∞} is univalent on the bounded component of $\mathbb{C} \setminus \gamma$, and hence $\gamma = \partial U_{\beta_\infty}$. It follows that $\partial U_{\beta''_n} \to \partial U_{\beta_\infty}$ with respect to the Hausdorff metric. This contradicts the fact that $\{\beta''_n\}_{n\in\mathbb{N}} \subset \{\beta'_n\}_{n\in\mathbb{N}}$. ☐

Proof of Theorem I (ii). Our proof is divided into the three steps which we mentioned at the beginning of this section. Recall that we restricted β to B_r (or $\overline{B_r}$) and $r > 0$ is small enough for the statements of Lemma 5.2 and Lemma 5.3 to hold.

Step 1: By the Riemann mapping theorem and Carathéodory's theorem, for $\beta \in \overline{B_r}$, we can take a homeomorphism $\rho_\beta : \widehat{\mathbb{C}} \setminus \mathbb{D} \to \widehat{\mathbb{C}} \setminus U_\beta$ which is conformal in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and satisfies $\rho_\beta(\infty) = \infty$ and $\rho_\beta(1) = 1$. By Lemma 2.1, we can extend ρ_β into a K^2-quasiconformal mapping $\hat{\rho}_\beta$ of $\widehat{\mathbb{C}}$ fixing 1 and ∞, where K is as in Lemma 5.3. By Proposition 2.28, for any sequence $\{\beta_n\}_{n\in\mathbb{N}} \subset B_r$ with $\beta_n \to \beta_\infty \in \overline{B_r}$ as $n \to \infty$, there exists a subsequence $\{\beta'_n\}_{n\in\mathbb{N}} \subset \{\beta_n\}_{n\in\mathbb{N}}$ such that $\hat{\rho}_{\beta'_n} \to \sigma$ locally uniformly on $\mathbb{C}$, where σ is a K^2-quasiconformal mapping of $\widehat{\mathbb{C}}$ fixing 1 and ∞. It follows from Lemma 5.4 that $\sigma|_{\widehat{\mathbb{C}}\setminus\mathbb{D}} = \rho_{\beta_\infty}$, and hence $\hat{\rho}_{\beta'_n}|_{\widehat{\mathbb{C}}\setminus\mathbb{D}} = \rho_{\beta'_n} \to \rho_{\beta_\infty}$ locally uniformly on $\mathbb{C} \setminus \mathbb{D}$. This implies that the set of the all limit functions of $\{\rho_{\beta_n}\}_{n\in\mathbb{N}}$ contains only ρ_{β_∞}, and hence $\rho_{\beta_n} \to \rho_{\beta_\infty}$ locally uniformly on $\mathbb{C} \setminus \mathbb{D}$. Therefore, ρ_β depends continuously on $\beta \in \overline{B_r}$. The map $f_\beta \circ \rho_\beta|_{\mathbb{S}^1} : \mathbb{S}^1 \to \partial D_\beta$ is a homeomorphism, where D_β is as in Lemma 5.2. From the standard theory about the rotation number, there exists a unique $\theta_\beta \in [0, 1)$ such that for

$$L_\beta(z) := \frac{e^{2\pi i\theta_\beta} z}{f_\beta(1)},$$

the rotation number of $L_\beta \circ f_\beta \circ \rho_\beta|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ is the θ which was fixed at the beginning (see [BraF, p.103, Theorem 3.20]). By the construction, L_β depends continuously on $\beta \in \overline{B_r}$. For $\beta \in \overline{B_r}$, we define

$$\tilde{F}_\beta(z) := L_\beta \circ f_\beta \circ \rho_\beta(z) \quad (z \in \mathbb{C} \setminus \mathbb{D}).$$

The Schwarz reflection principle shows that if $r > 0$ is small enough, then there exists an $l > 1$ such that for any β, $\tilde{F}_\beta$ is extended to a holomorphic map $\hat{F}_\beta$ in $\{z \mid |z| > 1/l\}$. Henceforward, we fix a small enough $r > 0$ so that such extension goes well and the statements of Lemma 5.2 and Lemma 5.3 hold. Set

$$A_l := \{z \mid 1/l < |z| < l\}.$$

By the construction, $\hat{F}_\beta|_{A_l}$ depends continuously on $\beta \in \overline{B_r}$, and hence the family $\{\hat{F}_\beta|_{A_l}\}_{\beta\in B_r}$ satisfies the assumption of Lemma 2.4. By Lemma 2.4, there exists a k-quasisymmetric mapping $s_\beta : \mathbb{S}^1 \to \mathbb{S}^1$ for $\beta \in B_r$ such that

$$s_\beta \circ \hat{F}_\beta|_{\mathbb{S}^1} \circ s_\beta^{-1} = R_\theta, \quad s_\beta(1) = 1,$$

where $k > 1$ is independent of β and $R_\theta(z) = e^{2\pi i\theta}z$. By the Beurling-Ahlfors extension, we can extend s_β as a homeomorphism $\hat{s}_\beta : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ which is an M-quasiconformal mapping in $\mathbb{D}$ with $s_\beta(0) = 0$, where M depends only on k, and hence M is independent of β (see Section 2.4). Since $\hat{F}_\beta|_{\mathbb{S}^1} = \tilde{F}_\beta|_{\mathbb{S}^1}$ depends continuously on β, one can show that s_β depends continuously on $\beta \in B_r$. Then it follows from the way of its extension that $\hat{s}_\beta$ also depends continuously on β. For $\beta \in B_r$, we define F_β as follows:

$$F_\beta(z) := \begin{cases} \tilde{F}_\beta(z) & (z \in \mathbb{C} \setminus \mathbb{D}) \\ \hat{s}_\beta^{-1} \circ R_\theta \circ \hat{s}_\beta(z) & (z \in \mathbb{D}). \end{cases}$$

Since $\tilde{F}_\beta|_{\mathbb{S}^1} = \hat{F}_\beta|_{\mathbb{S}^1} = s_\beta^{-1} \circ R_\theta \circ s_\beta$, F_β is continuous. By definition, F_β is locally M-quasiconformal in $\mathbb{C} \setminus (\mathbb{S}^1 \cup \{\rho_\beta^{-1}(c_\beta)\})$. Let V be a neighborhood of any $z \in \mathbb{S}^1 \setminus \{\rho_\beta^{-1}(1)(= 1)\}$, where $\hat{F}_\beta$ and F_β are homeomorphisms. By applying Rickman's lemma (see Section 2.4) to $U = V$, $C = V \setminus \mathbb{D}$, $\phi = \hat{F}_\beta$, and $\Phi = F_\beta$, F_β is M-quasiconformal on V. Thus F_β is an M-quasiregular mapping on $\mathbb{C}$. By the construction, F_β satisfies the following properties:

(1) $F_\beta(0) = 0$, $F_\beta(\mathbb{D}) = (\mathbb{D})$, and $F_\beta|_{\mathbb{S}^1}$ is a critical circle map;

(2) F_β and R_θ are quasiconformally conjugate on $\mathbb{D}$;

(3) F_β depends continuously on $\beta \in B_r$;

(4) The constant M is independent of $\beta \in B_r$.

Thus, we achieve the goal of Step 1.

Step 2: From the construction, we can apply Lemma 2.8 to $F = F_\beta$, $U = \mathbb{D}$, $\phi = \hat{s}_\beta$, and $f = R_\theta$. Let μ_β be the F_β-invariant Beltrami coefficient which is constructed exactly in the same way as the construction of μ_F in the proof of Lemma 2.8. To be more precise, the definition of μ_β is as follows: Let $\mu_{\hat{s}_\beta}$ be a Beltrami coefficient in $\mathbb{D}$ which corresponds to $(\hat{s}_\beta)_{\overline{z}}/(\hat{s}_\beta)_z$ almost everywhere. We define $\mu_\beta(u)$ by $(F_\beta^n)^*\mu_{\hat{s}_\beta}(u)$ for almost all $u \in \bigcup_{n=0}^\infty F_\beta^{-n}(\mathbb{D})$. Otherwise, set $\mu_\beta := \mu_0 = 0$. Since M is independent of β, Lemma 2.8 shows that for a constant $M_1 \geq 1$ independent of β, there exists an M_1-quasiconformal mapping $\varphi_\beta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ fixing 0, 1, and ∞ such that

$$G_\beta := \varphi_\beta \circ F_\beta \circ \varphi_\beta^{-1} : \mathbb{C} \to \widehat{\mathbb{C}}$$

is meromorphic. By the construction, G_β has the only one zero 0 and the only one pole. Thus there exist an entire function $h(z)$ and non-zero constants b and p such that

$$G_\beta(z) = b\frac{z}{z-p}e^{h(z)}.$$

We can show that $h(z)$ is a polynomial of degree 1 as follows: When $|z|$ is large enough, we have

$$\phi_1 \circ G_\beta(z) = f_\beta \circ \phi_2(z), \tag{$*$}$$

where

$$\phi_1 := L_\beta^{-1} \circ \varphi_\beta^{-1}, \qquad \phi_2 := \rho_\beta \circ \varphi_\beta^{-1}.$$

By Proposition 2.26 (i) and (ii), ϕ_1 and ϕ_2 are quasiconformal mappings. It follows from Proposition 2.26 (iii) that there exist positive constants $K' > 1$, C_1, and C_2 such that

$$|\phi_1(z)| \geq C_1|z|^{1/K'}, \quad |\phi_2(z)| \leq C_2|z|^{K'} \quad \text{for } |z| \text{ large enough.}$$

From this and $|f_\beta(z)| \leq e^{|z|^2}$ $(|z| \to \infty)$, there exist positive constants A and N such that

$$\max_{|z|=R} e^{h(z)} \leq e^{AR^N} \quad \text{for } R > 0 \text{ large enough.}$$

Thus $h(z)$ is a polynomial. In addition, the relation $(*)$ implies that both of f_β and G_β have only one positive (or negative) sector in a punctured neighborhood of ∞ in the sense of [Za2, p.495]. Therefore, we deduce that $h(z)$ is a polynomial of degree 1.

By the construction, we have $G'_\beta(0) = e^{2\pi i\theta}$ and $G'_\beta(1) = 0$. Hence as in the proof of Proposition 5.1, we obtain for some $\alpha \in \mathbb{C} \setminus \{0, -1\}$,

$$G_\beta(z) = h_\alpha(z) = e^{2\pi i\theta} \frac{z}{1 - \frac{\alpha+1}{\alpha} z} e^{\alpha z}.$$

It follows from the construction that $h_\alpha (= G_\beta)$ has the Siegel disk $\triangle_\alpha = \varphi_\beta(\mathbb{D})$ centered at the origin (see Remark 2.9). Since φ_β is M_1-quasiconformal, the boundary $\partial\triangle_\alpha = \varphi_\beta(\mathbb{S}^1)$ is an M_1-quasicircle containing exactly one critical point 1 of h_α. Therefore, the argument above completes Step 2.

Step 3: From Step 2, we can define the surgery map

$$\mathcal{S} : B_r \to \mathbb{C} \setminus \{0, -1\}, \quad \beta \mapsto \alpha,$$

where $G_\beta = h_\alpha$. In order to show that $\mathcal{S}$ is continuous, we claim the following assertion, whose proof is similar to the argument in [KeZ, p.157, Section 5] or [Za1, p.218, Section 11]:

Assertion. *Let $\{\beta_n\}_{n\in\mathbb{N}} \subset B_r$ be any sequence with $\beta_n \to \beta_\infty \in B_r$ as $n \to \infty$. Then there exists a subsequence $\{\beta'_n\}_{n\in\mathbb{N}} \subset \{\beta_n\}_{n\in\mathbb{N}}$ such that $\mathcal{S}(\beta'_n) \to \mathcal{S}(\beta_\infty)$ as $n \to \infty$.*

Proof of the assertion. By Step 2 and Proposition 2.28, there exists a subsequence $\{\beta'_n\}_{n\in\mathbb{N}} \subset \{\beta_n\}_{n\in\mathbb{N}}$ and an M_1-quasiconformal mapping $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\varphi_{\beta'_n} \to \varphi$ locally uniformly on $\mathbb{C}$ (as $n \to \infty$). We define

$$\varsigma := \varphi \circ F_{\beta_\infty} \circ \varphi^{-1}, \quad \varsigma_n := \varphi_{\beta'_n} \circ F_{\beta'_n} \circ \varphi^{-1}_{\beta'_n}, \quad \varsigma_\infty := \varphi_{\beta_\infty} \circ F_{\beta_\infty} \circ \varphi^{-1}_{\beta_\infty}$$

If $\varsigma = \varsigma_\infty$, then $\mathcal{S}(\beta'_n) \to \mathcal{S}(\beta_\infty)$. The proof is completed in the case. Henceforth we suppose that $\varsigma \neq \varsigma_\infty$.

We can show that if $\varsigma \neq \varsigma_\infty$, then $\mu_{\beta'_n} \to \mu_{\beta_\infty}$ with respect to the spherical measure as follows: For a measurable set $E \subset \widehat{\mathbb{C}}$, let Area$(E)$ be the Lebesgue area of E in the spherical metric. In addition, we define

$$Q_n^\varepsilon := \{z \in \mathbb{C} \mid |\mu_{\beta'_n}(z) - \mu_{\beta_\infty}(z)| > \varepsilon\},$$

for $\varepsilon > 0$ and $n \geq 1$. It suffices to show that for any $\varepsilon > 0$ and any $C > 0$, if n is large enough, then Area$(Q_n^\varepsilon) < C$. By the definitions of $\mu_{\beta'_n}$ and μ_{β_∞}, we obtain

$$Q_n^\varepsilon \subset \bigcup_{k\geq 0} F^{-k}_{\beta'_n}(\mathbb{D}) \cup \bigcup_{k\geq 0} F^{-k}_{\beta_\infty}(\mathbb{D}). \tag{5.17}$$

Obviously, ς and ς_∞ are quasiconformally conjugate. It follows from $\varsigma \neq \varsigma_\infty$ and the argument similar to that in [Za1, p.201] or [KeZ, p.157, p.158] that for n large enough, there exist quasiconformal mappings $\xi_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that:

(i) ξ_n fixes 0, 1, and ∞;

(ii) ξ_n satisfies

$$\xi_n \circ \varsigma = \varsigma_n \circ \xi_n;$$

(iii) The complex dilatations of ξ_n are uniformly bounded.

Hence we have

$$\tau_n \circ F_{\beta_\infty} = F_{\beta'_n} \circ \tau_n,$$

where $\tau_n := \varphi_{\beta'_n}^{-1} \circ \xi_n \circ \varphi$. It follows from the construction that for every $n \geq 1$,

$$\tau_n(\mathbb{D}) = \mathbb{D}, \quad \tau_n(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}) = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}, \quad \tau_n(0) = 0, \quad \tau_n(\infty) = \infty,$$

and the complex dilatations of quasiconformal mappings τ_n are uniformly bounded. Thus from this, the fact that the area of the Riemann sphere is finite, and Proposition 2.26 (iv), we deduce that for any $\delta > 0$, there exists an integer $N \geq 1$ such that:

$$\text{Area}\left(\bigcup_{k\geq 0} F_{\beta_\infty}^{-k}(\mathbb{D}) \setminus \bigcup_{0\leq k\leq N} F_{\beta_\infty}^{-k}(\mathbb{D})\right) < \delta, \tag{5.18}$$

and for n large enough,

$$\text{Area}\left(\bigcup_{k\geq 0} F_{\beta'_n}^{-k}(\mathbb{D}) \setminus \bigcup_{0\leq k\leq N} F_{\beta'_n}^{-k}(\mathbb{D})\right) < \delta. \tag{5.19}$$

Since $F_{\beta'_n} \to F_{\beta_\infty}$ locally uniformly, we have for n large enough,

$$\text{Area}\left(\bigcup_{0\leq k\leq N} F_{\beta'_n}^{-k}(\mathbb{D}) \setminus \bigcup_{0\leq k\leq N} F_{\beta_\infty}^{-k}(\mathbb{D})\right) < \delta. \tag{5.20}$$

From the construction, $\hat{s}_{\beta'_n} \circ F_{\beta'_n}^N \to \hat{s}_{\beta_\infty} \circ F_{\beta_\infty}^N$ locally uniformly on $\bigcup_{0\leq k\leq N} F_{\beta_\infty}^{-k}(\mathbb{D})$ as $n \to \infty$. In addition, for almost all $z \in \bigcup_{0\leq k\leq N} F_{\beta_\infty}^{-k}(\mathbb{D})$ and large enough n, the complex dilatation of $\hat{s}_{\beta'_n} \circ F_{\beta'_n}^N$ at z and that of $\hat{s}_{\beta_\infty} \circ F_{\beta_\infty}^N$ at z are $\mu_{\beta'_n}(z)$ and $\mu_{\beta_\infty}(z)$ respectively. It follows from this and the construction that for n large enough,

$$\text{Area}\left(Q_n^\varepsilon \cap \bigcup_{0\leq k\leq N} F_{\beta_\infty}^{-k}(\mathbb{D})\right) < \delta. \tag{5.21}$$

From (5.17)–(5.21), we obtain

$$\text{Area}(Q_n^\varepsilon) < 4\delta.$$

Since $\delta > 0$ is arbitrary, we can take $4\delta = C$. This implies that $\mu_{\beta'_n} \to \mu_{\beta_\infty}$ with respect to the spherical measure.

From the argument above and Lemma 2.3, we have $\varphi_{\beta'_n} \to \varphi_{\beta_\infty}$ locally uniformly. It follows that $\varsigma = \varsigma_\infty$. On the other hand, we assumed that $\varsigma \neq \varsigma_\infty$. This is a contradiction, and hence we obtain $\varsigma = \varsigma_\infty$ and $\mathcal{S}(\beta'_n) \to \mathcal{S}(\beta_\infty)$ as $n \to \infty$. This completes the proof of the assertion. ■

The assertion implies that if $\beta_n \to \beta_\infty \in B_r$, then the set $\{\mathcal{S}(\beta_n)\}_{n\in\mathbb{N}}$ is bounded and has only one accumulation point $\mathcal{S}(\beta_\infty)$. It follows that $\mathcal{S}(\beta_n) \to \mathcal{S}(\beta_\infty)$ as $n \to \infty$, and hence $\mathcal{S}$ is continuous.

Finally, we show that $\mathcal{S}(\beta) \to -1$ as $\beta \to -1$. Recall that $\varphi_\beta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is an M_1-quasiconformal mapping fixing 0, 1, and ∞, where M_1 is independent of β, and ρ_β can be extended to a K^2-quasiconformal mapping of $\widehat{\mathbb{C}}$ fixing 1 and ∞, where K is as in Lemma 5.3. Thus $\{\varphi_\beta \circ \rho_\beta^{-1}\}_{\beta\in B_r}$ is a family of M_1K^2-quasiconformal mappings fixing 1 and ∞. In addition, since $h_{\mathcal{S}(\beta)}$ has a critical point $c_{\mathcal{S}(\beta)} = -1/(\mathcal{S}(\beta)+1) = \varphi_\beta \circ \rho_\beta^{-1}(-1/(\beta+1))$, it follows from Proposition 2.28, Proposition 2.29, and $-1/(\beta+1) \to \infty$ as $\beta \to -1$ that

$$\frac{-1}{\mathcal{S}(\beta)+1} = \varphi_\beta \circ \rho_\beta^{-1}\left(\frac{-1}{\beta+1}\right) \to \infty$$

as $\beta \to -1$. This shows that $\mathcal{S}(\beta) \to -1$ as $\beta \to -1$, and hence $\mathcal{S}(B_r)$ is uncountable. Moreover, by the construction, $\triangle_\alpha$ is an M_1-quasicircle containing exactly one critical point when $\alpha \in \mathcal{S}(B_r)$ (see Step 2). Thus we can take

$$\Omega_2 := \mathcal{S}(B_r).$$

Therefore, we have the desired result of Theorem I (ii). □

5.4 Proof of Theorem I (iii)

In this section, we show Theorem I (iii) by the quasiconformal surgery in Section 5.3. Let f_β be as in Section 5.3.

Lemma 5.5. *Let*

$$f := f_{-1+i}, \qquad D := \{z \mid |z| < |f(1)|\},$$

and let U be the component of $f^{-1}(D)$ which contains the origin. (Note that $f(0) = 0$.) Then $f|_U : U \to D$ is univalent. Moreover, ∂U is a piecewise smooth Jordan curve which contains exactly two critical points of f.

Proof. From the argument in the proof of Lemma 5.2, it follows that $f|_U : U \to D$ is univalent and ∂U is a piecewise smooth Jordan curve. What is left is to show that ∂U contains two critical points 1 and $c_{-1+i} = i$ of f. Let $0 < x < 1$. An easy calculation shows that

$$f(x) = xe^{-x+ix}\frac{-1+i}{-1+(1-x)i}, \quad f(ix) = xe^{-x-ix}\frac{-1-i}{(-1+x)+i},$$

and hence

$$|f(x)| = |f(ix)| = xe^{-x}\sqrt{\frac{2}{1+(1-x)^2}}.$$

Let

$$M(x) := |f(x)|^2 (= |f(ix)|^2) = \frac{2x^2e^{-2x}}{1+(1-x)^2}.$$

Then it follows that

$$M'(x) = \frac{4x(1-x)(1+x+(1-x)^2)}{(1+(1-x)^2)^2}e^{-2x}.$$

Thus we have $M'(x) > 0$ for $0 < x < 1$. Hence we obtain $|f(x)| = |f(ix)| < |f(1)| = |f(i)|$ for $0 < x < 1$. It follows from this that U contains $(0,1)$ and $\{ix \mid 0 < x < 1\}$. This implies that ∂U contains 1 and i. □

Proof of Theorem I (iii). By Lemma 5.5 and the quasiconformal surgery technique in Step 1 and Step 2 in Section 5.3, we can show Theorem I (iii). We omit the details. □

Chapter 6

Concluding remarks

In this chapter, we introduce the remaining questions.

6.1 Remarks on Theorem A

The set $S \setminus SF$ is not empty. For example, $f(z) = \sin z \in S \setminus SF$. We constructed a $T \in S_q \cap SF$ to prove best possibility of the Fatou-Shishikura inequality for $f \in S_q$ (see Theorem A). Thus there is the following question:

Question 2. *Suppose that non-negative integers $m_{\rm AB}, m_{\rm PB}, m_{\rm SD}$, and $m_{\rm Cr}$, and a positive integer q satisfy*

$$m_{\rm AB} + m_{\rm PB} + m_{\rm SD} + m_{\rm Cr} \leq q.$$

Is there a $T \in S_q \setminus SF$ such that

$$(n_{\rm AB}(T), n_{\rm PB}(T), n_{\rm SD}(T), n_{\rm Cr}(T)) = (m_{\rm AB}, m_{\rm PB}, m_{\rm SD}, m_{\rm Cr})?$$

6.2 Remarks on Theorems B–H

Can we extend Theorem B?

Question 3. *Let*

$$F_{\theta,c}(z) := e^{2\pi i\theta} z(1+cz)^{d-1} e^z \in \mathscr{S},$$

where $\theta \in \mathbb{R} \setminus \mathbb{Q}$, an integer $d \geq 2$, and $c \in \mathbb{C} \setminus \{0\}$. Does $F_{\theta,c}$ have a Siegel point at the origin if and only if $\theta \in \mathcal{B}$?

On the analogy of Theorem 1.4, Theorem 1.5, and Theorem D, it is natural to expect that all bounded type Siegel disks for functions in $\mathscr{S}$ are bounded by quasicircles containing critical points. However, as a counterexample, Zakeri gave the map

$$G : z \mapsto \lambda e^{z-\lambda}$$

in $\mathscr{S}$, where $\lambda = e^{2\pi i\theta}$ for $\theta \in \mathcal{D}(2)$ (see [Za2]). More precisely, the boundary of the Siegel disk of G centered at λ is unbounded and fails to be a Jordan curve. This is because G has no critical points (see [GrS]). On the other hand, many functions in $\mathscr{S}$ have critical points. Hence G may be an extreme case in $\mathscr{S}$. Therefore, we ask here the following question:

Question 4. *Is there a $g \in \mathscr{S}$ with a critical point and a bounded type Siegel disk centered at a point other than the origin, whose boundary fails to be a quasicircle containing critical points?*

Furthermore, on the analogy of Theorem E, Theorem G, and Theorem H, we also ask the following questions:

Question 5. *Let $\theta \in \mathcal{E} \setminus \mathcal{D}(2)$, let*

$$g_\alpha(z) := (e^{2\pi i\theta} z + \alpha z^2)e^z,$$

where $\alpha \in \mathbb{C} \setminus \{0\}$, and let $\triangle_\alpha$ be the Siegel disk of g_α centered at the origin. Is the boundary $\partial\triangle_\alpha$ a Jordan curve which is not a quasicircle and contains a critical point for all $\alpha \in \mathbb{C} \setminus \{0\}$? Is there an α such that $J(g_\alpha)$ has positive Lebesgue measure?

Question 6. *Let $g_\theta(z) := e^{2\pi i\theta} z e^z$ for any $\theta \in \mathcal{E} \setminus \mathcal{D}(2)$. Is the Siegel disk of g_θ centered at the origin bounded by a Jordan curve which is not a quasicircle and contains the critical point -1?*

Question 7. *Is there an $f \in \mathscr{S}_{2,1}$ such that $J(f)$ has Lebesgue measure zero and f has a Cremer fixed point at the origin?*

6.3 Remarks on Theorem I

Let

$$h_\alpha(z) := e^{2\pi i\theta} \frac{z}{1 - \frac{\alpha+1}{\alpha} z} e^{\alpha z}$$

for any irrational number θ and any $\alpha \in \mathbb{C} \setminus \{0, -1\}$, and let $\triangle_\alpha$ be the Sigel disk centered at the origin whenever h_α has it. In Theorem I, we deal with the one parameter family $\{h_\alpha\}_{\alpha \in \mathbb{C} \setminus \{0,-1\}}$, where $\theta \in \mathcal{D}(2)$. By Theorem I, h_α has the bounded type Siegel disk $\triangle_\alpha$ bounded by a quasicircle containing critical points for uncountably many α. However, there are many parameters α left. We ask the following questions for any $\theta \in \mathcal{D}(2)$:

Question 8. *Are $\triangle_\alpha$ bounded by quasicircles containing at least one critical point of h_α for all $\alpha \in \mathbb{C} \setminus \{0, -1\}$?*

Question 9. *Is there an uncountable set Ω_3 such that for $\alpha \in \Omega_3$, $\triangle_\alpha$ is bounded by a quasicircle containing exactly two critical points?*

On the analogy of Theorem F, we ask the following question:

Question 10. *Are there an irrational number θ and an $\alpha \in \mathbb{C} \setminus \{0, -1\}$ such that $\partial\triangle_\alpha$ is a quasicircle without critical points?*

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