Admissible homomorphisms and equivariant relations between weighted projective lines

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- $\mathbf{p} = (p_1, p_2, \cdots, p_t)$: a sequence consisting of positive integers, $p = \text{l.c.m.}(p_1, \cdots, p_t)$.
- $\mathbb{L}(\mathbf{p}) = \langle \vec{x}_1, \vec{x}_2, \cdots, \vec{x}_t | p_1\vec{x}_1 = p_2\vec{x}_2 = \cdots = p_t\vec{x}_t \rangle$.
- $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_t)$: a sequence of pairwise distinct points on the projective line $\mathbb{P}^1_{\mathbf{k}}$, normalized such that $\lambda_1 = \infty, \lambda_2 = 0, \lambda_3 = 1$.
- $S(\mathbf{p}; \boldsymbol{\lambda}) = \mathbf{k}[X_1, X_2, \cdots, X_t]/I$, where the ideal I is generated by $X_i^{p_i} - (X_2^{p_2} - \lambda_i X_1^{p_1})$ for $3 \le i \le t$. $S(\mathbf{p}; \boldsymbol{\lambda})$ is $\mathbb{L}(\mathbf{p})$-graded by means of $\deg X_i = \vec{x}_i$ for $1 \le i \le t$.
- By [Geigle-Lenzing], there has an equivalence

$$\frac{\text{mod}^{\mathbb{L}(\mathbf{p})}\text{-}S(\mathbf{p};\boldsymbol{\lambda})}{\text{mod}_0^{\mathbb{L}(\mathbf{p})}\text{-}S(\mathbf{p};\boldsymbol{\lambda})} \xrightarrow{\sim} \text{coh-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda}).$$

Trichotomy for WPLs

- *dualizing element* of $\mathbb{L}(\mathbf{p})$: $\vec{\omega} = (t-2)\vec{c} - \sum_{i=1}^t \vec{x}_i$;
- a group homomorphism: $\delta\colon \mathbb{L}(\mathbf{p}) \to \mathbb{Z}$; $\quad \vec{x}_i \mapsto \frac{p}{p_i}$.

Then we have the following trichotomy for WPLs according to $\delta(\vec{\omega}) < 0, = 0$ or > 0 respectively:

- domestic: $(), (p), (p_1, p_2), (2,2,p_3), (2,3,3), (2,3,4), (2,3,5)$;
- tubular: $(2,3,6), (3,3,3), (2,4,4), (2,2,2,2)$;
- wild: all other weight types.

Motivation

- $\text{Aut}(\text{coh-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda}))$: the automorphism group of $\text{coh-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda})$;
- $\text{Aut}(\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda}))$: the subgroup of $\text{Aut}(\text{coh-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda}))$ fixing the structure sheaf.

By [Lenzing-Meltzer], there is a split-exact sequence

$$1 \to \mathbb{L}(\mathbf{p}) \to \text{Aut}(\text{coh-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda})) \to \text{Aut}(\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda})) \to 1.$$

According to [Lenzing], there is a dominance graph for domestic type

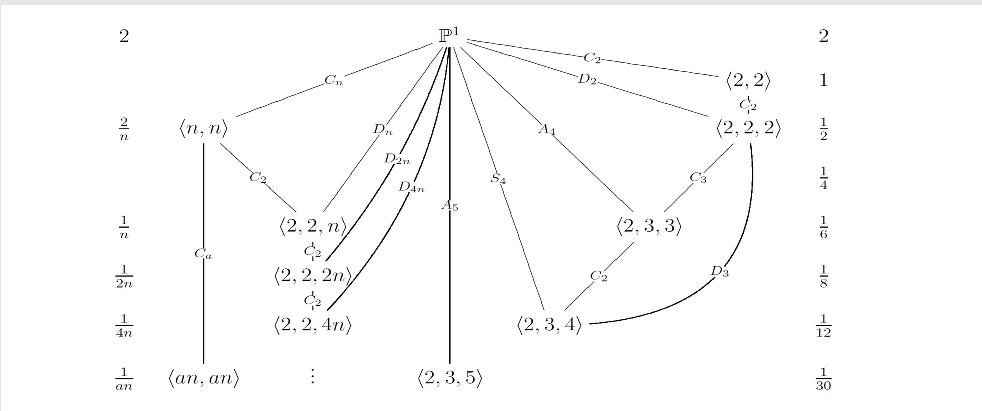


Figure 1. Dominance graph for positive Euler characteristic

- For each $1 \le i \le t$, define

$$\pi_i\colon \mathbb{L}(\mathbf{p}) \to \mathbb{Z}/p_i\mathbb{Z};\ \vec{x}_j \mapsto \delta_{i,j}\bar{1}.$$

An infinite subgroup $H \subseteq \mathbb{L}(\mathbf{p})$ is effective if $\pi_i(H) = \mathbb{Z}/p_i\mathbb{Z}$ for each i.

- For each element $\vec{x} = \sum_{i=1}^t l_i\vec{x}_i + l\vec{c}$ in its normal form, that is $0 \le l_i < p_i$ and $l \in \mathbb{Z}$, set

$$\text{mult}(\vec{x}) := \max\{l+1, 0\}.$$

Definition ([Chen-Chen]). A group homomorphism $\pi\colon \mathbb{L}(\mathbf{p}) \to \mathbb{L}(\mathbf{q})$ is admissible provided that the following conditions are satisfied

1. the subgroup $\text{im}\pi \subseteq \mathbb{L}(\mathbf{q})$ is effective;
2. for each $\vec{z} \in \text{im}\pi$, we have $\sum_{\vec{x}\in\pi^{-1}(\vec{z})} \text{mult}(\vec{x}) = \text{mult}(\vec{z})$.

A finite subgroup H of $\mathbb{L}(\mathbf{p})$ is called of *Cyclic type* if

$$H = \langle \frac{p_i}{n}\vec{x}_i - \frac{p_j}{n}\vec{x}_j \rangle;\ \text{ for some } i \ne j \text{ and } n \,|\, \text{g.c.d.}(p_i, p_j);$$

it is called of *Klein type* if

$$H = \langle \frac{p_i}{2}\vec{x}_i - \frac{p_j}{2}\vec{x}_j, \frac{p_i}{2}\vec{x}_i - \frac{p_k}{2}\vec{x}_k \rangle;\ \text{ for some pairwise distinct } i,\ j,\ k.$$

Main Theorem

Let H be a finite subgroup of $\mathbb{L}(\mathbf{p})$. Then the following statements are equivalent:

(1) there exists an admissible homomorphism $\pi\colon \mathbb{L}(\mathbf{p}) \to \mathbb{L}(\mathbf{q})$ with $\ker\pi = H$.

(2) for any parameter sequence $\boldsymbol{\lambda}$, there exists a parameter sequence $\boldsymbol{\mu}$, s.t.

$$(\text{coh-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda}))^H \xrightarrow{\sim} \text{coh-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{q};\boldsymbol{\mu}).$$

Moreover, $\ker\pi$ in (1) is either of *Cyclic type* or of *Klein type*. More precisely, up to a permutation isomorphism, one of the following holds:

- $\ker\pi = \langle \frac{p_1}{n}\vec{x}_1 - \frac{p_2}{n}\vec{x}_2 \rangle$ with $n \,|\, \text{g.c.d.}(p_1, p_2)$; in this case,

$$\mathbf{q} = (\frac{p_1}{n}, \frac{p_2}{n}, \underbrace{p_3, \cdots, p_3}_{n \text{ times}}, \cdots, \underbrace{p_t, \cdots, p_t}_{n \text{ times}});$$

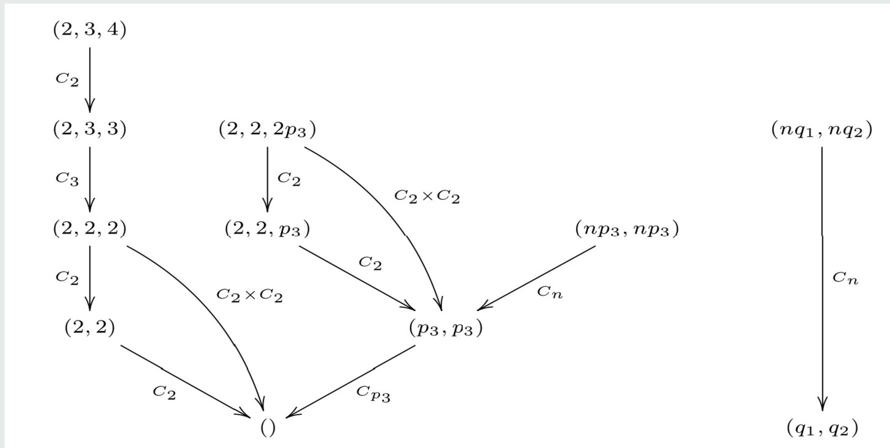
- $\ker\pi = \langle \frac{p_1}{2}\vec{x}_1 - \frac{p_2}{2}\vec{x}_2, \frac{p_1}{2}\vec{x}_1 - \frac{p_3}{2}\vec{x}_3 \rangle$ with p_1, p_2, p_3 even; in this case,

$$\mathbf{q} = (\frac{p_1}{2}, \frac{p_1}{2}, \frac{p_2}{2}, \frac{p_2}{2}, \frac{p_3}{2}, \frac{p_3}{2}, \underbrace{p_4, \cdots, p_4}_{4 \text{ times}}, \cdots, \underbrace{p_t, \cdots, p_t}_{4 \text{ times}}).$$

Admissible homomorphisms between domestic types

$\mathbf{p}$	$\mathbf{q}$	$\ker\pi$	$\pi(\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_t)$
(nq_1, nq_2)	(q_1, q_2)	$\langle q_1\vec{x}_1 - q_2\vec{x}_2 \rangle$	$(\vec{z}_1, \vec{z}_2)$
$(2,2,p_3)$	(p_3, p_3)	$\langle \vec{x}_1 - \vec{x}_2 \rangle$	$(\vec{d}, \vec{d}, \vec{z}_1 + \vec{z}_2)$
$(2,2,2p_3)$	$(2,2,p_3)$	$\langle \vec{x}_2 - p_3\vec{x}_3 \rangle$	$(\vec{z}_1 + \vec{z}_2, \vec{d}, \vec{z}_3)$
$(2,2,2p_3)$	(p_3, p_3)	$\langle \vec{x}_1 - \vec{x}_2, \vec{x}_1 - p_3\vec{x}_3 \rangle$	$(2\vec{d}, 2\vec{d}, \vec{z}_1 + \vec{z}_2)$
$(2,3,3)$	$(2,2,2)$	$\langle \vec{x}_2 - \vec{x}_3 \rangle$	$(\vec{z}_1 + \vec{z}_2 + \vec{z}_3, \vec{d}, \vec{d})$
$(2,3,4)$	$(2,3,3)$	$\langle \vec{x}_1 - 2\vec{x}_3 \rangle$	$(\vec{d}, \vec{z}_2 + \vec{z}_3, \vec{z}_1)$

Equivariant relations between domestic types



Here, a weight symbol (a,b,c) stands for the (isoclass of the) weighted projective line $\mathbb{P}^1_{\mathbf{k}}(a,b,c)$; an arrow $\mathbb{P}^1_{\mathbf{k}}(\mathbf{p}) \xrightarrow{H} \mathbb{P}^1_{\mathbf{k}}(\mathbf{q})$ stands for an equivalence $(\text{coh-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{p}))^H \xrightarrow{\sim} \text{coh-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{q})$; and the symbol C_m (resp. $C_2 \times C_2$) stands for a finite subgroup of $\mathbb{L}(\mathbf{p})$ of Cyclic type with order m (resp. of Klein type).

Admissible homomorphisms between tubular types

$\mathbf{p}$	$\mathbf{q}$	$\ker\pi$	$\pi(\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_t)$
$(2,2,2,2)$	$(2,2,2,2)$	$\langle \vec{x}_1 - \vec{x}_2 \rangle$	$(\vec{d}, \vec{d}, \vec{z}_1 + \vec{z}_2, \vec{z}_3 + \vec{z}_4)$
		$\langle \vec{x}_1 - \vec{x}_2, \vec{x}_1 - \vec{x}_3 \rangle$	$(2\vec{d}, 2\vec{d}, 2\vec{d}, \vec{z}_1 + \vec{z}_2 + \vec{z}_3 + \vec{z}_4)$
$(3,3,3)$	$(3,3,3)$	$\langle \vec{x}_1 - \vec{x}_2 \rangle$	$(\vec{d}, \vec{d}, \vec{z}_1 + \vec{z}_2 + \vec{z}_3)$
$(4,4,2)$	$(4,4,2)$	$\langle 2\vec{x}_1 - \vec{x}_3 \rangle$	$(\vec{z}_3, \vec{z}_1 + \vec{z}_2, \vec{d})$
	$(2,2,2,2)$	$\langle \vec{x}_1 - \vec{x}_2 \rangle$	$(\vec{d}, \vec{d}, \vec{z}_1 + \vec{z}_2 + \vec{z}_3 + \vec{z}_4)$
		$\langle 2\vec{x}_1 - 2\vec{x}_2 \rangle$	$(\vec{z}_1, \vec{z}_2, \vec{z}_3 + \vec{z}_4)$
		$\langle 2\vec{x}_1 - 2\vec{x}_2, 2\vec{x}_1 - \vec{x}_3 \rangle$	$(\vec{z}_1 + \vec{z}_2, \vec{z}_3 + \vec{z}_4, 2\vec{d})$
$(6,3,2)$	$(3,3,3)$	$\langle 3\vec{x}_1 - \vec{x}_3 \rangle$	$(\vec{z}_1, \vec{z}_2 + \vec{z}_3, \vec{d})$
	$(2,2,2,2)$	$\langle 2\vec{x}_1 - \vec{x}_2 \rangle$	$(\vec{z}_1, \vec{d}, \vec{z}_2 + \vec{z}_3 + \vec{z}_4)$

Equivariant relations between tubular types

Assume $\mathbb{L}(\mathbf{p})$ and $\mathbb{L}(\mathbf{q})$ are both of tubular type. Let H be a finite subgroup of $\mathbb{L}(\mathbf{p})$. Then all the equivalences of the form $(\text{coh-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda}))^H \xrightarrow{\sim} \text{coh-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{q};\boldsymbol{\mu})$ for some $\boldsymbol{\lambda}, \boldsymbol{\mu}$ are classified as the following table.

$(\mathbf{p};\boldsymbol{\lambda})$	$(\mathbf{q};\boldsymbol{\mu})$	H	μ
$(2,2,2,2;\lambda)$	$(2,2,2,2;\mu)$	$\langle \vec{x}_1 - \vec{x}_2 \rangle$ or $\langle \vec{x}_3 - \vec{x}_4 \rangle$	$\Gamma((\frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1})^2)$
		$\langle \vec{x}_1 - \vec{x}_3 \rangle$ or $\langle \vec{x}_2 - \vec{x}_4 \rangle$	$\Gamma((\frac{\sqrt{1-\lambda}+1}{\sqrt{1-\lambda}-1})^2)$
		$\langle \vec{x}_1 - \vec{x}_4 \rangle$ or $\langle \vec{x}_2 - \vec{x}_3 \rangle$	$\Gamma((\frac{\sqrt{\lambda}+\sqrt{\lambda-1}}{\sqrt{\lambda}-\sqrt{\lambda-1}})^2)$
		$\langle \vec{x}_i - \vec{x}_j, \vec{x}_i - \vec{x}_k \rangle$	$\Gamma(\lambda)$
$(3,3,3)$	$(3,3,3)$	$\langle \vec{x}_i - \vec{x}_j \rangle$	
$(4,4,2)$	$(4,4,2)$	$\langle 2\vec{x}_1 - \vec{x}_3 \rangle, \langle 2\vec{x}_2 - \vec{x}_3 \rangle$	
	$(2,2,2,2;\mu)$	$\langle \vec{x}_1 - \vec{x}_2 \rangle$	$\Gamma(-1)$
		$\langle 2\vec{x}_1 - 2\vec{x}_2 \rangle$	
		$\langle 2\vec{x}_1 - 2\vec{x}_2, 2\vec{x}_1 - \vec{x}_3 \rangle$	
$(6,3,2)$	$(3,3,3)$	$\langle 3\vec{x}_1 - \vec{x}_3 \rangle$	
	$(2,2,2,2;\mu)$	$\langle 2\vec{x}_1 - \vec{x}_2 \rangle$	$\Gamma(\omega)$

Here, $\Gamma(\lambda) = \{\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}\}$, $\omega = \frac{1+\sqrt{-3}}{2}$

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