ACYLINDRICAL HYPERBOLICITY OF SOME ARTIN GROUPS

MOTOKO KATO AND SHIN-ICHI OGUNI

1. Introduction

Definition 1.1. Let Γ be a finite simple graph. Let $V(\Gamma)$ be the vertex set and $E(\Gamma)$ the edge set. We suppose that every edge e is labeled by an integer $\mu(e) \geq 2$. The *Artin group* A_Γ associated with Γ is defined by the following presentation

$$A_\Gamma = \langle V(\Gamma) \mid \underbrace{s_e t_e s_e t_e \cdots}_{\text{length } \mu(e)} = \underbrace{t_e s_e t_e s_e \cdots}_{\text{length } \mu(e)} \qquad (e \in E(\Gamma)) \rangle, \tag{1.1}$$

where s_e and t_e are two endpoints of e. We call Γ the *defining graph* of A_Γ.

Typical examples of Artin groups are free abelian groups, free groups, and braid groups. If we add relations $v^2 = 1$ for all $v \in V(\Gamma)$ to (1.1), we get the associated *Coxeter group* W_Γ. If W_Γ is finite, A_Γ is called Artin groups of *finite type*. In this article, we mainly treat Artin groups of *infinite type*, that is, A_Γ with infinite W_Γ. A_Γ is said to be *reducible* if Γ can be decomposed as a join of two subgraphs Γ_1 and Γ_2 such that any edge between a vertex of Γ_1 and a vertex of Γ_2 is labeled by 2. By definition, a reducible Artin group can be decomposed into a direct product of non-trivial subgroups. If A_Γ is not reducible, we say that A_Γ is *irreducible*.

We consider the following problem ([2], see also [3, Conjecture B]).

Problem 1.2. Are irreducible Artin groups of infinite type acylindrically hyperbolic?

Problem 1.2 is solved affirmatively for various families of Artin groups. For example,

- Artin groups associated with graphs that are not joins [2];
- Euclidean Artin groups: A_Γ such that W_Γ acts geometrically on the Euclidean space [1];
- *Two-dimensional* Artin groups: A_Γ such that every triangle in Γ with edge labels μ_1, μ_2, μ_3 satisfies $\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} \leq 1$ [5].

On the first result, Charney and Morris-Wright [2] showed acylindrical hyperbolicity of Artin groups of infinite type associated with graphs that are not joins, by studying clique-cube complexes. Clique-cube complexes are CAT(0) cube complexes, on which Artin groups are acting isometrically and cocompactly. We generalized their result and treated Artin groups associated to graphs that are not cones.

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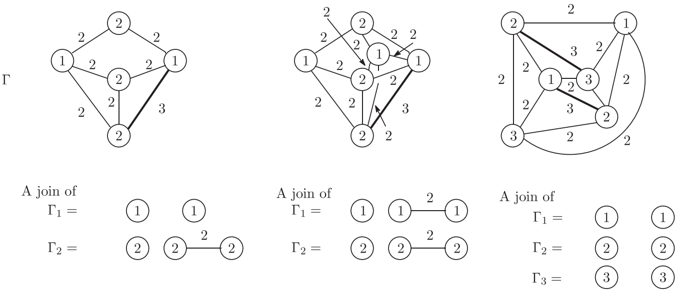

FIGURE 1. New examples of $\Gamma = \Gamma_1 * \Gamma_2$ such that A_Γ is acylindrically hyperbolic.

2. MAIN RESULT

Theorem 2.1 ([4]). Let A_Γ be an Artin group associated with Γ, where Γ has at least three vertices. Suppose that Γ is not a cone. Then, the following are equivalent:

(1) A_Γ is irreducible, that is, Γ cannot be decomposed as a join of two subgraphs such that all edges between them are labeled by 2;
(2) A_Γ has a WPD contracting element with respect to the isometric action on the clique-cube complex;
(3) A_Γ is acylindrically hyperbolic;
(4) A_Γ is directly indecomposable, that is, it cannot be decomposed as a direct product of two nontrivial subgroups.

According to Theorem 2.1, many Artin groups of infinite type, including Artin groups associated with graphs in Figure 1, are acylindrically hyperbolic. Under assumptions of Theorem 2.1, it is known that the center $Z(A_\Gamma)$ is trivial when A_Γ is reducible. Theorem 2.1 also gives an alternative proof for this fact.

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(M. Kato) FACULTY OF EDUCATION, UNIVERSITY OF THE RYUKYUS, 1 SEMBARU, NISHIHARA-CHO, NAKAGAMI-GUN, OKINAWA 903-0213, JAPAN
Email address: katom@edu.u-ryukyu.ac.jp

(S. Oguni) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, EHIME UNIVERSITY, 2-5 BUNKYO-CHO, MATSUYAMA, EHIME 790-8577, JAPAN
Email address: oguni.shinichi.mb@ehime-u.ac.jp