Virasoro Action on Schur Q-function

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Abstract

Schur Q-function was introduced by Schur as a symmetric polynomial describing the irreducible index of the projective representation of a symmetric group. A formula for Schur Q-functions is presented which describes the action of the Virasoro operators. For strict partition, we prove a formula for each $L_k Q_\lambda$ and $L_{-k} Q_\lambda$ $(k \geq 1)$, where L_k is the Virasoro operator. The present paper is a résumé of [1] and [2].

1 Schur Q-function

A partition is an integer sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell), \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$, whose size is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$. The number of nonzero parts is the length of λ, denoted by $\ell(\lambda)$. Let $\mathcal{SP}(n)$ be the set of partitions of n into distinct parts. We call a $\lambda \in \mathcal{SP}(n)$ strict partition of n. Let $V = \mathbb{C}[t_j; j \geq 1, \text{ odd}]$. This is decomposed as $V = \bigoplus_{n=0}^{\infty} V(n)$, where $V(n)$ is the space of homogeneous polynomials of degree n, according to the counting deg $t_j = j$. An inner product of V is defined by $\langle F, G\rangle = F(2\widetilde{\partial})\bar{G}(t)\Big|_{t=0}$, where $2\widetilde{\partial} = \left(2\partial_1, \frac{2}{3}\partial_3, \frac{2}{5}\partial_5, \ldots\right)$ with $\partial_j = \frac{\partial}{\partial t_j}$.

Schur Q-functions are defined in our context as follows. Put $\xi(t,u) = \sum_{j\geq 1,\text{odd}} t_j u^j$ and define $q_n(t) \in V(n)$ by

$$\mathrm{e}^{\xi(t,u)} = \sum_{n=0}^{\infty} q_n(t) u^n.$$

For integers a, b with $a > b > 0$, define

$$Q_{ab}(t) := q_a(t)q_b(t) + 2\sum_{i=1}^{b}(-1)^i q_{a+i}(t)q_{b-i}(t), \quad Q_{ba}(t) := -Q_{ab}(t).$$

Finally, the Q-function labelled by the strict partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2m})\,(\lambda_1 > \lambda_2 > \cdots > \lambda_{2m} \geq 0)$ is defined by

$$Q_\lambda(t) = Q_{\lambda_1\lambda_2\ldots\lambda_{2m}}(t) = \mathrm{Pf}\left(Q_{\lambda_i\lambda_j}\right)_{1\leq i,j\leq 2m}.$$

The Q-function $Q_\lambda(t)$ is homogeneous of degree $|\lambda|$. It is known that $\{Q_\lambda(t); |\lambda| = n\}$ forms an orthogonal basis for $V(n)$, with respect to the above inner product.

2 Action of the Virasoro algebra

For a positive odd integer j, put $a_j = \sqrt{2}\partial_j$ and $a_{-j} = \frac{j}{\sqrt{2}} t_j$ so that they satisfy the Heisenberg relation as operators on V :

$$[a_j, a_i] = j\delta_{j+i,0}.$$

For an integer k, put

$$L_k = \frac{1}{2}\sum_{j\in\mathbb{Z}_{\text{odd}}} : a_{-j}a_{j+2k} : + \frac{1}{8}\delta_{k,0},$$

where

$$: a_j a_i := \begin{cases} a_j a_i & \text{if } j \leq i, \\ a_i a_j & \text{if } j > i \end{cases}$$

is the normal ordering.

Theorem 1. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2m})$ be a strict partition. Then for any $k \geq 1$

$$L_k Q_\lambda = \sum_{i=1}^{2m} (\lambda_i - k)\, Q_{\lambda - 2k\epsilon_i},$$

where $\lambda - 2k\epsilon_i = (\lambda_1, \ldots, \lambda_i - 2k, \ldots, \lambda_{2m})$.

Theorem 2. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a positive integer sequence. Then

$$L_{-k} Q_\alpha = \sum_{i=1}^{\ell} (\alpha_i + k)\, Q_{\alpha + 2k\epsilon_i} + \frac{1}{2} \sum_{i=0}^{k-1} (-1)^i (k - i) Q_{\alpha, 2k-i, i}, \quad k \geq 1.$$

Theorem 1 and Theorem 2 completely describe the reduced Fock representation of the Virasoro algebra. Consider the Lie subalgebra $\mathfrak{g} = \sum_{|k| \leq 1} \mathbb{C} l_k$ which is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Let $\mathcal{ESP}$ be the set of the strict partitions whose parts are all even numbers, and let V^{even} be the subspace of V spanned by the Q_λ for $\lambda \in \mathcal{ESP}$.

Corollary 1. The space V^{even} is invariant under the action of $\mathfrak{g}$.

References

[1] Kazuya Aokage, Eriko Shinkawa, and Hiro Fumi Yamada. Pfaffian identities and virasoro operators. *Letters in Mathematical Physics*, Vol. 110, No. 6, pp. 1381–1389, 2020.

[2] Kazuya Aokage, Eriko Shinkawa, and Hiro Fumi Yamada. Virasoro action on the q-functions. *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA)*, Vol. 17, 2021.

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