Facial achromatic number of triangulations on the sphere

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A *graph* consists of a set of vertices and a set of edges. All graphs considered in this paper are finite, undirected and simple unless we particularly mention it. If a graph G is drawn on a closed surface F^2 without crossing edges, then G is said to be *embedded* on F^2. Each component of $F^2 - G$ is called a *face* of G. A *triangulation* G on F^2 is a simple graph embedded on F^2 such that each face is triangular. In particular, if the degree of any vertex of a triangulation is even, then such a triangulation is called an *even triangulation*.

Let G be a graph. An *n-coloring* $c : V(G) \to \{1, 2, \dots, n\}$ is a function which assigns a color from $\{1, \dots, n\}$ to each vertex of G such that any adjacent vertices receive different colors. The *chromatic number* of G is the minimum number n such that G has a proper n-coloring, and it is denoted by $\chi(G)$.

A *complete n-coloring* of G is an n-coloring such that each pair of colors appears on at least one edge. A $\chi(G)$-coloring of G is a complete $\chi(G)$-coloring since if there is a pair (i, j) of colors which does not appear on any edge, we can obtain a proper $(\chi(G) - 1)$-coloring of G by recoloring all vertices with color j by color i, a contradiction. We define the *achromatic number* of G, denoted $\psi(G)$, to be the maximum number n for which G has a complete n-coloring.

Since $\psi(G) \geq \chi(G)$ for any graph G, there are some studies considering which graph does satisfy $\psi(G) = \chi(G)$. Hara [1] completely characterized triangulations on a closed surface with achromatic number 3, as follows. A *complete tripartite graph* K_{n_1,n_2,n_3} is a graph satisfying the following conditions: (i) vertices of the graph are decomposed into three disjoint sets and the number of each sets are n_1, n_2 and n_3, (ii) no two vertices in the same set are adjacent, and (iii) for any two vertices in the other two sets, they are adjacent.

Theorem 1 (Hara [1]). *Let G be a triangulation on a closed surface. Then $\psi(G) = 3$ if and only if G is isomorphic to $K_{n,n,n}$ for some $n \geq 1$.*

In this paper, we introduce a new coloring of embedded graphs, called a facial complete coloring, which is an expansion of the complete coloring.

Let G be a graph embedded on a surface. An n-coloring $c : V(G) \to \{1, 2, \dots, n\}$ is a *facial t-complete n-coloring* if every t-tuple of colors appears on the boundary of some face of G. The *facial t-achromatic number* of G, denoted by $\psi_t(G)$, is the maximum number n such that G has a facial t-complete n-coloring. When $t = 1$, a facial t-complete coloring is just a coloring using each color at least once, and if $t = 2$, then a coloring is a complete coloring.

Remember that every graph G has a complete n-coloring for some $n \geq \chi(G)$. However, for any $n \geq 3$, there exists a triangulation G on the sphere which has no facial 3-complete n-coloring. On the other hand, every even triangulation G on the sphere has at least one facial 3-complete n-coloring for some $n \geq \chi(G)$, since it is 3-colorable [2]. Thus, in this paper, we principally focus on the facial 3-achromatic number of even triangulations on the sphere.

Since $\psi_3(G) \geq 3$ for any even triangulation G on the sphere, we shall consider to characterize the graphs which hold $\psi_3(G) = 3$. The *double wheel DW_n* for $n \geq 3$ is a triangulation on the sphere which is obtained from the cycle C_n by adding two vertices x and y and joining them to all vertices of C_n (see the left of Figure 1). When n is even, DW_n is an even triangulation on the sphere. We can completely characterize even triangulations on the sphere with facial 3-achromatic number equal to 3, as follows. The following is also an analog of Theorem 1.

Theorem 2. *Let G be an even triangulation on the sphere. The facial* 3*-achromatic number of G is exactly* 3 *if and only if G is isomorphic to the double wheel DW_{2n} for $n \geq 2$ or one of the two graphs shown in the center and the right in Figure* 1.

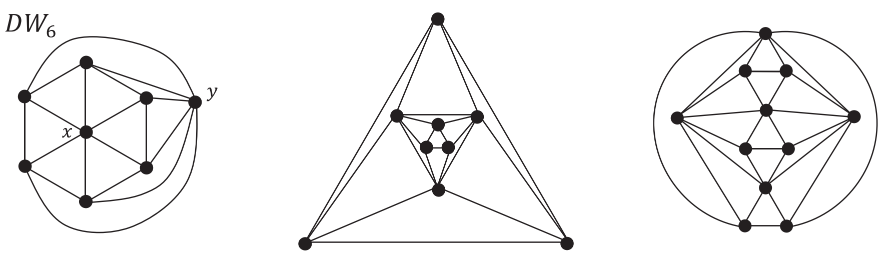

Figure 1: The double wheel DW_6 and graphs G with $\psi_3(G) = 3$

References

[1] S. Hara, Triangulations of closed surfaces with achromatic number 3, *Yokohama Math. J.* **47** (1999), 225–229.

[2] M.T. Tsai and D.B. West, A new proof of 3-colorability of Eulerian triangulations, *Ars Math. Contemp.* **4** (2011), 73–77.