Quasi-isometric embeddings from mapping class groups of nonorientable surfaces

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1 Introduction

Let $S = S^b_{g,p}$ be the connected orientable surface of genus g with b boundary components and p punctures, and $N = N^b_{g,p}$ the connected nonorientable surface of genus g with b boundary components and p punctures. In the case where $b = 0$ or $p = 0$, we drop the suffixes that denotes 0, excepting g, from $S^b_{g,p}$ and $N^b_{g,p}$. If we are not interested in whether a given surface is orientable or not, we denote the surface by F. The *mapping class group* $\mathrm{Mod}(F)$ of F is the group of isotopy classes of homeomorphisms on F which are orientation-preserving if F is orientable and preserve ∂F pointwise. For orientable surfaces S, if we consider also orientation-reversing homeomoriphisms, then we call it the *extended* mapping class group and write $\mathrm{Mod}^{\pm}(S)$. Quasi-isometry classification of finitely generated groups is a key issue in geometric group theory.

Definition 1.1. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f\colon X \to Y$ is a *quasi-isometric embedding* if there exist $\lambda_1 \geq 1$ and $\lambda_2 \geq 0$ such that

$$\frac{1}{k_1} d_X(x_1, x_2) - k_2 \leq d_Y(f(x_1), f(x_2)) \leq k_1 d_X(x_1, x_2) + k_2.$$

Furthermore, a quasi-isometric ambedding $f\colon X \to Y$ is a *quasi-isometry* if there exists $\lambda \geq 0$ such that for any $y \in Y$, there exists $x \in X$ such that $d_Y(y, f(x)) \leq \lambda$.

Let $j\colon S^{2b}_{g-1,2p} \to N^b_{g,p}$ be the orientation double covering of a nonorientable surface $N^b_{g,p}$ and $J\colon S^{2b}_{g-1,2p} \to S^{2b}_{g-1,2p}$ the deck transformation.

Lemma 1.2. ([2, Theorem 1], [7, Lemma 3], [5, Theorem 1.1]) *For all but* $(g, p, b) = (1, 0, 0)$, $(2, 0, 0)$, *the orientation double covering* j *induces an injective homomorphism* $\iota\colon \mathrm{Mod}(N^b_{g,p}) \hookrightarrow \mathrm{Mod}(S^{2b}_{g-1,2p})$. *Moreover, the image of* $\mathrm{Mod}(N^b_{g,p})$ *given by* ι *consists of the isotopy classes of orientation-preserving homeomorphisms of* $S^{2b}_{g-1,2p}$ *which commute with* J.

2 Main result

In this section, we state the main theorem (Theorem 2.1). and give the idea of the proof of Theorem 2.1.

Theorem 2.1. *For all but* $(g, p) = (2, 0)$*, the injective homomorphism* $\iota\colon \mathrm{Mod}(N^b_{g,p}) \hookrightarrow \mathrm{Mod}(S^{2b}_{g-1,2p})$ *is a quasi-isometric embedding.*

To show Theorem 2.1, we use the following results.

Proposition 2.2. *([4], [6]) For any finite type orientable surface* S*, the mapping class group* $\mathrm{Mod}(S)$ *and the extended mapping class group* $\mathrm{Mod}^{\pm}(S)$ *are semihyperbolic.*

Proposition 2.3. *([1]) Let* G *be a semihyperbolic group. Then any centralizer* H *of* G *is quasi-isometrically embedded in* G*.*

We can deduce Theorem 2.1 by the fact that $\mathrm{Mod}(N_{g,p})$ is realized as an index 2 subgroup of the centralizer of $[J]$ in the extended mapping class group $\mathrm{Mod}^{\pm}(S_{g-1,2p})$, and Propositions 2.2 and 2.3.

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