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Effort Allocations in Elimination Tournaments with Different Fatigue Parameters for Each Stage[1)]

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Abstract

In this paper, we study an elimination tournament with n stages by 2^n symmetric players, generalizing Sela's [2022] model. The probability that a player will win in a race against an opponent is determined by the amount of effort invested by each player (Tullock 1980). Each player has a common amount of effort budget: the budget constraint for a given stage is the budget constraint for the preceding stage, discounted by the fatigue parameter specific to each stage of the effort spent in that stage. We analyze the optimal allocation of effort for each player, and show that the optimal allocation of effort at a stage maximizes the expected payoff independent of the fatigue parameters at earlier stages and dependent only on the budget constraint at that stage. Also, if there is regularity in the fatigue parameters, ensure that the fatigue parameters should always be constant in order to make the tournament more exciting.

I Introduction

An elimination tournament is a contest in which competitors (or competing teams) compete against each other at each stage, with only the winner advancing to the next stage and the losers being eliminated from the contest, repeatedly to determine a single winner of the tournament. In reality, this method is used to determine winners in a variety of situations, from sports such as martial arts and soccer to board games such as chess and shogi. As another example in economic activity, Rosenbaum [1984] noted that personnel data from 1962 to 1975 of one large company in the U. S. showed that employee promotion patterns resembled an elimination tournament. In other words, if an employee is not promoted at each selection stage, there is little chance of promotion beyond that stage, but being promoted at each stage does not necessarily mean that the employee will continue to be promoted thereafter. Such promotion patterns are called promotion tournaments and are the subject of analysis in contract theory (see, e.g., DeVaro and Gürtler [2020]).

The Tullock contest is a model proposed in Tullock [1980]. In this model, the probability that player i will win against player j can be expressed as $\frac{x^i}{x^i+x^j}$ where x^i is the amount of effort expended

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by player i in a stage. Sela [2022] presents a Tullock model in which all players have a common fatigue parameter α for all stages in an elimination tournament. There, it is shown that if α is larger than $\frac{1}{n-t+1}$, the players distribute their efforts equally in the first t stages when $t \leq n-1$. In particular, if α is greater than $\frac{1}{2}$, they distribute their efforts equally in the first $n-1$ stages and consume all remaining effort in the last stage, which is lower. The intuitive explanation for this result is that if the fatigue parameter is small, the budget to be passed on to the next period is large, so effort is consumed up to the budget constraint, but if the fatigue parameter is large, the budget for the next period is small, so consumption is tried to be conserved.

In reality, however, the value of α may change due to changes in the degree of tension and the accumulation of fatigue at different stages. In such a case, what is the impact of the different α values on players' decision making at each stage? To answer this question, this paper extends Sela [2022] model by finding a subgame-perfect equilibrium when the fatigue parameter in stage t is set to α_t.

The structure of this paper is as follows. First, Section 2 presents a contest model of an elimination tournament with different fatigue parameters for each stage. Then, in Section 3, we derive the subgame perfect equilibrium by backward induction. Section 4 also discusses the case of regularity in fatigue parameters, from the tournament organizer's perspective. Finally, concluding remarks are presented in Section 5.

Ⅱ The Model

Consider a case where 2^n players compete in elimination tournament with n stages to determine a single eventual winner. Assume that all players have an equal gain normalized to 1 for winning the tournament. Also, all players have an equal amount of effort budget constraint, which is denoted by $v > 0$ in the first stage. Player i consumes x_1^i effort from v in the first stage, and then the budget constraint in stage $t+1$ for $1 \leq t \leq n-1$ is expressed as $v_{t+1}^i = v_t^i - \alpha_t x_t^i$ where $\alpha_t \in (0, 1]$ is a fatigue parameter. The probability that player i will win against player j can be expressed as $\frac{x^i}{x^i+x^j}$ as in the standard Tullock contest. Note that the opportunity cost of effort is not considered. Because of this assumption, players use up all their remaining budget in the final stage since no effort can be spent on any objects other than this tournament.

Ⅲ The Equilibrium Analysis

Under the setting above, the following proposition holds.

Proposition 1. *In an elimination tournament with n stages and symmetric 2^n players, where the players have the common fatigue parameters with each stage, the optimal effort exertion for each player x_t is $\frac{v-\sum_{i=1}^{t-1}\alpha_i x_i}{(n-t+1)\alpha_t}$ for any $t(=1, 2, \cdots, n-1)$ if $\alpha_t \geq \frac{1}{n-t+1}$ is satisfied; and $x_t = v - \sum_{i=1}^{t-1}\alpha_i x_i$, otherwise.*

To prove this proposition, we consider the problem of maximizing the expected payoff of the semifinal stage under the distribution of effort in the final stage and generalize it by backward induction.

Proof. Since opportunity cost is not considered, effort consumption by the finalists in the final stage equals the budget constraint. i.e.,

$$x_n = v - \sum_{i=1}^{n-1} \alpha_i x_i \tag{1}$$

Therefore, the expected payoff is,

$$u_n = \frac{v - \sum_{i=1}^{n-1} \alpha_i x_i}{v - \sum_{i=1}^{n-1} \alpha_i x_i + v - \sum_{i=1}^{n-1} \alpha_i y_i} \tag{2}$$

where x_i is a finalist's effort in i stage for $i = 1, \ldots, n-1$ while y_i is his opponent's.
Hereafter, $v - \sum_{i=1}^{t} \alpha_i x_i$ and $v - \sum_{i=1}^{t} \alpha_i y_i$ denoted as V_x^t and V_y^t, respectively.

If a player wins in the semi-final stage, his expected payoff is represented by (2). Denote the effort of that player's opponent in the semi-final stage by $\hat{y}_{n-1}$. This player chooses x_{n-1} such that the following maximization problem is satisfied:

$$\max_{x_{n-1}} \frac{x_{n-1}}{x_{n-1} + \hat{y}_{n-1}} u_n. \tag{3}$$

Here, the semi-final stage effort y_{n-1} of the opponent in the final stage and the semi-final stage effort $\hat{y}_{n-1}$ of the opponent in the semi-final stage are distinct, but due to the symmetry of the players, we obtain $\hat{y}_{n-1} = y_{n-1}$. Then, the first-order condition (FOC) is:

$$\frac{y_{n-1}}{(x_{n-1} + y_{n-1})^2} \frac{V_x^{n-1}}{V_x^{n-1} + V_y^{n-1}} + \frac{x_{n-1}}{x_{n-1} + y_{n-1}} \frac{-\alpha_{n-1} V_y^{n-1}}{(V_x^{n-1} + V_y^{n-1})^2} = 0$$

$$\Leftrightarrow \frac{1}{(x_{n-1} + y_{n-1})(V_x^{n-1} + V_y^{n-1})} \left(\frac{y_{n-1} V_x^{n-1}}{x_{n-1} + y_{n-1}} - \frac{\alpha_{n-1} x_{n-1} V_y^{n-1}}{V_x^{n-1} + V_y^{n-1}} \right) = 0 \tag{4}$$

From symmetry we have $x_i = y_i$ for $i = 1, \ldots, n-1$. So, by solving

$$\frac{1}{2} \left(v - \sum_{i=1}^{n-1} \alpha_i x_i \right) - \frac{1}{2} \alpha_{n-1} x_{n-1} = 0,$$

we get

$$x_{n-1} = \frac{v - \sum_{i=1}^{n-2} \alpha_i x_i}{2\alpha_{n-1}}. \tag{5}$$

Because of the existence of a budget constraint, this player allocates this amount of effort when (5) can be paid, and provides effort up to the budget constraint when it cannot, so

$$x_{n-1} = \min \left\{ \frac{v - \sum_{i=1}^{n-2} \alpha_i x_i}{2\alpha_{n-1}}, v - \sum_{i=1}^{n-2} \alpha_i x_i \right\}$$

i.e.,

$$x_{n-1} = \begin{cases} \frac{v - \sum_{i=1}^{n-2} \alpha_i x_i}{2\alpha_{n-1}} & \text{for } \alpha_{n-1} \geq \frac{1}{2} \\ v - \sum_{i=1}^{n-2} \alpha_i x_i & \text{for } \alpha_{n-1} < \frac{1}{2} \end{cases} \tag{6}$$

From (6), the expected payoff of the $n-1$ stage in either case is

$$u_{n-1} = \left(\frac{v - \sum_{i=1}^{n-2} \alpha_i x_i}{v - \sum_{i=1}^{n-2} \alpha_i x_i + v - \sum_{i=1}^{n-2} \alpha_i y_i} \right)^2 . \tag{7}$$

Now we generalize what we have stated above by backward induction. That is, we show that the following holds for all t satisfying $1 \leq t \leq n-1$,

$$x_t = \begin{cases} \frac{v - \sum_{i=1}^{t-1} \alpha_i x_i}{(n-t+1)\alpha_t} & \text{for } \alpha_t \geq \frac{1}{n-t+1} \\ v - \sum_{i=1}^{t-1} \alpha_i x_i & \text{for } \alpha_t < \frac{1}{n-t+1} \end{cases} \tag{8}$$

and

$$u_t = \left(\frac{v - \sum_{i=1}^{t-1} \alpha_i x_i}{v - \sum_{i=1}^{t-1} \alpha_i x_i + v - \sum_{i=1}^{t-1} \alpha_i y_i} \right)^{n-t+1} . \tag{9}$$

Under the hypothesis that equalities (8) and (9) hold for a certain t+1, the player's maximization problem in the t-th stage is

$$\max_{x_t} \frac{x_t}{x_t + \hat{y}_t} \left(\frac{v - \sum_{i=1}^{t} \alpha_i x_i}{v - \sum_{i=1}^{t} \alpha_i x_i + v - \sum_{i=1}^{t} \alpha_i y_i} \right)^{n-t} , \tag{10}$$

where the effort of that player's opponent in the t stage is denoted by $\hat{y}_t$. Since symmetry is assumed for all players, $\hat{y}_t = y_t$is valid. Then, the FOC is

$$\begin{aligned} &\frac{y_t}{(x_t+y_t)^2} \left(\frac{V_x^t}{V_x^t + V_y^t} \right)^{n-t} + \frac{x_t}{x_t+y_t}(n-t) \left(\frac{V_x^t}{V_x^t + V_y^t} \right)^{n-t-1} \frac{(-\alpha_t) V_y^t}{(V_x^t + V_y^t)^2} = 0 \\ \Leftrightarrow &\frac{V_x^{t^{n-t-1}}}{(x_t+y_t)\left(\frac{V_x^t}{V_x^t+V_y^t}\right)^{n-t}} \left\{ \frac{y_t V_x^t}{x_t+y_t} - \frac{(n-t)\alpha_t x_t V_y^t}{V_x^t + V_y^t} \right\} = 0. \end{aligned} \tag{11}$$

From symmetry we have $x_i = y_i$for $i = 1, \ldots, t$, so we obtain

$$\frac{1}{2}\left(v - \sum_{i=1}^{t-1} \alpha_i x_i - \alpha_t x_t \right) - \frac{1}{2}(n-t)\alpha_t x_t = 0,$$

which implies that

$$x_t = \frac{v - \sum_{i=1}^{t-1} \alpha_i x_i}{(n-t+1)\alpha_t}. \tag{12}$$

Due to the budget constraint, this player allocates this amount of effort when (12) can be paid, and provides effort up to the budget constraint when it cannot, so (8) follows. Thus, it is shown that (9) holds. □

This result is consistent with Sela [2022]. If the fatigue parameter is different for each stage, the effort allocation in one stage is not affected by the fatigue parameters in subsequent stages. On the other hand, effort allocation in one stage naturally affects budget constraints in subsequent stages. Decisions at that stage are then optimally allocated according to the magnitude of the fatigue parameter at that stage under the given budget constraint. In other words, when α is sufficiently small, effort is consumed using the full budget because the amount of effort returned in the next stage is large, and conversely, when α is sufficiently large, the available effort is conserved. The intuitive explanation for this conclusion is that since past payments cannot be recovered in the present, the players try to maximize their expected payoffs in the future within the given budget constraint at present. Also, if $\alpha_i > \frac{1}{2}$ holds for any i , $x_t = \frac{v}{n\alpha_t}$ holds for any t satisfying $1 \leq i \leq n-1$, and $x_n = \frac{v}{n}$ is consumed in the final stage.

Ⅳ Fatigue Parameters with Regularity

In this section, from the tournament organizer's perspective, we seek conditions for the tournament to be exciting when the fatigue parameters change regularly as the stages progress and when the regularity can be manipulated. Here, "exciting tournament" can be interpreted as players allocating more effort as the tournament stage progresses to the end of the tournament. For example, in the case of a sporting event, a heated match would attract spectators by conserving energy in the early stages and putting in more effort in the latter stages. The monotonically decreasing budget constraint at each stage with respect to t and equation (8) show that the actual effort allocation chosen will always decrease in the later stages. However, if we focus on the players' subjective allocation of effort rather than the objective amount of effort, we may be able to arrange an "exciting tournament" by having the players allocate their efforts to the full budget constraint over the later stages by manipulating the fatigue parameters.

Assume that the fatigue parameter is expressed as follows:

$$\alpha_t = k_0(t-1) + b \tag{13}$$

where k_0 is the sensitivity of α_t to t, and b satisfies $0 < b \leq 1$. The meaning of equation (8) now is that the smaller of V_x^{t-1} and $\frac{V_x^{t-1}}{(n-t+1)\alpha_t}$ is allocated as the amount of effort in stage t. Thus, as mentioned earlier, the conditions for preserving the budget in the early stages and striving for the full budget at the end of the stage are that a certain integer j exists and the following is satisfied:

$$\begin{cases} k_0(t-1) + b > \frac{1}{n-t+1} & \text{for } 1 \leq t \leq j \\ k_0(t-1) + b < \frac{1}{n-t+1} & \text{for } j+1 \leq t \leq n-1 \end{cases} \tag{14}$$

As illustrated in Figure 1, when n is relatively small, equation (14) allows b to be set so that the intersection can be between j and $j+1$, even if k_0 is large. However, for large n, $\frac{1}{n-t+1}$ hardly increases with increasing t, so it flattens out and k_0 must be very close to zero to have an intersection between j and $j+1$. Furthermore, given that tournament organizers need to successfully control j such that equation (14) is satisfied, it is still desirable for k_0 to be close to zero.

Figures 1 and 2 both illustrate only the case where k_0 is positive, but the same argument can be applied where k_0 is negative. That is, when n is large, if k_0 is small and one tries to draw a line so that there is an intersection between j and $j+1$, b will be larger than 1, which violates the

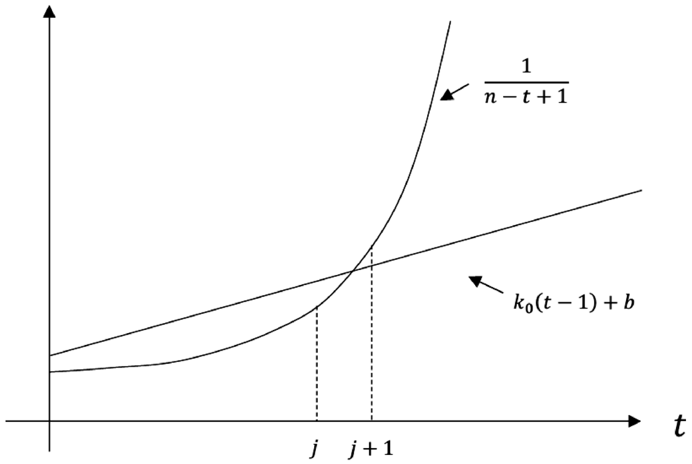

Figure 1 Equation (14) with n relatively small.

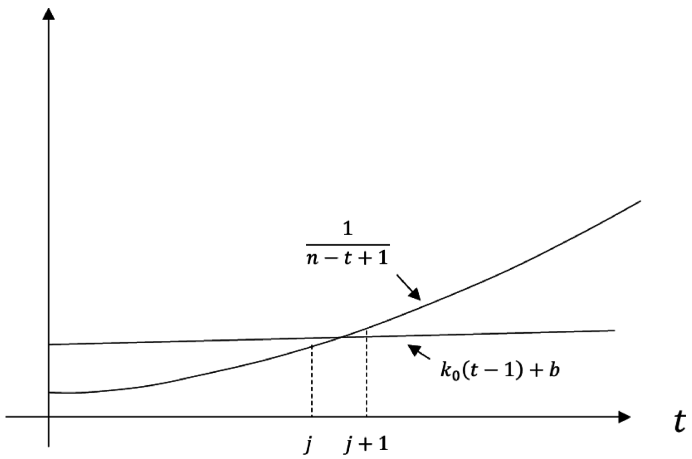

Figure 2 Equation (14) with n relatively large.

condition. Therefore, we know that k_0 should be close to zero regardless of its positive or negative value, and from this we obtain the following proposition.

Proposition 2. *When organizers try to make a tournament exciting, they should set the fatigue parameter to be a constant that does not depend on t.*

The implication of this proposition is that it is preferable for the organizers to avoid fatigue differences between competitions within a tournament. The conclusion from this model is consistent with the design of many real-life tournament games, even though their primary purpose may be otherwise. For example, the World Baseball Classic has restrictions on the number of pitches and pitching intervals, and the tournament matches played at Bellator MMA are evenly spaced.

V Conclusion

This paper is an attempt to extend Sela [2022] by introducing different fatigue parameters at each stage. Here, we make some comments on the parts that could be improved for a more realistic analysis. First, the setting on the number of players could be generalized. In this paper, we follow Sela [2022] and assume 2^n players, but this situation is not always the case in actual sports competitions. If a player has no opponents at a given stage, he or she is given a seeding, but in most cases, it is the strong players who are given the seeding. Our current setting is not aligned with the reality of the situation in which the stronger players can save their effort budgets by seeding.

Second, the symmetry assumption could be relaxed. In a real contest, it is unlikely that all players are homogeneous. Both "strong" and "weak" players are likely to participate, and these differences may manifest themselves in the size of budget constraints and the probability of the winner determination. Finally, the fatigue parameters might be better if they do not depend solely on the number of stages. In reality, they are more likely to accumulate after a series of races, so the fatigue parameter at a given stage may be an endogenous variable determined by the allocation of effort in the previous stages. Overcoming these problems is important for economic analysis to get closer to reality, and will be an issue to be addressed in the future.

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