

H-BASIS IN GEOMETRIC STRUCTURES WITH A DENSE/CODENSE INDEPENDENT SUBSET

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ABSTRACT. We discuss H-basis in geometric structures with a dense/codense independent subset, and algebraic n -gons in geometric structures to calculate the ranks of H -structures in trivial/non-trivial independence of base geometric structures.

1. NOTATIONS AND ALGEBRAIC DIMENSION

Let T be a complete L -theory and let \mathcal{M} be a sufficiently saturated model of T . a, b, c, \dots denote elements of \mathcal{M} . $\bar{a}, \bar{b}, \bar{c}, \dots$ denote finite tuples of \mathcal{M} . A, B, C denote small subsets of \mathcal{M} . We write $\bar{a} \in \text{acl}(A)$ if $|\{\sigma(\bar{a}) : \sigma \in \text{Aut}(\mathcal{M}/A)\}|$ is finite. The algebraic closure of A .

Let $i (\geq 1)$ be a natural number. We put $\bar{a}_{\leq i} := \bar{a}_1, \dots, \bar{a}_i$. $\bar{a}_{< i} = \bar{a}_1, \dots, \bar{a}_{i-1}$. $\bar{a}_{< 1} := \emptyset$.

We say that $a_{\leq n} = a_1, a_2, \dots, a_n$ is algebraically independent over B if $a_i \notin \text{acl}(Ba_{< i})$ for each $i \leq n$.

We say that $(\mathcal{M}, \text{acl}(\ast))$ has Steinitz exchange property if $a \in \text{acl}(Bb) \setminus \text{acl}(B)$ implies $b \in \text{acl}(Ba)$.

We say that $(\mathcal{M}, \text{acl}(\ast))$ is geometric if it has Steinitz exchange property and eliminates \exists^∞ .

Assume that $(\mathcal{M}, \text{acl}(\ast))$ has Steinitz exchange property. Then for any $a_{\leq n} = a_1, a_2, \dots, a_n$ and B , after renumbering indices, there exists unique $m (\leq n)$ such that

$$\begin{aligned} a_i &\notin \text{acl}(Ba_{< i}) \text{ for each } i \leq m \\ a_j &\in \text{acl}(Ba_{\leq m}) \text{ for each } j > m \end{aligned}$$

We write $m = \dim(a_{\leq n}/B)$, the dimension of $a_{\leq n}$ over B .

Basic properties on dimension

- (1) If $A \subseteq B$, then $\dim(\bar{a}/A) \geq \dim(\bar{a}/B)$.
- (2) Transitivity: If $A \subseteq B \subseteq C$, then $\dim(\bar{a}/A) = \dim(\bar{a}/C)$ iff $\dim(\bar{a}/A) = \dim(\bar{a}/B)$ and $\dim(\bar{a}/B) = \dim(\bar{a}/C)$.
- (3) Sub-additivity: $\dim(\bar{a}\bar{b}/A) = \dim(\bar{a}/A) + \dim(\bar{b}/A\bar{a})$.
- (4) Finite coding: There exists a finite tuple $\bar{b} \subseteq B$ such that $\dim(\bar{a}/B) = \dim(\bar{a}/\bar{b})$.

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The independence relation by dimension

We write $\bar{a} \downarrow_{\bar{b}}^{\dim} \bar{c}$ if $\dim(\bar{a}/\bar{b}) = \dim(\bar{a}/\bar{b}\bar{c})$.

Symmetry: $\bar{a} \downarrow_{\bar{b}}^{\dim} \bar{c}$ implies $\bar{c} \downarrow_{\bar{b}}^{\dim} \bar{a}$.

Proof. By using sub-additivity, we have $\dim(\bar{c}/\bar{a}\bar{b}) + \dim(\bar{a}/\bar{b}) = \dim(\bar{a}\bar{c}/\bar{b}) = \dim(\bar{a}/\bar{b}\bar{c}) + \dim(\bar{c}/\bar{b})$. So $\dim(\bar{a}/\bar{b}\bar{c}) = \dim(\bar{a}/\bar{b})$ implies $\dim(\bar{c}/\bar{a}\bar{b}) = \dim(\bar{c}/\bar{b})$.

Monotonicity: if $\bar{a} \downarrow_{\bar{b}}^{\dim} \bar{c}$ and $\bar{a}_0 \subseteq \bar{a}$, then $\bar{a}_0 \downarrow_{\bar{b}}^{\dim} \bar{c}$.

Proof. By symmetry $\bar{c} \downarrow_{\bar{b}}^{\dim} \bar{a}$. By transitivity $\bar{c} \downarrow_{\bar{b}}^{\dim} \bar{a}_0$. By symmetry again, $\bar{a}_0 \downarrow_{\bar{b}}^{\dim} \bar{c}$.

Now we define $A \downarrow_B^{\dim} C$ if $\dim(\bar{a}/B) = \dim(\bar{a}/BC)$ for any finite tuple $\bar{a} \subseteq A$.

We introduce the imaginarr element $e \in \mathcal{M}^{\text{eq}}$ if $e = \bar{a}/E$, where $E(\bar{x}, \bar{y})$ is an \emptyset -definable equivalence relation with $\text{lh}(\bar{a}) = \text{lh}(\bar{x}) = \text{lh}(\bar{y})$ and some $\bar{a} \subseteq \mathcal{M}$.

For $e \in \mathcal{M}^{\text{eq}}$ and $A \subseteq \mathcal{M}^{\text{eq}}$ we write $e \in \text{acl}^{\text{eq}}(A)$ if $|\{\sigma(e) : \sigma \in \text{Aut}(\mathcal{M}^{\text{eq}}/A)\}|$ is finite.

The independence calculus: See [A].

A *symmetric* ternary relation $* \downarrow_* *$ on \mathcal{M}^{eq} has the independence calculus if the following 8 conditions hold:

- (1) Normality: $A \downarrow_B C$ implies $A \downarrow_B BC$.
- (2) Invariance: $A \downarrow_B C$ and $ABC \equiv A'B'C'$ imply $A' \downarrow_{B'} C'$
- (3) Monotonicity: $A \downarrow_B C$ and $A_0 \subseteq A$ imply $A_0 \downarrow_B C$
- (4) *Transitivity*: If $B \subseteq C \subseteq D$, then $A \downarrow_B D$ iff $A \downarrow_B C$ and $A \downarrow_C D$
- (5) Extension: There exists $A' \equiv_B A$ such that $A' \downarrow_B C$.
- (6) Finite character: If $\bar{a} \downarrow_B C$ for any finite tuple $\bar{a} \subseteq A$, then $A \downarrow_B C$.
- (7) *Local character*: For any $\bar{a}, A \subseteq \mathcal{M}$, there exists $A_0 \subseteq A$ such that $|A_0| \leq |T|$ and $\bar{a} \downarrow_{A_0} A$.
- (8) Anti-reflexivity: $\bar{a} \downarrow_A \bar{a}$ implies $\bar{a} \in \text{acl}^{\text{eq}}(A)$.

symmetric \Leftrightarrow *transitive* \Leftrightarrow *local character* holds modulo other properties of the independence calculus. We have the following: *stable* \Rightarrow *simple* \Rightarrow *rosy* (i.e. having the independence calculus) \Leftarrow *o-minimal*

superstable ($U < \infty$) \Rightarrow *supersimple* ($SU < \infty$) \Rightarrow *superrosy* ($U^p < \infty$).

strongly minimal ($0 < U \leq \text{RM} = 1, \text{deg}_{\text{RM}} = 1$) \Rightarrow $SU = 1 \Rightarrow U^p = 1 \Leftarrow$ *o-minimal*.

2. NON-TRIVIALITY AND ALGEBRAIC n -GONS

Assume that $(\mathcal{M}, \text{acl}(*))$ has Steinitz exchange property and put $\downarrow = \downarrow^{\dim}$.

We say that $a \in \mathcal{M}$ is *non-trivial* if there exists a_2, a_3, \bar{c} such that $a \downarrow_{\bar{c}} a_2, a \downarrow_{\bar{c}} a_3, a_2 \downarrow_{\bar{c}} a_3$ and $a \not\downarrow_{\bar{c}} a_2, a_3$. Then we say that a, a_2, a_3 is an *algebraic triangle* over \bar{b} .

We say that $a_{\leq n} = a_1, \dots, a_n$ is an *algebraic n -gon* over A if $\dim(a_{\leq n}/A) = n-1$ and $\dim(a_{\leq n} \setminus \{a_i\}/A) = n-1$ for each $1 \leq i \leq n$.

Then a_i, a_j, a_k is an algebraic triangle over $Aa_{\leq n} \setminus \{a_i, a_j, a_k\}$, because $a_i \not\downarrow_{Aa_{\leq n} \setminus \{a_i, a_j, a_k\}} a_j, a_k$. So any point of n -gon is non-trivial.

Fact 2.1. *If $a \in \mathcal{M}$ is non-trivial, then for each $n < \omega$, there exists $b_{\leq n-1}, A$ such that $ab_{\leq n-1}$ is an algebraic n -gon over A .*

Proof. The case $n = 3$ is clear. By induction hypothesis, assume that $ab_{\leq n-1}$ is an algebraic n -gon over A . As b_{n-1} is non-trivial, there exist b_n, b_{n+1}, \bar{c} such that $b_{n-1}b_nb_{n+1}$ is an algebraic triangle over \bar{c} . By an automorphism fixing b_{n-1} , we may assume that $b_nb_{n+1}\bar{c} \downarrow_{b_{n-1}} b_{\leq n-2}A$.

CLAIM: $ab_{\leq n-2}b_nb_{n+1}$ is an algebraic $(n+1)$ -gon over $A\bar{c}$.

Subclaim 1: $\dim(ab_{\leq n-2}b_nb_{n+1}/A\bar{c}) = n$.

As $b_{n+1} \in \text{acl}(b_{n-1}b_n\bar{c})$ and $b_{n-1} \in \text{acl}(ab_{\leq n-2}A)$, we have $b_{n+1} \in \text{acl}(ab_{\leq n-2}b_nA\bar{c})$. On the other hand, we have $b_n\bar{c} \downarrow_{b_{n-1}} b_{\leq n-2}A$, $b_n \downarrow_{\bar{c}} b_{n-1}$ and $a \in \text{acl}(b_{\leq n-1}A)$, we have $b_n \downarrow_{\bar{c}} ab_{\leq n-2}A$. As $\bar{c} \downarrow_{b_{n-1}} ab_{\leq n-2}A$ and $\bar{c} \downarrow_{b_{n-1}}$, we have $\bar{c} \downarrow_A ab_{\leq n-2}$. Therefore $\dim(ab_{\leq n-2}b_nb_{n+1}/A\bar{c}) = n$.

Subclaim 2: $ab_{\leq n-2}b_n$ is independent over $A\bar{c}$. Similarly for $ab_{\leq n-2}b_{n+1}$.

By $b_n \notin \text{acl}(b_{n-1}\bar{c})$ and $b_nb_{n+1}\bar{c} \downarrow_{b_{n-1}} b_{\leq n-2}A$, we have $b_n \notin \text{acl}(b_{\leq n-1}A\bar{c}) = \text{acl}(ab_{\leq n-2}A\bar{c})$ as $ab_{\leq n-1}$ is an algebraic n -gons over A .

We use the following : We have $b_nb_{n+1} \downarrow_{\bar{c}} A$ by $b_nb_{n+1}\bar{c} \downarrow_{b_{n-1}} A$ and $b_{n-1} \downarrow_A A$. Note that $b_n \downarrow_{A\bar{c}} b_{n+1}$.

Subclaim 3: $b_{\leq n-2}b_nb_{n+1}$ is independent over $A\bar{c}$.

Since $b_{n-1}b_nb_{n+1}$ is an algebraic triangle over \bar{c} and $b_n \downarrow_{b_{n-1}\bar{c}} b_{\leq n-2}A$, we see that $b_n \downarrow_{\bar{c}} b_{\leq n-2}A$. So $b_{\leq n-2}b_n$ is independent over $A\bar{c}$. By $b_nb_{n+1}\bar{c} \downarrow_{b_{n-1}} b_{\leq n-2}A$ and $b_{n-1} \downarrow_A b_{\leq n-2}$, we have $b_nb_{n+1}b_{n-1}\bar{c} \downarrow_A b_{\leq n-2}$. If we had $b_{n+1} \in \text{acl}(b_nb_{\leq n-2}A\bar{c})$, we would have $b_{n+1} \in \text{acl}(b_nA\bar{c})$. As we have $b_nb_{n+1} \downarrow_{\bar{c}} A$, $b_{n+1} \in \text{acl}(b_n\bar{c})$ follows, a contradiction to $b_n \downarrow_{\bar{c}} b_{n+1}$.

Subclaim 4: $ab_{\leq n-2}b_nb_{n+1} \setminus \{b_j\}$ is independent over $A\bar{c}$ for each $1 \leq j \leq n-2$.

As $b_{n-1} \downarrow_A ab_{\leq n-2} \setminus \{b_j\}$ and $b_nb_{n+1}\bar{c} \downarrow_{b_{n-1}} ab_{\leq n-2}A$, we have $b_nb_{n+1}b_{n-1}\bar{c} \downarrow_A ab_{\leq n-2} \setminus \{b_j\}$. So $ab_{\leq n-2} \setminus \{b_j\}b_n$ is independent over $A\bar{c}$. If we had $b_{n+1} \in \text{acl}(b_nab_{\leq n-2} \setminus \{b_j\})$, we would have $b_{n+1} \in \text{acl}(b_nA\bar{c})$, a contradiction. \square

3. H-STRUCTURE

Let \mathcal{M} be a sufficiently saturated model of a complete L -theory T . Suppose that $(\mathcal{M}, \text{acl}(\ast))$ has Steinitz exchange property. We add a unary predicate $H(x)$ for a dense/codense independent subset. L_H denotes $L \cup \{H\}$.

Definition 3.1. We say that $(\mathcal{M}, H(\mathcal{M}))$ is an H -structure if

- (1) $H(\mathcal{M})$ is *independent*: If $a_1, \dots, a_n \in H(\mathcal{M})$ are distinct, they are independent over \emptyset .
- (2) *density*: If $A \subset \mathcal{M}$ is finite dimensional and $p(x)$ is a unary non-algebraic L -type over A , then there exists $a \in H(\mathcal{M})$ such that $a \models p(x)$.
- (3) *codensity*: If $A \subset \mathcal{M}$ is finite dimensional and $q(x)$ is a unary non-algebraic L -type over A , then there exists $b \notin \text{acl}(AH(\mathcal{M}))$ such that $b \models q(x)$.

For $A \subset \mathcal{M}$ we say A is H -independent in \mathcal{M} if $A \perp_{H(A)} H(\mathcal{M})$, where $H(A) = H(\mathcal{M}) \cap A$. Note that if A is H -independent, then $\text{acl}(A)$ is also H -independent. tp denotes L -type and tp_H denotes L_H -type.

Fact 3.2. (1) For any $(M, H(M))$, there exists a sufficiently saturated H -structure $(M, H(M)) \subseteq (\mathcal{M}, H(\mathcal{M}))$ with M is H -independent in \mathcal{M} .
(2) Suppose that $(\mathcal{M}, H(\mathcal{M}))$ and $(\mathcal{N}, H(\mathcal{N}))$ are H -structures. If H -independent tuples $\bar{a} \subset \mathcal{M}, \bar{b} \subset \mathcal{N}$ with $\text{tp}(\bar{a}, H(\bar{a})) = \text{tp}(\bar{b}, H(\bar{b}))$, then $\text{tp}_H(\bar{a}) = \text{tp}_H(\bar{b})$. In particular, all H -structures are elementarily equivalent, let T^{ind} be the common theory. If T eliminates \exists^∞ , then T^{ind} is axiomatizable such that all $|T|^+$ -saturated model of T^{ind} are H -structures.

The following theories eliminate \exists^∞ : strongly minimal theories, $\text{SU} = 1$ theories, dense o -minimal theories and the p -adics in a single sort.

Let T be the theory of infinite dimensional countable vector space V over a finite field. Put $H(V) := \{v_i : i < \omega\}$ a basis of V . Then $(V, H(V))$ is a model of T^{ind} but not H -structure because it does not satisfy codense property. Put $H_j(V) := \{v_i : i > j\}$. Then $(V, H_j(V))$ is a model of T^{ind} but not H -structure and $(V, H_j(V))$ is not isomorphic to $(V, H_k(V))$ for $j \neq k < \omega$, so T^{ind} is not ω -categorical. Put $H_{\text{even}}(V) = \{v_{2i} : i < \omega\}$. Then $(V, H_{\text{even}}(V))$ is an H -structure.

4. H -BASIS

Let $A \subset \mathcal{M}$ be H -independent. For any $\bar{a} \subset \mathcal{M}$ we can take a finite tuple $\bar{h} \subset H(\mathcal{M})$ such that $\bar{a} \perp_{A\bar{h}} H(\mathcal{M})$ and $A \cap \bar{h} = \emptyset$. Suppose that \bar{h} is minimal length. We show the uniqueness of \bar{h} up to permutation.

Take such another \bar{h}' and let $\bar{h}_1 := \bar{h} \cap \bar{h}', \bar{h} = \bar{h}_1 \bar{h}_2$ and $\bar{h}' = \bar{h}_1 \bar{h}'_2$. As $H(\mathcal{M})$ is an independent subset and $\bar{h} \cap A = \bar{h}' \cap A = \emptyset$, we have $\bar{h}_2 \perp_{H(A)\bar{h}_1} \bar{h}'_2$. As A is H -independent, $A\bar{h}_1 \perp_{H(A)\bar{h}_1} \bar{h}_2 \bar{h}'_2$, we have $\bar{h}_2 \perp_{A\bar{h}_1} \bar{h}'_2$.

Let $\bar{a} = \bar{a}_1 \bar{a}_2$ be such that $\bar{a}_1 \perp_{AH(\mathcal{M})}$ and $\bar{a}_2 \in \text{acl}(\bar{a}_1 AH(\mathcal{M}))$. Note that $\bar{a}_2 \in \text{acl}(\bar{a}_1 A\bar{h}_1 \bar{h}_2) \setminus \text{acl}(\bar{a}_1 A\bar{h}_1)$ and $\bar{a}_2 \in \text{acl}(\bar{a}_1 A\bar{h}_1 \bar{h}'_2) \setminus \text{acl}(\bar{a}_1 A\bar{h}_1)$ by minimality of \bar{h} and \bar{h}' . Note that $\bar{h}'_2 \not\perp_{\bar{a}_1 A\bar{h}_1} \bar{h}_2$ witnessed by \bar{a}_2 . By $\bar{a}_1 \perp_{CH(\mathcal{M})}$, we have $\dim(\bar{h}'_2 / A\bar{h}_1 \bar{h}_2) = \dim(\bar{h}'_2 / \bar{a}_1 A\bar{h}_1 \bar{h}_2) < \dim(\bar{h}'_2 / \bar{a}_1 A\bar{h}_1) = \dim(\bar{h}'_2 / A\bar{h}_1)$, $\bar{h}'_2 \not\perp_{A\bar{h}_1} \bar{h}_2$, a contradiction to the independency of $H(\mathcal{M})$.

We write $HB(\bar{a}/A) := \bar{h}$, which is called H -basis of $\text{tp}(\bar{a}/A)$, where A is H -independent. Note that $HB(\bar{a}/A) \in \text{acl}_H(\bar{a}, A)$ and $HB(\bar{a}/A) = HB(\bar{a}/\text{acl}(A))$.

Fact 4.1. (1) $HB(\bar{a}\bar{b}/A) = HB(\bar{a}/A)HB(\bar{b}/A\bar{a}HB(\bar{a}/A))$ for any H -independent set A .

(2) If $A \subseteq B$ are H -independent, then $HB(\bar{a}/A) \subseteq HB(\bar{a}/B)H(B)$.

Proof. (1): Put $\bar{h}_1 := HB(\bar{a}/A), \bar{h}_2 = HB(\bar{b}/\bar{a}A\bar{h}_1)$ and $\bar{h} = HB(\bar{a}\bar{b}/A)$.

As $\bar{a} \perp_{A\bar{h}_1} H(\mathcal{M})$, $\bar{a}A\bar{h}_1$ is H -independent, so we can consider $\bar{h}_2 = HB(\bar{b}/\bar{a}A\bar{h}_1)$.

Claim 1: $\bar{h} \subseteq \bar{h}_1 \bar{h}_2$.

As $\bar{a} \perp_{A\bar{h}_1} H(\mathcal{M})$ and $\bar{b} \perp_{\bar{a}A\bar{h}_1 \bar{h}_2} H(\mathcal{M})$, we have $\bar{a}\bar{b} \perp_{A\bar{h}_1 \bar{h}_2} H(\mathcal{M})$ as desired.

Claim 2: $\bar{h}_1 \bar{h}_2 \subseteq \bar{h}$.

As $\bar{a}\bar{b} \perp_{A\bar{h}} H(\mathcal{M})$, we have $\bar{a} \perp_{A\bar{h}} H(\mathcal{M})$, so $\bar{h}_1 \subseteq \bar{h}$ follows. On the other hand,

we have $\bar{b} \downarrow_{\bar{a}A\bar{h}} H(\mathcal{M})$, so $\bar{b} \downarrow_{\bar{a}A\bar{h}_1\bar{h}} H(\mathcal{M})$, $\bar{h}_2 \subseteq \bar{h}$ follows.

(2): Put $\bar{h} = HB(\bar{a}/B)$. As $\bar{a} \downarrow_{B\bar{h}} H(\mathcal{M})$, we have $\bar{a}B \downarrow_{B\bar{h}} H(\mathcal{M})$. As B is H -independent, $B\bar{h} \downarrow_{H(B)\bar{h}} H(\mathcal{M})$. So we get $\bar{a} \downarrow_{H(B)\bar{h}A} H(\mathcal{M})$. So $HB(\bar{a}/A) \subseteq H(B)\bar{h} = H(B)HB(\bar{a}/B)$. \square

Question 4.2. *If $A \subseteq B$ are H -independent, then $HB(\bar{a}/B) \subseteq HB(\bar{a}/A)$?*

Fact 4.3. *Let $(\mathcal{M}, H(\mathcal{M}))$ be an H -structure.*

- (1) *Suppose that A is H -independent, then $\text{acl}(A) = \text{acl}_H(A)$.*
- (2) *$\text{acl}(AHB(A)) = \text{acl}_H(A)$ for any $A \subset \mathcal{M}$.*

Proof. (1): $\text{acl}(A) \subseteq \text{acl}_H(A)$ is clear.

We show that $a \notin \text{acl}(A)$ implies $a \notin \text{acl}_H(A)$.

The case that $a \notin \text{acl}(AH(\mathcal{M}))$: As $a \downarrow_A H(\mathcal{M})$ and A is H -independent, we see that Aa is H -independent. By extension property take $(a_i : i < \omega)$ realizations of $\text{tp}(a/A)$ which are acl -independent over $AH(\mathcal{M})$. As A is H -independent, we see that Aa_i is H -independent. By Fact 3.2 (2), we see that $\text{tp}_H(a_i/A) = \text{tp}_H(a/A)$ as desired.

The case that $a \in \text{acl}(AH(\mathcal{M}))$: Take $\bar{b} \in H(\mathcal{M})$ such that $a \in \text{acl}(A\bar{b})$. By coheir property take $(\bar{b}_i : i < \omega)$ acl -independent realizations in $H(\mathcal{M})$ of $\text{tp}(\bar{b}/A)$. Let a_i be such that $a_i\bar{b}_i \models \text{tp}(a\bar{b}/A)$. As $A\bar{b}_i$ is H -independent and $Aa_i\bar{b}_i \subseteq \text{acl}(A\bar{b}_i)$, $Aa_i\bar{b}_i$ is H -independent. By Fact 3.2 (2), $\text{tp}_H(a_i\bar{b}_i/A) = \text{tp}_H(a\bar{b}/A)$ follows, as desired.

(2): $\text{acl}_H(A) \subseteq \text{acl}_H(AHB(A))$ is clear. By (1) and H -independence of $AHB(A)$, we have $\text{acl}_H(AHB(A)) = \text{acl}(AHB(A))$. As we have $HB(A) \in \text{acl}_H(A)$, we see $\text{acl}(AHB(A)) \subseteq \text{acl}_H(A)$. \square

Proposition 4.4. *Let $b \in \text{acl}(h_1, \dots, h_n A)$ where A is H -independent, $\bar{h} = h_1, \dots, h_n \subset H(\mathcal{M})$ and n is minimal. Then $\text{acl}_H(bA) = \text{acl}_H(\bar{h}A)$.*

Proof. We have $\text{acl}_H(bA) = \text{acl}(bAHB(bA))$ by 4.3 (2). By 4.1 (1), $HB(Ab) = HB(A)HB(b/AHB(A))$ follows. Since A is H -independent, we see $\text{acl}(HB(A)) = \text{acl}(H(A))$. On the other hand we have $b \downarrow_{A\bar{h}} H(\mathcal{M})$. By minimality of \bar{h} , $\bar{h} = HB(b/A)$ follows. Thus $\text{acl}(HB(Ab)) = \text{acl}(H(A)HB(b/\text{acl}(A))) = \text{acl}(H(A)\bar{h})$. So we have $\text{acl}_H(bA) = \text{acl}(bA\bar{h}) = \text{acl}(A\bar{h})$. As $A\bar{h}$ is H -independent, $\text{acl}(A\bar{h}) = \text{acl}_H(A\bar{h})$ follows. Therefore we have $\text{acl}_H(bA) = \text{acl}_H(\bar{h}A)$. \square

5. RANK IN $(\mathcal{M}, H(\mathcal{M}))$, WHERE \mathcal{M} IS STRONGLY MINIMAL

Let X_H be an L_H -definable set over A . We say that X_H is small if $X \subset \text{acl}(AH(\mathcal{M}))$. Otherwise, we say that X_H is large.

Let X be an L -definable set over A . By codensity of H , X is infinite iff X is large. Clearly $H(x)$ is small.

Fact 5.1. *In a sufficiently saturated H -structure $(\mathcal{M}, H(\mathcal{M}))$, let X_H be an L_H -definable set in \mathcal{M} . Then there exists an L -definable set X in \mathcal{M} such that $X_H \triangle X$ is small.*

Recall that $X \triangle Z \subseteq (X \triangle Y) \cup (Y \triangle Z)$.

The case that the base theory is strongly minimal

(1) large L_H -type is unique: If X_H, Y_H be large L_H -definable, then $X_H \triangle Y_H$ is

small. (Because there exist infinite L -definable sets X, Y in \mathcal{M} such that $X_H \triangle X, Y_H \triangle Y$ are small and $X \triangle Y$ is finite.) This argument holds if there exists the *unique* non-algebraic 1-type in geometric L -structures.

(2) $H(x)$ has the unique non-algebraic type over H -independent set A : Suppose that $b, b' \in H(\mathcal{M})$ such that $\text{tp}_H(b/A), \text{tp}_H(b'/A)$ are non-algebraic. Clearly $b, b' \notin \text{acl}(A)$. As $\text{tp}(AbH(Ab)) = \text{tp}(Ab'H(Ab'))$ (by strong minimality) and Ab, Ab' are H -independent, we have $\text{tp}_H(Ab) = \text{tp}_H(Ab')$.

(3) It is shown that $(\mathcal{M}, H(\mathcal{M}))$ is ω -stable by counting of types over countable sets. By (2) $RM_H(H(x)) = 1$.

The case that the base theory is strongly minimal and trivial

(1) Suppose that $b \in \text{acl}(AH(\mathcal{M}))$. Then by triviality either $b \in \text{acl}(A)$ or $b \in \text{acl}(h) \setminus \text{acl}(A)$ for some $h \in H(\mathcal{M})$. So we see that $RM_H(b/A) = RM_H(h/A) = RM_H(h) = 1$. So $RM_H(\text{small type}) \leq 1$. As any large type has the unique large extension, $RM_H(\text{large type}) \leq 2$.

(2) If $\text{acl}(a) \setminus \text{acl}(\emptyset)$ is finite for all non-algebraic $a \in \mathcal{M}$, $RM_H(x = x) = 1$.

(3) If $\text{acl}(a) \setminus \text{acl}(\emptyset)$ is infinite for all non-algebraic $a \in \mathcal{M}$, $RM_H(x = x) = 2$.

The case that the base theory is strongly minimal and non-trivial

(1) Suppose that $b \in \text{acl}(AH(\mathcal{M}))$, where A is H -independent. Take $h \subset H(\mathcal{M})$ be minimal length such that $b \in \text{acl}(Ah)$. Then $\text{acl}_H(Ab) = \text{acl}_H(A\bar{h})$. So $RM_H(b/A) = |\bar{h}|$.

(2) Let $a_{\leq n}$ be an algebraic n -gon over H -independent set A . By density, we may assume that $a_{\leq n-1} \subseteq H(\mathcal{M})$. Then $a_n \notin H(\mathcal{M})$ follows. Then we see that $HB(a_n/A) = a_{\leq n-1}$ and $\text{acl}_H(Aa_n) = \text{acl}(Aa_{\leq n-1})$ by Proposition 4.4. So we see that $RM_H(a_n/A) = n - 1$.

(3) Suppose that $b \notin \text{acl}(AH(\mathcal{M}))$. If $b \in \text{acl}(BH(\mathcal{M}))$, then $RM_H(b/B) < \omega$. Large extension of $\text{tp}(b/A)$ over B is unique, we see that $RM_H(b/A) = \omega$.

6. AMPLENESS AND TRIVIALITY

$\bar{a}_{\leq i}$ denotes $\bar{a}_0, \bar{a}_1, \dots, \bar{a}_i$, $\bar{a}_{< i}$ denotes $\bar{a}_0, \bar{a}_1, \dots, \bar{a}_{i-1}$ and $\bar{a}_{< 0} = \emptyset$.

We say that (\mathcal{M}, \perp) is n -ample over \bar{c} if there exist $\bar{c}, \bar{a}_{\leq n}$ such that for any $0 \leq i < n$

- (1) $\bar{a}_{i+1} \perp_{\bar{c}\bar{a}_i} \bar{a}_{< i}$
- (2) $\text{acl}^{\text{eq}}(\bar{c}\bar{a}_{< i}\bar{a}_{i+1}) \cap \text{acl}^{\text{eq}}(\bar{c}\bar{a}_{< i}\bar{a}_i) = \text{acl}^{\text{eq}}(\bar{c}\bar{a}_{< i})$
- (3) $\bar{a}_n \not\perp_{\bar{c}} \bar{a}_0$

If (\mathcal{M}, \perp) has weak canonical bases and $\bar{a}_{\leq n+1} = \bar{a}_0, \bar{a}_1, \dots, \bar{a}_{n+1}$ is $(n+1)$ -ample over \bar{c} , then $\bar{a}_1, \dots, \bar{a}_{n+1}$ is n -ample over $\bar{c}\bar{a}_0$, so $(n+1)$ -ampleness implies n -ampleness.

n -dimensional free pseudospace ($(n+1)$ -many sorts and n -many incident relations) is n -ample but not $(n+1)$ -ample. we have one-basedness \Leftrightarrow non-1-ampleness and CM-triviality \Leftrightarrow non-2-ampleness.

For any sequence $(\bar{a}_i : i \in I)$, we write $\bar{a}_{\neq i} := (\bar{a}_j : j \in I \setminus \{i\})$.

We say that $(\bar{a}_i : i \in I)$ is independent over A if $\bar{a}_i \perp_A \bar{a}_{\neq i}$.

We say that $(\bar{a}_i : i \in I)$ is pairwise independent over A if $\bar{a}_i \perp_A \bar{a}_j$ for any $i \neq j \in I$.

We say that T is *trivial* if any pairwise independent sequence in \mathcal{M}^{eq} is independent.

We say that $(\mathcal{M}, \text{acl}(\ast))$ is trivial if $\text{acl}(A) = \bigcup_{a \in A} \text{acl}(\{a\})$ for any $A \subset \mathcal{M}$.

If $(\mathcal{M}, \text{acl}(\ast))$ is not trivial, there exists $a, b, c, B \subset \mathcal{M}$ such that $a \in \text{acl}(bcB) \setminus (\text{acl}(bB) \cup \text{acl}(cB))$. (Take a minimal size $A =: Bc$ such that $\text{acl}(A) = \bigcup_{a \in A} \text{acl}(\{a\})$ and if $b \notin \text{acl}(A)$, then $\emptyset \neq \text{acl}(Ab) \setminus (\text{acl}(A) \cup \text{acl}(b))$. Take $a \in \text{acl}(Bcb) \setminus (\text{acl}(Bc) \cup \text{acl}(b))$. As $|A| = |Bb|$, we have $\text{acl}(Bb) = \bigcup_{d \in Bb} \text{acl}(\{d\})$. So $a \notin \text{acl}(Bb)$, as desired.)

The triviality of $(\mathcal{M}, \text{acl}(\ast))$ coincides with the triviality of $(\mathcal{M}, \downarrow)$.

7. A LOVELY PAIR AND AMPLENESS

Let $L_P = L \cup \{P(x)\}$, where $P(x)$ is a new unary predicate. Let $\kappa \geq |T|^+$. We say that $(M, P(M))$ is a κ -lovely pair if

- (1) $P(M) \prec M \models T$
- (2) Coheir property: For any $A \subset M$ with $|A| < \kappa$ and any finitary L -type $p(\bar{x})$ over A with $\bar{a} \downarrow_{P(A)} A$ and $\bar{a} \models p(\bar{x})$, then $p(\bar{x})$ is realized in $P(M)$
- (3) Extension property: For any $A \subset M$ with $|A| < \kappa$ and any finitary L -type $p(\bar{x})$ over A there exists $\bar{a} \models p(\bar{x})$ such that $\bar{a} \downarrow_A AP(M)$.

If $(\mathcal{M}, \downarrow)$ has independence calculus, then a κ -lovely pair exists. κ -lovely pairs are elementarily equivalent, T_P denotes the theory of lovely pair, where $T = \text{Th}(\mathcal{M})$. Any $|T|^+$ -saturated model of T_P is a lovely pair. If T is simple, T_P is axiomatizable iff T is low and has weak non-finite cover property. If T is simple and T_P is axiomatizable, then T_P is also simple. [B-YPiV].

- Fact 7.1.** (1) If T is simple, one-based and T_P is axiomatizable, then T_P is simple and one-based. [B-YPiV].
- (2) If T is stable, trivial and does not have finite cover property, then T is n -ample iff T_P is n -ample for any $n \geq 1$ assuming a nice characterization of forking in T_P : In general for $A, B, C \subset \mathcal{M}$, we have

$$A \downarrow_C^P B \Leftrightarrow AP \downarrow_{CP} BP, \text{Cb}(AC/P) \downarrow_{\text{Cb}(C/P)} \text{Cb}(BC/P).$$

The nice characterization of forking in T_P is as follows: For any P -independent subsets A, B, C ,

$$A \downarrow_C^P B \Leftrightarrow AP \downarrow_{CP} BP, AC \downarrow_C BC.$$

[CM-PP].

- (3) If T has SU-rank one with QE, then T is one-based iff $\text{acl} = \text{acl}_P$ in T_P iff T_P has SU-rank ≤ 2 iff T_P is model-complete iff $(\mathcal{M}, \text{acl})$ is modular over $P(\mathcal{M})$. [V1].
- (4) If T is geometric (=having Steinitz exchange property for acl and elimination of \exists^∞), then T is weakly locally modular iff $(\mathcal{M}, \text{acl})$ is modular over $P(\mathcal{M})$ iff $\text{acl} = \text{acl}_P$ in T_P . [BV1].

- (5) If T eliminates \exists^∞ and $U^p(T) = 1$, then $U_P^p(T_P) \leq \omega$. [B].
 Moreover if T is weakly local modular, then $U_P^p(T_P) \leq 2$. [BV1].

Question 7.2. *If T eliminates \exists^∞ , $U^p(T) = 1$ and T is weakly local modular, then is T_P weakly locally modular? There is a proof containing serious gaps in [BB-WKAP]. (We do not know whether $A \downarrow_B^p C$ and $AP \downarrow_{BP}^p CP$ imply $A \downarrow_B^{P,p} C$ or not. And we can not find the reason for $\bar{a} \downarrow_{\bar{a}_2 \bar{a}'}^p C$ in the proof of Theorem 4.2 in [BB-WKAP].)*

8. A DENSE INDEPENDENT SUBSET IN SUPERSIMPLE STRUCTURES AND AMPLENESS

This section is almost due to [BV4]. We assume that $(\mathcal{M}, \downarrow)$ is supersimple and let $T = \text{Th}(\mathcal{M})$. Fix a partial unary type $\Pi(x)$ over \emptyset and let $H(x)$ be a new predicate.

We say that $(M, H(M))$ is H -structure associated to Π if

- (1) If $h \in H(M)$, then $\models \Pi(h)$.
- (2) $h_1, \dots, h_n \in H(M)$ are distinct, then $\{h_1, \dots, h_n\}$ is \downarrow -independent.
- (3) Density: Let $\bar{b} \subset M$ and let $\Pi(x) \subset p(x)$ be a complete type over \bar{b} such that if $a \models p(x)$ then $a \downarrow \bar{b}$. Then there exists $h \in H(M)$ such that $h \models p(x)$.
- (4) Codensity: Let $\bar{b} \subset M$ and let $p(x)$ be a complete type over \bar{b} . Then there exists $a \in M$ such that $a \models p(x)$ and $a \downarrow_{\bar{b}} H(M)$.

An H -structure associated to $\Pi(x)$ exists. H -structures associated to $\Pi(x)$ are elementarily equivalent, T_Π^{ind} denotes the common theory of H -structure associated to $\Pi(x)$. T_Π^{ind} is axiomatizable if T has two conditions (1) (2):

- (1) For each formula $\varphi(x, \bar{y})$ there exists a formula $\psi(\bar{y})$ such that (there exists a such that $a \models \varphi(x, \bar{b})$ and $a \downarrow \bar{b}$) iff $\bar{b} \models \psi(\bar{y})$.
- (2) Let $\varphi(\bar{x}, \bar{y})$ and $\psi(\bar{y}, \bar{z})$ be formulas. The following condition on \bar{c} is type-definable: For any $\bar{b} \models \psi(\bar{y}, \bar{c})$ there exists $\bar{a} \models \varphi(\bar{x}, \bar{b})$ we have $\bar{a} \downarrow_{\bar{c}} \bar{b}$.

Any $|T|^+$ -saturated model of T_Π^{ind} is a H -structure associated to $\Pi(x)$. If T is supersimple and T_Π^{ind} is axiomatizable, then T_Π^{ind} is supersimple.

Fact 8.1. (1) [BV4] *Let $(\mathcal{M}, H(\mathcal{M}))$ be an H -structure associated to $\Pi(x)$, where \mathcal{M} is supersimple. Then*

$$\bar{a} \downarrow_A^H B \Leftrightarrow \bar{a} \downarrow_{AH(\mathcal{M})} BH(\mathcal{M}), HB(\bar{a}/A) = HB(\bar{a}/B).$$

- (2) [BV3] *Let $(\mathcal{M}, H(\mathcal{M}))$ be an H -structure, where $\text{SU}(\mathcal{M}) = 1$. Suppose that $A = \text{acl}_H(A)$.*

- (a) $a \in A$ iff $\text{SU}_H(a/A) = 0$
- (b) $a \in \text{acl}(AH(\mathcal{M})) \setminus A$ iff $\text{acl}_H(aA) = \text{acl}(AHB(a/A))$ iff $\text{SU}_H(a/A) = |HB(a/A)|$.
- (c) *If a is trivial over A , then $a \in \text{acl}(AH(\mathcal{M})) \setminus A$ then $\text{SU}_H(a/A) = 1$.*
- (d) *If a is trivial over A and $a \notin \text{acl}(AH(\mathcal{M}))$ then $\text{SU}_H(a/A) = 1$.*

(e) If a is non-trivial over A , then $a \notin \text{acl}(AH(\mathcal{M}))$ iff $\text{SU}_H(a/A) = \omega$.

Proof. (c) : By triviality there exists $h \in H(\mathcal{M})$ such that $HB(a/A) = h$. The conclusion follows from (b).

(d) : If $a \not\downarrow_A^H B$ with $B = \text{acl}_H(B)$, by $HB(a/B) \subseteq HB(a/A) = \emptyset$, we have $a \not\downarrow_{AH(\mathcal{M})}^H BH(\mathcal{M})$. So $a \in \text{acl}(BH(\mathcal{M}))$. By triviality and $a \notin \text{acl}(AH(\mathcal{M}))$ we have $a \in \text{acl}(B) \setminus \text{acl}(AH(\mathcal{M}))$. So $\text{SU}_H(a/B) = 0$ follows by (a). Therefore we have $\text{SU}_H(a/A) = 1$. \square

Fact 8.2. (1) [BV3] *There exists a non-trivial one-based strongly minimal theory T whose T^{ind} is not one-based. $(V, +, 0, H)$ be a vector space over \mathbb{Q} , where $H(V) = \{v_i \in V : i < \omega\}$. Then $(V, +, 0, H) \models T^{\text{ind}}$. Let $u \in V \setminus H(V)$. Put $t = u + v_1, t' = u + v_2$. Then $HB(tt') = v_1v_2$ because $\text{acl}_H(tt') = \text{acl}(v_1v_2t')$ as $t - t' = v_1 - v_2$ and $tt' \downarrow_{v_1v_2}^H H(V)$ and $tt' \not\downarrow_{v_i}^H H(V)$. We have $t \downarrow_u^H ut'$ as $v_1 \downarrow_u^H v_2$. If T^{ind} was one-based and $\text{acl}_H(t) \cap \text{acl}_H(ut') = \text{acl}_H(\emptyset)$, then $t \downarrow_{\emptyset} ut'$. On the other hand $\text{RM}_H(t/ut') = \text{RM}_H(v_1/ut') \leq 1$ and $\text{RM}_H(t/t') = \text{RM}_H(v_1v_2/t') = 2$. So $t \not\downarrow_{t'}^H ut'$, a contradiction.*

- (2) *If T is one-based and $T_{x=x}^{\text{ind}}$ is axiomatizable, then T is trivial iff $T_{x=x}^{\text{ind}}$ is one-based.*
(3) *For any partial type $\Pi(x)$ over \emptyset , if $T_{\Pi(x)}^{\text{ind}}$ is axiomatizable, T is n -ample iff $T_{\Pi(x)}^{\text{ind}}$ is n -ample for any $n \geq 2$.*

Let $\text{SU}(T) = \omega^\alpha$. Let $\text{cl}(A) := \{x \in \mathcal{M} : \text{SU}(x/A) < \omega^\alpha\}$. We say that $\text{cl}(\ast)$ is trivial if $\text{cl}(A) = \bigcup_{a \in A} \text{cl}(\{a\})$ for any $A \subset \mathcal{M}$.

Fact 8.3. *Suppose that SU -rank is continuous. Let $\Pi(x)$ be the union of all the types over \emptyset of $\text{SU} = \omega^\alpha$. Assume that T_{Π}^{ind} is axiomatizable and T is one-based. Then $\text{cl}(\ast)$ on \mathcal{M} is trivial iff T_{Π}^{ind} is one-based.*

9. TRIVIALITY AND T^{ind}

This section is due to [BV3].

Fact 9.1. (1) *If T is strongly minimal and trivial, then $\text{RM}_H(T^{\text{ind}}) \leq 2$.*

- (2) *If T is strongly minimal and non-trivial, then $\text{RM}_H(T^{\text{ind}}) = \omega$.*
(3) *If $\text{SU}(T) = 1$ and trivial, then $\text{SU}_H(T^{\text{ind}}) = 1$.*
(4) *If $\text{SU}(T) = 1$ and non-trivial, then $\text{SU}_H(T^{\text{ind}}) = \omega$.*
(5) *If $\text{UP}(T) = 1$ and trivial, then $\text{UP}_H^p(T^{\text{ind}}) = 1$.*
(6) *If $\text{UP}(T) = 1$ and non-trivial, then $\text{UP}_H^p(T^{\text{ind}}) = \omega$.*

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