Fano 4-folds with nef tangent bundle in positive characteristic

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1. Introduction

The positivity of the tangent bundle imposes strong restrictions on the geometry of varieties (cf. Hartshorne Conjecture [2]). We study Fano variety with positive tangent bundle, in particular study the Campana-Peternell Conjecture.

Campana-Peternell Conjecture [3]

Any complex smooth Fano variety X with nef tangent bundle is a homogeneous variety.

Campana-Peternell conjecture holds for varieties of

- dim$X \leq 5$,
- others (e.g. a toric variety...etc [4]).

Our purpose of this work is to give a classification of Fano 4-folds with nef tangent bundle and Picard number greater than one in positive characteristic [1].

Notation :

T_X : tangent bundle of X, $\mathbb{P}^n$: projective n-space

$\rho(X)$: Picard number, Q^n : quadric hypersurface in $\mathbb{P}^{n+1}$

Main Theorem

Let X be a smooth Fano 4-fold over $k = \overline{k}$. If T_X is nef and $\rho(X) > 1$, then X is isomorphic to one of the following:

(i) $\mathbb{P}^3 \times \mathbb{P}^1$, (ii) $Q^3 \times \mathbb{P}^1$, (iii) $\mathbb{P}^2 \times \mathbb{P}^2$,
(iv) $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$, (v) $\mathbb{P}(T_{\mathbb{P}^2}) \times \mathbb{P}^1$,
(vi) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, (vii) $\mathbb{P}(\mathcal{N})$.

($\mathcal{N}$ is a null-correlation bundle on $\mathbb{P}^3$ cf. [1])

2. Known results

To give a classification of Fano varieties with $\rho(X) > 1$, it is common to study extremal contractions. If T_X is nef, there exists an extremal contraction in positive characteristic.

Existence of contractions in positive characteristic [5]

Let X be a smooth Fano variety over $k = \overline{k}$ (char $k > 0$) with nef T_X. Let $R \subset \mathrm{NE}(X)$ be an extremal ray. Then the contraction $f : X \to Y$ of R exists and the following hold:

(1) f is smooth,
(2) any fiber F of f is a smooth Fano with nef T_F,
(3) Y is a smooth Fano with nef T_Y,
(4) $\rho(X) = \rho(Y) + 1$ and $\rho(F) = 1$.

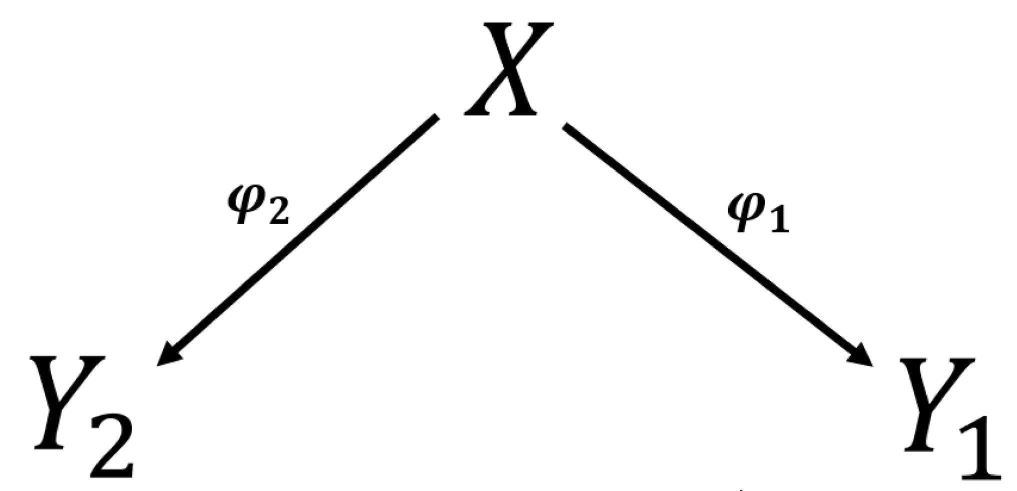
Classification results up to three dimensions [6]

Let X be a smooth Fano n-fold over $k = \overline{k}$ with nef T_X. If $n \leq 3$, then X is isomorphic to one of the following:

(i) $\mathbb{P}^n$, (ii) Q^n, (iii) $\mathbb{P}^2 \times \mathbb{P}^1$, (iv) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, (v) $\mathbb{P}(T_{\mathbb{P}^2})$.

3. Methods

Since $\rho(X) > 1$ and $\mathrm{NE}(X)$ is simplicial, contraction of extremal ray φ_1 and φ_2 exist. By (2) and (3), target Y_i and fiber F_i of φ_i are restricted to classification results up to dimension 3. Moreover, since $\rho(F_i) = 1$, F_i only is $\mathbb{P}^1$, $\mathbb{P}^2$, $\mathbb{P}^3$ or Q^3.



Apply with respect to each case.
$(Y_1, F_1, Y_2, F_2) = (\mathbb{P}^3, \mathbb{P}^1, \mathbb{P}^2, \mathbb{P}^2)$...etc.

To determine the structure of the manifold, it is examined using FT-manifolds.

FT-manifold

Let X be a smooth projective variety with nef T_X. X is an FT-manifold if every extremal contraction of X is a $\mathbb{P}^1$-bundle.

Example of an FT-manifold

$\mathbb{P}^1$, Q^2, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}(T_{\mathbb{P}^2})$ are FT-manifolds.

Structure theorem onto an FT-manifold [1], [7]

Let X be a smooth Fano variety with nef T_X. Assume that $f : X \to Y$ is an extremal contraction onto an FT-manifold Y. Then $X \simeq Y \times Z$. (Z : variety)

According to the above result, all that remains are the cases where a smooth Fano 4-fold X with nef T_X admits a $\mathbb{P}^2$ bundle structure on $\mathbb{P}^2$ or $\mathbb{P}^1$-bundle structure on $\mathbb{P}^3$ and Q^3.

Results of special cases [1, Proposition 3.1, 3.3]

- X admits $\mathbb{P}^2$-bundle on $\mathbb{P}^2 \Rightarrow X \simeq \mathbb{P}^2 \times \mathbb{P}^2$,
- X admits $\mathbb{P}^1$-bundle on $\mathbb{P}^3 \Rightarrow X \simeq \mathbb{P}^1 \times \mathbb{P}^3$ or $\mathbb{P}(\mathcal{N})$.

As a consequence, the only remaining case is that X admits two $\mathbb{P}^1$-bundle structure on Q^3.

Another $\mathbb{P}^1$-bundle structure on Q^3 [1, Proposition 3.4]

Assume that $\varphi_1 : X \to Q^3$ is a $\mathbb{P}^1$-bundle. Then X doesn't admit another $\mathbb{P}^1$-bundle structure on Q^3.

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