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Phase plane analysis for p-ultradiscrete system: infinite types of branching conditions

By

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Abstract

The p-ultradiscrete procedure (ultradiscretization with parity variables) enables to ultradiscretize an equation with subtraction. Its feature is that a solution may have an infinite branches under certain conditions. Recently, an "approximative" technique by which such infinite branches may reduce to finite ones is proposed. In this article, a complicated situation for solutions of the p-ultradiscrete hard-spring equation is investigated, in which an infinite types of branching conditions appear. The approximative technique fairly summarizes the solutions and extends an understanding of the structure of solutions.

§ 1. Introduction

Ultradiscretization (UD) is a limiting procedure which transforms a given difference equation into a piecewise linear equation [1]. If we write a dependent variable of the given equation as x_n, we first replace it by

$$x_n = e^{X_n/\varepsilon}, \tag{1.1}$$

where X_n is a new dependent variable and $\varepsilon > 0$ is a parameter. Then, we apply $\varepsilon \log$ to both sides of the equation and take the limit $\varepsilon \to +0$. By using a key formula

$$\lim_{\varepsilon \to +0} \varepsilon \log \left(e^{\frac{X}{\varepsilon}} + e^{\frac{Y}{\varepsilon}} \right) = \max(X, Y), \tag{1.2}$$

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addition, multiplication, and division for x_n are replaced with max operation, addition, and subtraction for X_n, respectively. Namely, the resulting piecewise linear equation is written in max-plus algebra [2]. If we assign integer values to initial values and system parameters for the resulting equation, its dependent variable takes integer values only. Therefore, it can be regarded as time evolution rule of a cellular automaton [3]. Hence, UD may be a procedure which transforms a difference equation into a cellular automaton. The procedure of UD has been applied to many difference equation and its solutions and the constructed cellular automata have been studied. Also, various applications have been reported (for example, [4]–[14]). From this perspective, UD acts as a mediator between continuous (differential or difference equation) and discrete (cellular automaton) mathematics. Since cellular automata have good compatibility with digital computers, UD may be a field with high future growth potential.

However, UD has a strong restriction called "negative difficulty." That is, a given difference equation must be subtraction-free for taking the limit (1.2). Solutions must be positive for applying (1.1). Some attempts have been reported to solve this issue [15]–[18]. The UD with parity variables (pUD, p-ultradiscretization) is one of such attempts [19]. In this method, the "parity (sign)" and "amplitude" variables are introduced. The method enables us to treat a difference equation with subtraction or negative-valued solutions. Its review will be given in the next section. This method has been applied some equations and studied sequentially unlike other ones (for example, [20], [21]). In this meaning, pUD is one of the active methods to aiming to solve the negative difficulty.

A feature of pUD is appearance of an "indeterminate solution." That is, uniqueness of solution may be lost in a specific situation, in exchange for handling subtraction in a equation or negative values for its solution. For example, the successive solution X_{n+1} is just restricted by an inequality such as $X_{n+1} \leq F(X_n, X_{n-1})$, which can be take an arbitrary value as long as it satisfies this inequality. As a result, the solution has an infinite number of branches at indeterminate step. As might be expected, such indeterminacy makes analysis of a solution difficult. On the other hand, it is necessary for capturing all solutions of the p-ultradiscrete system. For example, the p-ultradiscrete analog of the Airy function is constructed through a specific choice of indeterminate solutions [22]. For this usefulness, we cannot discard indeterminate solutions at all. Hence, it is an important problem how to express the indeterminate solutions. Recently, a new approximative expression for indeterminate solutions casts light on this problem in [23]. In this method, a kind of coarse graining is introduced. That is, a p-ultradiscrete system is reinterpreted as the mapping which maps a set on the phase plane to the other. By this method, an infinite number of branches can be reduced to a *finite* number of branches, and indeterminate solutions are efficiently understood than before. In this article, we shall report additional contents for [24], which gives another application of the approximative method to p-ultradiscrete analog of the hard-spring equation [25].

The remainder of this article is organized as follows. In Section 2, we review the hard-spring equation and its p-ultradiscrete analog. We express the p-ultradiscrete equation in the conditional explicit forms which include indeterminate cases. Then, in Section 3, we illustrates the transition of the amplitude on the phase plane. A complicated situation is mainly discussed, which was omitted in [24]. Although the indeterminate solutions still have an infinite number of branches after approximative expression, its complexness is relieved. Finally, we give concluding remarks in Section 4.

§ 2. Ultradiscrete hard-spring equation with parity variables

We consider the hard-spring equation

$$\frac{d^2x}{dt^2} + kx + lx^3 = 0, \tag{2.1}$$

where $x = x(t)$ and $k, l > 0$ are constants. This equation has the conserved quantity

$$H(t) = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 + \frac{1}{2}lx^4 = 0. \tag{2.2}$$

Therefore, (2.1) is an integrable equation. Its integrable discrete analog is presented in the Japanese book [26] as

$$x_{n+1} - 2x_n + x_{n-1} + 2c_1\delta^2\left(x_{n+1} + x_{n-1}\right) + 4c_2\delta^2 x_n + 2c_3\delta^2 x_n^2\left(x_{n+1} + x_{n-1}\right) = 0. \tag{2.3}$$

This difference equation (2.3) is also integrable because it has the conserved quantity

$$H_n = \frac{{x_n}^2 - 2x_n x_{n-1} + {x_{n-1}}^2}{2\delta^2} + c_1({x_n}^2 + {x_{n-1}}^2) + 2c_2 x_n x_{n-1} + c_3 {x_n}^2 {x_{n-1}}^2. \tag{2.4}$$

See [26] for details.

We shall ultradiscretize (2.3) with parity variables. We introduce the sign variable ξ_n and the "amplitude" variable X_n for x_n by

$$\xi_n = \frac{x_n}{|x_n|}, \qquad \mathrm{e}^{\frac{X_n}{\varepsilon}} = |x_n|. \tag{2.5}$$

We define a function s as

$$s(\xi) = \begin{cases} 1 & (\xi = +1) \\ 0 & (\xi = -1), \end{cases} \tag{2.6}$$

and we replace x_n with

$$x_n = \xi_n \mathrm{e}^{\frac{X_n}{\varepsilon}} = \big(s(\xi_n) - s(-\xi_n)\big)\mathrm{e}^{\frac{X_n}{\varepsilon}}. \tag{2.7}$$

We transform the positive parameters c_i and δ by

$$\mathrm{e}^{\frac{\alpha_i}{\varepsilon}} = c_i \ (i = 1, 2, 3), \qquad \mathrm{e}^{\frac{\Delta}{\varepsilon}} = \delta. \tag{2.8}$$

If we substitute (2.7) and (2.8) into (2.3), we have

$$\begin{aligned}
&\big(s(\xi_{n+1}) - s(-\xi_{n+1})\big)\mathrm{e}^{\frac{X_{n+1}}{\varepsilon}} - 2\big(s(\xi_n) - s(-\xi_n)\big)\mathrm{e}^{\frac{X_n}{\varepsilon}} + \big(s(\xi_{n-1}) - s(-\xi_{n-1})\big)\mathrm{e}^{\frac{X_{n-1}}{\varepsilon}}\\
&\quad + 2\mathrm{e}^{\frac{\alpha_1+2\Delta}{\varepsilon}}\left\{\big(s(\xi_{n+1}) - s(-\xi_{n+1})\big)\mathrm{e}^{\frac{X_{n+1}}{\varepsilon}} + \big(s(\xi_{n-1}) - s(-\xi_{n-1})\big)\mathrm{e}^{\frac{X_{n-1}}{\varepsilon}}\right\}\\
&\quad + 4\mathrm{e}^{\frac{\alpha_2+2\Delta}{\varepsilon}} 2\big(s(\xi_n) - s(-\xi_n)\big)\mathrm{e}^{\frac{X_n}{\varepsilon}}\\
&\quad + 2\mathrm{e}^{\frac{\alpha_3+2\Delta+2X_n}{\varepsilon}}\left\{\big(s(\xi_{n+1}) - s(-\xi_{n+1})\big)\mathrm{e}^{\frac{X_{n+1}}{\varepsilon}} + \big(s(\xi_{n-1}) - s(-\xi_{n-1})\big)\mathrm{e}^{\frac{X_{n-1}}{\varepsilon}}\right\} = 0.
\end{aligned} \tag{2.9}$$

For simplicity of notation, we write X_{n+1}, X_n, X_{n-1} as X_+, X, X_-, respectively, and put $\alpha_i + 2\Delta = \widehat{\alpha_i}$. Then, we move the negative terms to the other side of the equation and take the limit $\varepsilon \to +0$. Here, we shall utilize a formula

$$\lim_{\varepsilon\to+0} \varepsilon \log\left(s(\xi)\mathrm{e}^{\frac{A}{\varepsilon}} + \mathrm{e}^{\frac{B}{\varepsilon}}\right) = \max(S(\xi) + A, B), \tag{2.10}$$

where a function S is defined by

$$S(\xi) = \begin{cases} 0 & (\xi = +1) \\ -\infty & (\xi = -1). \end{cases} \tag{2.11}$$

The resulting equation

$$\begin{aligned}
&\max\big[S(\xi_{n+1}) + X_+,\ S(\xi_{n+1}) + \widehat{\alpha_1} + X_+,\ S(\xi_{n+1}) + \widehat{\alpha_3} + 2X + X_+,\\
&\qquad S(-\xi_n) + X,\ S(\xi_n) + \widehat{\alpha_2} + X,\\
&\qquad S(\xi_{n-1}) + X_-,\ S(\xi_{n-1}) + \widehat{\alpha_1} + X_-, S(\xi_{n-1}) + \widehat{\alpha_3} + 2X + X_-\big]\\
&= \max\big[S(-\xi_{n+1}) + X_+,\ S(-\xi_{n+1}) + \widehat{\alpha_1} + X_+,\ S(-\xi_{n+1}) + \widehat{\alpha_3} + 2X + X_+,\\
&\qquad S(\xi_n) + X,\ S(-\xi_n) + \widehat{\alpha_2} + X,\\
&\qquad S(-\xi_{n-1}) + X_-,\ S(-\xi_{n-1}) + \widehat{\alpha_1} + X_-,\ S(-\xi_{n-1}) + \widehat{\alpha_3} + 2X + X_-\big]
\end{aligned} \tag{2.12}$$

has the implicit form of $\max[\dots] = \max[\dots]$. We shall rewrite this form into explicit forms with introducing some cases. Here, we consider only the case $\widehat{\alpha_2} > 0$, which we are focusing in this article, and give the result only (See [24] for details). We introduce notations

$$a = \max\big[0,\ \widehat{\alpha_1},\ \widehat{\alpha_3} + 2X\big], \tag{2.13}$$

which includes the dependent variable X, and

$$\begin{cases} \mathrm{C}_g : X + \widehat{\alpha_2} > X_- + a \\ \mathrm{C}_e : X + \widehat{\alpha_2} = X_- + a \\ \mathrm{C}_\ell : X + \widehat{\alpha_2} < X_- + a, \end{cases} \tag{2.14}$$

which are used to describe the condition for each case. These notations are convenient to illustrate the cases for the amplitude on X_-v.s. X plane (See Figure 1). The amplitude of the next step X_+ is calculated from ξ_{n-1}, ξ_n, X_{n-1} and X_n as follows.
(i) $\xi_{n-1} = \xi_n = \xi_{n+1}$: a solution does not exist.
(ii) $\xi_{n-1} = \xi_n = -\xi_{n+1}$: we have the following amplitude.

$$X_+ = \begin{cases} X + \widehat{\alpha_2} - a & (\mathrm{C}_g) \\ X_- & (\mathrm{C}_e) \\ X_- & (\mathrm{C}_\ell) \end{cases} \tag{2.15}$$

(iii) $-\xi_{n-1} = \xi_n = \xi_{n+1}$: we have the following amplitude including indeterminate solutions.

$$X_+ \leq X_- \quad (\mathrm{C}_e) \tag{2.16}$$

$$X_+ = X_- \quad (\mathrm{C}_\ell) \tag{2.17}$$

(iv) $\xi_{n-1} = -\xi_n = \xi_{n+1}$: we have the following amplitude including indeterminate solutions.

$$X_+ = X + \widehat{\alpha_2} - a \qquad (\mathrm{C}_g) \tag{2.18}$$

$$X_+ \leq X_- \qquad (\mathrm{C}_e) \tag{2.19}$$

We comment that we need further cases to obtain the expression without max in a:

$$a = \begin{cases} \max\left[0, \widehat{\alpha_1}\right] & \left(X < \frac{\max\left[0,\widehat{\alpha_1}\right] - \widehat{\alpha_3}}{2}\right) \\ \max\left[0, \widehat{\alpha_1}\right] = \widehat{\alpha_3} + 2X & \left(X = \frac{\max\left[0,\widehat{\alpha_1}\right] - \widehat{\alpha_3}}{2}\right) \\ \widehat{\alpha_3} + 2X & \left(X > \frac{\max\left[0,\widehat{\alpha_1}\right] - \widehat{\alpha_3}}{2}\right). \end{cases} \tag{2.20}$$

Because of this, $X_+ = X + \widehat{\alpha_2} - a$ is not a line but a polygonal line on the phase plane:

$$X_+ = \begin{cases} X + \widehat{\alpha_2} - \max\left[0, \widehat{\alpha_1}\right] & \left(X < \frac{\max\left[0,\widehat{\alpha_1}\right] - \widehat{\alpha_3}}{2}\right) \\ -\frac{\max\left[0,\widehat{\alpha_1}\right] + \widehat{\alpha_3}}{2} + \widehat{\alpha_2} & \left(X = \frac{\max\left[0,\widehat{\alpha_1}\right] - \widehat{\alpha_3}}{2}\right) \\ -X + \widehat{\alpha_2} - \widehat{\alpha_3} & \left(X > \frac{\max\left[0,\widehat{\alpha_1}\right] - \widehat{\alpha_3}}{2}\right). \end{cases} \tag{2.21}$$

Based on this evolution rule, we analyze solutions on the phase plane for the amplitude. Each condition appears on the phase plane as a domain, for example, $\{(X_-, X) \mid \mathrm{C}_g\}$ (see Figure 1 (a)). Note that the polygonal line which gives the boundary draws the shape ">." The coordinate of the indifferentiable point $P1$ is

$$P1\left(-\frac{\max[0,\widehat{\alpha_1}]+\widehat{\alpha_3}}{2}+\widehat{\alpha_2},\ \frac{\max[0,\widehat{\alpha_1}]-\widehat{\alpha_3}}{2}\right). \tag{2.22}$$

Then, we calculate X_+ and examine the mapped domain $\{(X, X_+)\}$. Here, mirror-image domains about the line $X = X_-$ become important on the X v.s. X_+ plane. Such domains are shown in Figure 1 (b). The boundary is observed as the polygonal line of the shape "∧" whose indifferentiable point $P2$ is

$$P2\left(\frac{\max[0,\widehat{\alpha_1}]-\widehat{\alpha_3}}{2},-\frac{\max[0,\widehat{\alpha_1}]+\widehat{\alpha_3}}{2}+\widehat{\alpha_2}\right). \tag{2.23}$$

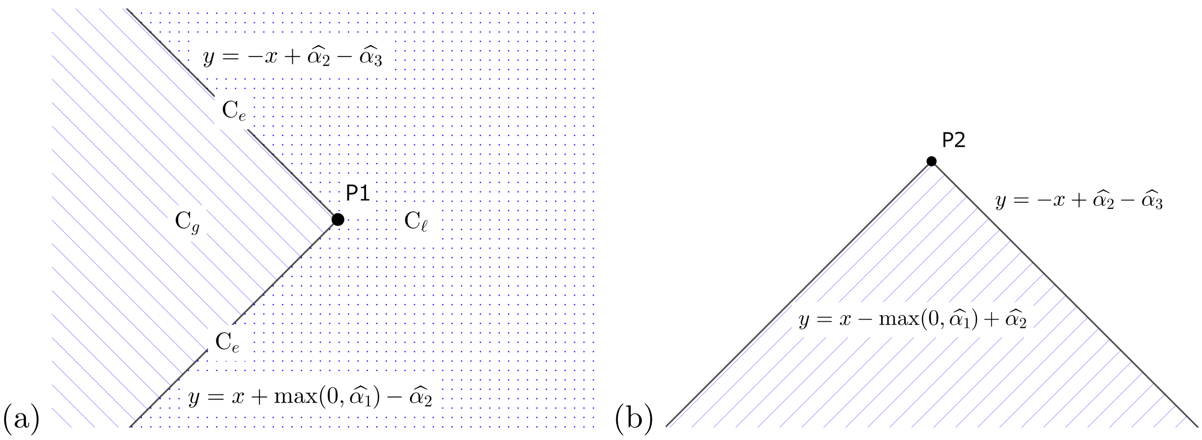

Figure 1. (a) conditions C_g, C_e, C_ℓ and domains (b) mirror-image domains

By the result in [24], the positional relationship between ">" and "∧" is quite important to classify the behavior of solutions. In this article, we focus on the "overlapping" type of relationship as shown in Figure 2. It was found in [24] that this positional type appears if and only if $\widehat{\alpha_2} > 0$ and $\widehat{\alpha_2} > \widehat{\alpha_1}$.

Note that we discuss only the amplitude variables on the above phase plane. For including information of the sign variables, in [24], four amplitude phase planes which correspond to the four pairs of signs $(\xi_n, \xi_{n+1}) = (+,+), (+,-), (-,+), (-,-)$, respectively, were introduced. If we use the $(\xi_n e^{X_n})$ v.s. $(\xi_{n+1} e^{X_{n+1}})$ coordinates, and place the four planes on the corresponding quadrants (See Fig. 8 in [24]), we can discuss the global dynamics of the solutions. However, in this article, we mainly discuss the amplitude phase plane, because the behavior of solutions for "overlapping" type becomes rather complicated on the $(\xi_n e^{X_n})$ v.s. $(\xi_{n+1} e^{X_{n+1}})$ plane.

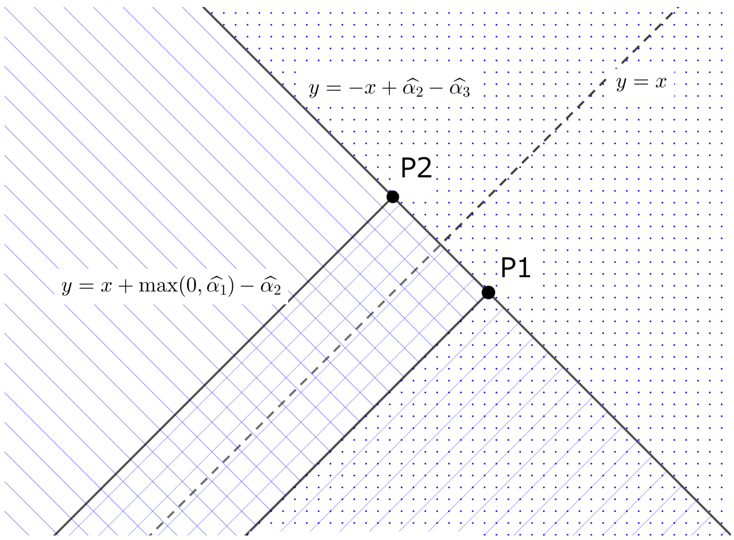

Figure 2. "overlapping" type of domains

§ 3. Transition on the amplitude phase plane

§ 3.1. Notation

For explanation in this section, we define some objects on the amplitude phase plane. We shall use (x, y) as the generic coordinate for the phase plane. We denote the midpoint between $P1$ and $P2$ as $P0$, whose coordinate is

$$P0\left(\frac{\widehat{\alpha_2}-\widehat{\alpha_3}}{2},\frac{\widehat{\alpha_2}-\widehat{\alpha_3}}{2}\right). \tag{3.1}$$

We define points $P(2k-1)$ $(k=1,2,3,\dots)$ whose coordinate $(P(2k-1)_x, P(2k-1)_y)$ is given by

$$P(2k-1)_x=\frac{\max[0,\widehat{\alpha_1}]-\widehat{\alpha_2}}{2}(k-1)-\frac{\max[0,\widehat{\alpha_1}]+\widehat{\alpha_3}}{2}+\widehat{\alpha_2}, \tag{3.2}$$

$$P(2k-1)_y=\frac{\max[0,\widehat{\alpha_1}]-\widehat{\alpha_2}}{2}(k-1)+\frac{\max[0,\widehat{\alpha_1}]-\widehat{\alpha_3}}{2} \tag{3.3}$$

and $P(2k)$ $(k=1,2,3,\dots)$ whose coordinate is $(P(2k-1)_y, P(2k-1)_x)$. Note that $\{P(2k-1)\}$ and $\{P(2k)\}$ are on $L4$ and $L5$ (defined by (3.15) and (3.14)), respectively, and that $P(2k)$ and $P(2k-1)$ for a fixed k are symmetric about $y=x$. Note that we may omit the round brackets for P if we substitute a specific value to k. For example, $P(2k-1)\big|_{k=1}=P1$, $P(2k)\big|_{k=1}=P2$, which are consistent with (2.22), (2.23), respectively.

We further define "open" half lines

$$L_{P(2k-1)} = \{(x, y) \mid x = P(2k-1)_x,\ y < P(2k-1)_y\}\,, \tag{3.4}$$

$$L_{P(2k)} = \{(x, y) \mid x < P(2k)_x,\ y = P(2k)_y\}\,. \tag{3.5}$$

Here, "open" means that each half line does not include its end point. Moreover, we define open segments

$$P(2k-1)P(2k+1) = \{(x, y) \in L4 \mid P(2k+1)_x < x < P(2k-1)_x\}\,, \tag{3.6}$$

$$P(2k)P(2k+2) = \{(x, y) \in L5 \mid P(2k+2)_x < x < P(2k)_x\}\,. \tag{3.7}$$

These objects are illustrated in Figure 3.

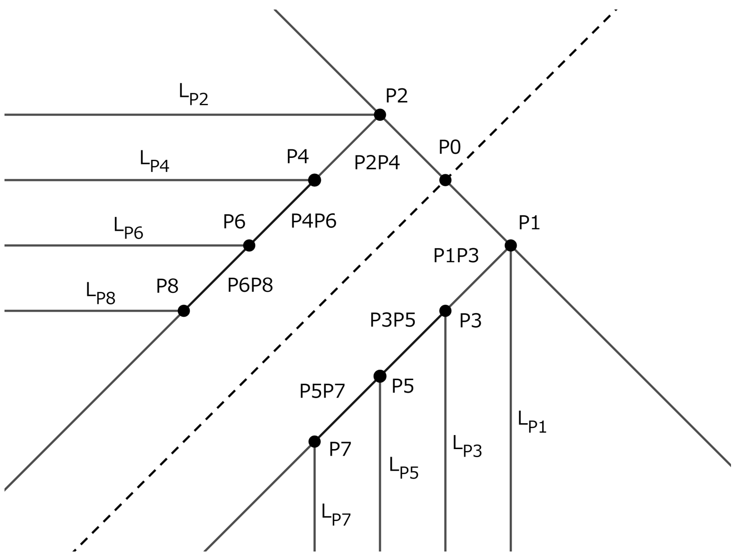

Figure 3. Points, half lines, and "open segments"

Finally, we define the following domains.

$$S1 = \{(x, y) \mid y > x,\ y > -x + \widehat{\alpha_2} - \widehat{\alpha_3}\} \tag{3.8}$$

$$S2 = \{(x, y) \mid y > P2_y,\ y < -x + \widehat{\alpha_2} - \widehat{\alpha_3}\} \tag{3.9}$$

$$S5 = \{(x, y) \mid x > P1_x,\ y < -x + \widehat{\alpha_2} - \widehat{\alpha_3}\} \tag{3.10}$$

$$S6 = \{(x, y) \mid y < x,\ y > -x + \widehat{\alpha_2} - \widehat{\alpha_3}\} \tag{3.11}$$

$$L1 = \{(x, y) \mid y = x\} \tag{3.12}$$

$$L2 = \{(x, y) \mid y = -x + \widehat{\alpha_2} - \widehat{\alpha_3},\ x < P2_x\} \tag{3.13}$$

$$L5 = \{(x, y) \mid y = x - \max[0, \widehat{\alpha_1}] + \widehat{\alpha_2},\ x < P1_y\} \tag{3.14}$$

$$L4 = \{(x, y) \mid y = x + \max[0, \widehat{\alpha_1}] - \widehat{\alpha_2},\ x < P1_x\} \tag{3.15}$$

$$L7 = \{(x, y) \mid y = -x + \widehat{\alpha_2} - \widehat{\alpha_3},\ x > P1_x\} \tag{3.16}$$

$$P0P1 = \left\{(x, y) \mid y = -x + \widehat{\alpha_2} - \widehat{\alpha_3},\ \frac{\widehat{\alpha_2} - \widehat{\alpha_3}}{2} < x < P1_x\right\} \tag{3.17}$$

$$P0P2 = \left\{(x, y) \mid y = -x + \widehat{\alpha_2} - \widehat{\alpha_3},\ P1_x < x < \frac{\widehat{\alpha_2} - \widehat{\alpha_3}}{2}\right\} \tag{3.18}$$

See Figure 4 in which these domains are illustrated. In the approximative method, we trace the time evolution of the amplitude as transition among these domains, not among points. Its detail is explained in the following subsections.

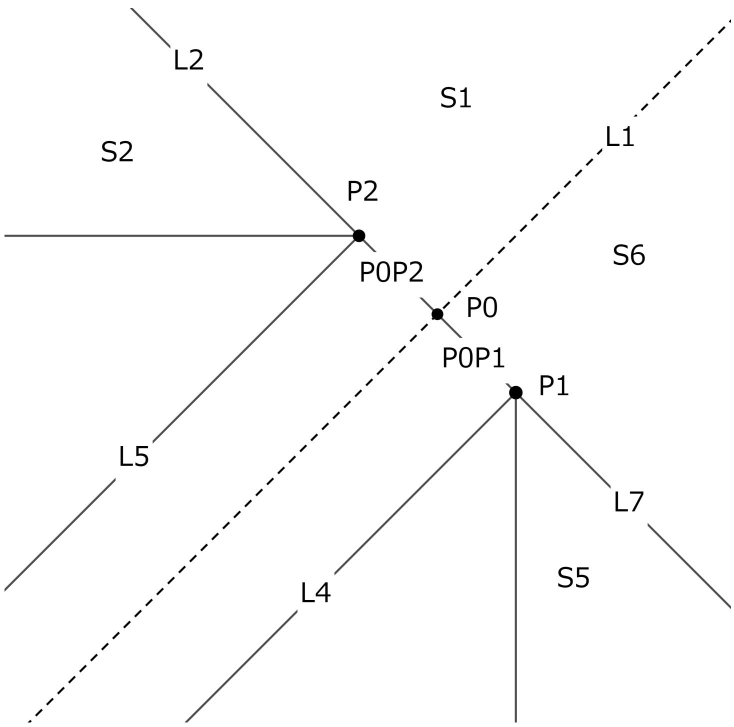

Figure 4. domains

§ 3.2. $\xi_{n-1} = \xi_n = \xi_{n+1}$

This case has no solution in the "overlapping" relationship.

§ 3.3. $\xi_{n-1} = \xi_n = -\xi_{n+1}$

Transition of the amplitude on the X_- v.s. X phase plane is given in Table 3 of [24]. First, we study the case $\mathrm{C}_\ell : X < X_- + a - \widehat{\alpha_2}$ ($a = \max\bigl[0,\ \widehat{\alpha_1},\ \widehat{\alpha_3} + 2X\bigr]$). Such points are on the right of "$>$" (see Figure 1 (a)). In this case, we have $X_{n+1} = X_-$, and therefore the unique mapping $(X_-, X) \mapsto (X, X_-)$, which is the reflection about the line $X = X_-$, is obtained. Therefore, the right of "$>$" is mapped to the upper of "$\wedge$." As specific cases, $L_{P(2k-1)}$ is mapped to $L_{P(2k)}$.

Second, we study the case $\mathrm{C}_e : X = X_- + a - \widehat{\alpha_2}$. Such points are on $L4 \cup L2 \cup P0P1 \cup P0P2 \cup P0 \cup P1 \cup P2$. We again have the unique reflection about the line $X = X_-$. As preparation to discuss other cases, we illustrate the approximative method in terms of (3.6) and (3.7). In this method, the mapping is considered to map a set on the phase plane to the other. By (3.6) and (3.7), $L4 \mapsto L5$ is segmentalized as $P(2k-1)P(2k+1) \mapsto P(2k)P(2k+2)$ ($k = 1, 2, 3, \dots$). We shall need this segmentation in later cases. We also obtained $P(2k-1) \mapsto P(2k)$. Moreover, we obtain $P0P1 \mapsto P0P2$, $P0P2 \mapsto P0P1$, $P1 \mapsto P2$, and $P2 \mapsto P1$, which are bijections. Finally, $P0$ is a fixed point.

Third, we study the case $\mathrm{C}_g : X > X_- + a - \widehat{\alpha_2}$. Such points are on the left of "$>$." In this case, we have $X_{n+1} = X - a + \widehat{\alpha_2}$, and therefore the non-injective mapping $(X_-, X) \mapsto (X, X - a + \widehat{\alpha_2}) \in L5 \cup L7$, which is independent in X_-, is obtained. If we introduce horizontal band domains (see Figure 5)

$$HB_k = \{(x, y) \mid (\tilde{X}, \tilde{Y}) \in P(2k-1)P(2k+1),\ x < \tilde{X},\ y = \tilde{Y}\}, \tag{3.19}$$

any points $(X_-, X) \in HB_k$, that is, $(X_-, \tilde{Y})$ is mapped to the point $(\tilde{Y}, \tilde{Y} - a + \widehat{\alpha_2})$ on $P(2k)P(2k+2)$. This mapping can be visually understood as follows. The point $(X_-, \tilde{Y})$ is temporally mapped to a mirror image about $X = X_-$, and the mirror image is further projected to vertical direction onto $P(2k)P(2k+2)$. We often use this "reflection and projection" hereafter. By the approximative method, we have $HB_k \mapsto P(2k)P(2k+2)$. If we further introduce

$$HB'_1 = \{(x, y) \mid (\tilde{X}, \tilde{Y}) \in P0P1,\ x < \tilde{X},\ y = \tilde{Y}\}, \tag{3.20}$$

we obtain the approximative mapping $HB'_1 \mapsto P0P2$. Similarly, introducing

$$HB'_2 = \{(x, y) \mid (\tilde{X}, \tilde{Y}) \in P0P2,\ x < \tilde{X},\ y = \tilde{Y}\}, \tag{3.21}$$

we obtain $HB'_2 \mapsto P0P1$.

Now, we have traced the image of all points on the X_- v.s. X plane for $\xi_{n-1} = \xi_n = -\xi_{n+1}$.

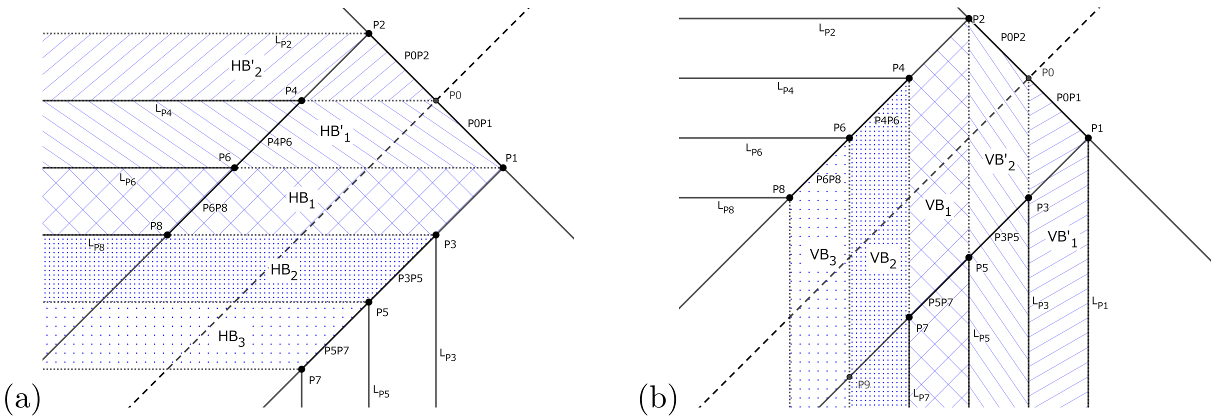

Figure 5. (a) “horizontal band” domains (b) “vertical band” domains

§ 3.4. $-\xi_{n-1} = \xi_n = \xi_{n+1}$

First, we check the case $C_\ell : X < X_- + a - \widehat{\alpha_2}$. As a consequence, we have the same results as the first case in Subsection 3.3.

Second, we consider the case $C_e : X = X_- + a - \widehat{\alpha_2}$. In this case, we have the indeterminate solution $X_+ \leq X_-$. We firstly consider a point on $L4$. If we take a point on a open segment $P(2k-1)P(2k+1)$, it can be mapped on the open segment $P(2k)P(2k+2)$ by reflection and projection or on the vertical-band domains (see Figure 5 (b))

$$VB_k = \{(x, y) \mid (\tilde{X}, \tilde{Y}) \in P(2k)P(2k+2),\ x = \tilde{X},\ y < \tilde{Y}\}. \tag{3.22}$$

By approximative method, $P(2k-1)P(2k+1) \mapsto P(2k)P(2k+2) \cup VB_k$. Secondly, a point on $P0P1$ is mapped on $P0P2$ by reflection and projection or VB'_2 defined by

$$VB'_2 = \{(x, y) \mid (\tilde{X}, \tilde{Y}) \in P0P2,\ x = \tilde{X},\ y < \tilde{Y}\}. \tag{3.23}$$

Points $P0$ and $P1$ are respectively mapped on $P0 \cup P3 \cup P0P3 \cup L_{P3}$ and $P0 \cup P5 \cup P2P5 \cup L_{P5}$, where we define open segments

$$P(2k)P(2k+3) = \{(x, y) \mid x = P(2k)_x,\ P(2k+3)_y < y < P(2k)_y\}, \tag{3.24}$$

and $P0P3$ and $P2P5$ are obtained their specific cases for $k = 0, 1$. Thirdly, a point on $P0P2$ is mapped on $P0P1$ by reflection and projection or VB'_1 defined by

$$VB'_1 = \{(x, y) \mid (\tilde{X}, \tilde{Y}) \in P0P1,\ x = \tilde{X},\ y < \tilde{Y}\}. \tag{3.25}$$

Similarly, a point on $P2$ is mapped on $P1 \cup L_{P1}$. Fourthly, we consider a point on the half line $L2$. It is mapped on $L7$ by reflection and projection or on $S5$.

§ 3.5. $\xi_{n-1} = -\xi_n = \xi_{n+1}$

First, we check the case C_e. We have the same results as the second case in Subsection 3.4.

Second, we check the case C_g. We have the same results as the third case in Subsection 3.3.

§ 3.6. Examples of solutions

In this section, some examples of solutions are illustrated. When $\xi_{n-1} \neq \xi_n$ and $(X_-, X) \in L4 \cup L2 \cup P0P1 \cup P0P2 \cup P0 \cup P1 \cup P2$, we encounter indeterminate solutions. By choosing a possible pair of sign and amplitude, the successive solutions are calculated. However, we may encounter indeterminate solutions again. Its indeterminacy may generally be different from the first one.

We use notation $Z_n = (\xi_{n-1}, \xi_n, X_{n-1}, X_n)$ for representing a pair of successive two points. Here, we assign initial values and study the initial value problem. In the first example, we do not encounter any indeterminate solution. We start from $Z_1 = (+, +, X_0, X_1)$ and $(X_0, X_1) \in S1$. We obtain the following transition:

$$\begin{aligned} Z_1 &\mapsto Z_2 = (+, -, X_1, X_0), (X_1, X_0) \in S6, \\ &\mapsto Z_3 = (-, -, X_0, X_1), (X_0, X_1) \in S1, \\ &\mapsto Z_4 = (-, +, X_1, X_0), (X_1, X_0) \in S6, \\ &\mapsto Z_5 = (+, +, X_0, X_1) = Z_1. \end{aligned}$$

Note that these transitions are unique. In "approximative method" which was proposed in [24], information on a domain in the phase plane to which a pair of amplitude (X_{n-1}, X_n) belongs is more important than values of amplitudes. Therefore, we reinterpret this mapping that from a domain to another one, not from a point to another. For convenience, we introduce further short expression, "$Z_n \in (\xi_{n-1}, \xi_n, domain)$." For example, the above examples are written as

$$\begin{aligned} Z_1 \in (+, +, S1) &\mapsto Z_2 \in (+, -, S6), \\ &\mapsto Z_3 \in (-, -, S1), \\ &\mapsto Z_4 \in (-, +, S6), \\ &\mapsto Z_5 \in (+, +, S1). \end{aligned}$$

In the concept of approximative method, this transition should be expressed as

$$(+, +, S1) \mapsto (+, -, S6) \mapsto (-, -, S1) \mapsto (-, +, S6) \mapsto (+, +, S1) \cdots .$$

We combine use of the original form $Z_n = (\xi_{n-1}, \xi_n, X_{n-1}, X_n)$ and the short form $Z_n \in (\xi_{n-1}, \xi_n, domain)$ for understanding the discussion. Since all evolutions are unique in this example, advantage of this approximation is not clear. However, it is convenient to understand indeterminate solutions as shown in the next example.

> The authors comment that they expressed the information of two signs (ξ_{n-1}, ξ_n) by some shapes in [24]. That is, a pair of signs $(+,+)$ is represented by a white circle, $(+,-)$ by a black circle, $(-,-)$ by a black-painted diamond, and $(-,+)$ by a white-painted diamond. The information of the amplitudes is represented by writing the set which a pair of two amplitudes belongs into the shape. For example, the above Z_1–Z_4 are represented as follows.
>
> 
>
>
> Moreover, by using these shapes, the transition diagrams were drawn in [24].

In the second example, we encounter a infinite number of indeterminate solutions but they are just two types. We start from $Z_1 = (+,-,X_0,X_1)$ and $(X_0,X_1) \in L2$. Note that this initial value belongs to the domain $(+,-,L2)$, and that actually $X_1 = -X_0 + \widehat{\alpha_2} - \widehat{\alpha_3}$ holds. At the next step, we encounter indeterminate solutions. Hence, in the usual sense, an infinite number of pairs of sign and amplitude can be chosen. However, we classify such solutions into four types as

$$Z_2 = (-,+,X_1,X_0),\ (X_1,Y_2) \in L7, \tag{3.26}$$

$$Z_2' = (-,+,X_1,\mathbf{Y_2}),\ (X_1,\mathbf{Y_2}) \in S5\ (\mathbf{Y_2} < -X_1 + \widehat{\alpha_2} - \widehat{\alpha_3}), \tag{3.27}$$

$$Z_2'' = (-,-,X_1,\mathbf{Y_2}),\ (X_1,\mathbf{Y_2}) \in S5, \tag{3.28}$$

$$Z_2''' = (-,-,X_1,X_0),\ (X_1,X_0) \in L7. \tag{3.29}$$

Here, $\mathbf{Y_2}$ denotes a chosen value for X_2 which is less than $-X_1 + \widehat{\alpha_2} - \widehat{\alpha_3}$. Note that $X_2 = -X_1 + \widehat{\alpha_2} - \widehat{\alpha_3} = X_0$ holds in (3.26) and (3.29). We calculate successive evolutions for each type. The results are as follows.
(i) When we choose Z_2,

$$\begin{aligned} Z_2 &\mapsto Z_3 = (+,+,X_0,X_1),\ (X_0,X_1) \in L2,\\ &\mapsto Z_4 = (+,-,X_1,X_0),\ (X_1,X_0) \in L7,\\ &\mapsto Z_5 = (-,-,X_0,X_1),\ (X_0,X_1) \in L2,\\ &\mapsto Z_6 = Z_2. \end{aligned}$$

(ii) When we choose Z_2',

$$\begin{aligned} Z_2' &\mapsto Z_3' = (+,+,\mathbf{Y_2},X_1),\ (\mathbf{Y_2},X_1) \in S2, \\ &\mapsto Z_4' = Z_4, \\ &\mapsto Z_5' = Z_5, \\ &\mapsto Z_6' = Z_6 = Z_2. \end{aligned}$$

(iii) When we choose Z_2'',

$$\begin{aligned} Z_2'' &\mapsto Z_3'' = (-,+,\mathbf{Y_2},X_1),\ (\mathbf{Y_2},X_1) \in S2, \\ &\mapsto Z_4'' = Z_4, \\ &\mapsto Z_5'' = Z_5, \\ &\mapsto Z_6'' = Z_6 = Z_2. \end{aligned}$$

(iv) When we choose Z_2''',

$$\begin{aligned} Z_2''' &\mapsto Z_3''' = (-,+,X_0,X_1), (X_0,X_1) \in L2, \\ &\mapsto Z_4''' \text{ becomes indeterminate.} \end{aligned}$$

We again classify indeterminate solutions into four types as

$$\begin{aligned} \hat{Z}_4 &= (+,+,X_1,X_0),\ (X_1,X_0) \in L7, \\ \hat{Z}_4' &= (+,-,X_1,X_0),\ (X_1,X_0) \in L7, \\ \hat{Z}_4'' &= (+,+,X_1,\hat{\mathbf{Y}}_\mathbf{2}),\ (X_1,\hat{\mathbf{Y}}_\mathbf{2}) \in S5, \\ \hat{Z}_4''' &= (+,-,X_1,\hat{\mathbf{Y}}_\mathbf{2}),\ (X_1,\hat{\mathbf{Y}}_\mathbf{2}) \in S5. \end{aligned}$$

Here, $\hat{\mathbf{Y}}_\mathbf{2}$ denotes a chosen value for X_4 which is less than $-X_1 + \widehat{\alpha_2} - \widehat{\alpha_3}$ and may be different from $\mathbf{Y_2}$. If we choose $Z_4''' = \hat{Z}_4$, we obtain the following evolution:

$$\begin{aligned} Z_4''' &= \hat{Z}_4, \\ &\mapsto Z_5''' = (+,-,X_0,X_1),\ (X_0,X_1) \in L2, \end{aligned}$$

which is identical to the initial value Z_1. For the other cases, we obtain

$$\begin{aligned} Z_4''' &= \hat{Z}_4'(= Z_4), \\ &\mapsto \hat{Z}_5' = Z_5, \\ &\mapsto \hat{Z}_6' = Z_2, \end{aligned}$$

and

$$\begin{aligned} Z_4''' &= \hat{Z}_4'', \\ &\mapsto \hat{Z}_5'' = (+,-,\hat{\mathbf{Y}}_\mathbf{2},X_1),\ (\hat{\mathbf{Y}}_\mathbf{2},X_1) \in S2, \\ &\mapsto \hat{Z}_6'' = Z_2, \end{aligned}$$

and

$$\begin{aligned} Z_4''' &= \hat{Z}_4''', \\ &\mapsto \hat{Z}_5''' = (-, -, \hat{\mathbf{Y}}_\mathbf{2}, X_1), \ (\hat{\mathbf{Y}}_\mathbf{2}, X_1) \in S2, \\ &\mapsto \hat{Z}_6''' = Z_2. \end{aligned}$$

Note that the domains $L2$, $L7$, $S2$, and $S5$ appear with all pair of signs (ξ_{n-1}, ξ_n). This means that these solution-orbits are closed in these domains. The behavior of this solution is summarized by the transition diagram in Figure 6. Four arrows which mean the indeterminate solutions leave from $(+, -, L2)$ and $(-, +, L2)$. Note that, although $\mathbf{Y}_\mathbf{2}$ and $\hat{\mathbf{Y}}_\mathbf{2}$ denote an infinite number of values in usual sense, they are expressed by one domain $S5$ or $S2$.

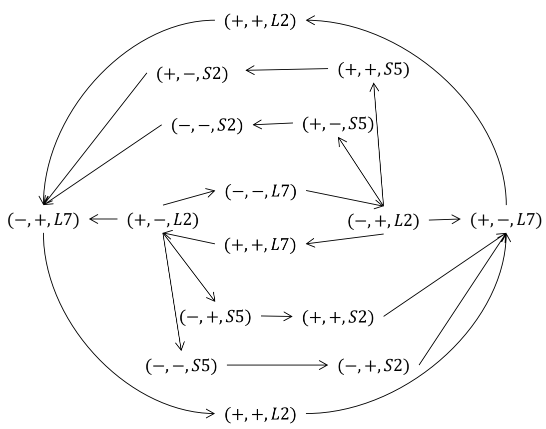

Figure 6. Example of transition diagram

In the third example, we study the most complicated case. We study the transition for the domain on the amplitude phase plane

$$\{(x, y) \mid y < P2_y, \ y \leq -x + \widehat{\alpha_2} - \widehat{\alpha_3}, \ x < P1_x\}, \tag{3.30}$$

whose discussion has been omitted in [24]. Note that we have already introduced some subsets included in (3.30). We first consider the horizontal-band domains defined by (3.19), (3.20), and (3.21). If we start from a point $(X_0, X_1) \in HB_k$, we obtain

$$\begin{aligned} (\pm, \pm, HB_k), (\pm, \mp, HB_k) &\mapsto (\pm, \mp, P(2k)P(2k+2)), (\mp, \pm, P(2k)P(2k+2)) \\ &\mapsto (\mp, \pm, P(2k-4)P(2k-2)), (\pm, \mp, P(2k-4)P(2k-2)) \\ &\mapsto \cdots . \end{aligned}$$

If we put it into words, a horizontal band is mapped to the corresponding open segment, and after this, goes up to every other open segment in order, and reach P2P4 or P4P6.

Note that both HB_1' and $P4P6$ are mapped to $P0P2$, and similarly, both HB_2' and $P2P4$ are mapped to $P0P1$. Moreover, the solution $(\xi_{n-1}, \xi_n, X_{n-1}, X_n)$ on an open segment has different signs, that is, $\xi_{n-1}\xi_n = -1$. Therefore, a point $(X_0, X_1) \in HB_k$ will be eventually included in one of $(\pm, \mp, P0P1)$, $(\pm, \mp, P0P2)$. Then, at the next step, we encounter indeterminate solutions.

Next, we define the "bottom" part of each vertical band by

(3.31)
$$B(VB_k) = \{(x, y) \mid (\tilde{X}, \tilde{Y}) \in P(2k)P(2k+2),\ x = \tilde{X},\ y < \tilde{X} + \max[0, \widehat{\alpha_1}] - \widehat{\alpha_2}\},$$
(3.32)
$$B(VB_1') = \{(x, y) \mid (\tilde{X}, \tilde{Y}) \in P0P1,\ x = \tilde{X},\ y < \tilde{X} + \max[0, \widehat{\alpha_1}] - \widehat{\alpha_2}\},$$
(3.33)
$$B(VB_2') = \{(x, y) \mid (\tilde{X}, \tilde{Y}) \in P0P2,\ x = \tilde{X},\ y < \tilde{X} + \max[0, \widehat{\alpha_1}] - \widehat{\alpha_2}\}.$$

Then, we start from a point $(X_0, X_1) \in B(VB_k)$. We obtain

$$(\pm, \pm, B(VB_k)), (\pm, \mp, B(VB_k)) \mapsto (\pm, \mp, HB_k), (\mp, \pm, HB_k),$$

and $(X_0, X_1) \in B(VB_k')$ $(k = 1, 2)$ are also mapped to the corresponding horizontal bands, respectively. The transition hereafter is reduced to the case of starting from the horizontal band, which we have already discussed.

If we start from a point on the half line $L_{P(2k-1)}$, $L_{P(2k)}$ $(k = 2, 3, 4, \dots)$, the solution shows similar behavior to starting from the next horizontal or vertical band. This result is similar if a starting point is on the (vertical) open segments $P(2k)P(2k+3)$ defined by (3.24) or (horizontal) open segments $P(2k-1)P(2k+4)$ defined by

$$P(2k-1)P(2k+4) = \{(x, y) \mid P(2k+4)_x < x < P(2k-1)_x,\ y = P(2k-1)_y\} \tag{3.34}$$

or the points $P(2k)$ $(k = 2, 3, 4, \dots)$. (To be exact, some segments have intersection points and such points are discussed with overlapping. However, it is not essential to understand the behavior of the solutions.)

The domains $P0P1$, $P0P2$, $P0$, $P1$, $P2$ and $L4$ have not been discussed. We start from $(+, -, P0P2)$ or $(+, -, X_0, X_1)$. The next step becomes indeterminate solutions. All candidates are $(-, \pm, X_1, Y)$ where $Y \le -X_1 + \widehat{\alpha_2} - \widehat{\alpha_3}$. In this example, we choose $(-, +, P1P3)$ (or $(-, +, X_1, X_1 + \max[0, \widehat{\alpha_1}] - \widehat{\alpha_2})$), because this is one of the candidates which have not been discussed yet. Then, we again encounter indeterminate solutions and choose $(+, -, P5P7)$. In this way, we can choose "the open segment two leftward," that is $(\pm, \mp, P(2k-1)P(2k+1)) \mapsto (\mp, \pm, P(2k+3)P(2k+5))$, and encounter new types of indeterminate solutions in infinitely many times (Note that such solutions were classified as "indeterminate type with unbounded diagram" in [25]).

This is a remarkable feature of the "overlapping" positional relationship, since it is not observed in the other positional relationship. If we choose another value as an indeterminate solution, we obtain unique evolution for some steps. However, the solution orbit shall definitely pass $(\pm, \mp, P0P1)$ or $(\pm, \mp, P0P2)$ after several steps and the next step becomes indeterminate solutions. Hence, we cannot avoid re-encountering indeterminate solutions.

§ 4. Concluding Remarks

This article is a continuation of [24] in which the "approximative method [23]" was applied to the p-ultradiscrete hard-spring equation [25]. In general p-ultradiscrete equations, uniqueness of the solution may be lost under certain specific conditions. Namely, indeterminate solutions with an infinite number of branches appear. However, approximative method makes it possible to reduce an infinite number of branches to some finite number of ones. As a result, the behavior of ultradiscrete solutions was summarized by finite size of transition diagrams.

In this article, we have focused on the "overlapping" case, which comes from the positional relationship between two characteristic polygonal lines on the phase plane. Analysis of this case was omitted in [24] because it is rather complicated. If we contrast this case with the results in [25], it corresponds the situation that infinitely many branching conditions which are independent from the initial values appear. In a way, this is a local perspective. Even by using approximative method, it is impossible to draw the transition diagram with finite size. However, the phase plane analysis provides us a global perspective and we have found that the behavior of solutions is rather organized. Since a mechanism of appearance of infinite branching conditions was unclear in [25], this study has moved toward a deeper understanding of the ultradiscrete solutions.

We believe that the approximative method is effective for other p-ultradiscrete systems. It is a future problem to study other p-ultradiscrete equation by means of this technique.

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