THE FREE PROBABILISTIC ANALYSIS OF RATIONAL FUNCTIONS AND q-DEFORMATION

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ABSTRACT. We present methods of dealing with non-commutative rational functions in free probability theory. First, we upgrade the convergence in $*$-distribution of non-commutative random variables to the convergence in distribution of spectral measures of self-adjoint non-commutative rational functions. Second, we show equivalence between the rationality of operators generated by free semicircles and finite rank commutators with right annihilation (as well as creation) operators, which is a free probabilistic analog of the conjecture by Connes in the context of non-commutative geometry that is solved by Duchamp and Reutenauer. Aiming to extend the second result for other tuples of operators, we prove the existence of dual and conjugate systems characterized by the adjoint of non-commutative derivatives. For the further analysis of q-Gaussians, we show a strong convergence result of q-Gaussians for $-1 < q < 1$. This is the thesis for the author's Ph.D. course at Kyoto University.

CONTENTS

1. Introduction

Free probability has been successfully developed since Dan Voiculescu established this theory with his seminal results on random matrices [84, 88]. The original motivation of free probability is to study free products of operator algebras from a probabilistic viewpoint, and the free analog of independence, so-called free independence, plays a significant role in this theory. Moreover, phenomenons of asymptotic freeness enable us to connect free probability theory with Random Matrix Theory, which has made this theory more attractive. Typical examples of asymptotic freeness are independent Haar unitary random matrices and Gaussian Unitary Ensemble (GUE), which converge in joint distribution to free Haar unitaries and respectively free semicircles. From the aspect of operator algebras, reduced free group C*-algebras $\mathrm{C}^*_{\mathrm{red}}(\mathbb{F}_d)$ and free group von Neumann algebras $L(\mathbb{F}_d)$ are typical examples of operator algebras generated by freely independent distributions, which have not been fully understood yet (e.g. isomorphism problem of free group von Neumann algebras). Voiculescu showed some properties of such von Neumann algebras by introducing the notion of free entropy [87], which is also one of the applications of free probability. Nowadays, free probability has been applied to many fields; operator algebra, Random Matrix, combinatorics, random graphs, quantum information, deep learning,...etc.

Our starting point of the study is the relation between free probability and non-commutative rational functions which are the generalization of rational functions in a non-commutative setting. From convergence in non-commutative distribution, we can see that spectral measures of self-adjoint non-commutative polynomials also converge in distribution, but it is not straightforward that we can replace non-commutative polynomials with non-commutative rational functions. This is because operators obtained by non-commutative rational functions are often unbounded and might not be well-defined. Well-definedness of evaluations of any non-commutative rational functions is discussed in Mai-Speicher-Yin [60]. They proved the equivalent conditions for well-defined evaluations which involve maximality of the free entropy dimension and more generally the quantity Δ defined in Connes-Shlyakhtenko [29], and they also proved that this evaluation induces the embedding of non-commutative rational functions into affiliated operators. This embedding question was affirmatively answered long ago in [56], whose goal was to provide an answer to the Atiyah conjecture for some groups including the free groups. The first main result of this thesis focuses on how to see convergence in distribution of non-commutative rational functions evaluated in given operators in a finite von Neumann algebra. In particular, we deal with the convergence of the empirical eigenvalue distribution of random matrices. Let us also mention that there are many natural random matrix models involving the inverse operation and that this is an important topic nowadays, see e.g. [36, 58]. One goal of Section 3 is to provide a unified approach to the study of the limiting spectral distribution under such generality. Therefore, the natural questions are:

- can we make sense of random matrix models involving inverses?
- do they converge towards their natural limiting candidates, whose properties have been unveiled recently?

These questions have been phrased by Speicher during a meeting at MFO in 2019, [80]. Partial answers have been given under some assumptions such as bounded evaluation and specific random matrix models, cf. [93, 36, 94].

Theorem 1.1 (Theorem 3.10)**.** *Let $X^N = (X_1^N, \dots, X_{d_1}^N)$ be a d_1-tuple of deterministic self-adjoint matrices and let $U^N = (U_1^N, \dots, U_{d_2}^N)$ be a d_2-tuple of deterministic unitary matrices. Further, let R be a non-degenerate square (matrix-valued) non-commutative rational expression in $d = d_1 + d_2$ variables which is self-adjoint of type (d_1, d_2) in the sense of Definition 2.34. Suppose that the following conditions are satisfied:*

(i) (X^N, U^N) converges in $*$-distribution towards a d-tuple of non-commutative random variables (x, u) in some tracial W^*-probability space $(\mathcal{M}, \tau)$ satisfying $\Delta(x, u) = d$.
(ii) For N large enough $R(X^N, U^N)$ is well-defined, i.e., there exists $N_0 \in \mathbb{N}$ such that $(X^N, U^N) \in \mathrm{dom}_{M_N(\mathbb{C})}(R)$ for all $N \geq N_0$.

Then $(x, u) \in \mathrm{dom}_{\widetilde{\mathcal{M}}}(R)$, so that $R(x, u)$ is well-defined, and the empirical measure of $R(X^N, U^N)$ converges in law towards the analytic distribution of $R(x, u)$.

For a more concrete analysis of non-commutative rational functions, Helton-Mai-Speicher [44] established analysis of non-commutative rational functions via operator-valued Cauchy transform. Arizmendi-Cébron-Speicher-Yin [2] showed how to compute atoms of spectral measures of non-commutative rational functions evaluated in freely independent distribution and show free independence gives the minimal weight of atoms (i.e. unavoidable atoms) when each distribution is given. Hoffmann-Mai-Speicher [46] developed a way to compute the inner rank of matrices in non-commutative random variables by using free probability.

These studies give us reasonable motivations to find further connections between free probability and non-commutative rational functions. As the next step, we focus on a characterization of non-commutative rational functions in terms of Hankel operators. Such attempts are classically known as Kronecker's theorem [52]. In terms of functional analysis, this theorem says that a L^∞-function f on the unit circle is a rational function if and only if $[P, f]$ is a finite rank operator for the Riesz projection P (i.e. the projection on to the Hardy space). A non-commutative analog of this theorem was conjectured by Connes in his book [28]. In this book, he constructed a counterpart of P (denoted by F) from the action of free groups on trees. With a K-theoretic application of the operator F, he also asserted that an operator a in the free group C*-algebra is in a kind of rational closure of the free group algebra if and only if $[F, a]$ is a finite rank operator. His conjecture was proved by Duchamp and Reutenauer[34] based on the theory of non-commutative rational series. Their result is also extended to the unbounded case by Linnell [57].

In the second main result, we show an analog of Duchamp-Reutenauer [34] for free semicircle distributions. While independent Gaussian distributions are fundamental distributions in classical probability theory, free semicircle distributions are the objects of center in free probability theory. This can be seen in the free central limit theorem, the free analog of the Wick formula, and characterizations by free cumulants and free difference quotients. In particular, as well as independent Gaussians are represented on the symmetric Fock space, free semicircle distributions can be represented as sums of left creation and annihilation operators on the full Fock space. Then the right annihilation (as well as creation) operators $r_1^*, \ldots, r_d^*$ play a similar role as the operator F. Namely, we have the following theorem for free semicircles:

Theorem 1.2 (Theorem 4.1)**.** *Let a be in a von Neumann algebra $\mathrm{W}^*(s)$ generated by free semicircles $s = (s_1, \ldots, s_d)$. Then $\{[r_i^*, a]\}_{i=1}^d$ are finite rank operators on $\mathcal{F}_0(H)$ if and only if $a \in C_{\mathrm{div}}(s)$. In addition, we have*

$$C_{\mathrm{div}}(s) = C_{\mathrm{rat}}(s) \subset \overline{\mathbb{C}\langle s\rangle}$$

where $\overline{\mathbb{C}\langle s\rangle}$ is the norm closure of non-commutative polynomials $\mathbb{C}\langle s\rangle$ in $\mathrm{W}^*(s)$.

In this theorem, we consider two notions of rationality based on the argument in Duchamp-Reutenauer, division closure $C_{\mathrm{div}}(s)$ and rational closure $C_{\mathrm{rat}}(s)$ of $\mathbb{C}\langle s\rangle$ in $\mathrm{W}^*(s)$ (see Definition 2.42). We can also show an analog of Linnell's result for unfounded non-commutative rational functions in free semicircle.

In free probability, we see $r_1^*, \ldots, r_d^*$ as a dual system for free semicircles. A dual system for a given tuple $(X_1, \ldots, X_d)$ of non-commutative random variables is defined as a tuple $(D_1, \ldots, D_d)$ of operators such that $[D_i, X_j] = \delta_{i,j} P_1$ where P_1 is the projection onto the cyclic vector in the GNS representation. The notion of dual systems is introduced by Voiculescu [89] with the closely related notion of conjugate systems which are defined by a tuple of vectors $(\partial_1^*(1 \otimes 1), \ldots, \partial_d^*(1 \otimes 1))$ in the GNS representation where $\partial_1, \ldots, \partial_d$ are the free (partial) difference quotients. Both notions characterize how close given non-commutative distributions are to free semicircle distributions, and they are used to define non-microstate free entropy. Based on the motivation to study dual systems and non-commutative rational functions, we tried to investigate examples of dual systems and conjugate systems, and we focused on the q-deformation of free semicircles.

The q-defomation of free semicircle distributions is known as q-Gaussians (or q-semicircle distributions), which are introduced by Frisch-Bourret [38] and later represented as operators and the vacuum state on the q-defomed Fock space by Bożejko-Kümmerer-Speicher [11]. More precisely, they can be written as $A_i = l_i + l_i^*$ where $\{l_i\}_{i=1}^d$ are left creation operators that satisfy q-Canonical Commutation Relations (q-CCR, see [9]):

$$l_i^* l_j - q l_j l_i^* = \delta_{i,j} I.$$

One of the interesting objects to this topic is the von Neumann algebras generated by the q-Gaussians $A = (A_1, \ldots, A_d)$, so-called q-Gaussian von Neumann algebras. The q-Gaussian von Neumann algebras have been studied for many years. One of the basic questions is whether and how those algebras depend on q. The extreme cases $q = 1$ and $q = -1$ are easy to understand and they are in any case different from the q in the open interval $-1 < q < 1$. The central case $q = 0$ is generated by free semicircular elements and free probability tools give easily that this case is isomorphic to the free group factor. So the main question is whether the q-Gaussian algebras are, for $-1 < q < 1$, isomorphic to the free group factor.

Over the years it has been shown that these algebras share many properties with the free group factors. For instance, for all $-1 < q < 1$ the q-Gaussian algebras are

II_1-factors, non-injective, prime, and have strong solidity. Here is an incomplete list of papers proving these properties [11], [78], [75], [67], [73], [3]. There are also some random matrix models for q-Gaussians in [77], and [72].

A partial answer to the isomorphism problem was achieved by A. Guionnet and D. Shlyakhtenko [39], who proved that the q-Gaussian algebras are isomorphic to the free group factors for small $|q|$ (where the size of the interval depends on d and goes to zero for $d \to \infty$). However, it is still open whether this is true for all $-1 < q < 1$.

In the third main result of this thesis, we compute a dual system and from this also a conjugate system for q-Gaussians:

Theorem 1.3. *Let $d \in \mathbb{N}$ be finite and $-1 < q < 1$ and consider corresponding q-Gaussians $A = (A_1, \dots, A_d)$. Then there exists a normalized dual system and thus also a conjugate system for the q-Gaussians $A = (A_1, \dots, A_d)$. Furthermore, the conjugate system is Lipschitz conjugate.*

Let us point out that the existence of a conjugate system for the q-Gaussians was shown for small $|q|$ by Y. Dabrowski [31], and Guionnet and Shlyakhtenko proved their isomorphism by using this result and the free monotone transport. We remark that they consider right annihilation operators as a different version of dual systems, which are operators such that their commutators with q-Gaussians are equal to certain Hilbert-Schmidt operators. On the other hand, our approach starts by finding the concrete formula for dual systems which are operators whose commutators with q-Gaussians are exactly the orthogonal projection onto the vacuum vector. Our argument is based on the recursion induced by the definition of dual systems, and it allows us to give a precise combinatorial formula involving crossing partitions. We want to call the attention of the reader to the fact that our formulas for the dual system and the conjugate operators contain a factor of the form $q^{m(m-1)/2}$ as coefficients for elements in the m-particle space, in contrast to previous works where such coefficients were usually of the form q^m. Since all other exponents arising from norm estimates are only linear in m, this quadratic exponent in m is in the end responsible for the fact that our estimates work for all q in the interval $(-1, 1)$.

Having the existence of conjugate systems for all q with $-1 < q < 1$ has then, by general results, many consequences for all such q; like, for any $-1 < q < 1$, non-Γ of q-Gaussian algebras, by [31], or that any non-constant self-adjoint rational function over q-Gaussians has no atom in its distribution, by [59], [60]. In Lemma 37 of [31], algebraic freeness of non-commutative power series over q-Gaussians is proved.

There are also quite some applications of the fact that our conjugate system is Lipschitz conjugate. By [31], the existence of a Lipschitz conjugate system and Connes embeddability (which is given for our q-Gaussians, for all q) imply the maximality of the micro-states free entropy dimension. As a consequence of this or a direct application of Theorem 1.3 in [33], we can recover the fact that $\mathrm{W}^*(A)$ has no Cartan subalgebra for any $-1 < q < 1$, which has been already shown by S. Avsec [3] by other methods. Furthermore, the paper by M. Banna and T. Mai [4] gives us Hölder continuity of cumulative distribution functions of non-commutative polynomials in the q-Gaussians.

Let us collect in the following corollary the most important consequences of our result.

Corollary 1.4. *For all $-1 < q < 1$ we have the following properties.*

1) The division closure of the q-Gaussians in the unbounded operators affiliated to $\mathrm{W}^*(A_1, \ldots, A_d)$ is isomorphic to the free field. This implies that any non-commutative rational function r in d non-commuting variables can be applied to the q-Gaussians, yielding a (possibly unbounded) operator $r(A_1, \ldots, A_d)$. If r is not the zero rational function, then this operator has trivial kernel; i.e., for any self-adjoint r which is different from a constant the corresponding distribution has no atoms.

2) There is no non-zero non-commutative power series $\sum_{w\in[d]^*} \alpha_w A^w$ of radius of convergence $R > \|A_i\|$ such that $\sum_{w\in[d]^*} \alpha_w A^w = 0$.

3) For any self-adjoint non-commutative polynomial $Y = p(A_1, \ldots, A_d)$ over $A_1, \ldots, A_d$, the cumulative distribution function $\mathcal{F}_Y$ of the distribution Y is Hölder continuous with exponent $\frac{1}{2^{\deg Y}-1}$ where $\deg Y$ is the degree of p.

4) The q-Gaussian operators have finite non-microstates free Fisher information and maximal microstates free entropy dimension,

$$\Phi^*(A_1, \ldots, A_d) < \infty, \quad \textit{and} \quad \delta_0(A_1, \ldots, A_d) = d.$$

5) $\mathrm{W}^*(A_1, \ldots, A_d)$ does not have property Γ, i.e., there is no non-trivial central sequence.

6) $\mathrm{W}^*(A_1, \ldots, A_d)$ does not have a Cartan subalgebra.

We also remark that the existence of a conjugate system for q-Gaussians was extended to the non-tracial and twisted cases by Kumar-Skalski-Wasilewski [53], Kumar [54] and Yang [92] , which were applied to show the factoriality of q and twisted Araki-Woods algebra.

Now, we go back to the convergence of non-commutative random variables in joint distribution. It is obvious that as $q \to q_0$, q-Gaussians converge to q_0-Gaussians in joint distribution. In fact, it can be upgraded to so-called strong convergence, which is the fourth main result of the thesis. The concept of strong convergence has attracted substantial attention in the fields of free probability and Random Matrix Theory. This interest is largely driven by the tendency of particular multiple random matrices (for instance, independent GUE [42] and Haar unitary [24]) to demonstrate strong convergence to free random variables when their size goes to infinity. Recently, it has also been possible to involve systematically the composition with smooth functions, even at the level of strong convergence, cf. [26, 69].

As an example of applications, once we can show strong convergence, we can estimate the operator norm of a polynomial in random matrices with sufficiently large sizes by computing that of their limit, which can be applied to show additivity violation of the minimum output entropy in quantum information [25]. Moreover, strong convergence of random matrices has applications to operator algebras. An early application is due to Haagerup and Thorbjørnsen [42] who prove that $\mathrm{Ext}(\mathrm{C}^*_{\mathrm{red}}(\mathbb{F}_2))$ is not a group. Another application was found by Hayes [43] to reformulate the Peterson-Thom conjecture for free group factors, and it was subsequently proved by Belinschi-Capitaine [5], and Bordenave-Collins [14].

Strong convergence also appears in group and quantum group theory. In particular, Brannan [16] proved strong convergence of the free orthogonal quantum groups. His idea is to prove the Haagerup-type inequality which is also known as RD (Rapid Decay) property and to combine it with convergence in non-commutative distribution.

Now, we state our main result of strong convergence of q-Gaussians.

Theorem 1.5 (Theorem 6.1)**.** *For any $-1 < q_0 < 1$, strong convergence of q-Gaussians $A^{(q)} = (A_1^{(q)}, \ldots, A_d^{(q)})$ holds at q_0, i.e. for any non-commutative polynomial P,*

$$\lim_{q \to q_0} \tau[P(A^{(q)})] = \tau[P(A^{(q_0)})],$$

$$\lim_{q \to q_0} \|P(A^{(q)})\| = \|P(A^{(q_0)})\|.$$

Thanks to strong convergence and functional calculus, we can also show the convergence of spectrums in the Hausdorff distance. We can generalize our result in a more general setting where operators satisfy "uniform RD property" and convergence in non-commutative $*$-distribution. Let us mention that this theorem also holds if we replace non-commutative polynomials with bounded non-commutative rational functions evaluated in the limit operators $A^{(q)}$. This follows from the result of Yin [93] saying that once we have strong convergence, we can extend it to the level of non-commutative rational functions.

We conclude the introduction with remarks on future perspectives. Here, we list possible questions on the main results of the thesis:

- Does Theorem 4.1 hold for other tuples of operators with dual systems? Possible candidates are q-Gaussians. However, solving this problem is technically hard since the combinatorial methods for q-Gaussians are much more difficult than for free semicircles.
- Is there any characterization of an operator $a \in \mathrm{W}^*(s)$ such that $[r_i^*, a]$ is a compact operator or Schatten class operator for each $i \in \{1, \ldots, d\}$? In terms of the theory of Hakel operators, we can say this operator belongs to a kind of Vanishing Mean Oscillation (VMO) or Besov spaces.
- When the number of generators is finite and q moves in $(-1, 1)$, are all q-Gaussian von Neumann algebras isomorphic? Unfortunately, we are not able to use our result to add anything to the isomorphism problem. However, the fact that the free entropy dimension is maximal for all q in the whole interval is another indication that they might all be isomorphic to the free group factor.

This thesis is separated into the preliminary part and the main results. In the preliminary part, we introduce fundamental facts for the proof of the main results. Subsequently, we prove our main results introduced in this section.

Acknowledgement

The author would like to thank his supervisor Prof. Benoit Collins for his continuous support during his Ph.D. study. He also thanks his colleague Junichiro Matsuda and co-authors Tobias Mai, Félix Parraud, Roland Speicher, and Sheng Yin for a lot of useful discussions and checking the proofs in detail.

The problem considered in Section 3 appeared in the context of the MSc studies of the author (cf [62]), and related questions were discussed during the visit of T. Mai in Kyoto in 2019. The authors would like to thank G. Cébron, A. Connes, M. de la Salle and R. Speicher for useful discussions.

About Section 4, his Ph.D. supervisor B. Collins gave a talk about our paper [27] and received comments from A. Connes and R. Speicher on this occasion. Their feedback was the starting point of this problem. The author wants to thank them both for sharing their thoughts and references.

The study of Section 5 started during the stay of the first author at Saarland University as a program of Kyoto University Top Global Project, which the second author hosted. During the visit, Y. Ueda asked the author about this topic in an online workshop, which motivated us to study the dual system of q-Gaussians. We want to thank him for giving us a chance to explore this topic.

The basic idea of Section 6 was found when the author visited UCSD in March. He would like to thank David Jekel and Prof. Kemp for hosting his visit and inspiring discussions. He would like to thank his supervisor Prof. Collins as well as Prof. Hayes, for recommending him to write this note.

During the Ph.D. study, the author was supported by JST, the establishment of university fellowships towards the creation of science technology innovation, Grant Number JPMJFS2123 and by JSPS Research Fellowships for Young Scientists, JSPS KAKENHI Grant Number JP 22J12186, JP 22KJ1817.

2. Preliminaries

2.1. **Free probability.**

2.1.1. *Free independence and free semicircle distribution.* We start with the definition of non-commutative probability spaces which are fundamental spaces to deal with non-commutative probability. We refer to several standard textbooks [85, 68, 61] for the basic knowledge of this area. The key observation is that classical random variables are determined by their joint moments in some cases, including the case when they have bounded support (Riesz-Markov-Kakutani representation theorem). The idea of non-commutative probability is to extract the information of joint distribution as a linear functional on non-commutative $*$-polynomials. Non-commutative $*$-polynomials are polynomials over $\mathbb{C}$ in non-commutative formal variables $x_1, \dots, x_d$ and $x_1^*, \dots, x_d^*$. If we define the operation $*$ by $(x_i^*)^* = x_i$, then this $*$ can be extended to the involution (i.e. anti-linear, idempotent, $(ab)^* = b^*a^*$) of the unital algebra of non-commutative $*$-polynomials $\mathbb{C}\langle x_1, \dots, x_d, x_1^*, \dots, x_d^*\rangle$. Thus it turns out to be natural to consider a unital $*$-algebra with a state as the framework of non-commutative probability.

Definition 2.1. A (non-commutative) $*$-probability space $(\mathcal{A}, \phi)$ is a couple of a unital $*$-algebra $\mathcal{A}$ and state ϕ, which is a linear map $\phi : \mathcal{A} \to \mathbb{C}$ such that $\phi(1) = 1$ and $\phi(a^*a) \geq 0$ for any $a \in \mathcal{A}$. Non-commutative $*$-probability spaces $(\mathcal{A}, \phi)$ are differently called in the following contexts.

- C*-probability space: $\mathcal{A}$ is a unital C*-algebra and ϕ is a state on $\mathcal{A}$.
- W*-probability space: $\mathcal{A}$ is a von Neumann algebra and ϕ is a state on $\mathcal{A}$ that is normal, i.e. W*-continuous on the unit ball $\{a \in \mathcal{A}; \|a\| \leq 1\}$.

We often assume the following two properties on the state, which hold in many examples.

Definition 2.2. A state ϕ on a $*$-algebra $\mathcal{A}$ is faithful if $\phi(a^*a) > 0$ for any $a \neq 0$, and ϕ is tracial if $\phi(ab) = \phi(ba)$ for any $a, b \in \mathcal{A}$.

Example 2.3. Here, we give two examples of (C* or W*) non-commutative probability spaces with faithful normal traces. The first example is the matrix algebra $M_n(\mathbb{C})$, and the faithful trace is given by normalized trace:

$$\mathrm{tr}A = \frac{1}{n}\sum_{i=1}^{n} a_{ii}, \quad A = (a_{ij}) \in M_n(\mathbb{C}).$$

The second example is the operator algebra constructed from a group. Let G be a discrete group with the unit e and consider l^2-space $l^2(G)$. We define the unitary operator $\lambda(g)$ on $l^2(G)$ by

$$\lambda(g)\delta_h = \delta_{gh}$$

where $\{\delta_h\}_{h\in G}$ is the canonical basis on $l^2(G)$. This induces $*$-representation of the group G and we call this representation the left regular representation of G. We obtain C*-algebra $\mathrm{C}^*_{\mathrm{red}}(G)$ by taking a norm closure of the image of the group algebra $\lambda(\mathbb{C}[G])$ and call it reduced group C*-algebra of G. Moreover, if we take the weak closure (or double commutant) of $\lambda(\mathbb{C}[G])$, we obtain von Neumann algebra $L(G)$ and we call it the group von Neumann algebra of G. Both operator algebras have the same canonical faithful trace τ_e defined by

$$\tau_e(X) = \langle X\delta_e, \delta_e\rangle.$$

As well as classical probability theory, the convergence of random variables is an important phenomenon in non-commutative probability theory. In a non-commutative setting, we have a state instead of a probability measure and see the convergence in distribution as the convergence in moments.

Definition 2.4. Let $(\mathcal{A}_n, \phi_n)_{n\in\mathbb{N}}$ and $(\mathcal{A}_\infty, \phi_\infty)$ be $*$-probability spaces, and $X^{(n)} = (X_1^{(n)}, \ldots, X_d^{(n)})_{n\in\mathbb{N}}$ and $X^{(\infty)} = (X_1^{(\infty)}, \ldots, X_d^{(\infty)})$ be d-tuples of non-commutative random variables (not necessarily self-adjoint) in $\mathcal{A}_n$ and $\mathcal{A}_\infty$. Then, we say $X^{(n)}$ converges in $*$-distribution (or joint distribution) to $X^{(\infty)}$ if for any non-commutative $*$-polynomial P, we have

$$\lim_{n\to\infty} \phi_n[P(X^{(n)})] = \phi_\infty[P(X^\infty)].$$

Now, we define free independence which is a crucial notion in free probability.

Definition 2.5. For a $*$-probability space $(\mathcal{A}, \phi)$, a family of unital $*$-subalgebras $\{\mathcal{A}_i\}_{i\in I}$ are freely independent if for any $n \in \mathbb{N}$ and $a_k \in \mathcal{A}_{i_k}$ with $i_k \neq i_{k+1}$ $(k = 1, \ldots, n)$, we have

$$\phi\left[\prod_{k=1}^{n} \mathring{a}_k\right] = 0$$

where $\mathring{a} = a - \phi(a)$. In particular, we say that a set of elements $\{x_i\}_{i\in I} \in \mathcal{A}$ are freely independent if $*$-subalgebras generated by each x_i are freely independent.

Example 2.6. The definition of free independence above is derived from the free product of groups. To see this, let $\{G_i\}_{i\in I}$ be discrete groups and e_i be the identity of G_i for each $i \in I$. We define the free product $*_{i\in I}G_i$ of G_i's by

$$*_{i\in I}G_i = \{e\} \cup \{g_1g_2\cdots g_n; n \in \mathbb{N},\ g_k \in G_{i_k} \setminus \{e_{i_k}\},\ i_k \neq i_{k+1} \text{ for any } k\},$$

where e is the identity of $*_{i\in I}G_i$. A multiplication of $*_{i\in I}G_i$ is defined by connecting as words and reducing them according to the group structure of G_i's (see chapter

1 in [85]). Then one can see that $\{L(G_i)\}_{i\in I}$ are freely independent in a W*-probability space $(L(*_{i\in I}G_i), \tau_e)$ since a trace of any reduced word except for the identity is always 0.

There are several important non-commutative distributions. If we take the free group $\mathbb{F}_d$ with generators $g_1, \dots, g_d$, then the unitary operators $\lambda(g_1), \dots, \lambda(g_d)$ behave as freely independent Haar distributions in the W*-probability space $(L(\mathbb{F}_d), \tau_e)$. In this thesis, we focus on another important distribution, so-called the semicircle distribution.

Definition 2.7. The probability measure on $\mathbb{R}$ whose density function is given by

$$\frac{1}{2\pi}\sqrt{4-x^2}\ 1_{[-2,2]}dx,$$

is called the (standard) semicircle distribution.

Remark 2.8. The odd moments of the standard semicircle distribution are zero since the density is an even function and the $2n$-th moment is given by the n-th Catalan number:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Note that Catalan numbers satisfy the following recursion:

$$C_0 = 1, \quad C_1 = 1, \quad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$

In free probability, the semicircle distribution behaves like the Gaussian distribution. One of the reasons for this is that we have a free probabilistic analog of the central limit theorem as follows

Theorem 2.9. *Let $(\mathcal{A}, \phi)$ be a $*$-probability space and $\{a_i\}_{i=1}^{\infty}$ be a sequence of non-commutative real random variables $a_i^* = a_i$ which are identically distributed, i.e. $\phi(a_1^n) = \phi(a_i^n)$ for any $i \in \mathbb{N}$, and a_i satisfies $\phi(a_i) = 0$ and $\phi(a_i^2) = 1$. We set*

$$s_N = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} a_i.$$

Then we have the convergence in moments of s_N to the semicircle distribution, i.e. for each $k \in \mathbb{N}$

$$\lim_{N\to\infty}\phi(s_N^k) = \frac{1}{2\pi}\int_{\mathbb{R}} x^k\sqrt{4-x^2}\ 1_{[-2,2]}dx.$$

2.1.2. *Free semicircles and Non-crossing partitions.* In many parts of this thesis, we focus on freely independent semicircle distributions (or merely free semicircles). From a combinatorial point of view, free probability is deeply connected with non-crossing partitions (c.f. free cumulants), and in particular joint moments of free semicircles can be computed by counting non-crossing pair partitions. Here, we give the definition of partitions and how to compute joint moments of free semicircles.

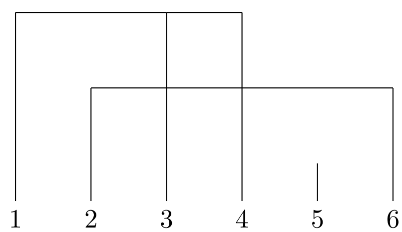
Definition 2.10. A partition π on the vertex set $[n] = \{1, \dots, n\}$ is a family of subsets $\{V_k\}_{k=1}^{l}$ such that $\bigsqcup_{k=1}^{l} V_k = [n]$ (disjoint union). We say that each V_k is a block of π. If all blocks of π have two elements, we say π is a pair partition.

Let $P(n)$ (and $P_2(n)$) denote a set of partitions (resp. pair partitions) on $[n]$. We define a notion of crossings in a partition.

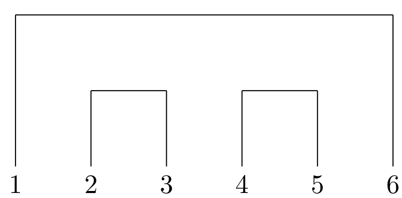
Definition 2.11. A crossing of a partition π is a tuple of vertices $a < b < c < d$ such that there exist different blocks V_k and V_l with $a, c \in V_k$ and $b, d \in V_l$. If π has no crossings, we say π is non-crossing.

Let $NC(n)$ (and $NC_2(n)$) denote a set of non-crossing partitions (resp. non-crossing pair partitions) on $[n]$. In the following, we give examples of partitions and non-crossing partitions.

Example 2.12. We consider two partitions on $[6]$, $\pi_1 = \{\{1,3,4\},\{2,6\},\{5\}\}$ and $\pi_2 = \{\{1,6\},\{2,3\},\{4,5\}\}$. By arranging n-vertices in increasing order and connecting vertices in the same block of a given partition with appropriate height, we can draw a partition and count the number of crossings in the drawing. For two examples π_1, π_2, we can draw them as follows:



and



Then the number of crossings is 2 for π_1 and 0 for π_2. Also, π_2 is a non-crossing pair partition, i.e. $\pi_2 \in NC_2(6)$.

Remark 2.13. By the combinatorial argument (using the recursion, for example), we can see that the number of non-crossing partitions is given by Catalan numbers:

$$\#NC(n) = \#NC_2(2n) = C_n.$$

Now, we consider a tuple of freely independent semicircles $s = (s_1, \ldots, s_d)$ in a $*$-probability space $(\mathcal{A}, \tau)$. In fact, the joint distribution of free semicircles is characterized by non-crossing pair partitions.

Theorem 2.14. *For any $k \in \mathbb{N}$ and $i_1, \ldots, i_k \in \{1, \ldots, d\}$, we have*

$$\tau[s_{i_1} s_{i_2} \cdots s_{i_n}] = \sum_{\pi \in NC_2(n)} \prod_{(k,l) \in \pi} \delta_{i_k, i_l}.$$

One can check this theorem by free cumulants or an analog of the Wick formula of which we will see the q-version.

We have another characterization of free semicircles. Recall that the joint distribution of independent Gaussians can be computed recursively by integration by part. Similarly, the joint distribution of free semicircles can be computed recursively by a kind of integration by part with respect to free partial difference quotients. We define free partial difference quotients $\partial_1, \dots, \partial_d$ by linear operators from $\mathbb{C}\langle x_1, \dots, x_d\rangle$ to $\mathbb{C}\langle x_1, \dots, x_d\rangle^{\otimes 2}$ such that for each monomial P, we have

$$\partial_i P = \sum_{P = Ax_iB} A \otimes B$$

where the sum is taken over all possible decomposition of $P = Ax_iB$. Note that free partial difference quotients satisfy a kind of Leibniz rule:

$$\partial_i(PQ) = (P \otimes 1)\partial_i Q + (\partial_i P)(1 \otimes Q).$$

Then we have a kind of integration by part for free semicircles, which is one of the basic observations to introduce non-microstate free entropy.

Theorem 2.15. *For any non-commutative polynomial P, we have*

$$\tau[s_i P(s)] = (\tau \otimes \tau)[\partial_i P(s)].$$

We will see a q-version of this theorem in Section 5. In terms of operator algebra, free semicircles generate free group von Neumann algebra, which can be generalized as the following theorem.

Theorem 2.16 (cf. Theorem 6 in [61])**.** *Let $(\mathcal{M}, \tau)$ be a tracial* W**-probability space and $X = (X_1, \dots, X_d)$ be a tuple of self-adjoint operators on $\mathcal{M}$. If X satisfies*

- *Each spectral distribution X_i with respect to τ has no atom,*
- *X are freely independent with respect to τ,*

then W$^*(X)$ *is isomorphic to the free group von Neumann algebra $L(\mathbb{F}_d)$.*

2.1.3. *Asymptotic freeness.* We explain the asymptotic freeness by taking the Gaussian Unitary ensemble as an example. A random matrix is a matrix whose entries are random variables.

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N.$$

Then the *empirical eigenvalue distribution* of X is defined as a random measure,

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i},$$

where δ_a is a Dirac measure on a for $a \in \mathbb{R}$.

Definition 2.17. Let $N \in \mathbb{N}$ and $x_{ij}(1 \leq i \leq j \leq N), y_{ij}(1 \leq i < j \leq N)$ are independent random variables which are identically distributed with the standard normal distribution. Put $z_{ii} = x_{ii}$ for $i \in \{1, \dots, N\}$ and $z_{ij} = \frac{1}{\sqrt{2}}(x_{ij} + iy_{ij}) = \overline{z_{ji}}$ for $1 \leq i < j \leq N$. Then $Z^{(N)} = \frac{1}{\sqrt{N}}(z_{ij})$ is called a *GUE (Gaussian Unitary Ensemble) random matrix.*

Voiculescu proved the following convergence result for independent GUE random matrices which is the foundation of the connection between free probability and Random Matrix Theory.

Theorem 2.18 (Theorem 3.3 in [84])**.** *Independent GUE random matrices $Z_1^{(N)}, \dots, Z_d^{(N)}$ satisfy*

$$\lim_{N\to\infty} E \circ \mathrm{tr}[Z_{i_1}^{(N)} \cdots Z_{i_n}^{(N)}] = \tau(s_{i_1} \cdots s_{i_n}),$$

where $s_1, \dots, s_d$ are free semicircular elements with respect to τ.

Remark 2.19. When $d = 1$ the above theorem tells us that the average of the empirical eigenvalue distribution $\mu_{Z^{(N)}}$ of a GUE random matrix converges in distribution to the semicircular distribution. In other words, we have for any compactly supported continuous function on $\mathbb{R}$,

$$\lim_{N\to\infty} E[\int_{\mathbb{R}} f d\mu_{Z^{(N)}}] = \frac{1}{2\pi}\int_{\mathbb{R}} f(x)\sqrt{4-x^2}1_{[-2,2]}dx,$$

where E is the expectation for $Z^{(N)}$. It is known as Wigner's semicircle law that this can be improved in almost sure convergence (see [81, Theorem 2.4.2]).

Remark 2.20. We identify $Z_N^1, \dots, Z_N^d$ with elements in a non-commutative probability space $(M_N(L), E\circ \mathrm{tr}_N)$, where $L = \bigcap_{p\geq 1} L^p(\Omega)$ for a probability space $(\Omega, \mathcal{F}, \mu)$. Then the behavior of $Z_1^{(N)}, \dots, Z_d^{(N)}$ with respect to $E\circ \mathrm{tr}$ gets closer to that of freely independent elements with respect to their state. Such phenomenon is known as *asymptotic freeness*. More precisely, asymptotic freeness can be stated as follows. Let $(\mathcal{A}_N, \phi_N)$ be a non-commutative probability space for each $N\in\mathbb{N}$ and $\{a_i^{(N)}\}_{i\in I}\subset \mathcal{A}_N$. Then $\{a_i^{(N)}\}_{i\in I}$ is *asymptotically free* as $N\to\infty$ if there exists a non-commutative probability space $(\mathcal{A},\phi)$ and $\{a_i\}_{i\in I}\subset\mathcal{A}$ such that $\{a_i\}$ are freely independent and that we have

$$\lim_{N\to\infty}\phi_N(a_{i_1}^{(N)}a_{i_2}^{(N)}\cdots a_{i_m}^{(N)}) = \phi(a_{i_1}a_{i_2}\cdots a_{i_m}),$$

for any $m\in\mathbb{N}$ and $i_1,\dots.i_m\in I$.

Since independent GUE random matrices behave like freely independent semicircles and are constructed by independent Gaussian random variables, one might expect that there is a kind of integration by part for independent GUE random variables. Indeed, such a formula exists and is called the Dyson-Schwinger equation. It can be stated as follows.

Theorem 2.21 (cf. [40])**.** *Let $Z^{(N)} = (Z_1^{(N)},\dots,Z_d^{(N)})$ be a tuple of independent GUE random matrices. Then for any non-commutative polynomial P, we have*

$$E\circ\mathrm{tr}[Z_i^{(N)}P(Z^{(N)})] = E\circ(\mathrm{tr}\otimes\mathrm{tr})[\partial_i P(Z^{(N)})].$$

2.1.4. *Conjugate system and free Fisher information.* In this section, we deal with a tracial W*-probability space $(\mathcal{M},\tau)$ where τ is faithful, and define dual systems and conjugate systems for tuples of self-adjoint operators.

For a von Neuman subalgebra $\mathcal{A}$ of $\mathcal{M}$, we consider the Hilbert space $L^2(\mathcal{A},\tau)$ which is the completion of a pre-Hilbert space $\mathcal{A}$ with the inner product defined by

$$\langle x, y\rangle = \tau(x^*y).$$

Recall that joint moments of free semicircles are characterized by a kind of integration by part with respect to free difference quotients. We use this characterization to see how close given non-commutative real random variables are to free semicircles.

Definition 2.22 (Definition 3.1 in [89])**.** For a tuple of self-adjoint operators $X = (X_1,\dots,X_d)\in\mathcal{M}^d$, a tuple of vectors $(\xi_1,\dots,\xi_d)\in L^2(\mathrm{W}^*(X),\tau)^d$ is called a conjugate system for X if $\xi = \partial_i^*(1\otimes 1)$ for all i, in other words, ξ_i's satisfy for any non-commutative polynomial P, we have

$$\langle P(X), \xi\rangle = \langle \partial_i P(X), 1\otimes 1\rangle.$$

There is a similar notion as conjugate systems, so-called dual systems which are a bit easier to compute than conjugate systems.

Definition 2.23. For a tuple of operators $X = (X_1, \dots, X_d) \in \mathcal{M}^d$, a tuple of densely defined operators $(D_1, \dots, D_d)$ on $L^2(\mathrm{W}^*(X), \tau)$ is called a normalized dual system for X if $\mathbb{C}\langle X\rangle \subset \mathrm{dom}(D_i)$, and $D_i 1 = 0$, and $1 \in \mathrm{dom}\, D_i^*$, and $[D_i, X_j] = \delta_{i,j} P_1$ for all i, j where P_1 is the one rank projection onto the trace vector $1 \in L^2(\mathrm{W}^*(X), \tau)$.

In the above definition, we use "normalized" for the condition $D_i 1 = 0$ because this is an additional condition on the original definition by Voiculescu. Conjugate systems and dual systems are closely related by the following theorem by Shlyakhtenko.

Theorem 2.24 (Theorem 1 in [76])**.** *For self-adjoint elements $X = (X_1, \dots, X_d) \in \mathcal{M}^d$, the existence of a conjugate system $(\xi_1, \dots, \xi_d)$ is equivalent to the existence of a normalized dual system $(D_1, \dots, D_d)$. In this case we have for each $i \in [d]$*

$$\xi_i = D_i^* 1.$$

Proof. For the reader's convenience, let us give a proof of this theorem. Let us compute

$$\tau(X_{j_n} \cdots X_{j_1} D_i^* 1) = \langle X_{j_n} \cdots X_{j_1} D_i^* 1, 1\rangle$$

for $j_n, \dots, j_1 \in [d]$. Note that $X_1, \dots, X_d$ are self-adjoint, and we have

$$\begin{aligned}
\langle X_{j_n} \cdots X_{j_1} D_i^* 1,\ 1\rangle &= \langle 1, D_i X_{j_1} \cdots X_{j_n}\rangle \\
&= \langle 1, X_{j_1} D_i X_{j_2} \cdots X_{j_n}\rangle + \delta_{ij_1} \overline{\tau(X_{j_2} \cdots X_{j_n})} \\
&= \cdots \\
&= \sum_{k=1}^{n} \delta_{ij_k} \overline{\tau(X_{j_1} \cdots X_{j_{k-1}}) \tau(X_{j_{k+1}} \cdots X_{j_n})} + \langle 1, X_{j_1} \cdots X_{j_n} D_i 1\rangle \\
&= \sum_{k=1}^{n} \delta_{ij_k} \tau(X_{j_n} \cdots X_{j_{k+1}}) \tau(X_{j_{k-1}} \cdots X_{j_1}) + \langle 1, X_{j_1} \cdots X_{j_n} D_i 1\rangle,
\end{aligned}$$

where the last term is equal to 0 since we required $D_i 1 = 0$. This implies that $(D_1^* 1, \dots, D_d^* 1)$ forms the conjugate system.

For the converse direction, we consider the unbounded operators $D_1, \dots, D_d$ defined by $D_i = (\mathrm{id} \otimes \tau)\partial_i$. We can check for any non-commutative polynomial Q

$$\begin{aligned}
[D_i, X_j]Q &= D_i X_j Q - X_j D_i Q \\
&= \delta_{i,j} \tau(Q) + X_j (\mathrm{id} \otimes \tau)\partial_i Q - X_j D_i Q \\
&= \delta_{i,j} P_1(Q)
\end{aligned}$$

where we use the Leibniz rule of ∂_i at X_j.

Then the existence of the conjugate system implies $1 \otimes 1 \in \mathrm{dom}\, \partial_i^*$ and therefore $1 \in \mathrm{dom}\, D_i^*$. □

We remark that the condition $1 \in \mathrm{dom}(D_i^*)$ implies that $\mathbb{C}\langle X\rangle \subset \mathrm{dom}(D_i^*)$ and hence that D_i is a closable operator. This can be seen by a similar computation as above, for $Q(X) \in \mathbb{C}\langle X\rangle$ and $j_n, \ldots, j_1 \in [d]$:

$$\begin{aligned}\langle D_i Q(X), X_{j_n}\cdots X_{j_1}\rangle &= \langle D_i X_{j_1}\cdots X_{j_n} Q(X), 1\rangle \\ &\quad - \sum_{k=1}^{n} \tau(X_{j_1}\cdots X_{j_{k-1}})\langle P_1 X_{j_{k+1}}\cdots X_{j_n} Q(X), 1\rangle\end{aligned}$$

where each term is a bounded operator with respect to $Q(X)$, since $X \in \mathcal{M}^d$ and $1 \in \mathrm{dom}(D_i^*)$. Thus $\mathbb{C}\langle X\rangle \subset \mathrm{dom}(D_i^*)$. We also consider an analytic condition on a conjugate system, the so-called Lipschitz condition introduced by Dabrowski.

Definition 2.25. We say the conjugate system $(\xi_1, \ldots, \xi_d)$ is Lipschitz conjugate (see Definition 1 in [31] or Section 2.4 in [4]) if $\xi_i \in \mathrm{dom}(\overline{\partial_j})$ and $\overline{\partial_j}\xi_i \in \mathrm{W}^*(X)\overline{\otimes}\mathrm{W}^*(X)$ for any $i, j \in [d]$, where $\overline{\partial_j}$ is the closure of ∂_j and we consider a von Neumann algebra tensor product for $\mathrm{W}^*(X)\overline{\otimes}\mathrm{W}^*(X)$.

The existence of a Lipschitz conjugate system for given self-adjoint operators $X_1, \ldots, X_d$ implies analytic properties of not only distributions but also generated von Neumann algebra $\mathrm{W}^*(X_1, \ldots, X_d)$. Here, we give a list of those properties (the properties 1), 2), 3) hold without the Lipschitz condition):

1) The division closure of the $X_1, \ldots, X_d$ in the unbounded operators affiliated to $\mathrm{W}^*(X_1, \ldots, X_d)$ is isomorphic to the free field. This implies that any non-commutative rational function r in d non-commuting variables can be applied to the $X_1, \ldots, X_d$, yielding a (possibly unbounded) operator $r(X_1, \ldots, X_d)$. If r is not the zero rational function, then this operator has a trivial kernel; i.e., for any self-adjoint r which is different from a constant the corresponding distribution has no atoms (see [59], [60]).

2) There is no non-zero non-commutative power series $\sum_{w\in[d]^*} \alpha_w X^w$ (see Section 2.3 for notations) of radius of convergence $R > \|X_i\|$ (i.e. $\sum_{n=0}^{\infty}\sum_{|w|=n} |\alpha_w| R^n < \infty$) such that $\sum_{w\in[d]^*} \alpha_w X^w = 0$ (See Lemma 37 [32]).

3) $\mathrm{W}^*(X_1, \ldots, X_d)$ does not have property Γ, i.e., there is no non-trivial central sequence (see [31]).

3) For any self-adjoint non-commutative polynomial $Y = p(X_1, \ldots, X_d)$ over $X_1, \ldots, X_d$, the cumulative distribution function $\mathcal{F}_Y$ of the distribution Y is Hölder continuous with exponent $\frac{1}{2^{\deg Y}-1}$ where $\deg Y$ is the degree of p. Even if we don't assume the Lipschitz condition, we have Hölder continuity with exponent $\frac{2}{3(2^{\deg Y}-1)}$ (see [4]).

4) If $\mathrm{W}^*(X_1, \ldots, X_d)$ is Connes embeddable, the operators $X_1, \ldots, X_d$ have the maximal microstates free entropy dimension

$$\delta_0(X_1, \ldots, X_d) = d,$$

and $\mathrm{W}^*(X_1, \ldots, X_d)$ does not have a Cartan subalgebra (see [32], [33]).

2.2. **Non-commutative rational functions and evaluation in operators.**

In this section, we introduce the notion of non-commutative rational functions and their linearization which we use in Section 3. We also explain the fundamental theorem for non-commutative rational power series which we use in Section 4.

2.2.1. *Non-commutative rational functions.*

We start from the notion of non-commutative rational expression. Let $x = (x_1, \ldots, x_d)$ be a tuple of non-commutative formal variables. We define non-commutative rational expressions (over $\mathbb{C}$) as possible combinations of addition, multiplication, inversion of $\mathbb{C}$ variables in x with parentheses which determine the order of operations. At this stage, we just consider symbolic expressions. For example, $(2x_1x_2 + x_3^{-1})x_1$ and $(x_1 - x_1)^{-1}$ are non-commutative rational expressions, although $(x_1 - x_1)^{-1}$ is not well-defined if we evaluate it in any tuple of unital algebra.

One can also define *matrix-valued non-commutative rational expressions*; see Definition 2.1 in [49]. Those are possible combinations of symbols $A \otimes 1$ and $A \otimes x_j$ for $j = 1, \ldots, d$, for each rectangular matrix A over $\mathbb{C}$ of arbitrary size, with $+$, $\cdot$, $^{-1}$, and $()$, where the operations are required to be compatible with the matrix sizes. Notice that $\otimes$ has only symbolic meaning here, but will turn into the ordinary tensor product (over $\mathbb{C}$) under evaluation as will be defined below.

Let us enumerate the rules which allow to recursively compute for every matrix-valued non-commutative rational expression R the domain $\mathrm{dom}_{\mathcal{A}}(R)$ of R for every unital complex algebra $\mathcal{A}$ and evaluations $R(X)$ of R at any point $X \in \mathrm{dom}_{\mathcal{A}}(R)$; note that the evaluation $R(X)$ of a $p \times q$ matrix-valued non-commutative rational expression R and every point $X \in \mathrm{dom}_{\mathcal{A}}(R)$ belongs to $M_{p\times q}(\mathbb{C}) \otimes \mathcal{A} \cong M_{p\times q}(\mathcal{A})$.

- If $R = A \otimes 1$ for some $A \in M_{p\times q}(\mathbb{C})$, then R is a $p \times q$ matrix-valued non-commutative rational expression with $\mathrm{dom}_{\mathcal{A}}(R) := \mathcal{A}^d$ and $R(X) := A \otimes 1_{\mathcal{A}}$ for every $X \in \mathcal{A}^d$.
- If $R = A \otimes x_j$ for some $A \in M_{p\times q}(\mathbb{C})$ and $1 \leq j \leq d$, then R is a $p \times q$ matrix-valued non-commutative rational expression with $\mathrm{dom}_{\mathcal{A}}(R) := \mathcal{A}^d$ and $R(X) := A \otimes X_j$ for every $X = (X_1, \ldots, X_d) \in \mathcal{A}^d$.
- If R_1, R_2 are $p \times q$ matrix-valued non-commutative rational expressions, then $R_1 + R_2$ is a $p \times q$ matrix-valued non-commutative rational expression with $\mathrm{dom}_{\mathcal{A}}(R_1 + R_2) := \mathrm{dom}_{\mathcal{A}}(R_1) \cap \mathrm{dom}_{\mathcal{A}}(R_2)$ and $(R_1 + R_2)(X) := R_1(X) +_{\mathcal{A}} R_2(X)$ for every $X \in \mathrm{dom}_{\mathcal{A}}(R_1 + R_2)$, where $+_{\mathcal{A}}$ stands for the addition $M_{p\times q}(\mathcal{A}) \times M_{p\times q}(\mathcal{A}) \to M_{p\times q}(\mathcal{A})$.
- If R_1, R_2 are $p \times q$ respectively $q \times r$ matrix-valued non-commutative rational expressions, then $R_1 \cdot R_2$ is a $p \times r$ matrix-valued non-commutative rational expression with $\mathrm{dom}_{\mathcal{A}}(R_1 \cdot R_2) := \mathrm{dom}_{\mathcal{A}}(R_1) \cap \mathrm{dom}_{\mathcal{A}}(R_2)$ and $(R_1 \cdot R_2)(X) := R_1(X) \cdot_{\mathcal{A}} R_2(X)$ for every $X \in \mathrm{dom}_{\mathcal{A}}(R_1 \cdot R_2)$, where $\cdot_{\mathcal{A}}$ stands for the matrix mutliplication $M_{p\times q}(\mathcal{A}) \times M_{q\times r}(\mathcal{A}) \to M_{p\times r}(\mathcal{A})$.
- If R is a $p \times p$ matrix-valued non-commutative rational expression, then
$$\mathrm{dom}_{\mathcal{A}}(R^{-1}) := \{X \in \mathrm{dom}_{\mathcal{A}}(R) \mid R(X) \text{ is invertible in } M_p(\mathcal{A})\}$$
and $R^{-1}(X) := R(X)^{-1}$ for every $X \in \mathrm{dom}_{\mathcal{A}}(R^{-1})$.

Note that the (scalar-valued) non-commutative rational expressions which we have introduced before belong to the strictly larger class of 1×1 matrix-valued non-commutative rational expressions; see Remark 2.11 in [49].

For the reader's convenience, we introduce two types of matrix-valued non-commutative rational expressions which are important in a practical sense.

- A non-commutative rational expression evaluated in formal tensor products of matrices and formal variables like as
$$R = r(A_1 \otimes x_1, A_2 \otimes x_2, \ldots, A_d \otimes x_d)$$
where r is a (scalar-valued) non-commutative rational expression and $A_i \in M_p(\mathbb{C})$ for $1 \leq i \leq d$. In other words, in this case we amplify formal variables by matrices and then consider their (scalar-valued) rational expression.

- A matrix which consists of (scalar-valued) non-commutative rational expressions
$$R = (r_{ij})_{1\leq i\leq p, 1\leq j\leq q}.$$
This can be seen as a $p \times q$ matrix-valued non-commutative rational expression by identifying with $\sum_{ij}(a_i \otimes 1)r_{ij}(b_j \otimes 1)$ where $a_i \in M_{p\times 1}(\mathbb{C})$ and $b_j \in M_{1\times q}(\mathbb{C})$ are standard basis of $\mathbb{C}^p$ and $\mathbb{C}^q$. We will implicitly use this viewpoint later (for example, in the proof of Proposition 3.4).

A class of matrix-valued non-commutative polynomial expressions are affine linear pencils. An *affine linear pencil (in d variables with coefficients from $M_k(\mathbb{C})$)* is a $k \times k$ matrix-valued non-commutative polynomial expression of the form
$$A = A_0 \otimes 1 + A_1 \otimes x_1 + \cdots + A_d \otimes x_d$$
with coefficient matrices $A_0, A_1, \ldots, A_d$ belonging to $M_k(\mathbb{C})$. Notice, once again, that we omit the parentheses for better readability as each syntactically valid placement of parentheses will produce the same result under evaluation. If $\mathcal{A}$ is any unital complex algebra and $X \in \mathcal{A}^d$, then
$$A(X) = A_0 \otimes 1_{\mathcal{A}} + A_1 \otimes X_1 + \cdots + A_d \otimes X_d \in M_k(\mathbb{C}) \otimes \mathcal{A} \cong M_k(\mathcal{A}).$$

Of particular interest are matrix-evaluations. To collect meaningful rational expressions, we consider the evaluation of rational expressions in square matrices of all sizes. For each matrix-valued non-commutative rational expression R, we put
$$\operatorname{dom}_{M(\mathbb{C})}(R) := \coprod_{N=1}^{\infty} \operatorname{dom}_{M_N(\mathbb{C})}(R),$$
i.e., $\operatorname{dom}_{M(\mathbb{C})}(R)$ is the subset of all square matrices over $\mathbb{C}$ where evaluation of R is well-defined. A matrix-valued non-commutative rational expression R is said to be *non-degenerate* if it satisfies $\operatorname{dom}_{M(\mathbb{C})}(R) \neq \emptyset$. In the sequel, we will make use of the following important fact.

Theorem 2.26 (Remark 2.3 in [49])**.** *Let R be a non-degenerate matrix-valued non-commutative rational expression. Then there exists some $N_0 = N_0(R) \in \mathbb{N}$ such that $\operatorname{dom}_{M_N(\mathbb{C})}(R) \neq \emptyset$ for all $N \geq N_0$.*

Two non-degenerate matrix-valued non-commutative rational expressions R_1, R_2 are called $M(\mathbb{C})$-*evaluation equivalent* if the condition $R_1(X) = R_2(X)$ is satisfied for all $X \in \operatorname{dom}_{M(\mathbb{C})}(R_1) \cap \operatorname{dom}_{M(\mathbb{C})}(R_2)$.

One can construct a skew field by evaluating (scalar-valued) non-commutative rational expressions in scalar-valued matrices. For a non-degenerate non-commutative rational expression r, we denote by $[r]$ its equivalence class of non-commutative rational expressions with respect to $M(\mathbb{C})$-evaluation equivalence. We endow the set of all such equivalence classes with the arithmetic operations $+$ and $\cdot$ defined by $[r_1]+[r_2] := [r_1+r_2]$ and $[r_1]\cdot[r_2] := [r_1\cdot r_2]$. Notice that the arithmetic operations are indeed well-defined as one has $\operatorname{dom}_{M(\mathbb{C})}(r_1)\cap\operatorname{dom}_{M(\mathbb{C})}(r_2) \neq \emptyset$ for any two non-degenerate scalar-valued non-commutative rational expressions r_1 and r_2; see the footnote on page 52 of [50], for instance. It is known (see Proposition 2.2 in [50]) that the set of all equivalence classes of non-commutative rational expressions with respect to $M(\mathbb{C})$-evaluation equivalence endowed with the arithmetic operations $+$ and $\cdot$ forms a skew field, so-called the free field $\mathbb{C}\langle\!\langle x_1, \ldots, x_d \rangle\!\rangle$ which was originally introduced by Amitsur [1].

2.2.2. *Linearization of non-commutative rational functions.* Let us recall the following terminology that was introduced in [44, Definition 4.10].

Definition 2.27 (Formal linear representation)**.** Let R be a $p \times q$ matrix-valued non-commutative rational expression in the variables $x_1, \dots, x_d$. A *formal linear representation $\rho = (u, A, v)$ of R (of dimension k)* consists of an affine linear pencil

$$A = A_0 \otimes 1 + A_1 \otimes x_1 + \cdots + A_d \otimes x_d$$

in d variables and with coefficients $A_0, A_1, \dots, A_d$ from $M_k(\mathbb{C})$ and matrices $u \in M_{p\times k}(\mathbb{C})$ and $v \in M_{k\times q}(\mathbb{C})$, such that the following condition is satisfied: for every unital complex algebra $\mathcal{A}$, we have that $\mathrm{dom}_{\mathcal{A}}(R) \subseteq \mathrm{dom}_{\mathcal{A}}(A^{-1})$ and for each $X \in \mathrm{dom}_{\mathcal{A}}(R)$ it holds true that $R(X) = uA(X)^{-1}v$, where $A(X) \in M_k(\mathcal{A})$.

Note that we use here a different sign convention by requiring $R(X) = uA(X)^{-1}v$ instead of $R(X) = -uA(X)^{-1}v$; this, however, does not affect the validity of the particular results that we will take from [44]. Furthermore, as we will exclusively work with formal linear representations for matrix-valued non-commutative rational expressions, we will go without specifying them as matrix-valued formal linear representations like it was done in [44].

It follows from [44, Theorem 4.12] that indeed every matrix-valued non-commutative rational expression R admits a formal linear representation $\rho = (u, A, v)$. For the reader's convenience, we include with Theorem 2.28 the precise statement as well as its constructive proof. In doing so, we will see that Algorithm 4.11 in [44], on which the proof of Theorem 4.12 in the same paper relies, provides a formal linear representation $\rho = (u, A, v)$ of the $p \times q$ matrix-valued non-commutative rational expression R with the additional property that the dimension k of ρ is larger than $\max\{p, q\}$ and that both u and v have maximal rank; we will call such ρ *proper.* Note that if R is a scalar-valued rational expression, then a proper formal linear representation ρ simply means that u and v are non-zero vectors. In general, due to the restriction $k \geq \max\{p, q\}$, we have that the rank of u is p and the rank of v is q for any proper formal linear representation $\rho = (u, A, v)$ of R. This notion of proper formal linear representation will be important in the sequel.

Theorem 2.28 (Theorem 4.12 in [44])**.** *Every matrix-valued non-commutative rational expression admits a formal linear representation in the sense of Definition 2.27 which is also proper.*

Proof. Here, we give the algorithm which inductively builds a proper linear representation of any matrix-valued non-commutative rational expression. For $R = A\otimes 1$ or $R = A \otimes x_j$ for some $A \in M_{p\times q}(\mathbb{C})$ and $1 \leq j \leq d$ we have

$$R(X) = \begin{pmatrix} I_p & 0_{p\times q} \end{pmatrix} \begin{pmatrix} I_p \otimes 1_{\mathcal{A}} & -R(X) \\ 0_{q\times p} & I_q \otimes 1_{\mathcal{A}} \end{pmatrix}^{-1} \begin{pmatrix} 0_{p\times q} \\ I_q \end{pmatrix}$$

where $I_p \in M_p(\mathbb{C})$ is an identity matrix. Clearly, we obtain a proper formal linear representation in this way.

If the $p\times q$ matrix-valued non-commutative expressions R_1 and R_2 admit proper formal linear representations (u_1, A_1, v_1) and (u_2, A_2, v_2) then we have

$$(R_1 + R_2)(X) = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \left(\begin{array}{c|c} A_1(X) & 0_{k_1\times k_2} \\ \hline 0_{k_2\times k_1} & A_2(X) \end{array}\right)^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

This gives us a proper formal linear representation since $(u_1\ u_2)$, resp. $(v_1\ v_2)^T$ is of rank p, resp. q.

If R_1, R_2 are $p \times q$, resp. $q \times r$ matrix-valued non-commutative rational expressions and admit formal linear representations (u_1, A_1, v_1), resp. (u_2, A_2, v_2) of dimension k_1, resp. k_2 then we have

$$(R_1 \cdot R_2)(X) = \begin{pmatrix} u_1 & 0_{p\times k_2} \end{pmatrix} \left(\begin{array}{c|c} A_1(X) & -v_1 u_2 \\ \hline 0_{k_2\times k_1} & A_2(X) \end{array}\right)^{-1} \begin{pmatrix} 0_{k_1\times r} \\ v_2 \end{pmatrix}.$$

We obtain a proper formal linear representation since $(u_1\ 0_{p\times k_2})$, resp. $(0_{k_1\times r}\ v_2)^T$ is of rank p, resp. q.

If R is a $p \times p$ matrix-valued non-commutative rational expression which admits a formal linear representation (u, A, v) of dimension k, then we have

$$R^{-1}(X) = \begin{pmatrix} I_p & 0_{p\times k} \end{pmatrix} \left(\begin{array}{c|c} 0_{p\times p} & u \\ \hline v & A(X) \end{array}\right)^{-1} \begin{pmatrix} -I_p \\ 0_{k\times p} \end{pmatrix},$$

where X belongs to an appropriate domain for each step. It is clear that this gives us a proper formal linear representation.

Finally, since all matrix-valued non-commutative rational expressions can be represented by finitely many of the above steps, any matrix-valued non-commutative rational expression has such a formal linear representation which is proper. □

In the non-degenerate case, formal linear representations are connected with the concept of representations for non-commutative rational functions which is used, for instance, in [21, 22]; this will be addressed in Remark 2.30 and Remark 2.33. Before, we need to recall the following terminology.

Definition 2.29 (Inner rank and fullness)**.** Let $\mathcal{R}$ be a ring. For $A \in M_{n\times m}(\mathcal{R})$ we define the *inner rank* $\rho_{\mathcal{R}}(A)$ by

$$\rho_{\mathcal{R}}(A) = \min\{r \in \mathbb{N} \mid A = BC,\ B \in M_{n\times r}(\mathcal{R}),\ C \in M_{r\times m}(\mathcal{R})\},$$

and $\rho_{\mathcal{R}}(0) = 0$. In addition we call A *full* if $\rho_{\mathcal{R}}(A) = \min\{n, m\}$.

Remark 2.30. (i) Let A be a matrix over non-commutative polynomials in a tuple $x = (x_1, \dots, x_d)$ of formal variables. According to Theorem 7.5.13 in [23] (see also A.2 in [59]), we have

$$\rho_{\mathbb{C}\langle x\rangle}(A) = \rho_{\mathbb{C}\,\langle\!\!\!<\, x \,\rangle\!\!\!>}(A).$$

For this reason, we just say A is full, for a square matrix A over the non-commutative polynomials, without mentioning which algebra we consider.

(ii) Let A be an affine linear pencil in x with coefficients taken from $M_k(\mathbb{C})$. We may view A as an element in $M_k(\mathbb{C}) \otimes \mathbb{C}\langle x\rangle \cong M_k(\mathbb{C}\langle x\rangle)$, i.e., $A = A(x)$ is considered as a matrix over the ring $\mathbb{C}\langle x\rangle$. We notice that if there exists a tuple $X \in M_N(\mathbb{C})^d$ such that $A(X)$ is invertible in $M_k(\mathbb{C}) \otimes M_N(\mathbb{C}) \cong M_{kN}(\mathbb{C})$, or equivalently, if $\mathrm{dom}_{M(\mathbb{C})}(A^{-1}) \neq \emptyset$, then A must be full. In fact, if A is not full, then any factorization $A = BC$ with $B \in M_{k\times r}(\mathbb{C}\langle x\rangle)$ and $C \in M_{r\times k}(\mathbb{C}\langle x\rangle)$ for $r = \rho(A) < k$ yields under evaluation $A(X) = B(X)C(X)$ at any point $X \in M_N(\mathbb{C})^d$, so that $A(X)$ is never invertible.

On the other hand, if A is full, then A is invertible as a matrix over $\mathbb{C}\,\langle\!\!\!<\, x \,\rangle\!\!\!>$. Indeed, fullness and invertibility are equivalent for any skew field (see Lemma 5.20 in [59]).

(iii) Now, let R be a non-degenerate matrix-valued non-commutative rational expression. From Theorem 2.28, we know that there exists a formal linear representation $\rho = (u, A, v)$; in particular, we have that

$$\begin{aligned}\operatorname{dom}_{M(\mathbb{C})}(R) &\subseteq \operatorname{dom}_{M(\mathbb{C})}(A^{-1})\\ &= \coprod_{N=1}^{\infty} \{X \in M_N(\mathbb{C})^d \mid A(X) \text{ invertible in } M_{kN}(\mathbb{C})\}.\end{aligned}$$

Since R is non-degenerate, we find $X \in \operatorname{dom}_{M(\mathbb{C})}(R)$; from the aforementioned inclusion and (ii), we infer that A is a full matrix.

(iv) Suppose that R is a non-degenerate $p \times p$ matrix-valued non-commutative rational expression such that R^{-1} is non-degenerate as well. Let $\rho = (u, A, v)$ be a formal linear representation of R; we associate to ρ the affine linear pencil

$$\tilde{A} := \begin{pmatrix} 0_{p\times p} & u \\ v & A \end{pmatrix}.$$

We claim that both A and $\tilde{A}$ are full. For A, we already know from (iii) that this is true. To check fullness of $\tilde{A}$, we use that R^{-1} is non-degenerate, which guarantees the existence of some $X \in \operatorname{dom}_{M(\mathbb{C})}(R^{-1})$. Since in particular $X \in \operatorname{dom}_{M(\mathbb{C})}(R)$, we get as ρ is a formal linear representation of R that $A(X)$ is invertible and $R(X) = uA(X)^{-1}v$. Because $X \in \operatorname{dom}_{M(\mathbb{C})}(R^{-1})$, we know that $R(X)$ is invertible. Hence, by the Schur complement formula, it follows that the matrix $\tilde{A}(X)$ is invertible. Thanks to (ii), this implies that the affine linear pencil $\tilde{A}$ is full.

The following lemma explains that non-degenerate matrix-valued non-commutative rational expressions induce in some very natural way matrices over the free field.

Lemma 2.31. *Let R be a $p \times q$ matrix-valued non-commutative rational expression in d formal variables. If R is non-degenerate, then $x = (x_1, \dots, x_d) \in \operatorname{dom}_{\mathbb{C}\langle\!\langle x_1,\dots,x_d\rangle\!\rangle}(R)$ and consequently $R(x) \in M_{p\times q}(\mathbb{C}\langle\!\langle x\rangle\!\rangle)$.*

Proof. Let us denote by $\mathfrak{R}_0$ the set of all non-degenerate matrix-valued non-commutative rational expressions R which have the property $x \in \operatorname{dom}_{\mathbb{C}\langle\!\langle x\rangle\!\rangle}(R)$. We want to show that $\mathfrak{R}_0$ consists of all non-degenerate matrix-valued non-commutative rational expressions. In order to verify this assertion, we proceed as follows. Firstly, we notice that both $R_1 + R_2$ and $R_1 \cdot R_2$ belong to $\mathfrak{R}_0$ whenever we take $R_1, R_2 \in \mathfrak{R}_0$ for which the respective arithmetic operation is defined. Secondly, we consider some $R \in \mathfrak{R}_0$ which is of size $p \times p$ and has the property that R^{-1} is non-degenerate. By Theorem 2.28, there exists a formal linear representation $\rho = (u, A, v)$ of R, say of dimension k, and according to Remark 2.30 (iv) we have that both A and the associated affine linear pencil

$$\tilde{A} := \begin{pmatrix} 0_{p\times p} & u \\ v & A \end{pmatrix}$$

are full, i.e., $A(x) \in M_k(\mathbb{C}\langle x\rangle)$ and $\tilde{A}(x) \in M_{k+p}(\mathbb{C}\langle x\rangle)$ become invertible as matrices over the free field $\mathbb{C}\langle\!\langle x\rangle\!\rangle$. Since $x \in \operatorname{dom}_{\mathbb{C}\langle\!\langle x\rangle\!\rangle}(R)$ as $R \in \mathfrak{R}_0$, we get $R(x) = uA(x)^{-1}v$, because ρ is a formal linear representation of R. Putting these observations together, the Schur complement formula yields that $R(x) \in M_p(\mathbb{C}\langle\!\langle x\rangle\!\rangle)$ must be invertible, i.e., $x \in \operatorname{dom}_{\mathbb{C}\langle\!\langle x\rangle\!\rangle}(R^{-1})$ and thus $R^{-1} \in \mathfrak{R}_0$, as desired. ∎

Remark 2.32. With arguments similar to the proof of Lemma 2.31 as based on Remark 2.30 (iv), one finds that if R_1, R_2 are non-degenerate matrix-valued non-commutative rational expressions satisfying $R_1(x) = R_2(x)$, then $R_1 \sim_{M(\mathbb{C})} R_2$. In other words, matrix identities over $\mathbb{C}\langle\!\langle x \rangle\!\rangle$ are preserved under well-defined matrix evaluations.

Remark 2.33. In the scalar-valued case, the conclusion of Lemma 2.31 can be strengthened slightly. For that purpose, it is helpful to denote the formal variables out of which the non-commutative rational expressions are built by $\chi_1, \dots, \chi_d$ in order to distinguish them from the variables $x_1, \dots, x_d$ of the free skew field $\mathbb{C}\langle\!\langle x_1, \dots, x_d \rangle\!\rangle$; note that accordingly $x_j = [\chi_j]$ for $j = 1, \dots, d$. Now, if r is any scalar-valued non-commutative rational expression in the formal variables $\chi_1, \dots, \chi_d$, then Lemma 2.31 tells us that $x \in \mathrm{dom}_{\mathbb{C}\langle\!\langle x \rangle\!\rangle}(r)$ and $r(x) \in \mathbb{C}\langle\!\langle x \rangle\!\rangle$. Moreover, we have the equality $r(x) = [r]$. This can be shown with a recursive argument similar to the proof of Lemma 2.31; notice that if a non-degenerate rational expression r satisfies $r(x) = [r]$ and has the additional property that r^{-1} is non-degenerate, then $r(x) = [r]$ is invertible in $\mathbb{C}\langle\!\langle x \rangle\!\rangle$, which implies $x \in \mathrm{dom}_{\mathbb{C}\langle\!\langle x \rangle\!\rangle)}(r^{-1})$ with $r^{-1}(x) = [r]^{-1} = [r^{-1}]$.

This has the consequence that every formal linear representation $\rho = (u, A, v)$ of r satisfies $[r] = r(x) = uA(x)^{-1}v$. In the language of [21, 22], this means that the formal linear representation ρ of r induces a (pure and linear) representation of the corresponding non-commutative rational function $[r]$.

2.2.3. *Self-adjointness for matrix-valued non-commutative rational expressions.* When evaluations of matrix-valued non-commutative rational expressions R at points $X = (X_1, \dots, X_d) \in \mathrm{dom}_{\mathcal{A}}(R)$ for $*$-algebras $\mathcal{A}$ are considered, it is natural to ask for conditions which guarantee that the result $R(X)$ is self-adjoint, i.e., $R(X)^* = R(X)$. Those conditions shall concern the matrix-valued non-commutative rational expression R itself, but depending on the particular type of its the arguments $X_1, \dots, X_d$. The case when $X_1, \dots, X_d$ are all self-adjoint was discussed in [44, Section 2.5.7]. The following definition generalizes the latter to matrix-valued non-commutative rational expressions in self-adjoint and unitary variables.

Recall that an element X in a complex $*$-algebra $\mathcal{A}$ with unit 1_A is called *self-adjoint* if $X^* = X$, and $U \in \mathcal{A}$ is said to be *unitary* if $U^*U = 1_{\mathcal{A}} = UU^*$.

Definition 2.34 (Self-adjoint matrix-valued non-commutative rational expressions)**.** Let R be a square matrix-valued non-commutative rational expression in $d_1 + d_2$ formal variables which we denote by $x_1, \dots, x_{d_1}, u_1, \dots, u_{d_2}$. We say that R is *self-adjoint of type* (d_1, d_2), if for every unital complex $*$-algebra $\mathcal{A}$ and all tuples $X = (X_1, \dots, X_{d_1})$ and $U = (U_1, \dots, U_{d_2})$ of self-adjoint respectively unitary elements in $\mathcal{A}$, the following implication holds:

$$(X, U) \in \mathrm{dom}_{\mathcal{A}}(R) \qquad \Longrightarrow \qquad R(X, U)^* = R(X, U)$$

One comment on this definition is in order. The reader might wonder why the matrix-valued non-commutative rational expressions do not explicitly involve further variables $u_1^*, \dots, u_{d_2}^*$ serving as placeholder for the adjoints of $u_1, \dots, u_{d_2}$. In fact, for (scalar-valued) non-commutative rational expressions such an approach was presented, for instance, in the appendix of [36] (a version for non-commutative polynomials appears also in [83]); more precisely, non-commutative rational expressions in collections of self-adjoint variables x, non-self-adjoint variables y, and their adjoints y^* were considered. For our purpose, however, this has the slight disadvantage that non-degenerate non-commutative rational expressions of this kind (take $r(y, y^*) = (yy^* - 1)^{-1}$, for example) may have no unitary elements in their domain. On the other hand, there are non-commutative rational expressions (or even non-commutative polynomial expressions such as $yyy^* + y^*yy^*$) which are not self-adjoint on their entire domain but self-adjoint on unitaries.

The following example illustrates Definition 2.34 and highlights the effect of having two types of variables.

Example 2.35. $x_1+x_2^{-1}$, $i(u_1-u_1^{-1})$ and $u_1^{-1}x_1^{-1}u_1$ are self-adjoint non-commutative rational expressions since we have for self-adjoint elements X_1, X_2 and a unitary U_1 in their domain,

$$\begin{aligned} (X_1 + X_2^{-1})^* &= X_1^* + (X_2^*)^{-1} = X_1 + X_2^{-1} \\ [i(U_1 - U_1^{-1})]^* &= -i(U_1^* - (U_1^*)^{-1}) = i(U_1 - U_1^{-1}) \\ (U_1^{-1}X_1^{-1}U_1)^* &= U_1^*(X_1^*)^{-1}(U_1^*)^{-1} = U_1^{-1}X_1^{-1}U_1. \end{aligned}$$

However, $u_1 + u_2^{-1}$, $i(x_1 - x_1^{-1})$ and $x_1^{-1}u_1^{-1}x_1$ are not self-adjoint in our definition. So we need to be careful to the roles of formal variables when we consider self-adjoint rational expressions. For the matrix-valued case, the 2×2 respectively 1×1 matrix-valued non-commutative rational expressions

$$\begin{pmatrix} x_1^{-1} & u_1 \\ u_1^{-1} & x_2^{-1} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} u_1 & x_1 + iu_2 \end{pmatrix} \begin{pmatrix} 1 & -iu_1 \\ iu_1^{-1} & x_2 \end{pmatrix}^{-1} \begin{pmatrix} u_1^{-1} \\ x_1 - iu_2^{-1} \end{pmatrix}$$

are self-adjoint of type $(2, 1)$ and $(2, 2)$, respectively.

Like in [44, Definition 4.13] for the case of self-adjoint arguments, we can introduce self-adjoint formal linear representations; see also [36, Definition A.5] for the scalar-valued case.

Note that in order to make the machinery of self-adjoint linearizations ready for further applications, we will switch from now on to a more general situation.

Definition 2.36 (Self-adjoint formal linear representation)**.** Let R be a $p \times p$ matrix-valued non-commutative rational expression in d formal variables $x_1, \dots, x_d$. A tuple $\rho = (Q, w)$ consisting of an affine linear pencil

$$Q = A_0 \otimes 1 + \sum_{j=1}^{d} \left(B_j \otimes x_j + B_j^* \otimes x_j^*\right)$$

in the formal variables $x_1, \dots, x_d$ and $x_1^*, \dots, x_d^*$, with coefficients being (not necessarily self-adjoint) matrices $B_1, \dots, B_d$ in $M_k(\mathbb{C})$ for some $k \in \mathbb{N}$, some self-adjoint matrix $A_0 \in M_k(\mathbb{C})$ and some matrix $w \in M_{k\times p}(\mathbb{C})$ is called a *self-adjoint formal linear representation of R (of dimension k)* if the following condition is satisfied: for every unital complex $*$-algebra $\mathcal{A}$ and all tuples $X = (X_1, \dots, X_d)$ of (not necessarily self-adjoint) elements in $\mathcal{A}$, one has

$$X \in \mathrm{dom}_{\mathcal{A}}(R) \qquad \Longrightarrow \qquad (X, X^*) \in \mathrm{dom}_{\mathcal{A}}(Q^{-1})$$

and for every $X \in \mathrm{dom}_{\mathcal{A}}(R)$ for which $R(X)$ is self-adjoint, it holds true that

$$R(X) = w^*Q(X, X^*)^{-1}w.$$

We point out that in contrast to the related concept introduced in [44, Definition 4.13] the existence of a self-adjoint formal linear representation in the sense of the previous Definition 2.36 does not enforce R to be self-adjoint at any distinguished points in its domain. In fact, we have the following theorem which says that every square matrix-valued non-commutative rational expression admits a self-adjoint formal linear representation; this is analogous to [44, Theorem 4.14].

Like for formal linear representations, we will say that a self-adjoint formal linear representation $\rho = (Q, w)$ of a self-adjoint $p \times p$ matrix-valued non-commutative rational expression R is *proper* if the dimension k of ρ is larger than p and if w has full rank (i.e., the rank of w is p).

Theorem 2.37. *Every square matrix-valued non-commutative rational expression in d formal variables admits a self-adjoint formal linear representation in the sense of Definition 2.36 which is proper.*

Proof. Let $\rho = (v, Q, w)$ be a formal linear representation of R in the variables $x_1, \dots, x_d$ with the affine linear pencil Q being of the form

$$Q = A_0 \otimes 1 + \sum_{j=1}^{d} B_j \otimes x_j.$$

We consider $\widetilde{\rho} = (\widetilde{Q}, \widetilde{w})$ with the affine linear pencil

$$\widetilde{Q} = \widetilde{A}_0 \otimes 1 + \sum_{j=1}^{d} \big(\widetilde{B}_j \otimes x_j + \widetilde{B}_j^* \otimes x_j^*\big)$$

in the variables $x_1, \dots, x_d$, $x_1^*, \dots, x_d^*$ given by

$$\widetilde{A}_0 := \begin{pmatrix} 0 & A_0^* \\ A_0 & 0 \end{pmatrix}, \qquad \widetilde{B}_j := \begin{pmatrix} 0 & 0 \\ B_j & 0 \end{pmatrix}, \qquad \text{and} \qquad \widetilde{w} := \begin{pmatrix} \frac{1}{2} v^* \\ w \end{pmatrix}.$$

One verifies that $\widetilde{\rho} = (\widetilde{Q}, \widetilde{w})$ is a self-adjoint formal linear representation of R which is moreover proper whenever ρ is proper. □

Notice that if R is a $p \times p$ matrix-valued non-commutative rational expression in $d_1 + d_2$ formal variables $x_1, \dots, x_{d_1}$, $u_1, \dots, u_{d_2}$ which is self-adjoint of type (d_1, d_2), then each self-adjoint formal linear representation of R can be brought into the simplified form $\rho = (Q, w)$ with an affine linear pencil

$$Q = A_0 \otimes 1 + \sum_{j=1}^{d_1} A_j \otimes x_j + \sum_{j=1}^{d_2} \big(B_j \otimes u_j + B_j^* \otimes u_j^*\big)$$

in the formal variables $x_1, \dots, x_{d_1}$, $u_1, \dots, u_{d_2}$, $u_1^*, \dots, u_{d_2}^*$ with coefficients being self-adjoint matrices $A_0, A_1, \dots, A_{d_1}$ and (not necessarily self-adjoint) matrices $B_1, \dots, B_{d_2}$ in $M_k(\mathbb{C})$ for some $k \in \mathbb{N}$ and some matrix $w \in M_{k \times p}(\mathbb{C})$; indeed Theorem 2.37 yields a self-adjoint formal linear representation of R with an affine linear pencil in the formal variables $x_1, \dots, x_{d_1}$, $x_1^*, \dots, x_{d_1}^*$ and $u_1, \dots, u_{d_2}$, $u_1^*, \dots, u_{d_2}^*$, from which we obtain Q of the asserted form by replacing $x_1^*, \dots, x_{d_1}^*$ by $x_1, \dots, x_{d_1}$ and merging their coefficients. In particular, we have

$$(X, U) \in \operatorname{dom}_{\mathcal{A}}(R) \qquad \Longrightarrow \qquad (X, U, U^*) \in \operatorname{dom}_{\mathcal{A}}(Q^{-1})$$

and for every $(X, U) \in \operatorname{dom}_{\mathcal{A}}(R)$ it holds true that

$$R(X, U) = w^* Q(X, U, U^*)^{-1} w.$$

Example 2.38. We return to the self-adjoint non-commutative rational expressions presented in Example 2.35. Let us construct a self-adjoint formal linearization of $x_1 + x_2^{-1}$. Using the algorithm from [44] which we recalled in the proof of Theorem 2.28, we obtain first a formal linear representation

$$X_1 + X_2^{-1} = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -X_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & X_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Out of the latter, we construct with the help of Theorem 2.37 the self-adjoint formal linear representation

$$X_1 + X_2^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -X_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & X_2 \\ 1 & -X_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & X_2 & 0 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

The second example is $u_1 + u_1^{-1}$. Since we have for unitary U_1 in any $*$-algebra

$$U_1 + U_1^{-1} = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -U_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & U_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

we have a formal self-adjoint linearization

$$U_1 + U_1^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -U_1^* & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & U_1^* \\ 1 & -U_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & U_1 & 0 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

2.2.4. *Unbounded random variables.* In this subsection, we set $(\mathcal{M}, \tau)$ to be a tracial W^*-probability space (i.e., a von Neumann algebra $\mathcal{M}$ that is endowed with a faithful normal tracial state $\tau : \mathcal{M} \to \mathbb{C}$). The condition that τ is a trace is necessary since we are going to consider closed and densely defined operators affiliated with the von Neumann algebra $\mathcal{M}$. We will simply call these operators unbounded operators. In general, unbounded operators might not well-behave under either addition or composition. However, in the case of tracial W^*-probability space, they form a $*$-algebra, denoted by $\widetilde{\mathcal{M}}$, which provides us a framework in which one has well-defined evaluations of rational expressions.

In a language of probability, this framework allows us to consider random variables that may not have compact support or even finite moments. For a normal random variable X in a W^*-probability space $(\mathcal{M}, \tau)$, we know that X has finite moments of all orders and its analytic distribution μ_X determined by the moments (i.e., the probability measure associated to X by a representation theorem of Riesz) has a compact support. For an (unbounded) operator X in $\widetilde{\mathcal{M}}$, it may not have finite moments. But we could still associate a probability measure to X via the spectral theorem. We refer the interested reader to [66, 8] for more details on unbounded operators (which are also known as measurable operators as the non-commutative analogue of measurable functions, cf. [82]).

Let $\mathcal{P}(\mathcal{M})$ denote the set of self-adjoint projections in $\mathcal{M}$ and let $\widetilde{\mathcal{M}}_{sa}$ be the set of self-adjoint elements in $\widetilde{\mathcal{M}}$. Given an element $X \in \widetilde{\mathcal{M}}_{\mathrm{sa}}$, for a Borel set B on $\mathbb{R}$, we denote by $\mathbf{1}_B(X) \in \mathcal{P}(\mathcal{M})$ the spectral projection of X on B given by the spectral theorem (see, for example, [30]). Then we can associate a probability measure μ_X to X as follows.

Definition 2.39. For $X \in \widetilde{\mathcal{M}}_{\mathrm{sa}}$, we define its *analytic distribution* μ_X by

$$\mu_X(B) := \tau(\mathbf{1}_B(X)), \quad \text{for all Borel sets } B \subseteq \mathbb{R}.$$

Furthermore, we define the *cumulative distribution function* of X as the function $\mathcal{F}_X : \mathbb{R} \to [0,1]$ given by,

$$\mathcal{F}_X(t) := \int_{-\infty}^{t} 1 d\mu_X(s) = \tau(\mathbf{1}_{(-\infty,t]}(X)).$$

In particular, if we take $\mathcal{M} = L^\infty(\Omega, \mathbb{P})$ and $\tau = \mathbb{E}$ for some probability measure space $(\Omega, \mathbb{P})$, then $\widetilde{\mathcal{M}}$ is the $*$-algebra consisting of all measurable functions, i.e., classical random variables. Moreover, the analytic distribution and cumulative distribution defined above coincide with their classical counterparts.

Recall that for a probability measure μ on $\mathbb{R}$. A number $\lambda \in \mathbb{R}$ is called an *atom* of μ if $\mu(\{\lambda\}) \neq 0$. Thus for a random variable X in $\widetilde{\mathcal{M}}_{\mathrm{sa}}$, we say that $\lambda \in \mathbb{R}$ is an atom for X if λ is an atom for μ_X. Moreover, we see that X has an atom $\lambda \in \mathbb{R}$ if and only if $p_{\ker(\lambda - X)} \neq 0$, where $p_{\ker(\lambda-X)} \in \mathcal{P}(\mathcal{M})$ is the orthogonal projection onto the kernel of $\lambda - X$ (in the Hilbert space $L^2(\mathcal{M}, \tau)$). For an atom λ of X, we have

$$\mu_X(\{\lambda\}) = \tau(p_{\ker(\lambda-X)}).$$

A closely related notion is a rank defined via the image. That is, we define

$$\mathrm{rk}(X) := \tau(p_{\overline{\mathrm{im} X}}),$$

where $p_{\overline{\mathrm{im} X}}$ is the orthogonal projection onto the closure of the image of X. The following alternative description of this rank will be needed later:

$$\mathrm{rk}(X) = \inf\{\tau(r) \mid r \in \mathcal{P}(\mathcal{M}),\ rX = X\}. \tag{1}$$

Clearly, since $p_{\overline{\mathrm{im}(X)}} X = X$, we have $\inf\{\tau(r) \mid r \in \mathcal{P}(\mathcal{M}), rX = X\} \leq \tau(p_{\overline{\mathrm{im}(X)}}) = \mathrm{rk}(X)$. To see it is an equality, note that for any $r \in \mathcal{P}(\mathcal{M})$ satisfying $rX = X$, $\mathrm{im}(X) \subseteq \mathrm{im}(r)$, which implies that $p_{\overline{\mathrm{im}(X)}} \leq r$.

2.2.5. *The quantity* Δ. The regularity condition which we impose in Theorem 3.10 on the limit of the considered random matrix model involves the quantity Δ which was introduced by Connes and Shlyakhtenko in [29]. We briefly recall the definition. Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space and consider a tuple $x = (x_1, \dots, x_d)$ of (not necessarily self-adjoint) non-commutative random variables in $\mathcal{M}$. We denote by $\mathcal{F}(L^2(\mathcal{M}, \tau))$ the ideal of all finite rank operators on $L^2(\mathcal{M}, \tau)$ and by J Tomita's conjugation operator, i.e., the conjugate-linear map $J : L^2(\mathcal{M}, \tau) \to L^2(\mathcal{M}, \tau)$ that extends isometrically the conjugation $x \mapsto x^*$ on $\mathcal{M}$. We then put

$$\Delta(x) := d - \dim_{\mathcal{M} \overline{\otimes} \mathcal{M}^{\mathrm{op}}} \overline{\left\{ (T_1, \dots, T_d) \in \mathcal{F}(L^2(\mathcal{M}, \tau))^d \;\middle|\; \sum_{j=1}^{d} [T_j, J x_j^* J] = 0 \right\}}^{\mathrm{HS}},$$

where the closure is taken with respect to the Hilbert-Schmidt norm. Note that in contrast to [29], we do not require the set $\{x_1, \dots, x_d\}$ to be closed under the involution $*$; see also [60]. Despite this slight deviation from the setting of [29], the following result remains true.

Theorem 2.40 (Theorem 3.3 (e) in [29])**.** *Let $1 \leq k < d$ and suppose that the sets $\{x_1, \dots, x_k\}$ and $\{x_{k+1}, \dots, x_d\}$ are freely independent, then*

$$\Delta(x_1, \dots, x_d) = \Delta(x_1, \dots, x_k) + \Delta(x_{k+1}, \dots, x_d).$$

Further, we recall from [60, Corollary 6.4] that $\Delta(u) = d$ for every d-tuple u of freely independent Haar unitary elements in $(\mathcal{M}, \tau)$.

In the particular case of a d-tuple x consisting of self-adjoint operators in $\mathcal{M}$, Corollary 4.6 in [29] says that $d \geq \Delta(x) \geq \delta(x)$, where $\delta(x)$ denotes the so-called *microstates free entropy dimension* which was introduced by Voiculescu in [86, Definition 6.1]. Now, if the $x_1, \dots, x_d$ are freely independent, then Proposition 6.4 in [86] tells us that

$$\delta(x) = d - \sum_{j=1}^{d} \sum_{t \in \mathbb{R}} \mu_{x_j}(\{t\})^2,$$

where μ_{x_j} is the analytic distribution of the operator x_j in the sense of Definition 2.39. We infer that $\Delta(x_1, \dots, x_d) = d$ if $x_1, \dots, x_d$ are self-adjoint, freely independent and their individual analytic distributions $\mu_{x_1}, \dots, \mu_{x_d}$ are all non-atomic. For reference, we summarize these observations by the following corollary.

Corollary 2.41. *Let $x = (x_1, \dots, x_{d_1})$ be a d_1-tuple of self-adjoint and freely independent elements in $(\mathcal{M}, \tau)$ with $\mu_{x_1}, \dots, \mu_{x_{d_1}}$ being non-atomic. Further, let $u = (u_1, \dots, u_{d_2})$ be a d_2-tuple of freely independent Haar unitary elements in $(\mathcal{M}, \tau)$. Suppose that x and u are freely independent. Then $\Delta(x, u) = d_1 + d_2$.*

2.3. **Non-commutative rational power series and the fundamental theorem.** For $d \in \mathbb{N}$, $[d]^*$ denotes the set of words that consist of letters in $[d]$ with the empty word Ω. In other words, $[d]^*$ is the free semigroup with generators $[d]$ and the identity Ω.

We consider the algebra $\mathbb{C}\langle\langle X_1, \dots X_d \rangle\rangle$ of *non-commutative formal power series* with formal (non-commutative) variables $\{X_i\}_{i \in [d]}$ like as

$$\sum_{v \in [d]^*} \alpha_v X^v$$

where $X^v = X_{v_1} X_{v_2} \cdots X_{v_n}$ for $v = v_1 v_2 \cdots v_n \in [d]^*$ and $X^\Omega = 1$. Let $\mathbb{C}\langle X_1, \dots, X_d \rangle$ denote the subalgebra of non-commutative polynomials.

To define the notion of rational series, we give two definitions of rationality in a setting of unital algebras (over $\mathbb{C}$) as follows (see [7, Definition 6] or [60, Definition 4.6 and 4.8]).

Definition 2.42. Let $\mathcal{A}$ be a unital algebra and $\mathcal{B} \subset \mathcal{A}$ be a unital subalgebra of $\mathcal{A}$. We define the *division closure* of $\mathcal{B}$ in $\mathcal{A}$ as the smallest unital subalgebra $\mathcal{C}$ of $\mathcal{A}$ such that $\mathcal{C}$ contains $\mathcal{B}$ and satisfies

$$x \in \mathcal{C} \text{ is invertible in } \mathcal{A} \implies x^{-1} \in \mathcal{C}.$$

In addtion, we define the *rational closure* of $\mathcal{B}$ in $\mathcal{A}$ as the smallest (unital) subalgebra D of $\mathcal{A}$ such that $\mathcal{D}$ contains $\mathcal{B}$ and satisfies for any $n \in \mathbb{N}$,

$$X \in M_n(\mathcal{D}) \text{ is invertible in } M_n(\mathcal{A}) \implies X^{-1} \in M_n(\mathcal{D}).$$

Obviously, the division closure of any subalgebra is always contained in the rational closure of the same subalgebra, however, the converse is not necessarily true (Exercise 7.1.3 in [23]).

We will use facts for non-commutative rational series specialized for our notations. The proofs of these results can be found in [7, Chapter 1].

Definition 2.43. Let $Z = \sum_{v\in[d]^*} \alpha_v X^v \in \mathbb{C}\langle\langle X_1, \dots X_d\rangle\rangle$. We say Z is *recognizable* if there exists $m \in \mathbb{N}$ and a *linear representation* (λ, μ, γ) of dimension m which consists of a multiplicative map $\mu : [d]^* \to M_m(\mathbb{C})$ (i.e. $\mu(vw) = \mu(v)\mu(w)$ for any $v, w \in [d]^*$) and $\lambda, \gamma \in \mathbb{C}^m$ such that for any $v \in [d]^*$

$$\alpha_v = {}^t\lambda\mu(v)\gamma.$$

Let us say Z is *rational* if Z belong to the division closure of $\mathbb{C}\langle X_1, \dots, X_d\rangle$ in $\mathbb{C}\langle\langle X_1, \dots X_d\rangle\rangle$. Then the following theorem, known as the fundamental theorem, is crucial in this paper.

To state the fundamental theorem, we introduce two operations on $[d]^*$, which correspond with right and left annihilation operators (see Section 2.4).

Definition 2.44. Let 0 be a new letter. For $v \in [d]^* \sqcup \{0\}$ and $w \in [d]^*$, we define

$$vw^{-1} = \begin{cases} v' \text{ if } v = v'w, \ v' \in [d]^* \\ 0 \text{ otherwise} \end{cases}$$

and also define

$$w^{-1}v = \begin{cases} v' \text{ if } v = wv', \ v' \in [d]^* \\ 0 \text{ otherwise.} \end{cases}$$

For non-commutative power series, we define $X^0 = 0$. Then the following result is called the Fundamental theorem, which is a collection of several works by Fliess, Jacobi, Kleene, and Schützenberger.

Theorem 2.45 (Corollary 1.5.4 and Theorem 1.7.1 in [7])**.** *Let* $Z = \sum_{v\in[d]^*} \alpha_v X^v \in \mathbb{C}\langle\langle X_1, \dots X_d\rangle\rangle$. *Then the following are equivalent.*

(i) A $\mathbb{C}$-vector subspace of $\mathbb{C}\langle\langle X_1, \dots X_d\rangle\rangle$ generated by $\sum_{v\in[d]^*} \alpha_v X^{vw^{-1}}$ ($w \in [d]^*$) is finitely generated.

(ii) A $\mathbb{C}$-vector subspace of $\mathbb{C}\langle\langle X_1, \dots X_d\rangle\rangle$ generated by $\sum_{v\in[d]^*} \alpha_v X^{w^{-1}v}$ ($w \in [d]^*$) is finitely generated.

(iii) Z is recognizable.

(iv) Z is rational.

Moreover, if a non-commutative formal power series is recognizable and its linear representation (λ, μ, γ) has the minimal dimension, then μ is determined by its coefficients. This can be stated as follows.

Theorem 2.46 (Corollary 2.2.3 in [7])**.** *Suppose* $Z = \sum_{v\in[d]^*} \alpha_v X^v \in \mathbb{C}\langle\langle X_1, \dots X_d\rangle\rangle$ *is recognizable with a linear representation (λ, μ, γ) which has the minimal dimension m. Then there exist $\{u_k\}_{k=1}^K, \{w_l\}_{l=1}^L \subset [d]^*$ and $c_{ij}^{kl} \in \mathbb{C}$ such that for any $v \in [d]^*$ and $1 \le i, j \le m$*

$$\mu(v)_{ij} = \sum_{kl} c_{ij}^{kl} \alpha_{u_k v w_l}.$$

In addition, we use an operation between non-commutative formal power series. For $Z_1 = \sum_{v\in[d]^*} \alpha_v X^v, Z_2 = \sum_{v\in[d]^*} \beta_v X^v \in \mathbb{C}\langle\langle X_1, \dots X_d\rangle\rangle$, we define the *Hadamard product* $Z_1 \odot Z_2$ by

$$Z_1 \odot Z_2 = \sum_{v\in[d]^*} \alpha_v \beta_v X^v.$$

One of the connections between the Hadamard product and rationality can be stated as follows.

Theorem 2.47 (Theorem 1.5.5 in [7]). *If $Z_1, Z_2 \in \mathbb{C}\langle\langle X_1, \dots X_d\rangle\rangle$ are rational, then $Z_1 \odot Z_2$ is also rational.*

2.3.1. *Kronecker's theorem.* We need to recall Kronecker's theorem which basically tells us the equivalence between bounded rational functions and finite rank Hankel operators.

Let $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{C}$. We call a bounded operator H on $l^2(\mathbb{Z}_{\geq 0})$ the *Hankel operator* with respect to $\{\alpha_n\}_{n=0}^{\infty}$ if H satisfies

$$\langle He_m, e_n\rangle = \alpha_{m+n}$$

for any $m, n \in \mathbb{Z}_{\geq 0}$ where $\{e_m\}_{m=0}^{\infty}$ is the standard orthonormal basis of $l^2(\mathbb{Z}_{\geq 0})$. The following theorem is known as Kronecker's theorem for the studies of Hankel operators (see [52] and [70, Theorem 3.11]).

Theorem 2.48. *Let $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{C}$. Then a formal Laurent series (in z^{-1}) $a(z) = \sum_{n=0}^{\infty} \alpha_n z^{-n-1}$ is a rational function (i.e. $a(z) = \frac{P(z)}{Q(z)}$ for some polynomials $P(z), Q(z)$) such that all poles of $a(z)$ are contained in $\{z \in \mathbb{C} \mid |z| < 1\}$ if and only if $\{\alpha_n\}_{n=0}^{\infty}$ determines a finite rank Hankel operator. In this case, the number of poles on f is equal to the rank of the Hankel operator.*

Here, we explain a related recursion and estimate in Theorem 2.48 in order to explain Corollary 2.49, which we will use in the proof of Corollary 4.6. Indeed, if $a(z) = \sum_{n=0}^{\infty} \alpha_n z^{-n-1}$ is rational and the denominator of $a(z)$ written as $Q(z) = \sum_{k=0}^{m} \lambda_k z^k$ ($\lambda_m \neq 0$), then we have the following recursion for $\{\alpha_n\}_{n=0}^{\infty}$

$$\sum_{k=0}^{m} \lambda_k \alpha_{n+k} = 0,$$

where $\{\alpha_n\}_{n=0}^{m-1}$ are determined by the numerator of $a(z)$. This recursion is characterized by the poles of $a(z)$, and if we additionally assume $\lim_{n\to\infty} \alpha_n = 0$, we can see that all poles of $a(z)$ are contained in $\{z \in \mathbb{C} \mid |z| < 1\}$ (see the proof of [70, Theorem 3.11]). Moreover, this implies $|\alpha_n|$ is bounded above by Mc^n where $M > 0$ and $c = \max\{|p| \mid p \text{ is a pole of } a(z)\}$.

By replacing $a(z)$ by $za(z^{-1})$, we obtain the following estimate from the above observation, which is used in the proof of [34, Lemma 10].

Corollary 2.49. *Let $a(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a formal power series with $\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty$. If $a(z)$ is rational, then there exists $M > 0$ and $0 < c < 1$ such that we have for any $n \in \mathbb{N}$*

$$|\alpha_n| \leq Mc^n.$$

2.4. ***q*-CCR and related operators.**

2.4.1. *q-CCR and q-Gaussians.* Let $H_{\mathbb{R}}$ be a real Hilbert space and $H_{\mathbb{C}}$ denote its amplification $H_{\mathbb{C}} = H_{\mathbb{R}} \oplus iH_{\mathbb{R}}$. We start from the algebraic Fock space $\mathcal{F}_{\text{alg}}(H_{\mathbb{C}})$ defined by

$$\mathcal{F}_{\text{alg}}(H_{\mathbb{C}}) = \bigoplus_{k=0}^{\infty} H_{\mathbb{C}}^{\otimes k},$$

where $H_{\mathbb{C}}^{\otimes 0} = \mathbb{C}\Omega$ with $\|\Omega\| = 1$ and the direct sum means all finite linear spans of $H_{\mathbb{C}}^{\otimes k}$'s.

Let q be a parameter in $[-1, 1]$. We introduce q-inner product $\langle \cdot, \cdot \rangle_q$ into $\mathcal{F}_{\text{alg}}(H_{\mathbb{C}})$ defined by

$$\langle \xi_1 \otimes \cdots \otimes \xi_m, \eta_1 \otimes \cdots \otimes \eta_n \rangle_q = \delta_{m,n} \sum_{\pi \in S_m} q^{\text{inv}(\pi)} \prod_{i=1}^{m} \langle \xi_i, \eta_{\pi(i)} \rangle_{H_{\mathbb{C}}}$$

where S_m is the symmetric group with degree m and $\text{inv}(\pi) = \#\{(i,j) \in [m]^2;\ i < j,\ \pi(i) > \pi(j)\}$ is the number of inversions in π. Actually, we can see that this inner product satisfies positivity.

Theorem 2.50 (Proposition 1 in [9])**.** *For $-1 \le q \le 1$ and $\xi \in \mathcal{F}_{\text{alg}}(H_{\mathbb{C}})$, $\langle \xi, \xi \rangle_q \ge 0$. Moreover, when $-1 < q < 1$, $\langle \xi, \xi \rangle_q > 0$ for $\xi \neq 0$.*

By completing $\mathcal{F}_{\text{alg}}(H_{\mathbb{C}})$ with respect to this q-inner product after dividing it out by the kernel of the seminorm, we obtain a Hilbert space $\mathcal{F}_q(H_{\mathbb{C}})$, so-called q-Fock space.

For $\xi \in H_{\mathbb{C}}$, we define left creation operator $l(\xi)$ by

$$l(\xi)\xi_1 \otimes \cdots \otimes \xi_n = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.$$

We can also compute the adjoint operator $l(\xi)^*$ (annihilation operators).

$$l(\xi)^*\xi_1 \otimes \cdots \otimes \xi_n = \sum_{k=1}^{n} q^{k-1} \langle \xi_k, \xi \rangle \xi_1 \otimes \cdots \otimes \check{\xi}_k \otimes \cdots \otimes \xi_n$$

where $\check{\xi}_k$ means omission of this tensor component. Note that these operators satisfy q-Canonical Commutation Relations (q-CCR):

$$l(\xi)^* l(\eta) - q l(\eta) l(\xi)^* = \langle \eta, \xi \rangle I \quad (\xi, \eta \in H_{\mathbb{C}}).$$

We state a basic property of left creation operators.

Theorem 2.51 (Lemma 4 in [9])**.** *For $-1 \le q < 1$ and $\xi \in H_{\mathbb{C}}$, $l(\xi) \in B(\mathcal{F}_q(H_{\mathbb{C}}))$ and we have*

$$\|l(\xi)\| = \frac{\|\xi\|}{\sqrt{1-q}} \quad (0 \le q < 1)$$
$$\|l(\xi)\| = \|\xi\| \quad (-1 \le q \le 0).$$

Remark 2.52. Dykema and Nica [35] showed that for $|q| < 0.44$ and $\dim H_{\mathbb{R}} < \infty$, the C*-algebra $T_q(H_{\mathbb{R}})$ generated by left creation operators $l(\xi)$ ($\xi \in H_{\mathbb{R}}$) is isomorphic to $T_0(H_{\mathbb{R}})$ which is called the Cuntz-Toeplitz algebra (cf. Kuzmin [55]).

We can also consider the right version of these operators. Namely, we define the right creation operator $r(\xi)$ by

$$r(\xi)\xi_1 \otimes \cdots \otimes \xi_n = \xi_1 \otimes \cdots \otimes \xi_n \otimes \xi.$$

Then the right creation operators also satisfy q-CCR and commute with the left creation operators.

One of the main objects of the thesis is q-Gaussian operators defined by

$$A^{(q)}(\xi) = l(\xi) + l(\xi)^*,$$

and the von Neumann algebra, so-called the q-Gaussian von Neumann algebra defined by the double commutant,

$$\Gamma_q(H_{\mathbb{R}}) = \left\{ A^{(q)}(\xi) \ ; \ \xi \in H_{\mathbb{R}} \right\}''.$$

This von Neumann algebra admits a faithful normal tracial state τ (called the vacuum state or the Fock state) defined by

$$\tau(X) = \langle X\Omega, \Omega \rangle_q.$$

The fact that τ is a trace on $\Gamma_q(H_{\mathbb{R}})$ can be checked by the q-analog of Wick formula

Theorem 2.53 ([38])**.** *For $\xi_1, \dots, \xi_n \in H_{\mathbb{C}}$,*

$$\tau\left[A^{(q)}(\xi_1) \cdots A^{(q)}(\xi_1)\right] = \sum_{\pi \in P_2(n)} q^{\mathrm{cr}(\pi)} \prod_{\substack{(k,l)\in\pi \\ k<l}} \langle \xi_l, \xi_k \rangle_{H_{\mathbb{C}}},$$

where $\mathrm{cr}(\pi)$ is the number of crossings in the pair partition π.

Since we take ξ_i from real Hilbert space $H_{\mathbb{R}}$, we can see $\langle \xi_l, \xi_k \rangle_{H_{\mathbb{C}}} = \langle \xi_k, \xi_l \rangle_{H_{\mathbb{C}}}$. The traciality of τ follows from this property and the observation that $\mathrm{cr}(\pi)$ does not change by replacing vertices cyclically. As a consequence, the couple $(\Gamma_q(H_{\mathbb{R}}), \tau)$ forms a tracial W* probability space. In addition, $\Gamma_q(H_{\mathbb{R}})$ is generated by $A^{(q)} = \{A^{(q)}(e_i)\}_{i\in I}$ where $\{e_i\}_{i\in I}$ is a orthonormal basis of $H_{\mathbb{R}}$. From this viewpoint, we also use the notation W$^*(A^{(q)})$ for the q-Gaussian von Neumann algebra. Note that when $q = 0$, $A^{(0)}$ is a family of free semicircle distributions due to the q-Wick formula and this implies W$^*(A^{(0)})$ is isomorphic to the free group von Neumann algebra.

The basic question on the factoriality of $\Gamma_q(H_{\mathbb{R}})$ was initially studied by Bożko-Kümmerer-Speicher [11] and Sniady [78], and subsequently, Ricard [73] answered this question in full generality ($-1 < q < 1$ and $\dim H_{\mathbb{R}} \geq 2$). Moreover, it is known that they share several properties with free group von Neumann algebras. In particular, Guionnet and Shlyakhtenko [39] proved by using a free probabilistic technique (free transport) that they are actually isomorphic to the free group von Neumann algebras under certain conditions.

Theorem 2.54 ([39])**.** *Assume $\dim H_{\mathbb{R}} < \infty$ and $|q|$ is sufficiently small (the range depends on $\dim H_{\mathbb{R}}$). Then $\Gamma_q(H_{\mathbb{R}})$ is isomorphic to $\Gamma_0(H_{\mathbb{R}})$.*

In the above theorem, the condition $\dim H_{\mathbb{R}} < \infty$ is necessary. In fact, we have the following theorem.

Theorem 2.55 ([15], [19])**.** *When $\dim H_{\mathbb{R}} = \infty$, $\Gamma_q(H_{\mathbb{R}})$ is not isomorphic to $\Gamma_0(H_{\mathbb{R}})$ for any $q \neq 0$.*

Remark 2.56. To prove Theorem 2.55, they checked the invariant, so-called Akemann-Ostrand (AO) property for q-Gaussian von Neumann algebras. In fact, when $\dim H_{\mathbb{R}} = \infty$, $\Gamma_q(H_{\mathbb{R}})$ ($q \neq 0$) does not have AO property, while $\Gamma_0(H_{\mathbb{R}})$ has this property. When $\dim H_{\mathbb{R}} < \infty$, it is known that all $\Gamma_q(H_{\mathbb{R}})$ have AO property, and therefore it is still open whether $\Gamma_q(H_{\mathbb{R}})$ is isomorphic to $\Gamma_0(H_{\mathbb{R}})$ for any $-1 < q < 1$ or not.

2.4.2. *q-Wick polynomials and Haagerup type estimates for q-Gaussians.* In a non-commutative probabilistic framework, we would like to analyze the joint moments of q-Gaussians, and in principle, it is a study of combinatorics on crossing partitions. However, Fock representations and functional analysis often give us good control of joint moments without computing them explicitly.

Here, we explain one of such phenomenons, Bożejko's Haagerup-type inequality. The key observation is the isomorphism D between the GNS Hilbert space $L^2(\mathrm{W}^*(A^{(q)}), \tau)$ and the q-Fock space $\mathcal{F}_q(H_{\mathbb{C}})$ which is the extension of

$$L^2(\mathrm{W}^*(A^{(q)}), \tau) \supset \mathrm{W}^*(A^{(q)}) \ni x \mapsto x\Omega \in \mathcal{F}_q(H_{\mathbb{C}}).$$

Note that D is an isometry by definition of the vacuum state (we will see soon the surjectivity).

Let us assume $d = \dim H_{\mathbb{R}} < \infty$ and we take an orthonormal basis $\{e_1, \dots, e_d\}$ on $H_{\mathbb{R}}$ which also forms an orthonormal basis on $H_{\mathbb{C}}$. We associate the word $[d]^*$ with a basis $\{e_w\}_{w\in[d]^*}$ on $\mathcal{F}_{\mathrm{alg}}(H_{\mathbb{C}})$ by defining

$$e_\Omega = \Omega,$$
$$e_{w_1\cdots w_n} = e_{w_1} \otimes \cdots \otimes e_{w_n}.$$

By applying the isomorphism D^*, we obtain $e_w^{(q)} = D^*(e_w) \in L^2(\mathrm{W}^*(A^{(q)}), \tau)$. Actually, for each $w \in [d]^*$, $e_w^{(q)}$ is a (non-commutative) polynomial in q-Gaussians $A_i = A(e_i)$, which are determined by the recursion:

$$e_\Omega^{(q)} = 1,$$
$$e_{w_1}^{(q)} = A_{w_1},$$
$$e^{(q)}_{w_{n+1}w_n\cdots w_1} = A_{w_{n+1}} e^{(q)}_{w_n\cdots w_1} - \sum_{k=1}^{n} q^{n-k} \delta_{w_{n+1},w_k} e^{(q)}_{w_n\cdots \hat{w_k}\cdots w_1}.$$

Note that this recursion comes from the definition of q-Gaussian operator A and the formula of l and l^*. $\{e_w^{(q)}\}_{w\in[d]^*}$ are called q-Wick polynomials.

Example 2.57. When $q = 1$, $e_w^{(1)}$ is a product of Hermite polynomials in commutative variables $A_1, \dots, A_d$. In fact, for each w, we have

$$e_w^{(1)} = H_{k_1}(A_1) \cdots H_{k_d}(A_d)$$

where k_l $(l = 1, \dots, d)$ is the number of the letter l in w and $H_n(x)$ is a (normalized) Hermite polynomial defined by

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_{n+1}(x) = xH_n(x) - nH_{n-1}(x).$$

When $q = 0$, $e_w^{(0)}$ is a product of Chebyshev polynomials in non-commutative variables $A_1, \dots, A_d$. In this case, for each w, we write $w = i_1^{k_1} \cdots i_n^{k_n}$ with $i_j \neq i_{j+1}$ $(j = 1, \dots, n-1)$ and $k_j \geq 1$, and we have

$$e_w^{(0)} = U_{k_1}(A_{i_1}) \cdots U_{k_n}(A_{i_n})$$

where $U_n(x)$ is the (normalized) Chebyshev polynomials of the second kind defined by

$$U_0(x) = 1, \quad U_1(x) = x, \quad U_{n+1}(x) = U_n(x) - U_{n-1}(x).$$

To emphasize Chebyshev polynomials, we will use the notation U_w instead of $e_w^{(0)}$ in the subsequent sections. When $q = -1$, $e_w^{(q)}$ is 0 if w has the same latter. If $w = w_1 \cdots w_n$ consists of different letters, then $e_w^{(-1)}$ is just a product of anti-commutative variables $A_1, \ldots, A_d$ with the same order as w, i.e.

$$e_w^{(-1)} = A_{w_1} \cdots A_{w_n}.$$

Let $\|\cdot\|$ is the operator norm on $B(\mathcal{F}_q(H))$ and $\|\cdot\|_{\mathcal{F}_q(H)}$ (or simply $\|\cdot\|_q$) is the norm defined by $\sqrt{\langle \xi, \xi \rangle_q}$ for $\xi \in \mathcal{F}_q(H)$. Now, we state the Bożejoko's Haagerup-type inequality for q-Gaussians with $-1 < q < 1$.

Theorem 2.58 (Proposition 2.1 in [13]). *For each $k \in \mathbb{N}$ and $-1 < q < 1$, we have*

$$\left\| \sum_{|w|=k} \alpha_w e_w^{(q)} \right\| \leq (k+1) C_{|q|}^{\frac{3}{2}} \left\| \sum_{|w|=k} \alpha_w e_w^{(q)} \right\|_{\mathcal{F}_q(H_{\mathbb{C}})}$$

where $\alpha_w \in \mathbb{C}$ and $C_{|q|}^{-1} = \prod_{m=1}^{\infty} (1 - |q|^m)$.

Here, we revisit a proof of this inequality in the case $q = 0$ for the reader's convenience since this argument also appears in general q and it is crucial in the section.

Lemma 2.59 (Haagerup inequality). *Let $m \in \mathbb{Z}_{\geq 0}$ and $\{\alpha_v\}_{|v|=m}$ be a family of complex numbers. Then we have*

$$\left\| \sum_{|v|=m} \alpha_v U_v \right\| \leq (m+1) \left\| \sum_{|v|=m} \alpha_v \hat{U}_v \right\|_{\mathcal{F}_0(H)},$$

Proof. First, we show

$$\max\left\{ \left\| \sum_{|v|=m} \alpha_v l_v \right\|, \left\| \sum_{|v|=m} \alpha_v l_v^* \right\| \right\} \leq \left\| \sum_{|v|=m} \alpha_v \hat{U}_v \right\|_{\mathcal{F}_0(H)}$$

where $l_v = l_{v_1} l_{v_2} \cdots l_{v_m}$, $l_v^* = l_{v_1}^* l_{v_2}^* \cdots l_{v_m}^*$ with $l_{v_i} = l(e_{v_i})$ for $v = v_1 v_2 \cdots v_m$. Since we have for $\xi \in H^{\otimes n}$

$$\begin{aligned} \left\| \sum_{|v|=m} \alpha_v l_v \xi \right\|_{\mathcal{F}_0(H)}^2 &= \left\| \sum_{|v|=m} \alpha_v (e_v \otimes \xi) \right\|_{\mathcal{F}_0(H)}^2 \\ &= \sum_{|v|=m} |\alpha_v|^2 \|\xi\|_{\mathcal{F}_0(H)}^2 \end{aligned}$$

and $\sum_{|v|=m} \alpha_v l_v(\xi)$ and $\sum_{|v|=m} \alpha_v l_v(\eta)$ are orthogonal for $\xi \in H^{\otimes n}$ and $\eta \in H^{\otimes n'}$, $n \neq n'$, we have $\|\sum_{|v|=m} \alpha_v l_v\| \leq \|\sum_{|v|=m} \alpha_v \hat{U}_v\|_{\mathcal{F}_0(H)}$. Moreover, by taking involution, we have

$$\begin{aligned}\left\|\sum_{|v|=m}\alpha_v l_v^*\right\| &= \left\|\left(\sum_{|v|=m}\alpha_v l_v^*\right)^*\right\| \\ &= \left\|\sum_{|v|=m}\overline{\alpha_v} l_{v^*}\right\| \\ &\leq \sqrt{\sum_{|v|=m}|\alpha_{v^*}|^2} \\ &= \left\|\sum_{|v|=m}\alpha_v \hat{U}_v\right\|_{\mathcal{F}_0(H)}.\end{aligned}$$

In order to prove this lemma, we use the following characterization of U_v for $v = v_1 \cdots v_m \in [d]^*, v_i \in [d]$ (see Proposition 2.7 in [11])

$$U_v = \sum_{k=0}^m l_{v_1}\cdots l_{v_k} l_{v_{k+1}}^* \cdots l_{v_m}^*.$$

From this formula, we rewrite $\sum_{|v|=m}\alpha_v U_v$ by $\sum_{k=0}^m F^{(k)}$ where $F^{(k)}$ denotes

$$\sum_{\substack{|u|=k\\|v|=m-k}}\alpha_{uv} l_u l_v^*$$

for $k = 0,\dots,n$. We will show $\|F^{(k)}\| \leq \|\sum_{|v|=m}\alpha_v\hat{U}_v\|_{\mathcal{F}_0(H)}$ for any k. Since we have already proved this for $k = 0, m$ in the previous argument, we fix $k = 1,\dots,n-1$. In addition, since $F^{(k)}(\xi)$ and $F^{(k)}(\eta)$ are orthogonal when $\xi \in H^{\otimes n}$, $\eta \in H^{\otimes n'}$ where $n \neq n'$, it suffices to show that $\|F^{(k)}(\xi)\|_{\mathcal{F}_0(H)} \leq \|\sum_{|v|=m}\alpha_v\hat{U}_v\|_{\mathcal{F}_0(H)}\|\xi\|_2$ for $\xi \in H^{\otimes n}$ where $n \geq m-k$ (note that $F^{(k)}(\xi) = 0$ when $n < m-k$). Then we have

$$\begin{aligned}\|F^{(k)}\xi\|^2_{\mathcal{F}_0(H)} &= \langle \sum_{\substack{|u_1|=k\\|u_2|=n-k}}\alpha_{u_1u_2} l_{u_1} l_{u_2}^*\xi, \sum_{\substack{|v_1|=k\\|v_2|=n-k}}\alpha_{v_1v_2} l_{v_1} l_{v_2}^*\xi\rangle_0 \\ &= \sum_{\substack{|u_1|=|v_1|=k\\|u_2|=|v_2|=n-k}}\alpha_{u_1u_2}\overline{\alpha_{v_1v_2}}\langle l_{u_1}l_{u_2}^*\xi, l_{v_1}l_{v_2}^*\xi\rangle_0 \\ &= \sum_{\substack{|u_1|=|v_1|=k\\|u_2|=|v_2|=n-k}}\alpha_{u_1u_2}\overline{\alpha_{v_1v_2}}\langle e_{u_1}, e_{v_1}\rangle_{\mathcal{F}_0(H)}\langle l_{u_2}^*\xi, l_{v_2}^*\xi\rangle_0.\end{aligned}$$

Since $\{e_v\}_{v\in[d]^*}$ is an orthonormal basis of $\mathcal{F}_0(H)$, the last term is equal to

$$\begin{aligned}\sum_{\substack{|u|=k\\|u_2|=|v_2|=n-k}}\alpha_{uu_2}\overline{\alpha_{uv_2}}\langle l_{u_2}^*\xi, l_{v_2}^*\xi\rangle_0 &= \sum_{|u|=k}\langle \sum_{|u_2|=n-k}\alpha_{uu_2}l_{u_2}^*\xi, \sum_{|v_2|=n-k}\alpha_{uv_2}l_{v_2}^*\xi\rangle_0 \\ &= \sum_{|u|=k}\left\|\sum_{|v|=n-k}\alpha_{uv}l_v^*\xi\right\|^2_{\mathcal{F}_0(H)}.\end{aligned}$$

Since we have $\left\|\sum_{|v|=n-k} \alpha_{uv} l_v^*\right\| \leq \sqrt{\sum_{|v|=n-k} |\alpha_{uv}|^2}$, we obtain

$$\|F^{(k)}\xi\|_2^2 \leq \sum_{|u|=k} \sum_{|v|=n-k} |\alpha_{uv}|^2 \|\xi\|_{\mathcal{F}_0(H)}^2 = \left\|\sum_{|v|=m} \alpha_v \hat{U}_v\right\|_{\mathcal{F}_0(H)}^2 \|\xi\|_{\mathcal{F}_0(H)}^2.$$

Thus we conclude

$$\begin{aligned} \left\|\sum_{|v|=m} \alpha_v U_v\right\| &= \left\|\sum_{k=0}^m F^{(k)}\right\| \\ &\leq \sum_{k=0}^m \|F^{(k)}\| \\ &\leq (m+1) \left\|\sum_{|v|=m} \alpha_v \hat{U}_v\right\|_{\mathcal{F}_0(H)}. \end{aligned}$$

□

Remark 2.60. We remark that this inequality corresponds with the Haagerup inequality for free Haar unitaries,

$$\left\|\sum_{\substack{g\in\mathbb{F}_d \\ |g|=k}} \alpha_g \lambda(g)\right\| \leq (k+1) \left\|\sum_{\substack{g\in\mathbb{F}_d \\ |g|=k}} \alpha_g \lambda(g)\right\|_{l^2(\mathbb{F}_d)}.$$

2.4.3. *Generalization: twisted relations and Araki-Woods algebras.* Here, we briefly explain generalizations of q-CCR and q-Gaussian von Neumann algebra. To generalize q-CCR, we go back to the definition of the q-inner product

$$\langle \xi_1 \otimes \cdots \otimes \xi_m, \eta_1 \otimes \cdots \otimes \eta_n \rangle_q = \delta_{m,n} \sum_{\pi \in S_m} q^{\mathrm{inv}(\pi)} \prod_{i=1}^m \langle \xi_i, \eta_{\pi(i)} \rangle.$$

We see this inner product as the following form

$$\langle \xi_1 \otimes \cdots \otimes \xi_m, \eta_1 \otimes \cdots \otimes \eta_n \rangle_q = \langle \xi_1 \otimes \cdots \otimes \xi_m, P^{(n)}(\eta_1 \otimes \cdots \otimes \eta_n) \rangle_0,$$

where the operator $P^{(n)} : H_{\mathbb{C}}^{\otimes n} \to H_{\mathbb{C}}^{\otimes n}$ is defined by $P^{(n)} = \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} U_\pi$ and U_π is the unitary operator (with respect to 0-inner product) on $H_{\mathbb{C}}^{\otimes n}$ which permutes tensor components according to π, i.e. $U_\pi \eta_1 \otimes \cdots \otimes \eta_n = \eta_{\pi(1)} \otimes \cdots \otimes \eta_{\pi(n)}$. Note that we have the following identity in the group algebra of the Symmetric group,

$$\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} \pi = (1+T_1)(1+T_2+T_2T_1) \cdots (1+T_{n-1}+T_{n-1}T_{n-2}+\cdots+T_{n-1}\cdots T_1)$$

where $T_k = q(k, k+1)$. We identify U_π with π, and then T_k is identified with

$$\mathrm{Id}_{H_{\mathbb{C}}}^{\otimes k-1} \otimes qF \otimes \mathrm{Id}_{H_{\mathbb{C}}}^{\otimes n-1-k}$$

where $F \in B(H_{\mathbb{C}} \otimes H_{\mathbb{C}})$ is the flip operator $F(\xi \otimes \eta) = \eta \otimes \xi$. From the formula above, we have

$$P^{(n)} = (1+T_1)(1+T_2+T_2T_1) \cdots (1+T_{n-1}+T_{n-1}T_{n-2}+\cdots+T_{n-1}\cdots T_1).$$

Now, we replace the operator qF by a self-adjoint operator $T = T^* \in B(H_{\mathbb{C}} \otimes H_{\mathbb{C}})$ and define T_k, $P^{(n)}$, and the sesqui-linear form $\langle x, y\rangle_T = \langle x, \oplus_{n=0}^{\infty} P^{(n)} y\rangle_0$ in the same way. The problem is the positivity of the sesqui-linear form $\langle \cdot, \cdot\rangle_T$, i.e. $\langle x, x\rangle_T \geq 0$ for any $x \in \mathcal{F}_{\mathrm{alg}}(H_{\mathbb{C}})$. The sufficient condition was proved by Bożejko-Speicher [10] and Jorgensen-Schmitt-Werner [48].

Theorem 2.61. *If $T = T^* \in B(H_{\mathbb{C}} \otimes H_{\mathbb{C}})$ satisfies $\|T\| \leq 1$ and Yang-Baxter relation*

$$(T \otimes \mathrm{Id}_{H_{\mathbb{C}}})(\mathrm{Id}_{H_{\mathbb{C}}} \otimes T)(T \otimes \mathrm{Id}_{H_{\mathbb{C}}}) = (\mathrm{Id}_{H_{\mathbb{C}}} \otimes T)(T \otimes \mathrm{Id}_{H_{\mathbb{C}}})(\mathrm{Id}_{H_{\mathbb{C}}} \otimes T),$$

then the sesqui-linear form $\langle \cdot, \cdot\rangle_T$ satisfies the positivity condition. Moreover, if $\|T\| < 1$, then we have strict positivity, i.e. $\langle x, x\rangle_T > 0$ for any $x \neq 0$.

If the sesqui-linear form $\langle \cdot, \cdot\rangle_T$ satisfies the positivity, we obtain the Hilbert space $\mathcal{F}_T(H_{\mathbb{C}})$ by completing $\mathcal{F}_{\mathrm{alg}}(H_{\mathbb{C}})$, the so-called twisted Fock space. We consider the left creation operator $l(\xi)$ for $\xi \in H_{\mathbb{C}}$ by putting $\xi\otimes$ from the left. In particular, if we take an orthonormal basis $\{e_i\}_{i\in I}$ on $H_{\mathbb{C}}$, $\{l_i = l(e_i)\}_{i\in I}$ satisfy twisted CCR,

$$l_i^* l_j - \sum_{r,s\in I} t_{js}^{ir} l_r l_s^* = \delta_{i,j} I,$$

where I is the identity operator and $t_{js}^{ir} \in \mathbb{C}$ is determined from T by

$$t_{js}^{ir} = \langle T e_j \otimes e_s, e_i \otimes e_r\rangle_{H_{\mathbb{C}}^{\otimes 2}}.$$

Of course, this twisted CCR contains q-CCR, and it also contains mixed q_{ij}-CCR

$$l_i^* l_j - q_{ij} l_j l_i^* = \delta_{i,j} I.$$

We also remark that q-Gaussian operators $A^{(q)}(\xi)$ produce not only q-Gaussian von Neumann algebra $\Gamma_q(H_{\mathbb{R}})$ but also non-tracial von Neumann algebra by considering the double commutant

$$\Gamma_q(H) = \{A^{(q)}(\xi) : \xi \in H\}''$$

where $H \subset H_{\mathbb{C}}$ is a standard subspace of $H_{\mathbb{C}}$, in other words H is a real Hilbert subspace of $H_{\mathbb{C}}$ and satisfies

- H is closed as a real Hilbert subspace of $(H_{\mathbb{C}}, \mathrm{Re}\langle \cdot, \cdot\rangle_{H_{\mathbb{C}}})$,
- $H + iH$ is dense in $H_{\mathbb{C}}$,
- $H \cap iH = \{0\}$.

Since $\langle x, y\rangle_{H_{\mathbb{C}}} \neq \langle y, x\rangle_{H_{\mathbb{C}}}$ for $x, y \in H$ in general, the vacuum state is not necessary a trace on $\Gamma_q(H)$ and $\Gamma_q(H)$ can be non-tracial von Neumann algebra. For general H, $\Gamma_q(H)$ is called q-Araki-Wodds algebra introduced in Hiai [45], which is a q-deformation of the free Araki-Woods algebra introduced by Shlyakhtenko [74]. We can apply the same construction of q-Araki-Woods algebra to twisted CCR, and we call it a twisted Araki-Woods algebra. When $|q| < 1$ and more generally $\|T\| < 1$, the factoriality of q-Araki-Woods algebra, mixed q-Araki-Woods algebra and twisted Araki-Woods algebra was recently solved by Kumar-Skalski-Wasilewski [53], Kumar [54] and Yang [92] by proving the existence of the conjugate system as in Section 5 in the thesis.We won't go further on the non-tracial case since it is not the main object of this thesis.

3. Convergence for non-commutative rational functions evaluated in random matrices

This section is a part of the paper [27]. The main result of this section is to show the convergence in distribution of spectral measures of non-commutative rational functions evaluated in given non-commutative random matrices and to see how it works for random matrices. First, we consider the well-definedness of evaluations of non-commutative rational functions in random matrices.

3.1. Evaluations of non-degenerate matrix-valued non-commutative rational expressions.

By definition, every non-degenerate matrix-valued non-commutative rational expression has a non-empty domain when evaluations in matrices of sufficiently large size are considered. In this section, we show that actually much more is true. Namely, we establish that the assumptions of Theorem 3.10 are satisfied in very general situations.

3.2. Evaluations in random matrices.

The following result asserts, loosely spoken, that one can almost surely evaluate every non-degenerated matrix-valued non-commutative rational expression in "absolutely continuous" random matrix models, provided that their size is large enough. The precise statement reads as follows.

Theorem 3.1. *Let R be a matrix-valued non-commutative rational expression in $d = d_1 + d_2$ formal variables which is non-degenerate. Suppose that $\mu^N_{d_1,d_2}$ is a probability measure on $M_N(\mathbb{C})^{d_1}_{\mathrm{sa}} \times U_N(\mathbb{C})^{d_2}$ which is absolutely continuous with respect to the product measure of the Lebesgue measure on $M_N(\mathbb{C})_{\mathrm{sa}}$ and the Haar measure on $U_N(\mathbb{C})$. If (X^N, U^N) is a tuple of random matrices in $M_N(\mathbb{C})^{d_1}_{\mathrm{sa}} \times U_N(\mathbb{C})^{d_2}$ with law $\mu^N_{d_1,d_2}$, then there exists some $N_0 \in \mathbb{N}$ such that almost surely $(X^N, U^N) \in \mathrm{dom}_{M_N(\mathbb{C})}(R)$ for all $N \geq N_0$.*

Remark 3.2. For the validity of Theorem 3.1, it is essential to work over the field of complex numbers. In order to see this, consider the scalar-valued non-commutative rational expression $r = (x_1x_2 - x_2x_1)^{-1}$. Note that there are real matrices X_1, X_2 at which one can evaluate r, but in $M_N(\mathbb{R})$ for N odd there cannot exist symmetric real matrices X_1, X_2 at which evaluation $r(X_1, X_2)$ would be defined, since necessarily $\det(X_1X_2 - X_2X_1) = 0$ because

$$\det(X_1X_2 - X_2X_1) = \det\left((X_1X_2 - X_2X_1)^T\right) = -\det(X_1X_2 - X_2X_1).$$

It is consistent with this observation that the proof of Proposition 3.3, on which Theorem 3.1 relies, will make use of complex analysis techniques.

One can also see an algebraic description of the existence of a symmetric matrix in a domain of non-commutative rational expression in [90, Remark 6.7].

The proof of Theorem 3.1 relies on a study of evaluations of affine linear pencils. The first step is the following proposition, which requires some notation. Consider an affine linear pencil

$$Q = A_0 \otimes 1 + \sum_{j=1}^{d_1} A_j \otimes x_j + \sum_{j=1}^{d_2} B_j \otimes u_j \tag{2}$$

in the variables $x = (x_1, \ldots, x_{d_1})$ and $u = (u_1, \ldots, u_{d_2})$, say with coefficients $A_0, A_1, \ldots, A_{d_1}$ and $B_1, \ldots, B_{d_2}$ taken from $M_k(\mathbb{C})$. We regard Q as an element in

$$M_k(\mathbb{C}) \otimes \mathbb{C}\langle x, u\rangle \cong M_k(\mathbb{C}\langle x, u\rangle).$$

Given an d-tuple $Z = (Z', Z'')$ of matrices in $M_N(\mathbb{C})$, we consider the evaluation of Q at Z which is given by

$$Q(Z) := A_0 \otimes 1 + \sum_{j=1}^{d_1} A_j \otimes Z'_j + \sum_{j=1}^{d_2} B_j \otimes Z''_j,$$

where $Q(Z)$ lies in $M_k(\mathbb{C}) \otimes M_N(\mathbb{C}) \cong M_{kN}(\mathbb{C})$. Building on such evaluations, we associate to Q functions

$$\phi_Q^{(N)}: \ M_N(\mathbb{C})^d \longrightarrow \mathbb{C}, \quad Z \longmapsto \det(Q(Z))$$

for every $N \in \mathbb{N}$. Notice that $\phi_Q^{(N)}$ is a holomorphic commutative polynomial in the dN^2 complex matrix entries appearing in the tuple Z. This allows us to use the complex analysis machinery in order to relate $\phi_Q^{(N)}$ and its restriction to the real space $M_N(\mathbb{C})_{\mathrm{sa}}^{d_1} \times U_N(\mathbb{C})^{d_2}$.

Proposition 3.3. *Let Q be an affine linear pencil of the form* (2) *in $M_k(\mathbb{C}) \otimes \mathbb{C}\langle x, u\rangle$ and let $N \in \mathbb{N}$. If $\phi_Q^{(N)}|_{M_N(\mathbb{C})_{\mathrm{sa}}^{d_1} \times U_N(\mathbb{C})^{d_2}} \equiv 0$, then $\phi_Q^{(N)} \equiv 0$.*

Proof. Fix any $Z = (Z', Z'') \in M_N(\mathbb{C})^{d_1} \times M_N(\mathbb{C})^{d_2}$ and suppose that the d_2-tuple Z'' consists of invertible matrices. We write $Z' = X + iY$ with the tuples $X = (X_1, \dots, X_{d_1}), Y = (Y_1, \dots, Y_{d_1}) \in M_N(\mathbb{C})_{\mathrm{sa}}^{d_1}$ that are given by $X_j := \Re(Z'_j)$ and $Y_j := \Im(Z'_j)$ for $j = 1, \dots, d_1$. Further, for $j = 1, \dots, d_2$, we consider the polar decomposition $Z''_j = P_j U_j$ of Z''_j with a positive definite matrix $P_j \in M_N(\mathbb{C})$ and $U_j \in U_N(\mathbb{C})$. As the matrices $P_1, \dots, P_{d_2}$ are positive definite, we can define a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ by

$$\begin{aligned} f(z) := \phi_Q^{(N)}\big(&X_1 + zY_1, \dots, X_{d_1} + zY_{d_1}, \\ &\exp(-iz\log(P_1))U_1, \dots, \exp(-iz\log(P_{d_2}))U_{d_2}\big) \end{aligned}$$

for $z \in \mathbb{C}$. Due to the assumption that $\phi_Q^{(N)}|_{M_N(\mathbb{C})_{\mathrm{sa}}^{d_1} \times U_N(\mathbb{C})^{d_2}} \equiv 0$, we have that $f|_{\mathbb{R}} \equiv 0$. Thus, by the identity principle, it follows that f vanishes identically on $\mathbb{C}$. In particular, $\phi_Q^{(N)}(Z) = f(i) = 0$. This shows that $\phi_Q^{(N)}$ vanishes on all d-tuples $Z = (Z', Z'') \in M_N(\mathbb{C})^{d_1} \times M_N(\mathbb{C})^{d_2}$ satisfying the condition that Z'' consists of invertible matrices. Since those are dense in $M_N(\mathbb{C})^d$, the assertion follows. ☐

With the help of Proposition 3.3, we see that fullness of affine linear pencils Q can be detected by evaluations of Q at points in $M_N(\mathbb{C})_{\mathrm{sa}}^{d_1} \times U_N(\mathbb{C})^{d_2}$.

Proposition 3.4. *Let Q be an affine linear pencil of the form* (2) *in $M_k(\mathbb{C}) \otimes \mathbb{C}\langle x, u\rangle$ which is full. Then there exists $N_0 \in \mathbb{N}$ with the following property: for each $N \geq N_0$, we have that $\phi_Q^{(N)}|_{M_N(\mathbb{C})_{\mathrm{sa}}^{d_1} \times U_N(\mathbb{C})^{d_2}} \not\equiv 0$, i.e., one can find some d-tuple $(X^N, U^N) \in M_N(\mathbb{C})_{\mathrm{sa}}^{d_1} \times U_N(\mathbb{C})^{d_2}$ for which $Q(X^N, U^N)$ becomes invertible in $M_{kN}(\mathbb{C})$.*

Proof. First, we note that there exists some $N_0 \in \mathbb{N}$ such that $\phi_Q^{(N)} \not\equiv 0$ for all $N \geq N_0$. This fact is well-known (see Proposition 2.4 in [91], for instance), but we include the argument for the sake of completeness. Since Q is full, $Q(x)$ is invertible as a matrix over the free skew field $\mathbb{C}\langle\!\!\langle x, u\rangle\!\!\rangle$; see Remark 2.30 (ii). Its inverse $Q(x)^{-1} \in M_k(\mathbb{C}\langle\!\!\langle x, u\rangle\!\!\rangle)$ is represented by some non-degenerate $k \times k$ matrix-valued non-commutative rational expression R, i.e., we have $Q^{-1}(x) = R(x)$; this follows by applying Remark 2.33 entrywise. From Theorem 2.26, we know that there exists some $N_0 \in \mathbb{N}$ such that $\mathrm{dom}_{M_N(\mathbb{C})}(R) \neq \emptyset$ for all $N \geq N_0$. Thanks to Remark 2.32, the identity $Q(x,u)R(x,u) = I_k$ over $\mathbb{C}\langle\!\langle x,u\rangle\!\rangle$ continues to hold on $\mathrm{dom}_{M(\mathbb{C})}(R)$, and by applying determinants, we infer that $\phi_Q^{(N)} \not\equiv 0$ for all $N \geq N_0$, as desired.

Having this, Proposition 3.3 guarantees that $\phi_Q^{(N)}$ does not vanish identically on all of $M_N(\mathbb{C})_{\mathrm{sa}}^{d_1} \times U_N(\mathbb{C})^{d_2}$, as we wished to show. $\square$

In the next step, we involve the concrete random matrix model that we want to consider.

Proposition 3.5. *Let Q be an affine linear pencil of the form* (2) *in $M_k(\mathbb{C}) \otimes \mathbb{C}\langle x,u\rangle$ which is full. For $N \in \mathbb{N}$, let (X^N, U^N) be a random matrix in $M_N(\mathbb{C})_{\mathrm{sa}}^{d_1} \times U_N(\mathbb{C})^{d_2}$ with an absolutely continuous law μ_{d_1,d_2}^N like in Theorem 3.1. Then there exists $N_0 \in \mathbb{N}$ such that almost surely $Q(X^N, U^N)$ is invertible in $M_k(\mathbb{C}) \otimes M_N(\mathbb{C}) \cong M_{kN}(\mathbb{C})$ for all $N \geq N_0$.*

Proof. Thanks to Proposition 3.4, since Q is supposed to be full, there is an $N_0 \in \mathbb{N}$ such that none of the functions $\phi_A^{(N)}|_{M_N(\mathbb{C})_{\mathrm{sa}}^{d_1} \times U_N(\mathbb{C})^{d_2}}$ for $N \geq N_0$ can vanish identically. Notice that $M_N(\mathbb{C})_{\mathrm{sa}}^{d_1} \times U_N(\mathbb{C})^{d_2}$ is a real manifold of dimension dN^2. In suitable local charts, we see that $\phi_Q^{(N)}|_{M_N(\mathbb{C})_{\mathrm{sa}}^{d_1} \times U_N(\mathbb{C})^{d_2}}$ induces a real analytic function on an open subset of $\mathbb{R}^{dN^2}$ and can therefore vanish only on a set of Lebesgue measure 0. Due to the choice of μ_{d_1,d_2}^N, we conclude that, for each $N \geq N_0$, the random matrix $Q(X^N, U^N)$ is almost surely invertible in $M_{kN}(\mathbb{C})$. $\square$

Proof of Theorem 3.1. We define the set $\mathfrak{R}_0$ of all non-degenerate matrix-valued non-commutative rational expressions R for which the conclusion of Theorem 3.1 is true, i.e., there exists $N_0 \in \mathbb{N}$ such that almost surely $(X^N, U^N) \in \mathrm{dom}_{M_N(\mathbb{C})}(R)$ for all $N \geq N_0$. We have to prove that $\mathfrak{R}_0$ consists in fact of all non-degenerate matrix-valued non-commutative rational expressions.

Notice that obviously, all matrix-valued non-commutative polynomial expressions belong to $\mathfrak{R}_0$. Further, it is easily seen that both $R_1 + R_2$ and $R_1 \cdot R_2$ are in $\mathfrak{R}_0$ whenever we take $R_1, R_2 \in \mathfrak{R}_0$ for which the respective arithmetic operation makes sense. Therefore, it only remains to prove that if $R \in \mathfrak{R}_0$ is square and enjoys the property that R^{-1} is non-degenerate, then necessarily $R^{-1} \in \mathfrak{R}_0$. In order to verify this, we take any square matrix-valued non-commutative rational expression R belonging to $\mathfrak{R}_0$ for which R^{-1} is non-degenerate. Further, let $\rho = (v, Q, w)$ be a formal linear representation of R in the sense of Definition 2.27, say of dimension k; see Theorem 2.28.

By assumption, we have that R^{-1} is a non-degenerate matrix-valued non-commutative rational expression. Thus, Remark 2.30 (iv) gives us that the affine linear pencil in d variables with coefficients from $M_{k+p}(\mathbb{C})$ which is given by

$$\tilde{Q} := \begin{pmatrix} 0_{p\times p} & v \\ w & Q \end{pmatrix}$$

is full. Therefore, Proposition 3.5 tells us that an $N_0 \in \mathbb{N}$ exists such that almost surely $\tilde{Q}(X^N, U^N)$ is invertible in $M_{(k+p)N}(\mathbb{C})$ for all $N \geq N_0$. Since $R \in \mathfrak{R}_0$, we may suppose that (after enlarging N_0 if necessary) that at the same time almost surely $(X^N, U^N) \in \mathrm{dom}_{M_N(\mathbb{C})}(R)$ for all $N \geq N_0$. Because ρ is a formal linear representation, the latter implies that almost surely $Q(X^N,U^N)$ is invertible and $R(X^N,U^N) = vQ(X^N,U^N)^{-1}w$ for all $N \geq N_0$. Putting these observations together, we see, again with the help of the Schur complement formula, that almost surely $R(X^N,U^N)$ is invertible for all $N \geq N_0$. In other words, we have almost surely that $(X^N,U^N) \in \operatorname{dom}_{M_N(\mathbb{C})}(R^{-1})$ for all $N \geq N_0$. The latter means that $R^{-1} \in \mathfrak{R}_0$, as desired. ☐

3.3. **Evaluation in operators with maximal Δ.** It follows from [60, Theorem 1.1] that for any d-tuple $X = (X_1,\dots,X_d)$ of (not necessarily self-adjoint) operators in some W^*-probability space $(\mathcal{M},\tau)$ which satisfy the "regularity condition" $\Delta(X) = d$, where Δ stands for a quantity that was introduced in [29] and which we discussed in Section 2.2.5, then the canonical evaluation homomorphism

$$\operatorname{ev}_X:\ \mathbb{C}\langle x_1,\dots,x_d\rangle \to \mathcal{M}$$

which is determined by $1 \mapsto 1$ and $x_j \mapsto X_j$ for $j = 1,\dots,d$ extends to an injective homomorphism

$$\operatorname{Ev}_X:\ \mathbb{C}(\!\!<x_1,\dots,x_d>\!\!) \to \widetilde{\mathcal{M}}$$

into the $*$-algebra $\widetilde{\mathcal{M}}$ of all closed and densely defined operators affiliated with $\mathcal{M}$; see Section 2.2.4.

While the result of [60] addresses evaluations of non-commutative rational functions, it leaves open the question whether also all non-degenerate rational expressions can be evaluated; indeed, this is not immediate as the domain of a rational function is larger than the domain of any of its representing non-degenerate non-commutative rational expressions. This question is answered to the affirmative by the next theorem, which gives the conclusion even in the matrix-valued case. For that purpose, we will consider the canonical amplifications

$$\operatorname{Ev}_X^\bullet:\ M_\bullet(\mathbb{C}(\!\!<x_1,\dots,x_d>\!\!)) \to M_\bullet(\widetilde{\mathcal{M}}).$$

Theorem 3.6. *Let $X = (X_1,\dots,X_d)$ be a d-tuple of (not necessarily self-adjoint) operators in some tracial W^*-probability space $(\mathcal{M},\tau)$ satisfying $\Delta(X) = d$. Then, for every non-degenerate matrix-valued non-commutative rational expression R, we have that $X \in \operatorname{dom}_{\widetilde{\mathcal{M}}}(R)$ and $R(X) = \operatorname{Ev}_X^\bullet(R(x))$, where $R(x)$ is the matrix over $\mathbb{C}(\!\!<x_1,\dots,x_d>\!\!)$ associated to R via Lemma 2.31.*

Proof. The proof is similar to the proof of Theorem 3.1. Here, we consider the set $\mathfrak{R}_0$ of all non-degenerate matrix-valued non-commutative rational expressions r for which the conclusion of Theorem 3.6 is true, i.e., we have $X \in \operatorname{dom}_{\widetilde{\mathcal{M}}}(R)$ and $R(X) = \operatorname{Ev}_X^\bullet(R(x))$. We want to show that $\mathfrak{R}_0$ consists of all non-degenerate matrix-valued non-commutative rational expressions. This can be done in almost the same way as in Theorem 3.1, except some slight modification in the last step. Suppose that $R \in \mathfrak{R}_0$ is of size $p \times p$ and has the property that R^{-1} is non-degenerate. Consider a formal linear representation $\rho = (u,A,v)$ of r, say of dimension k. Like in the proof of Theorem 3.1, we deduce from Remark 2.30 (iv) that the associated affine linear pencil

$$\tilde{A} := \begin{pmatrix} 0 & u \\ v & A \end{pmatrix}$$

is full. Now, by applying [60, Theorem 5.6] instead of Proposition 3.5, we get that $\tilde{A}(X)$ is invertible. Having this, we can proceed again like in the proof of Theorem 3.1 and we arrive at $X \in \operatorname{dom}_{\widetilde{\mathcal{M}}}(R^{-1})$. Moreover, since $R(X) = \operatorname{Ev}_X^\bullet(R(x))$ by the assumption $R \in \mathfrak{R}_0$, we further get that $R^{-1}(X) = R(X)^{-1} = \mathrm{Ev}_X^\bullet(R(x))^{-1} = \mathrm{Ev}_X^\bullet(R(x)^{-1}) = \mathrm{Ev}_X^\bullet(R^{-1}(x))$; notice that $R(x)$ is invertible because Lemma 2.31 guarantees that $x \in \mathrm{dom}_{\mathbb{C}\langle\!\langle x \rangle\!\rangle}(R^{-1})$ as R^{-1} was assumed to be non-degenerate. In summary, we see that $R^{-1} \in \mathfrak{R}_0$. □

3.4. **Convergence in law of the spectral measure.**

3.5. **Estimate on the cumulative distribution function of the spectral measure of self-adjoint operators.** In this subsection, we simply list and prove a few properties that we need in the next subsection to prove Theorem 3.10 about the convergence of the empirical measure of a self-adjoint non-degenerate matrix-valued non-commutative rational expression evaluated in matrices towards the analytic distribution of the limiting operator.

Lemma 3.7 (Lemma 3.2 in [6])**.** *For $X \in \widetilde{\mathcal{M}}_{\mathrm{sa}}$ and $t \in \mathbb{R}$ we have*

$$\mathcal{F}_X(t) = \max\{\tau(p) \mid p \in \mathcal{P}(\mathcal{M}),\ p(t-X)p \geq 0\}.$$

The crux of the proof of Theorem 3.10 lies in the following two lemmas.

Lemma 3.8. *Let $X, Y \in \widetilde{\mathcal{M}}_{\mathrm{sa}}$, then*

$$\sup_{t\in\mathbb{R}} |\mathcal{F}_X(t) - \mathcal{F}_{X+Y}(t)| \leq \mathrm{rk}(Y).$$

Proof. We fix $t \in \mathbb{R}$. Let $r \in \mathcal{P}(\mathcal{M})$ be such that $rY = Y$ and $q \in \mathcal{P}(\mathcal{M})$ such that $q(t-X)q \geq 0$. Then if we set $p = q \wedge (1-r)$, we have $pY = 0$ and $pq = p$, thus

$$p(t-X-Y)p = p(t-X)p = pq(t-X)qp \geq 0.$$

Consequently

$$\mathcal{F}_{X+Y}(t) \geq \tau(p) \geq \tau(q) - \tau(r).$$

By taking the supremum over q and the infimum over r we get that

$$\mathcal{F}_{X+Y}(t) \geq \mathcal{F}_X(t) - \mathrm{rk}(Y).$$

Now let's assume that q is such that $q(t-X-Y)q \geq 0$, then similarly with $p = q \wedge (1-r)$,

$$p(t-X)p = p(t-X-Y)p = pq(t-X-Y)pq \geq 0.$$

Hence

$$\mathcal{F}_X(t) \geq \tau(p) \geq \tau(q) - \tau(r).$$

And once again by taking the supremum over q and the infimum over r we get that

$$\mathcal{F}_X(t) \geq \mathcal{F}_{X+Y}(t) - \mathrm{rk}(Y).$$

Hence the conclusion. □

The authors are indebted to Mikael de la Salle for indicating them the following lemma.

Lemma 3.9. *Let $p \in \mathcal{P}(\mathcal{M})$, $X \in \widetilde{\mathcal{M}}_{sa}$, then* $\mathrm{rk}(pXp) \leq \mathrm{rk}(X)$.

Proof. Let $q \in \mathcal{P}(\mathcal{M})$ be such that $qX = X$, $r = p \wedge (1-q)$, then $r + 1 - p$ is such that

$$(r+1-p)pXp = rpXp = rXp = rqXp = 0.$$

Consequently $(p-r)pXp = pXp$. And since $p \geq r$, $p-r$ is a self-adjoint projection, hence

$$\mathrm{rk}(pXp) \leq \tau(p-r) \leq \tau(q).$$

Hence the conclusion by taking the infimum over q. □

3.6. **Main result.** This subsection focuses on proving the convergence in law of the empirical measure of matrix-valued non-commutative rational expressions evaluated in matrices satisfying some assumptions. Theorem 3.10 is for deterministic matrices, but it can easily be extended to random matrices by applying this result almost surely.

Theorem 3.10. *Let $X^N = (X_1^N, \dots, X_{d_1}^N)$ be a d_1-tuple of deterministic self-adjoint matrices and let $U^N = (U_1^N, \dots, U_{d_2}^N)$ be a d_2-tuple of deterministic unitary matrices. Further, let R be a non-degenerate square matrix-valued non-commutative rational expression in $d = d_1 + d_2$ variables which is self-adjoint of type (d_1, d_2) in the sense of Definition 2.34. Suppose that the following conditions are satisfied:*

(i) (X^N, U^N) converges in $*$-distribution towards a d-tuple of non-commutative random variables (x, u) in some tracial W^*-probability space $(\mathcal{M}, \tau)$ satisfying $\Delta(x, u) = d$.

(ii) For N large enough $R(X^N, U^N)$ is well-defined, i.e., there exists $N_0 \in \mathbb{N}$ such that $(X^N, U^N) \in \mathrm{dom}_{M_N(\mathbb{C})}(R)$ for all $N \geq N_0$.

Then $(x, u) \in \mathrm{dom}_{\widetilde{\mathcal{M}}}(R)$, so that $R(x, u)$ is well-defined, and the empirical measure of $R(X^N, U^N)$ converges in law towards the analytic distribution of $R(x, u)$.

The fact that $(x, u) \in \mathrm{dom}_{\widetilde{\mathcal{M}}}(R)$ holds was established already in Theorem 3.6. Accordingly, the main statement of Theorem 3.10 is the convergence of the empirical measure of $R(X^N, U^N)$ towards the spectral measure of $R(x, u)$. This convergence result actually holds in a more general setting than the above theorem. We summarize it as the following proposition.

Proposition 3.11. *For each $N \in \mathbb{N}$, let $X^N = (X_1^N, \dots, X_d^N)$ be a d-tuple of non-commutative random variables in some tracial W^*-probability space $(\mathcal{M}^{(N)}, \tau^{(N)})$. Further, let X^N converge in $*$-distribution towards a d-tuple $X = (X_1, \dots, X_d)$ of non-commutative random variables in some tracial W^*-probability space $(\mathcal{M}, \tau)$. Let R be a square matrix-valued non-commutative rational expression in d variables such that, for all $N \in \mathbb{N}$ which are sufficiently large,*

(i) $X^N \in \mathrm{dom}_{\widetilde{\mathcal{M}^{(N)}}}(R)$ and $X \in \mathrm{dom}_{\widetilde{\mathcal{M}}}(R)$,

(ii) $R(X^N)$ and $R(X)$ are self-adjoint.

Then the analytic distribution of $R(X^N)$ converges in law towards the analytic distribution of $R(X)$.

Once Proposition 3.11 is shown, the statement on the convergence in Theorem 3.10 follows immediately. Indeed, the condition formulated in Item (i) of Proposition 3.11 is satisfied as we have $(X^N, U^N) \in \mathrm{dom}_{M_N(\mathbb{C})}(R)$ for all $N \geq N_0$ by Item (ii) of Theorem 3.10 and $(x, u) \in \mathrm{dom}_{\widetilde{\mathcal{M}}}(R)$ by Theorem 3.6; further, we have that $R(X^N, U^N)$ for all $N \geq N_0$ and $R(x, u)$ are self-adjoint thanks to Definition 2.34 as R is supposed to be self-adjoint of type (d_1, d_2), so that the condition in Item (ii) of Proposition 3.11 is fulfilled as well.

Let us provide an outline of the proof of Proposition 3.11. Let $\rho = (Q, w)$ be a self-adjoint formal linear representation of R in the sense of Definition 2.36 which is moreover proper as given by Theorem 2.37. Thanks to Lemma 3.8, we can ignore the singularity in 0 of $Q(X, X^*)^{-1}$. More precisely, as long as the spectral measure of $Q(X, X^*)$ has no atom at 0, we can use Lemma 3.8 to prove that the cumulative distribution function of $w^*Q(X,X^*)^{-1}w$ is close to the one of $w^*f_\varepsilon(Q(X^N,X^{N*}))w$ where f_ε is a continuous function which is equal to $t\mapsto t^{-1}$ outside of a neighborhood of 0 of size ε. Then we can use the convergence in $*$-distribution of X^N to show that the cumulative distribution function of $w^*f_\varepsilon(Q(X^N,X^{N*}))w$ converges towards the correct limit when we let N go to infinity and ε go to 0.

It is important to note that in this subsection, by convergence in $*$-distribution of X^N of non-commutative random variables $X^N=(X_1^N,\dots,X_d^N)$, we mean that the trace of any non-commutative $*$-polynomial P evaluated in X^N converges towards the trace of $P(X,X^*)$ where X is a d-tuple of non-commutative random variables in some tracial W^*-probability space. In particular, this does *not* exclude the case where the operator norm of X_i^N is not bounded over N. This forces us to do a few more computations since the convergence in law of the analytic measure of $P(X^N,X^{N*})$ towards the analytic measure of the limiting operator, while still true, is not immediate anymore.

Proof of Proposition 3.11. Let $\rho=(Q,w)$ be a proper self-adjoint formal linear representation (of dimension k) of R. If $p\in\mathbb{N}$ is the size of R, then since $k\ge p$ and w has full rank, there exists a matrix $T\in \mathrm{GL}_k(\mathbb{C})$ such that $w=Tw_0$ where $w_0\in M_{k\times p}(\mathbb{C})$ is the rectangular matrix whose diagonal coefficients are all 1, and non-diagonal coefficients are all 0. By replacing Q by T^*QT, one can assume without loss of generality that $w=w_0$.

Notice that by assumption $Q_N:=Q(X^N,X^{N*})$ is invertible in $\widetilde{\mathcal{M}}^{(N)}$ and $R(X^N)=w^*Q_N^{-1}w$. Further, we have also that $Q_\infty:=Q(X,X^*)$ is invertible in $\widetilde{\mathcal{M}}$ and $R(X)=w^*Q_\infty^{-1}w$. To prove the convergence in law, we need to prove that $\mathcal{F}_{w^*Q_N^{-1}w}(t)$ converges towards $\mathcal{F}_{w^*Q_\infty^{-1}w}(t)$ for $t\in\mathbb{R}$ such that the function $s\mapsto\mathcal{F}_{w^*Q_\infty^{-1}w}(s)$ is continuous in t. To do so, let $g:t\mapsto t^{-1}$ and $f_\varepsilon:\mathbb{R}\to\mathbb{R}$ be a continuous function such that on the complementary set of $[-\varepsilon,\varepsilon]$, $f_\varepsilon=g$. We have for any $t\in\mathbb{R}$,

$$\begin{aligned}\left|\mathcal{F}_{w^*Q_N^{-1}w}(t)-\mathcal{F}_{w^*Q_\infty^{-1}w}(t)\right| &\le \left|\mathcal{F}_{w^*f_\varepsilon(Q_N)w}(t)-\mathcal{F}_{w^*f_\varepsilon(Q_\infty)w}(t)\right| \\ &\quad+\left|\mathcal{F}_{w^*Q_N^{-1}w}(t)-\mathcal{F}_{w^*f_\varepsilon(Q_N)w}(t)\right| \\ &\quad+\left|\mathcal{F}_{w^*Q_\infty^{-1}w}(t)-\mathcal{F}_{w^*f_\varepsilon(Q_\infty)w}(t)\right|.\end{aligned}$$

Thus thanks to Lemma 3.8,

$$\begin{aligned}\left|\mathcal{F}_{w^*Q_N^{-1}w}(t)-\mathcal{F}_{w^*Q_\infty^{-1}w}(t)\right| &\le \left|\mathcal{F}_{w^*f_\varepsilon(Q_N)w}(t)-\mathcal{F}_{w^*f_\varepsilon(Q_\infty)w}(t)\right| \\ &\quad+\mathrm{rk}(w^*(f_\varepsilon-g)(Q_N)w) \\ &\quad+\mathrm{rk}(w^*(f_\varepsilon-g)(Q_\infty)w).\end{aligned}$$

Since $w=w_0$, we have that for any $X\in M_p(\widetilde{\mathcal{M}})$,

$$\begin{aligned}\mathrm{rk}(wXw^*) &= \mathrm{rk}\begin{pmatrix} X & 0_{p\times(k-p)} \\ 0_{(k-p)\times p} & 0_{k-p}\end{pmatrix} \\ &= \frac{p}{k}\,\mathrm{rk}(X).\end{aligned}$$

This implies that

$$\mathrm{rk}(w^*(f_\varepsilon-g)(Q_\infty)w)=\frac{k}{p}\times\mathrm{rk}(ww^*(f_\varepsilon-g)(Q_\infty)ww^*)\le\frac{k}{p}\times\mathrm{rk}((f_\varepsilon-g)(Q_\infty)),$$

where in the last inequality we used Lemma 3.9. Besides $\mathbf{1}_{[-\varepsilon,\varepsilon]}(Q_\infty)$ is a self-adjoint projection such that $\mathbf{1}_{[-\varepsilon,\varepsilon]}(Q_\infty)(f_\varepsilon - g)(Q_\infty) = (f_\varepsilon - g)(Q_\infty)$. Consequently with Tr_k the non-renormalized trace on $M_k(\mathbb{C})$ and τ the trace on $\mathcal{M}$,

$$\mathrm{rk}(w^*(f_\varepsilon - g)(Q_\infty)w) \leq \frac{1}{p}\,\mathrm{Tr}_k \otimes\tau\big(\ \mathbf{1}_{[-\varepsilon,\varepsilon]}(Q_\infty)\ \big).$$

Let h_ε be a continuous function which takes value 1 on $[-\varepsilon,\varepsilon]$, 0 outside of $[-2\varepsilon, 2\varepsilon]$ and in $[0,1]$ elsewhere, then

$$\mathrm{rk}(w^*(f_\varepsilon - g)(Q_\infty)w) \leq \frac{1}{p}\,\mathrm{Tr}_k \otimes\tau\big(\ h_\varepsilon(Q_\infty)\ \big). \tag{3}$$

Hence with similar computations we get

$$\begin{aligned}\Big|\mathcal{F}_{w^*Q_N^{-1}w}(t) - \mathcal{F}_{w^*Q_\infty^{-1}w}(t)\Big| \leq & \left|\mathcal{F}_{w^*f_\varepsilon(Q_N)w}(t) - \mathcal{F}_{w^*f_\varepsilon(Q_\infty)w}(t)\right| \\ & + \frac{1}{p}\,\mathrm{Tr}_k \otimes\tau\big(\ h_\varepsilon(Q_\infty)\ \big) \\ & + \frac{1}{p}\,\mathrm{Tr}_k \otimes\tau^{(N)}\big(\ h_\varepsilon(Q_N)\ \big).\end{aligned}$$

In order to use the Portmanteau theorem, we want to prove that the analytic distribution of $w^*f_\varepsilon(Q_N)w$ converges towards the analytic distribution of $w^*f_\varepsilon(Q_\infty)w$. However, since this self-adjoint operator is uniformly bounded over N, we simply need to prove the convergence of the moments. That is, that

$$\lim_{N\to\infty} \frac{1}{p}\,\mathrm{Tr}_p \otimes\tau^{(N)}\left((w^*f_\varepsilon(Q_N)w)^l\right) = \frac{1}{p}\,\mathrm{Tr}_p \otimes\tau\left((w^*f_\varepsilon(Q_\infty)w)^l\right)$$

for any l. The strategy consists in approximating f_ε by a polynomial then use the convergence in $*$-distribution of X^N. However, the fact that we did not assume the operator norm of the matrices X_i^N to be bounded over N, forces us to make additional estimates.

Let $C = \|Q_\infty\| + 1$, and h be a non-negative continuous function which takes value 0 on $[-C, C]$, 1 outside of $[-C-1, C+1]$ and in $[0,1]$ elsewhere. Let P_m be a polynomial such that $\|f_\varepsilon - P_m\|_{\mathcal{C}^0([-C-1,C+1])} \leq 1/m$. We set

$$B^N := \big(f_\varepsilon(Q_N) - P_m(Q_N)\big)(1 - h(Q_N)) \quad \text{and} \quad C^N := \big(f_\varepsilon(Q_N) - P_m(Q_N)\big)h(Q_N),$$

then

$$\begin{aligned}&\frac{1}{p}\,\mathrm{Tr}_p \otimes\tau^{(N)}\left((w^*f_\varepsilon(Q_N)w)^l\right) - \frac{1}{p}\,\mathrm{Tr}_p \otimes\tau^{(N)}\left((w^*P_m(Q_N)w)^l\right) \\ &= \sum_{i=1}^{l} \frac{1}{p}\,\mathrm{Tr}_p \otimes\tau^{(N)}\left((w^*f_\varepsilon(Q_N)w)^{i-1}w^*(B^N + C^N)w(w^*P_m(Q_N)w)^{l-i}\right).\end{aligned}$$

Thanks to the Cauchy-Schwarz inequality, we have for any i,

$$\begin{aligned}
&\left|\frac{1}{p}\operatorname{Tr}_p\otimes\tau^{(N)}\left((w^*f_\varepsilon(Q_N)w)^{i-1}w^*(B^N+C^N)w(w^*P_m(Q_N)w)^{l-i}\right)\right|\\
&\leq\left(\sqrt{\frac{1}{p}\operatorname{Tr}_p\otimes\tau^{(N)}\left(w^*B^Nww^*B^Nw\right)}+\sqrt{\frac{1}{p}\operatorname{Tr}_p\otimes\tau^{(N)}\left(w^*C^Nww^*C^Nw\right)}\right)\\
&\quad\times\sqrt{\frac{1}{p}\operatorname{Tr}_p\otimes\tau^{(N)}\left((w^*P_m(Q_N)w)^{2(l-i)}(w^*f_\varepsilon(Q_N)w)^{2(i-1)}\right)}\\
&\leq\left(\sqrt{\operatorname{Tr}_k\otimes\tau^{(N)}\left((B^N)^2\right)}+\sqrt{\operatorname{Tr}_k\otimes\tau^{(N)}\left((C^N)^2\right)}\right)\\
&\quad\times\left(\frac{1}{p}\operatorname{Tr}_p\otimes\tau^{(N)}\left((w^*P_m(Q_N)w)^{4(l-i)}\right)\right)^{1/4}\\
&\quad\times\left(\frac{1}{p}\operatorname{Tr}_p\otimes\tau^{(N)}\left((w^*f_\varepsilon(Q_N)w)^{4(i-1)}\right)\right)^{1/4}.
\end{aligned}$$

Since f_ε is bounded by a constant K, we have that

$$\frac{1}{p}\operatorname{Tr}_p\otimes\tau^{(N)}\left((w^*f_\varepsilon(Q_N)w)^{4(i-1)}\right)\leq K^{4(i-1)}.$$

Thanks to the convergence in $*$-distribution of X^N, and since the expression $w^*P_m(Q_N)w$ is a matrix of polynomials in X^N, we have

$$\lim_{N\to\infty}\frac{1}{p}\operatorname{Tr}_p\otimes\tau^{(N)}\left((w^*P_m(Q_N)w)^{4(l-i)}\right)=\frac{1}{p}\operatorname{Tr}_p\otimes\tau\left((w^*P_m(Q_\infty)w)^{4(l-i)}\right),$$

which means that

$$\lim_{N\to\infty}\frac{1}{p}\operatorname{Tr}_p\otimes\tau^{(N)}\left((w^*P_m(Q_N)w)^{4(l-i)}\right)\leq(K+1/m)^{4(l-i)}.$$

We also have

$$\operatorname{Tr}_k\otimes\tau^{(N)}\left((B^N)^2\right)\leq\frac{k}{m^2}.$$

Finally since f_ε is bounded, there exists an integer g such that for any $t\in\mathbb{R}$, $|f_\varepsilon(t)-P_m(t)|\leq(1+t^2)^g$, thus for any $r\geq 0$,

$$\begin{aligned}
\operatorname{Tr}_k\otimes\tau^{(N)}\left((C^N)^2\right)&\leq\frac{\operatorname{Tr}_k\otimes\tau^{(N)}\left((C^N)^2Q_N^{2r}\right)}{C^{2r}}\\
&\leq\frac{\operatorname{Tr}_k\otimes\tau^{(N)}\left((1+Q_N^2)^{2g}Q_N^{2r}\right)}{C^{2r}}.
\end{aligned}$$

And so for any $r\geq 0$,

$$\begin{aligned}
\lim_{N\to\infty}\operatorname{Tr}_k\otimes\tau^{(N)}\left((C^N)^2\right)&\leq\frac{\operatorname{Tr}_k\otimes\tau\left((1+Q_\infty^2)^{2g}Q_\infty^{2r}\right)}{C^{2r}}\\
&\leq k\left\|(1+Q_\infty^2)^{2g}\right\|\frac{(C-1)^{2r}}{C^{2r}}.
\end{aligned}$$

So by letting r go to infinity, we get

$$\lim_{N\to\infty}\operatorname{Tr}_k\otimes\tau^{(N)}\left((C^N)^2\right)=0.$$

By combining those results, we obtain

$$\limsup_{N\to\infty}\left|\frac{1}{p}\operatorname{Tr}_p\otimes\tau^{(N)}\left((w^*f_\varepsilon(Q_N)w)^l\right)-\frac{1}{p}\operatorname{Tr}_p\otimes\tau\left((w^*f_\varepsilon(Q_\infty)w)^l\right)\right|=\mathcal{O}(1/m).$$

Thus, by letting m go to infinity we get the convergence of the moments. This implies that the analytic distribution of $w^*f_\varepsilon(Q_N)w$ converges towards the analytic distribution of $w^*f_\varepsilon(Q_\infty)w$. Thanks to Portmanteau's theorem and Lemma 3.7, we have

$$\begin{aligned}\mathcal{F}_{w^*f_\varepsilon(Q_\infty)w}(t)&\geq\limsup_{N\to\infty}\mathcal{F}_{w^*f_\varepsilon(Q_N)w}(t)\\&\geq\liminf_{N\to\infty}\mathcal{F}_{w^*f_\varepsilon(Q_N)w}(t)\geq\lim_{s\to t,s<t}\mathcal{F}_{w^*f_\varepsilon(Q_\infty)w}(s).\end{aligned}$$

Consequently,

$$\begin{aligned}&\limsup_{N\to\infty}\left|\mathcal{F}_{w^*Q_N^{-1}w}(t)-\mathcal{F}_{w^*Q_\infty^{-1}w}(t)\right|\\&\leq\lim_{s\to t,s<t}\left|\mathcal{F}_{w^*f_\varepsilon(Q_\infty)w}(t)-\mathcal{F}_{w^*f_\varepsilon(Q_\infty)w}(s)\right|+\frac{2}{p}\operatorname{Tr}_k\otimes\tau(\ h_\varepsilon(Q_\infty)\ ),\end{aligned}$$

where we used the convergence in $*$-distribution of X^N once again in the last line, coupled with an argument similar to the one which let us prove the convergence of the moments of $w^*f_\varepsilon(Q_N)w$. But by using Lemma 3.8 one more time we have

$$\begin{aligned}&\left|\mathcal{F}_{w^*f_\varepsilon(Q_\infty)w}(t)-\mathcal{F}_{w^*f_\varepsilon(Q_\infty)w}(s)\right|\\&\leq\left|\mathcal{F}_{w^*Q_\infty^{-1}w}(t)-\mathcal{F}_{w^*Q_\infty^{-1}w}(s)\right|+2\operatorname{rk}(w^*(f_\varepsilon-g)(Q_\infty)w).\end{aligned}$$

Hence by using equation (3), we have that

$$\begin{aligned}&\limsup_{N\to\infty}\left|\mathcal{F}_{w^*Q_N^{-1}w}(t)-\mathcal{F}_{w^*Q_\infty^{-1}w}(t)\right|\\&\leq\lim_{s\to t,s<t}\left|\mathcal{F}_{w^*Q_\infty^{-1}w}(t)-\mathcal{F}_{w^*Q_\infty^{-1}w}(s)\right|+\frac{4}{p}\operatorname{Tr}_k\otimes\tau(\ h_\varepsilon(Q_\infty)\ ).\end{aligned}$$

Now, we assume that t is a point of continuity of the function $s\mapsto\mathcal{F}_{w^*Q_\infty^{-1}w}(s)$. Then, we have that $\lim_{s\to t,s\leq t}\left|\mathcal{F}_{w^*Q_\infty^{-1}w}(t)-\mathcal{F}_{w^*Q_\infty^{-1}w}(s)\right|=0$. Besides, by the dominated convergence theorem, $\lim_{\varepsilon\to 0}\operatorname{Tr}_k\otimes\tau(h_\varepsilon(Q_\infty))=\operatorname{Tr}_k\otimes\tau(\mathbf{1}_{\{0\}}(Q_\infty))$, which is equal to 0 since otherwise the distribution of Q_∞ would have an atom in 0, in contradiction to the invertibility of Q_∞; indeed, analogous to the proof of [60, Corollary 5.13], we notice that $Q_\infty\mathbf{1}_{\{0\}}(Q_\infty)=0$ and conclude from the latter that since Q_∞ is invertible over $\widetilde{\mathcal{M}}$ we necessarily have that $\mathbf{1}_{\{0\}}(Q_\infty)=0$ and hence $\mu_{Q_\infty}(\{0\})=\frac{1}{k}\operatorname{Tr}_k\otimes\tau(\mathbf{1}_{\{0\}}(Q_\infty))=0$. □

4. A characterization of rationality in free semicircular operators

This section is a part of the paper [63]. We fix an integer $d\in\mathbb{N}$, and we will focus on freely independent semicircle distributions $s=(s_1,\dots,s_d)$ represented on the full Fock space $\mathcal{F}_0(H)$ where H is a d-dimensional Hilbert space with an orthonormal basis $e_1,\dots,e_d$ (i.e. $s_i=l(e_i)+l(e_i)^*$ in Section 2.4). Our main interest in this section is commutators $[r_1^*,\cdot\,],\dots[r_d^*,\cdot\,]$ with the right annihilation operators $r_i^*=r(e_i)^*$.

Recall that there exists polynomials $\{U_w\}_{w\in[d]^*}$ in $s_1,\dots,s_d$ which forms an orthonormal basis on $L^2(\mathrm{W}^*(s),\tau)$ since $\{e_w\}_{w\in[d]^*}$ is an orthonormal basis on

$\mathcal{F}_0(H)$. For $a \in \mathrm{W}^*(s)$, we use the notation $\hat{a} \in L^2(\mathrm{W}^*(s), \tau)$ for the canonical embedding $\mathrm{W}^*(s) \subset L^2(\mathrm{W}^*(s), \tau)$. Namely, we identify U_w as a operator in $\mathrm{W}^*(s)$ and $\hat{U}_w$ with a vector in $L^2(\mathrm{W}^*(s), \tau)$. We put $U_0 = 0$. One should be careful that we use the same notation U_0 for $U_0 = 0$ and $U_0(x) = 1$ in the definition of the Chebyshev polynomials ($U_0(x)$ corresponds with U_Ω in our definition). Then by using the above notations and via the isomorphism between $L^2(\mathrm{W}^*(s), \tau)$ and $\mathcal{F}_0(H)$, we can see for each $i \in [d]$

$$l_i^*(\hat{U}_v) = \hat{U}_{i^{-1}v} \quad \text{and} \quad r_i^*(\hat{U}_v) = \hat{U}_{vi^{-1}},\ v \in [d]^*.$$

4.1. **A semicircle analog of Duchamp-Reutenauer's chracterization.** Let $C_{\mathrm{div}}(s)$ denote the division closure of $\mathbb{C}\langle s\rangle$ in $\mathrm{W}^*(s)$ and $C_{\mathrm{rat}}(s)$ denote the rational closure of $\mathbb{C}\langle s\rangle$ in $\mathrm{W}^*(s)$.

Let us state our main theorem again.

Theorem 4.1. *Let $a \in \mathrm{W}^*(s)$. Then $\{[r_i^*, a]\}_{i=1}^d$ are finite rank operators on $\mathcal{F}_0(H)$ if and only if $a \in C_{\mathrm{div}}(s)$. In addition, we have*

$$C_{\mathrm{div}}(s) = C_{\mathrm{rat}}(s) \subset \overline{\mathbb{C}\langle s\rangle}$$

where $\overline{\mathbb{C}\langle s\rangle}$ is the norm closure of $\mathbb{C}\langle s\rangle$ in $\mathrm{W}^*(s)$.

We basically follow the proof by G. Duchamp and C. Reutenauer [34]. The following two lemmas have important roles in proving our main theorem.

Lemma 4.2. *For any $i, j \in [d]$ and $k \in \mathbb{N}$, we have*

$$[r_i^*, U_k(s_j)] = \delta_{i,j} \sum_{l=1}^{k} U_{l-1}(s_j) P_\Omega U_{k-l}(s_j)$$

where P_Ω is the orthogonal projection onto $\hat{U}_\Omega = \Omega$.

Proof. This lemma is easily deduced from the property of Chebyshev polynomials (see [61, Exercise 10 in Section 8.8]) and a dual system (see [89, Semicircular Example 5.13]). However, we give a proof of this lemma for the purpose of self-containment.

First, we show for any $i, j \in [d]$

$$[r_i^*, s_j] = \delta_{i,j} P_\Omega.$$

For any $v \in [d]^*$ we have

$$\begin{aligned} [r_i^*, s_j]\hat{U}_v &= [r_i^*, l_j^* + l_j]\hat{U}_v \\ &= r_i^*(l_j^* + l_j)\hat{U}_v - (l_j^* + l_j)r_i^*\hat{U}_v \\ &= \hat{U}_{(j^{-1}v)i^{-1}} + \hat{U}_{(jv)i^{-1}} - \hat{U}_{j^{-1}(vi^{-1})} - \hat{U}_{j(vi^{-1})}. \end{aligned}$$

Note that $[r_i^*, s_j]\hat{U}_v = 0$ except for $v = \Omega$ and in this case we have

$$[r_i^*, s_j]\hat{U}_\Omega = \hat{U}_{ji^{-1}} = \delta_{i,j}\hat{U}_\Omega.$$

Then we can compute $[r_i^*, U_k(s_j)]$ by induction since by the Leibniz rule we have

$$\begin{aligned} [r_i^*, U_{k+1}(s_j)] &= [r_i^*, s_j U_k(s_j)] - [r_i^*, U_{k-1}(s_j)] \\ &= [r_i^*, s_j]U_k(s_j) + s_j[r_i^*, U_k(s_j)] - [r_i^*, U_{k-1}(s_j)] \\ &= \delta_{i,j} P_\Omega U_k(s_j) + s_j[r_i^*, U_k(s_j)] - [r_i^*, U_{k-1}(s_j)] \end{aligned}$$

and also have by the recursion formula of U_k

$$\begin{aligned} s_j \sum_{l=1}^{k} U_{l-1}(s_j) P_\Omega U_{k-l}(s_j) &= \sum_{l=1}^{k} U_l(s_j) P_\Omega U_{k-l}(s_j) + \sum_{l=2}^{k} U_{l-2}(s_j) P_\Omega U_{k-l}(s_j) \\ &= \sum_{l=2}^{k+1} U_{l-1}(s_j) P_\Omega U_{k+1-l}(s_j) + \sum_{l=1}^{k-1} U_{l-1}(s_j) P_\Omega U_{k-1-l}(s_j). \end{aligned}$$

By multiplying by $\delta_{i,j}$ and using the induction hypothesis,

$$s_j [r_i^*, U_k(s_j)] = \delta_{i,j} \sum_{l=2}^{k+1} U_{l-1}(s_j) P_\Omega U_{k+1-l}(s_j) + [r_i^*, U_{k-1}(s_j)],$$

which gives the asserted formula for $[r_i^*, U_{k+1}(s_j)]$. ☐

Lemma 4.3. *For $v, w \in [d]^*$ and $i \in [d]$, we have*

$$[r_i^*, U_v] \hat{U}_w = \hat{U}_{v(iw^*)^{-1}}$$

where w^* is the transpose of w, in other words $w^* = i_n^{k_n} i_{n-1}^{k_{n-1}} \cdots i_1^{k_1}$ when $w = i_1^{k_1} i_2^{k_2} \cdots i_n^{k_n}$.

Proof. By Lemma 4.2 and the fact that $[r_i^*, \cdot]$ is a derivation, we have for $v = i_1^{k_1} i_2^{k_2} \cdots i_n^{k_n}$,

$$\begin{aligned} & [r_i^*, U_v] \hat{U}_w \\ = & \left(\sum_{m=1}^{n} U_{k_1}(s_{i_1}) \cdots U_{k_{m-1}}(s_{i_{m-1}}) [r_i^*, U_{k_m}(s_{i_m})] U_{k_{m+1}}(s_{i_{m+1}}) \cdots U_{k_n}(s_{i_n}) \right) \hat{U}_w \\ = & \sum_{m=1}^{n} \sum_{j=1}^{k_m} \delta_{i,i_m} U_{k_1}(s_{i_1}) \cdots U_{k_{m-1}}(s_{i_{m-1}}) U_{j-1}(s_{i_m}) P_\Omega U_{k_m-j}(s_{i_m}) U_{k_{m+1}}(s_{i_{m+1}}) \cdots U_{k_n}(s_{i_n}) \hat{U}_w \\ = & \sum_{m=1}^{n} \sum_{j=1}^{k_m} \delta_{i,i_m} U_{i_1^{k_1} \cdots i_m^{j-1}} P_\Omega U_{i_m^{k_m-j} \cdots i_n^{k_n}} \hat{U}_w. \end{aligned}$$

Since we have

$$\begin{aligned} P_\Omega U_{i_m^{k_m-j} \cdots i_n^{k_n}} \hat{U}_w &= \langle U_{i_m^{k_m-j} \cdots i_n^{k_n}} \hat{U}_w, \Omega \rangle_{\mathcal{F}_0(H)} \Omega \\ &= \langle \hat{U}_w, \hat{U}_{i_n^{k_n} \cdots i_m^{k_m-j}} \rangle_{\mathcal{F}_0(H)} \Omega \end{aligned}$$

and $\{\hat{U}_w\}_{w \in [d]^*}$ is an orthonormal basis, we conclude

$$\begin{aligned} [r_i^*, U_v] \hat{U}_w &= \sum_{m=1}^{n} \sum_{j=1}^{k_m} \delta_{i,i_m} \langle \hat{U}_w, \hat{U}_{i_n^{k_n} \cdots i_m^{k_m-j}} \rangle_{\mathcal{F}_0(H)} \hat{U}_{i_1^{k_1} \cdots i_m^{j-1}} \\ &= \hat{U}_{v(iw^*)^{-1}}. \end{aligned}$$

☐

Next we associate elements $\sum_{v \in [d]^*} \alpha_v \hat{U}_v$ in $\mathcal{F}_0(H)$ with non-commutative formal power series $\sum_{v \in [d]^*} \alpha_v X^v$. Since $U_v U_w \neq U_{vw}$ in general, we cannot directly connect U_v with X^v while keeping a multiplicative structure. However, we can connect them by using a matrix representation, which may help us to prove our main theorem.

Lemma 4.4. *For each $i \in [d]$, we put*

$$S_i = E_{ii} \otimes \begin{pmatrix} s_i & -1 \\ 1 & 0 \end{pmatrix} + \sum_{j \neq i} E_{ji} \otimes \begin{pmatrix} s_i & -1 \\ 0 & 0 \end{pmatrix} \in M_d(\mathbb{C}) \otimes M_2(\mathrm{W}^*(s))$$

where $E_{ji} \in M_d(\mathbb{C})$ is a matrix whose (j,i) entry is 1 and other entries are 0. Then for $v = i_1^{k_1} i_2^{k_2} \cdots i_n^{k_n} \in [d]^* (i_1 \neq i_2 \neq \cdots \neq i_n)$ we have

$$U_v = \begin{pmatrix} 1 & 0 \end{pmatrix} ({}^t e_1 \otimes I_2)\ S_{i_1}^{k_1} S_{i_2}^{k_2} \cdots S_{i_n}^{k_n}\ (e \otimes I_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where I_2 is the identity matrix and $\{e_i\}_{i \in [d]} \subset \mathbb{C}^d$ is the standard basis of $\mathbb{C}^d$, and we put $e = \sum_{i=1}^d e_i$.

Proof. Since Chebyshev polynomials $U_n(X)$ satisfy for $n \in \mathbb{N}$

$$\begin{pmatrix} U_n(X) \\ U_{n-1}(X) \end{pmatrix} = \begin{pmatrix} X & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U_{n-1}(X) \\ U_{n-2}(X) \end{pmatrix},$$

we can show that

$$\begin{pmatrix} X & -1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} U_n(X) & -U_{n-1}(X) \\ U_{n-1}(X) & -U_{n-2}(X) \end{pmatrix}.$$

In particular, we have for any $i \in [d]$ and $n \in \mathbb{Z}_{\geq 0}$

$$U_{i^n} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s_i & -1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then we have for $v = i_1^{k_1} i_2^{k_2} \cdots i_n^{k_n}$

$$\begin{aligned} U_v &= U_{i_1^{k_1}} U_{i_2^{k_2}} \cdots U_{i_n^{k_n}} \\ &= \prod_{l=1}^n \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s_{i_l} & -1 \\ 1 & 0 \end{pmatrix}^{k_l} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \left[\prod_{l=1}^n P \begin{pmatrix} s_{i_l} & -1 \\ 1 & 0 \end{pmatrix}^{k_l} P \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

where we put $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Note that $P \begin{pmatrix} s_i & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} s_i & -1 \\ 0 & 0 \end{pmatrix}$. Since we have

$$S_i^n = E_{ii} \otimes \begin{pmatrix} s_i & -1 \\ 1 & 0 \end{pmatrix}^n + \sum_{j \neq i} E_{ji} \otimes P \begin{pmatrix} s_i & -1 \\ 1 & 0 \end{pmatrix}^n,$$

we obtain for $i_1 \neq i_2 \neq \cdots \neq i_n$

$$\begin{aligned} S_{i_1}^{k_1} S_{i_2}^{k_2} \cdots S_{i_n}^{k_n} &= \prod_{l=1}^n \left[E_{i_l i_l} \otimes \begin{pmatrix} s_{i_l} & -1 \\ 1 & 0 \end{pmatrix}^{k_l} + \sum_{j \neq i_l} E_{j i_l} \otimes P \begin{pmatrix} s_{i_l} & -1 \\ 1 & 0 \end{pmatrix}^{k_l} \right] \\ &= E_{i_1 i_n} \otimes \left[\begin{pmatrix} s_{i_1} & -1 \\ 1 & 0 \end{pmatrix}^{k_1} \prod_{l=2}^n P \begin{pmatrix} s_{i_l} & -1 \\ 1 & 0 \end{pmatrix}^{k_l} \right] \\ &\quad + \sum_{j \neq i_1} E_{j i_n} \otimes \left[\prod_{l=1}^n P \begin{pmatrix} s_{i_l} & -1 \\ 1 & 0 \end{pmatrix}^{k_l} \right]. \end{aligned}$$

Thus we conclude (note that $\begin{pmatrix}1 & 0\end{pmatrix} P = \begin{pmatrix}1 & 0\end{pmatrix}$),

$$\begin{aligned}\begin{pmatrix}1 & 0\end{pmatrix} ({}^te_1 \otimes I_2)\, S_{i_1}^{k_1} S_{i_2}^{k_2} \cdots S_{i_n}^{k_n}\, (e \otimes I_2) \begin{pmatrix}1\\0\end{pmatrix} &= \begin{pmatrix}1 & 0\end{pmatrix} \left[\prod_{l=1}^{n} P \begin{pmatrix}s_{i_l} & -1\\ 1 & 0\end{pmatrix}^{k_l} P\right] \begin{pmatrix}1\\0\end{pmatrix}\\ &= U_v.\end{aligned}$$

□

Next step is to show convergence of $\sum_{w\in[d]^*} \alpha_v S^v$ under certain assumptions. In order to estimate the operator norm of $\sum_{w\in[d]^*} \alpha_v S^v$, we use the Haagerup type inequality for the full Fock space which was proved by M. Bozėjko [13] in terms of the q-deformed Fock space.

Lemma 4.5. *Let us take $S_1, \ldots, S_d \in M_d(\mathbb{C}) \otimes M_2(\mathrm{W}^*(s))$ as in Lemma 4.4. Then we have for any $m \in \mathbb{Z}_{\geq 0}$*

$$\left\| \sum_{|v|=m} \alpha_v S^v \right\| \leq 4d^2(m+1) \left\| \sum_{|v|=m} \alpha_v \hat{U}_v \right\|_{\mathcal{F}_0(H)}.$$

Proof. When $m = 0, 1$, one can easily derive the above inequality from Lemma 4.4 and Lemma 2.59. Thus we may suppose from now on that $m \geq 2$. Recall from the proof of Lemma 4.4 that we have for $v = i_1^{k_1} i_2^{k_2} \cdots i_n^{k_n}$

$$S^v = E_{i_1 i_n} \otimes \left[\begin{pmatrix}s_{i_1} & -1\\ 1 & 0\end{pmatrix}^{k_1} \prod_{l=2}^{n} P \begin{pmatrix}s_{i_l} & -1\\ 1 & 0\end{pmatrix}^{k_l}\right] + \sum_{j\neq i_1} E_{j i_n} \otimes \left[\prod_{l=1}^{n} P \begin{pmatrix}s_{i_l} & -1\\ 1 & 0\end{pmatrix}^{k_l}\right].$$

Since we have

$$\begin{aligned}\begin{pmatrix}s_{i_1} & -1\\ 1 & 0\end{pmatrix}^{k_1} \prod_{l=2}^{n} P \begin{pmatrix}s_{i_l} & -1\\ 1 & 0\end{pmatrix}^{k_l} &= \begin{pmatrix}U_{i_1^{k_1}} & 0\\ U_{i_1^{k_1-1}} & 0\end{pmatrix} \left(\prod_{l=2}^{n-1} U_{i_l^{k_l}}\right) \begin{pmatrix}U_{i_n^{k_n}} & -U_{i_n^{k_n-1}}\\ 0 & 0\end{pmatrix}\\ &= \begin{pmatrix}U_v & -U_{vi_n^{-1}}\\ U_{i_1^{-1}v} & -U_{i_1^{-1}vi_n^{-1}}\end{pmatrix},\end{aligned}$$

S^v can be written as the following form,

$$S^v = E_{i_1 i_n} \otimes \begin{pmatrix}U_v & -U_{vi_n^{-1}}\\ U_{i_1^{-1}v} & -U_{i_1^{-1}vi_n^{-1}}\end{pmatrix} + \sum_{j\neq i_1} E_{j i_n} \otimes \begin{pmatrix}U_v & -U_{vi_n^{-1}}\\ 0 & 0\end{pmatrix}.$$

Thus we have

$$\begin{aligned}&\sum_{|v|=m} \alpha_v S^v\\ &= \sum_{i,j\in[d]} \sum_{|v|=m-2} \alpha_{ivj} \left(E_{ij} \otimes \begin{pmatrix}U_{ivj} & -U_{iv}\\ U_{vj} & -U_v\end{pmatrix} + \sum_{k\neq i} E_{kj} \otimes \begin{pmatrix}U_{ivj} & -U_{iv}\\ 0 & 0\end{pmatrix}\right)\\ &= \sum_{i,j\in[d]} E_{ij} \otimes \begin{pmatrix}\sum_{k\in[d]} \sum_{|v|=m-2} \alpha_{kvj} U_{kvj} & -\sum_{k\in[d]} \sum_{|v|=m-2} \alpha_{kvj} U_{kv}\\ \sum_{|v|=m-2} \alpha_{ivj} U_{vj} & -\sum_{|v|=m-2} \alpha_{ivj} U_v\end{pmatrix}\\ &= \sum_{i,j\in[d]} E_{ij} \otimes \begin{pmatrix}\sum_{|v|=m-1} \alpha_{vj} U_{vj} & -\sum_{|v|=m-1} \alpha_{vj} U_v\\ \sum_{|v|=m-2} \alpha_{ivj} U_{vj} & -\sum_{|v|=m-2} \alpha_{ivj} U_v\end{pmatrix}.\end{aligned}$$

Note that all entries of $\sum_{|v|=m} \alpha_v S^v$ are sums of U_v ($|v| = m,\ m-1,\ m-2$) whose coefficients are subsequences of $\{\alpha_v\}_{|v|=m}$. Therefore by Lemma 2.59, operator norms of all entries of $\sum_{|v|=m} \alpha_v S^v$ are bounded by $(m+1)\|\sum_{|v|=m} \alpha_v \hat{U}_v\|_{\mathcal{F}_0(H)}$ and we obtain a desired estimate by the triangle inequality. □

By the same argument in the Lemma 10 of [34], we have the following corollary.

Corollary 4.6. *Let $\{\alpha_v\}_{v\in[d]^*}$ be a family of complex numbers such that $\sum_{v\in[d]^*} |\alpha_v|^2 < \infty$ and $\sum_{v\in[d]^*} \alpha_v X^v$ is rational as a non-commutative formal power series. We put $a_m = \sum_{|v|=m} \alpha_v S^v \in M_d(\mathbb{C}) \otimes M_2(\mathrm{W}^*(s))$. Then $\sum_{m=0}^{\infty} a_m$ converges in the operator norm.*

Proof. Note that $\sum_{v\in[d]^*} \overline{\alpha_v} X^v$ is also rational (i.e. recognizable) by taking a complex conjugate of each entry of a linear representation of the recognizable series $\sum_{v\in[d]^*} \alpha_v X^v$. Since the Hadamard product of two rational series is also rational by Lemma 2.47, $\sum_{v\in[d]^*} |\alpha_v|^2 X^v$ is also rational as a non-commutative formal power series. By evaluating $X_1, X_2 \dots, X_d$ in one variable z (i.e. $X_1 = X_2 = \cdots = X_d = z$), we can use the argument of Kronecker (see Corollary 2.49) for the formal power series

$$\sum_{m=0}^{\infty} \left(\sum_{|v|=m} |\alpha_v|^2 \right) z^m.$$

Thus there exists $M > 0$ and $0 < c < 1$ such that

$$\sum_{|v|=m} |\alpha_v|^2 \le M c^m.$$

By using Lemma 2.59, we can estimate the operator norm of a_m as

$$\|a_m\| \le 4d^2(m+1)\sqrt{\sum_{|v|=m} |\alpha_v|^2} \le M'(m+1)c'^m$$

for some constant $M' > 0$ and $0 < c' < 1$. Thus $\sum_{m=0}^{\infty} a_m$ converges in operator norm. □

We also use the following technical lemma.

Lemma 4.7 (Lemma 11 in [34])**.** *Let $n \in \mathbb{N}$ and $\mathcal{A}$ be a Banach algebra. If $x \in M_n(\mathcal{A})$ satisfies $\lim_{m\to\infty} \|x^m\| = 0$, then we have*

(i) $\sum_{m=0}^{\infty} x^m$ converges in the operator norm to $(1-x)^{-1} \in M_n(\mathcal{A})$.
(ii) All entries of $(1-x)^{-1}$ belong to the division closure of the subalgebra generated by all entries of x in $\mathcal{A}$.

Proposition 4.8. *Let $a \in \mathrm{W}^*(s)$. If $\{[r_i^*, a]\}_{i=1}^d$ are finite rank operators on $\mathcal{F}_0(H)$, then $a \in C_{\mathrm{div}}(s)$.*

Proof. Let $\hat{a} = \sum_{v\in[d]^*} \alpha_v \hat{U}_v$ be the expansion of $\hat{a}$ and M be a $\mathbb{C}$-submodule of $\mathbb{C}\langle\langle X_1, \dots X_d\rangle\rangle$ generated by $\sum_{v\in[d]^*} \alpha_v X^{vw^{-1}}$ ($w \in [d]^*$). Thanks to Lemma 4.3, we have for each $i \in [d]$

$$[r_i^*, a]\hat{U}_w = \sum_{v\in[d]^*} \alpha_v \hat{U}_{v(iw^*)^{-1}}.$$

Note that the linear map from $\mathcal{F}_0(H)$ to $\mathbb{C}\langle\langle X_1, \dots X_d\rangle\rangle$ which maps $\hat{U}_v$ to X^v for each $v \in [d]^*$ is injective. Therefore M is finitely generated if $\{[r_i^*, a]\}_{i=1}^d$ are finite rank operators. Thus the non-commutative formal power series $\sum_{v\in[d]^*} \alpha_v X^v$ is a recognizable series by Theorem 2.45. In other words, there exists a linear representation which consists of a multiplicative morphism $\mu : [d]^* \to M_m(\mathbb{C})$ and vectors $\lambda, \gamma \in \mathbb{C}^m$ such that $\alpha_v = {}^t\lambda\mu(v)\gamma$. Moreover by choosing a linear representation such that its dimension is minimal, we may assume from Theorem 2.46 that there exists $\{u_k\}_{k=1}^K, \{w_l\}_{l=1}^L \subset [d]^*$ such that

$$\mu(v)_{ij} = \sum_{kl} c_{ij}^{kl} \alpha_{u_k v w_l}$$

for any $v \in [d]^*$ and $1 \le i, j \le m$. We put $V(X) = \sum_{i\in[d]} \mu(i)X_i$. Note that since μ is multiplicative, $V(X)$ satisfies

$$V(X)^m = \sum_{|v|=m} \mu(v)X^v$$

and, on the level of formal power series, we have

$$\begin{aligned} \sum_{v\in[d]^*} \alpha_v X^v &= {}^t\lambda \left[\sum_{m=0}^{\infty} V(X)^m\right] \gamma \\ &= {}^t\lambda[1 - V(X)]^{-1}\gamma. \end{aligned}$$

We evaluate $X = (X_1, \dots, X_d)$ in $S = (S_1, \dots, S_d)$, where the S_i's are defined like in Lemma 4.4. Then we can see that $\sum_{m=0}^{\infty} V(S)^m \in M_m(\mathbb{C})\otimes M_d(\mathbb{C})\otimes M_2(\mathrm{W}^*(s))$ converges in the operator norm from Corollary 4.6 since all entries of $\mu(v)$ are given by finite linear spans of α_{uvw} for some words u, w. Thus we can conclude

$$\begin{aligned} \sum_{v\in[d]^*} \alpha_v \hat{U}_v &= \begin{pmatrix} 1 & 0 \end{pmatrix} ({}^te_1 \otimes I_2) \sum_{v\in[d]^*} \alpha_v S^v (e\otimes I_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Omega \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} ({}^te_1 \otimes I_2){}^t\lambda[1 - V(S)]^{-1}\gamma(e\otimes I_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Omega. \end{aligned}$$

Note that $\lim_{m\to\infty} \|V(S)^m\| = 0$, and we can apply Lemma 4.7 to $V(S)$. Since Ω is a separating vector for $\mathrm{W}^*(s)$, we conclude $a \in C_{\mathrm{div}}(s)$. $\square$

Proof of Theorem 4.1. Let $\mathcal{A}$ be a subset of $\mathrm{W}^*(s)$ such that $\{[r_i^*, a]\}_{i=1}^d$ are finite rank operators on $\mathcal{F}_0(H)$ for any element $a \in \mathcal{A}$. We will show $\mathcal{A}$ is a subalgebra of $\mathrm{W}^*(s)$ which contains $\mathbb{C}\langle s\rangle$ and satisfies for any $n \in \mathbb{N}$,

$$X \in M_n(\mathcal{A}) \text{ is invertible in } M_n(\mathrm{W}^*(s)) \implies X^{-1} \in M_n(\mathcal{A}).$$

Note that $s_i \in \mathcal{A}$ for any $i \in [d]$ since $[r_i^*, s_j] = \delta_{i,j}P_\Omega$ is a finite rank operator for any $i, j \in [d]$. If $a, b \in \mathcal{A}$, then the following operators

$$\begin{aligned} [r_i^*, a + b] &= [r_i^*, a] + [r_i^*, b] \\ [r_i^*, ab] &= [r_i^*, a]b + a[r_i^*, b] \end{aligned}$$

are finite rank operators for each $i \in [d]$. Thus $\mathcal{A}$ is a subalgebra of $\mathrm{W}^*(s)$ which contains $\mathbb{C}\langle s\rangle$. Let $n \in \mathbb{N}$ be given and assume $X \in M_n(\mathcal{A})$ is invertible in $M_n(\mathrm{W}^*(s))$. Then we have for any $i \in [d]$ and $1 \le j, k \le n$

$$\begin{aligned} [r_i^*, {}^te_j X^{-1} e_k] &= {}^te_j[I_n \otimes r_i^*, X^{-1}]e_k \\ &= -{}^te_j X^{-1}[I_n \otimes r_i^*, X]X^{-1}e_k \end{aligned}$$

where $I_n \otimes r_i^* \in M_n(\mathbb{C}) \otimes B(\mathcal{F}_0(H)) \cong M_n(B(\mathcal{F}_0(H)))$ is the operator such that all diagonal entries are r_i^* and other entries are zero. Since $X \in M_n(\mathcal{A})$, all entries of $[I_n \otimes r_i^*, X]$ are finite rank operators and therefore $X^{-1} \in M_n(\mathcal{A})$. Since $C_{\mathrm{rat}}(s)$ is the smallest subalgebra satisfying above properties, we obtain $C_{\mathrm{rat}}(s) \subset \mathcal{A}$.

Moreover, we have $\mathcal{A} \subset C_{\mathrm{div}}(s) \subset \overline{\mathbb{C}\langle s\rangle}$ by Proposition 4.8 and thus $C_{\mathrm{rat}}(s) = C_{\mathrm{div}}(s) = \mathcal{A} \subset \overline{\mathbb{C}\langle s\rangle}$. □

Remark 4.9. Let us see what happens when we take r_i and consider $[r_i, a]$ instead of $[r_i^*, a]$ for $a \in \mathrm{W}^*(s)$. Indeed, for any $i \in [d]$, $[r_i, a]$ is a finite rank operator if and only if $[r_i^*, a]$ is also a finite rank operator since $[r_i + r_i^*, a] = 0$ and therefore $[r_i, a] = -[r_i^*, a]$ for any $a \in \mathrm{W}^*(s)$. This is deduced from the commutativity of the left multiplication with the right multiplication. One can also see this directly via the following equalities

$$\begin{aligned} [r_i + r_i^*, s_j] &= [r_i, s_j] + [r_i^*, s_j] \\ &= -([r_i^*, s_j])^* + \delta_{i,j} P_\Omega \\ &= -\delta_{i,j} P_\Omega^* + \delta_{i,j} P_\Omega = 0. \end{aligned}$$

for any $i, j \in [d]$ where we use $[a, b]^* = b^*a^* - a^*b^* = -[a^*, b^*]$ and $[r_i^*, s_j] = \delta_{i,j} P_\Omega$. Then we have $[r_i + r_i^*, a] = 0$ for any $a \in \mathbb{C}\langle s\rangle$ and thus for any $a \in \mathrm{W}^*(s)$.

Remark 4.10. We remark that a tuple of operators $(r_1^*, r_2^*, \ldots, r_d^*)$ is known as a dual system which is introduced by D. Voiculescu in [89].

In our setting, $(D_1, \ldots, D_d) \in B(\mathcal{F}_0(H))$ is called a dual system for s if we have for any $i, j \in [d]$

$$[D_i, s_j] = \delta_{i,j} P_\Omega.$$

From the proof of Lemma 4.2, $(r_1^*, r_2^*, \ldots, r_d^*)$ is obviously a dual system and we have

$$[D_i, a] = [r_i^*, a]$$

for each $i \in [d]$ and $a \in \mathrm{W}^*(s)$. Thus Theorem 4.1 holds if we change $\{r_i^*\}_{i=1}^d$ by any dual system for s.

One can also see that $(D_1, \ldots, D_d) \in B(\mathcal{F}_0(H))$ is a dual system for s if and only if $r_i^* - D_i$ belongs to $\mathrm{W}^*(s)'$ for each $i \in [d]$ where $\mathrm{W}^*(s)'$ is the commutant of $\mathrm{W}^*(s)$.

We have not proved Theorem 4.1 for a general tuple of operators with a dual system yet and we leave it for future works.

4.2. **Rationality criterion for affiliated operators.** In this section, we extend our main result in the previous section to affiliated operators, which follows results of Linnell [57]. Let us denote by $\widetilde{\mathrm{W}^*(s)}$ the $*$-algebra of closed densely defined (unbounded) linear operators affiliated with $\mathrm{W}^*(s)$. Note that any element $u \in \widetilde{\mathrm{W}^*(s)}$ can be written as $u = f^{-1}a = bg^{-1}$ by using some $a, b, f, g \in \mathrm{W}^*(s)$ where f, g are nonzero divisors (i.e. $fx, gx \neq 0$ for any $x \in \mathrm{W}^*(s)\backslash\{0\}$) and thus invertible in $\widetilde{\mathrm{W}^*(s)}$. For example, we can take $f = (1+uu^*)^{-1},\ a = (1+uu^*)^{-1}u,\ b = u(1+u^*u)^{-1},\ g = (1+u^*u)^{-1}$. We focus on bounded operators $\{fr_i^*b - ar_i^*g\}_{i=1}^d$ instead of commutators $\{[r_i^*, u]\}_{i=1}^d$. Note that we have formally $fr_i^*b - ar_i^*g = f[r_i^*, u]g$ since $u = f^{-1}a = bg^{-1}$.

The following lemma tells us that we can find a common denominator of two affiliated operators.

Lemma 4.11. *Let $u_1, u_2 \in \widetilde{\mathrm{W}^*(s)}$. Then there exist $a_1, a_2, b_1, b_2 \in \mathrm{W}^*(s)$ and $f, g \in \mathrm{W}^*(s)$ such that $u_1 = f^{-1}a_1 = b_1 g^{-1}$ and $u_2 = f^{-1}a_2 = b_2 g^{-1}$.*

Proof. Let $u_k = f_k^{-1}a_k = b_k g_k^{-1}$ for $k = 1, 2$ where $a_k, b_k, f_k, g_k \in \widetilde{\mathrm{W}^*(s)}$. Then we can write $f_1 f_2^{-1} = x^{-1}y$ for some $x, y \in \mathrm{W}^*(s)$. Note that $f_1^{-1} = (yf_2)^{-1}x$ and $f_2^{-1} = (xf_1)^{-1}y$ and $xf_1 = yf_2$. We put $f = xf_1 = yf_2$. Then we have $u_1 = f^{-1}xa_1$ and $u_2 = f^{-1}ya_2$. Similarly by representing $g_1^{-1}g_2 = xy^{-1}$ for some $x, y \in \mathrm{W}^*(s)$, we have $u_1 = b_1 x g^{-1}$ and $u_2 = b_2 y g^{-1}$ where $g = g_1 x = g_2 y$. □

We also use the following lemmas for bounded operators and affiliated operators (see [57]).

Lemma 4.12 (Lemma 2.1 in [57]). *Let $\theta : H \to K$ and $\phi : K \to H$ be bounded linear maps between Hilbert spaces.*

(i) If $\ker \phi = \{0\}$ and $\phi\theta$ has finite rank, then θ also has finite rank.

(ii) If $\mathrm{Im}\,\theta$ is dense in K and $\phi\theta$ has a finite rank, then ϕ also has a finite rank.

The following lemma is proved in Lemma 2.2 in [57] in terms of the free group, and the proof can be also applied to our setting.

Lemma 4.13 (Lemma 2.2 in [57]). *Let $\theta \in \mathrm{W}^*(s)$. If θ is a nonzero divisor, then $\ker \theta = \{0\}$ and $\mathrm{Im}\,\theta$ is dense in $\mathcal{F}_0(H)$.*

We define $R(s)$ and $R'(s)$ as subsets of $\widetilde{\mathrm{W}^*(s)}$. We say $u \in R(s)$ if $\{fr_i^*b - ar_i^*g\}_{i=1}^d$ are finite rank operators for any expression $u = f^{-1}a = bg^{-1}$ where $a, b, f, g \in \mathrm{W}^*(s)$. We say $u \in R'(s)$ if we can write $u = f^{-1}a = bg^{-1}$ for some $a, b, f, g \in \mathrm{W}^*(s)$ such that $\{fr_i^*b - ar_i^*g\}_{i=1}^d$ are finite rank operators. Note that we have $R(s) \subset R'(s)$ by definition.

To define rationality, we consider the division closure $D(s)$ of $\mathbb{C}\langle s\rangle$ in $\widetilde{\mathrm{W}^*(s)}$. From the results in [60], $D(s)$ forms the free skew field of fractions of $\mathbb{C}\langle s\rangle$. Note that the rational closure of $\mathbb{C}\langle s\rangle$ in $\widetilde{\mathrm{W}^*(s)}$ coincides with $D(s)$ since $D(s)$ is a skew field (see [60, Proposition 4.9]).

Remark 4.14. Let $i \in [d]$ and $u = f^{-1}a = bg^{-1} \in \widetilde{\mathrm{W}^*(s)}$ where $a, b, f, g \in \mathrm{W}^*(s)$ and assume $fr_i^*a - br_i^*g$ is a finite rank operator. Then thanks to Remark 4.9, we have

$$f(r_i^* + r_i)b - a(r_i^* + r_i)g = (fb - ag)(r_i^* + r_i) = 0$$

for any $i \in [d]$ where we use $fb = ag$. Thus $fr_ib - ar_ig$ is also a finite rank operator.

Let us state the main theorem in this section.

Theorem 4.15. *We have $R(s) = R'(s) = D(s)$ and $D(s) \cap \mathrm{W}^*(s) = C_{\mathrm{div}}(s)$. Moreover, for any $u \in D(s)$, there exists $a, b, f, g \in C_{\mathrm{div}}(s)$ such that $u = f^{-1}a = bg^{-1}$.*

In order to prove this theorem, first we show $R(s) = R'(s)$.

Lemma 4.16. *Let $u \in \widetilde{\mathrm{W}^*(s)}$ and assume $u = f^{-1}a = bg^{-1}$. If $\{fr_i^*b - ar_i^*g\}_{i=1}^d$ are finite rank operators, then $u \in R(s)$. In other words, we have $R(s) = R'(s)$.*

Proof. Let $u = f_1^{-1}a_1 = b_1g_1^{-1}$ where $a_1, b_1, f_1, g_1 \in \mathrm{W}^*(s)$. We need to show $f_1r_i^*b_1 - a_1r_i^*g_1$ is a finite rank operator for any $i \in [d]$. First we note that there exist $x, y \in \mathrm{W}^*(s)$ such that $ff_1^{-1} = x^{-1}y$. We infer that $xf = yf_1$ and $ya_1 = xff_1^{-1}a_1 = xff^{-1}a = xa$. Thus we obtain

$$y(f_1r_i^*b_1 - a_1r_i^*g_1) = yf_1r_i^*b_1 - ya_1r_i^*g_1 = x(fr_i^*b_1 - ar_i^*g_1).$$

On the other hand, since there exist some $x', y' \in \mathrm{W}^*(s)$ such that $g^{-1}g_1 = x'y'^{-1}$, we obtain in the same way as before that

$$(fr_i^*b_1 - ar_i^*g_1)y' = (fr_i^*b - ar_i^*g)x'.$$

By combining them, we have

$$\begin{aligned} y(f_1r_i^*b_1 - a_1r_i^*g_1)y' &= x(fr_i^*b_1 - ar_i^*g_1)y' \\ &= x(fr_i^*b - ar_i^*g)x'. \end{aligned}$$

Since $fr_i^*b - ar_i^*g$ is a finite rank operator for any $i \in [d]$ and y, y' are non-zero divisors, $f_1r_i^*b_1 - a_1r_i^*g_1$ is also a finite rank operator for any $i \in [d]$ by Lemmas 4.12 and 4.13; hence, we see that $u \in R(s)$. This shows $R'(s) \subset R(s)$ and thus we conclude $R(s) = R'(s)$. □

Remark 4.17. If $u \in R(s) \cap \mathrm{W}^*(s)$, since we can write $u = u1^{-1} = 1^{-1}u$, $\{[r_i^*, u]\}_{i=1}^d$ are finite rank operators. Thanks to Theorem 4.1, we have $u \in C_{\mathrm{div}}(s)$. On the other hand, if $u \in C_{\mathrm{div}}(s)$, then $\{[r_i^*, u]\}_{i=1}^d$ are finite rank operators, and thus $u \in R'(s)$ by the same theorem. By using Lemma 4.16, we have

$$R(s) \cap \mathrm{W}^*(s) = R'(s) \cap \mathrm{W}^*(s) = C_{\mathrm{div}}(s).$$

We will prove four lemmas in order to deduce that $R(s)$ is a $*$-subalgebra which is closed under taking inverse.

Lemma 4.18. *If $u_1, u_2 \in R(s)$, then $u_1 + u_2 \in R(s)$.*

Proof. By Lemma 4.11, we can write $u_k = f^{-1}a_k = b_kg^{-1}$ for $k = 1, 2$. Then $u_1 + u_2 = f^{-1}(a_1 + a_2) = (b_1 + b_2)g^{-1}$. Since $u_1, u_2 \in R(s)$ and for any $i \in [d]$

$$fr_i^*(b_1 + b_2) - (a_1 + a_2)r_i^*g = (fr_i^*b_1 - a_1r_i^*g) + (fr_i^*b_2 - a_2r_i^*g),$$

we see that $fr_i^*(b_1 + b_2) - (a_1 + a_2)r_i^*g$ is a finite rank operator for any $i \in [d]$. Therefore $u_1 + u_2 \in R(s)$ by Lemma 4.16. □

Lemma 4.19. *If $u_1, u_2 \in R(s)$, then $u_1u_2 \in R(s)$.*

Proof. Let us write $u_k = f_k^{-1}a_k = b_kg_k^{-1}$ where $a_k, b_k, f_k, g_k \in \mathrm{W}^*(s)$ for $k = 1, 2$. Let $a_1f_2^{-1} = x^{-1}y$ and $g_1^{-1}b_2 = pq^{-1}$ where $p, q, x, y \in \mathrm{W}^*(s)$. Then $u_1u_2 = f_1^{-1}a_1f_2^{-1}a_2 = (xf_1)^{-1}ya_2$ and $u_1u_2 = b_1g_1^{-1}b_2g_2^{-1} = b_1p(g_2q)^{-1}$. Since $xa_1 = yf_2$ and $g_1p = b_2q$, we have

$$xf_1r_i^*b_1p - ya_2r_i^*g_2q = x(f_1r_i^*b_1 - a_1r_i^*g_1)p + y(f_2r_i^*b_2 - a_2r_i^*g_2)q.$$

Since $u_1, u_2 \in R(s)$, this operator is a finite rank operator for any $i \in [d]$, and thus $u_1u_2 \in R(s)$ by Lemma 4.16. □

Lemma 4.20. *If $u \in R(s)$ is invertible, then $u^{-1} \in R(s)$.*

Proof. If $u \in R(s)$, then we can write $u = f^{-1}a = bg^{-1}$ and $fr_i^*b - ar_i^*g$ has a finite rank for any $i \in [d]$. In addition if u is invertible, we have $u^{-1} = a^{-1}f = gb^{-1}$. Since $ar_i^*g - fr_i^*b = -(fr_i^*b - ar_i^*g)$ for each $i \in [d]$, $u \in R(s)$ by Lemma 4.16. □

Lemma 4.21. *If $u \in R(s)$, then $u^* \in R(s)$.*

Proof. If $u \in R(s)$, then we can write $u = f^{-1}a = bg^{-1}$ and $fr_i^*b - ar_i^*g$ is a finite rank operator for any $i \in [d]$. Since $u^* = g^{*-1}b^* = a^*f^{*-1}$, we need to check that $g^*r_i^*a^* - b^*r_i^*f^*$ is a finite rank operator.

Since T^* is a finite rank operator if T is a finite rank operator on a Hilbert space and $fr_ib - ar_ig$ is a finite rank operator by Remark 4.14, $g^*r_i^*a^* - b^*r_i^*f^* = -(fr_ib - ar_ig)^*$ is also a finite rank operator for any $i \in [d]$. Thus we conclude $u^* \in R(s)$ by Lemma 4.16. □

Proof of Theorem 4.15. By Lemmas 4.18, 4.19, 4.20, we see that $R(s)$ is a subalgebra of $\widetilde{\mathrm{W}^*(s)}$ which contains $\mathbb{C}\langle s\rangle$ and is closed under taking inverse. Thus $D(s) \subset R(s)$.

Now, let $u \in R(s)$. Since $R(s)$ is also closed under the involution by Lemma 4.21, $a = (1+uu^*)^{-1}u$ and $f = (1+uu^*)^{-1}$ belong to $R(s) \cap \mathrm{W}^*(s) = C_{\mathrm{div}}(s)$ (see Remark 4.17) and therefore $u = f^{-1}a$ belongs to the division closure of $C_{\mathrm{div}}(s)$ in $\widetilde{\mathrm{W}^*(s)}$. Since $D(s)$ is the division closed subalgebra of $\widetilde{\mathrm{W}^*(s)}$ which contains $C_{\mathrm{div}}(s)$, it also contains the division closure of $C_{\mathrm{div}}(s)$ in $\widetilde{\mathrm{W}^*(s)}$ (both coincide actually). Thus $u \in D(s)$. □

In Theorem 4.15, we show an equivalent condition to $u \in D(s)$ by using bounded operators $\{fr_i^*b - ar_i^*g\}_{i=1}^d$ instead of commutators $\{[r_i^*, u]\}_{i=1}^d$. As we remark at the beginning of Section 4.2, both operators $fr_i^*b - ar_i^*g$ and $[r_i^*, u]$ are formally connected by $fr_i^*b - ar_i^*g = f[r_i^*, u]g$.

In the following proposition, we give another characterization of $u \in D(s)$ by using commutators $\{[r_i^*, u]\}_{i=1}^d$, which is an analogue of Proposition 1.2 in [57].

Proposition 4.22. *Let $u \in \widetilde{\mathrm{W}^*(s)}$. Then $u \in D(s)$ if and only if there exists a linear subspace M of finite codimension in $\mathcal{F}_0(H)$ such that $M \cap \bigcap_{i\in[d]} \mathrm{dom}(ur_i^*) = M \cap \mathrm{dom}(u)$ and $r_i^*u = ur_i^*$ on $M \cap \mathrm{dom}(u)$ for each $i \in [d]$, where $\mathrm{dom}(u)$ denotes the domain of u.*

Proof. We use the well-known fact that for any subspace M of finite codimension in a linear space H and for any linear map T on H, the preimage $T^{-1}(M)$ is also a subspace of finite codimension in H (since T induces an injective linear map from the quotient subspace $H/T^{-1}(M)$ to H/M which is finite-dimensional).

In addition, an intersection $M_1 \cap M_2$ of two subspaces M_1, M_2 of finite codimension in H is also a subspace of finite codimension (since $(M_1 + M_2)/M_2$ is isomorphic to $M_1/(M_1 \cap M_2)$ and the two quotient spaces H/M_1, $(M_1 + M_2)/M_2$ are finite-dimensional).

Now we suppose M is a subspace of finite codimension such that $r_i^*u = ur_i^*$ on $M \cap \mathrm{dom}(u) = M \cap \bigcap_{i\in[d]} \mathrm{dom}(ur_i^*)$ for any $i \in [d]$. We can write u as $u = f^{-1}a = bg^{-1}$ where $a, b, f, g \in \mathrm{W}^*(s)$. Note that $N = g^{-1}(M)$ is a subspace of finite codimension in $\mathcal{F}_0(H)$ such that $gN \subset M \cap \mathrm{dom}(u) = M \cap \bigcap_{i\in[d]} \mathrm{dom}(ur_i^*)$. Thus we have for any $\xi \in N$

$$(fr_i^*b - ar_i^*g)\xi = f(r_i^*ug\xi - ur_i^*g\xi) = 0.$$

Since N has a finite codimension in $\mathcal{F}_0(H)$, $fr_i^*b - ar_i^*g$ is a finite rank operator for each $i \in [d]$ and thus $u \in D(s)$ by Theorem 4.15.

On the other hand, if $u \in D(s)$, then by Theorem 4.15 there exists $a, f \in C_{\mathrm{div}}(s)$ such that $u = f^{-1}a$. Note that $f^{-1}a$ forms a closed operator even though we see it as a composition of unbounded operators (we do not have to take closure). Thus we can write $\mathrm{dom}(u) = \{\xi \in \mathcal{F}_0(H); a\xi \in f\mathcal{F}_0(H)\}$. Since $a, f \in C_{\mathrm{div}}(s)$, $[r_i^*, f]$, $[r_i, f]$, $[r_i^*, a]$ are finite rank operators for each $i \in [d]$. Then kernels of these operators have finite codimensions. We put for each $i \in [d]$

$$N_i = \ker[r_i^*, f] \cap \ker[r_i, f]$$

and define N as

$$N = \bigcap_{i \in [d]} N_i.$$

Note that N is a subspace of finite codimension in $\mathcal{F}_0(H)$ and there exists a subspace M_1 of finite codimension in $\mathcal{F}_0(H)$ such that

$$M_1 \cap f\mathcal{F}_0(H) \subset fN.$$

For example, we can take M_1 as a direct sum of fN and a complementary subspace of $f\mathcal{F}_0(H) \subset \mathcal{F}_0(H)$. Since $r_i fN = fr_iN$ for each $i \in [d]$, there exists a subspace M_2^i of finite codimension in $\mathcal{F}_0(H)$ for each $i \in [d]$ such that

$$M_2^i \cap r_i f\mathcal{F}_0(H) \subset f\mathcal{F}_0(H).$$

We can take M_2^i either as a direct sum of $r_i fN$ and complementary subspace of $f\mathcal{F}_0(H) \subset \mathcal{F}_0(H)$ as above, or as $(r_i^*)^{-1}(M_1)$. Then we put M by

$$M = a^{-1}[(\mathbb{C}\Omega)^\perp] \cap a^{-1}(M_1) \cap \bigcap_{i \in [d]} \ker[r_i^*, a] \cap \bigcap_{i \in [d]} (r_i r_i^* a)^{-1}(M_2^i).$$

Then M is obviously a subspace of finite codimension in $\mathcal{F}_0(H)$.

Let us show $M \cap \bigcap_{i \in [d]} \mathrm{dom}(ur_i^*) = M \cap \mathrm{dom}(u)$. If $\xi \in M \cap \mathrm{dom}(u)$, then $a\xi = f\eta$ for $\eta \in N$ since $a\xi \in M_1 \cap f\mathcal{F}_0(H) \subset fN$. Since $\xi \in \ker[r_i^*, a]$ and $\eta \in N$, we obtain $ar_i^*\xi = fr_i^*\eta$ which implies $\xi \in \mathrm{dom}(ur_i^*)$ for any $i \in [d]$. On the other hand, if $\xi \in M \cap \bigcap_{i \in [d]} \mathrm{dom}(ur_i^*)$, then $r_i^* a\xi = ar_i^*\xi \in f\mathcal{F}_0(H)$ for each $i \in [d]$. By multiplying by r_i, we have $r_i r_i^* a\xi \in M_2^i \cap r_i f\mathcal{F}_0(H) \subset f\mathcal{F}_0(H)$ for each $i \in [d]$. Then there exists $\eta_i \in \mathcal{F}_0(H)$ for each $i \in [d]$ such that $r_i r_i^* a\xi = f\eta_i$. Since $a\xi \in (\mathbb{C}\Omega)^\perp$, we have

$$a\xi = \sum_{i \in [d]} r_i r_i^* a\xi = f \sum_{i \in [d]} \eta_i$$

which implies $\xi \in \mathrm{dom}(u)$. Therefore we have $M \cap \bigcap_{i \in [d]} \mathrm{dom}(ur_i^*) = M \cap \mathrm{dom}(u)$.

In order to see $ur_i^* = r_i^* u$ on $M \cap \mathrm{dom}(u)$, we take $\xi \in M \cap \mathrm{dom}(u)$. Then, as shown above, there exists $\eta \in N$ such that $a\xi = f\eta$; hence $r_i^* u\xi = r_i^*\eta$ for each $i \in [d]$. Moreover since $ar_i^*\xi = fr_i^*\eta$, as shown above, we conclude $ur_i^*\xi = r_i^*\eta = r_i^* u\xi$. □

Remark 4.23. The proof of Proposition 4.22 also works when we take $\{r_i\}_{i \in [d]}$ instead of $\{r_i^*\}_{i \in [d]}$. In this case, we can say that there exists a subspace of finite codimension in $\mathcal{F}_0(H)$ such that $M \cap \mathrm{dom}(u) = M \cap \mathrm{dom}(ur_i)$ and $ur_i = r_i u$ on $M \cap \mathrm{dom}(u)$ for each $i \in [d]$.

5. A dual and conjugate system for q-Gaussians for all q

This section is a part of the paper [64]. In this section, we consider q-Gaussians $A_1, \dots, A_d$ ($A_i = l(e_i) + l^*(e_i)$) acting on the q-deformed Fock space $\mathcal{F}_q(H)$ where H is a d-dimensional Hilbert space with an orthonormal basis $e_1, \dots, e_d$. We abbreviate the upper index (q) for the notation of q-Gaussians $A^{(q)}$ as well q-Wick polynomials $e_w^{(q)}$ because we fixed $-1 < q < 1$ now.

One of the key to proving the existence of a dual and conjugate system is the free right annihilation operators $(R_1^*, \dots, R_d^*)$ which are defined by $R_i^* e_{wj} = \delta_{ij} e_w$ ($w \in [d]^*$, $j \in [d]$) and $R_i^* e_\Omega = 0$. Note that, only if we consider the full Fock space $\mathcal{F}_0(H)$, they are adjoint operators of the right creation operator which maps e_w to e_{wi} for $w \in [d]^*$. In the case $q \neq 0$, they are not adjoints of the right creation operators. For this reason, we use the notation R_i^* instead of r_i^* in this section. The operators l_i, l_i^* behave quite differently than the operators $R_i = (R_i^*)^*, R_i^*$; in particular, the latter is not the right version of the former. Whereas our operators l_i and l_j^* satisfy the q-commutation relations, this is not true for R_i and R_j^*; also there is no nice concrete formula for the action of R_j^* on the basic vectors $e_{j_n j_{n-1} \cdots j_1}$ (though, it is at least obvious that the n-particle space $H^{\otimes n}$ is mapped into the $(n+1)$-particle space $H^{\otimes(n+1)}$). Thus it is not directly clear how to determine the operator norm of those operators. However, by relying on the results of Bożejko, we are able to give an estimate for this in the following lemma.

Lemma 5.1. *For $-1 < q < 1$, the free right annihilation operators $R_1^*, \dots, R_d^*$ are bounded on $\mathcal{F}_q(H)$ with*

$$\|R_i^*\| \le \frac{1}{\sqrt{w(q)}}, \qquad \text{where} \qquad w(q)^2 = (1 - |q|^2)^{-1} \prod_{k=1}^{\infty} (1 - |q|^k)(1 + |q|^k)^{-1}.$$

Proof. Since R_i^* respects the orthogonality between different tensor powers in the algebraic Fock space, it suffices to restrict for the norm estimate to a fixed tensor power $m + 1 \in \mathbb{N}$; R_i^* connects then elements in $H^{\otimes(m+1)}$ with elements in $H^{\otimes m}$. By Theorem 1 in [12], we have

$$P^{(m)} \otimes 1 \le w(q)^{-1} P^{(m+1)}$$

and we can estimate

$$\begin{aligned}
\|R_i^* \sum_{|w|=m+1} \alpha_w e_w\|_q^2 &= \langle \sum_{|u|=m} \alpha_{ui} e_u, P^{(m)} \sum_{|v|=m} \alpha_{vi} e_v \rangle_{H^{\otimes m}} \\
&= \sum_{|v|=m} \sum_{\pi \in S_m} \alpha_{\pi(v)i} \overline{\alpha_{vi}} q^{|\pi|} \\
&= \langle \sum_{|u|=m+1} \alpha_u e_u, (P^{(m)} \otimes Q_i) \sum_{|v|=m+1} \alpha_v e_v \rangle_{H^{\otimes m+1}}
\end{aligned}$$

where $Q_i \in B(H)$ is the orthogonal projection onto $\mathbb{C} e_i$.

Then we have $P^{(m)} \otimes Q_i \le P^{(m)} \otimes 1 \le w(q)^{-1} P^{(m+1)}$ and

$$\|R_i^* \sum_{|w|=m+1} \alpha_w e_w\|_q^2 \le w(q)^{-1} \| \sum_{|w|=m+1} \alpha_w e_w\|_q^2.$$

□

5.1. **One variable case.** Consider first the case where $H = \mathbb{C}e$ with a unit vector e. Then $\{e_n\}_{n=0}^{\infty}$ defined by

$$e_0 = e_\Omega = \Omega, \quad e_n = e \otimes e \otimes \cdots \otimes e = e^{\otimes n}$$

forms an orthonormal basis of $\mathcal{F}_{\mathrm{alg}}(H)$. Then the q-deformed inner product is determined by $\langle e_n, e_m \rangle_q = \delta_{nm}[n]_q!$ where

$$[n]_q = \frac{1-q^n}{1-q}, \quad [n]_q! = [n]_q \cdot [n-1]_q \cdots [2]_q \cdot [1]_q.$$

Note that our q-Gaussian A satisfies

$$Ae_n = e_{n+1} + [n]_q e_{n-1}.$$

We would like to find an operator D such that $De_0 = 0$ and $[D, A] = P^{(0)}$. Then D needs to satisfy for $n \geq 1$

$$DAe_n = ADe_n,$$

and then we have

$$De_{n+1} = -[n]_q De_{n-1} + ADe_n.$$

Thus we obtain a recursion for De_n where $De_0 = 0$ and $De_1 = e_0$.

Example 5.2. For example, from this recursion, we can compute

$$\begin{aligned}
De_2 &= e_1 \\
De_3 &= e_2 - qe_0 \\
De_4 &= e_3 - q(1+q)e_1 \\
De_5 &= e_4 - q(1+q+q^2)e_2 + q^3(1+q)e_0 \\
De_6 &= e_5 - q(1+q^2+q^3)e_3 + q^3(1+q)(1+q+q^2)e_1.
\end{aligned}$$

Remark 5.3. In the case $q = -1$, we cannot define a linear operator D by using the recursion, since $e_2 = 0$ in $\mathcal{F}_{-1}(H)$ while $De_2 = e_1 \neq 0$.

Those examples suggest the following general explicit formula for De_n.

Proposition 5.4. *We define an unbounded operator D with the domain $\mathcal{F}_{\mathrm{alg}}(H)$ by linear extension of $De_0 = 0$ and*

$$De_n = \sum_{k=1}^{\lceil \frac{n}{2} \rceil} (-1)^{k-1} q^{\frac{k(k-1)}{2}} P_q(n-k, k-1) e_{n-2k+1}$$

for $n \in \mathbb{N}$, where $\lceil x \rceil$ is the ceiling function and $P_q(n,k) = \frac{[n]_q!}{[n-k]_q!}$. Then D satisfies $[D, A] = P^{(0)}$ on $\mathcal{F}_{\mathrm{alg}}(H)$.

Proof. We prove this by induction of n. Suppose we have the formula for $n = 2m-2, 2m-1$. then we compute

$$
\begin{aligned}
&-[2m-1]_q De_{2m-2} + ADe_{2m-1} \\
&\quad = -[2m-1]_q \sum_{k=1}^{m-1} (-1)^{k-1} q^{\frac{k(k-1)}{2}} P_q(2m-k-2, k-1) e_{2m-2k-1} \\
&\quad\quad + \sum_{k=1}^{m} (-1)^{k-1} q^{\frac{k(k-1)}{2}} P_q(2m-k-1, k-1) Ae_{2m-2k} \\
&\quad = \sum_{k=1}^{m} (-1)^{k-1} q^{\frac{k(k-1)}{2}} \ \{-[2m-1]_q P_q(2m-k-2, k-1) \\
&\quad\quad + [2m-2k]_q P_q(2m-k-1, k-1)\} e_{2m-2k-1} \\
&\quad\quad + \sum_{k=1}^{m} (-1)^{k-1} q^{\frac{k(k-1)}{2}} P_q(2m-k-1, k-1) e_{2m-2k+1}
\end{aligned}
$$

We also have

$$
\begin{aligned}
&-[2m-1]_q P_q(2m-k-2, k-1) + [2m-2k]_q P_q(2m-k-1, k-1) \\
&\quad = (-[2m-1]_q [2m-2k]_q + [2m-k-1]_q [2m-2k]_q) P_q(2m-k-2, k-2) \\
&\quad = -q^{2m-k-1} [k]_q [2m-2k]_q P_q(2m-k-2, k-2).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
& -[2m-1]_q De_{2m-2} + ADe_{2m-1} \\
= & \sum_{k=1}^{m-1} (-1)^k q^{\frac{k(k+1)}{2}} q^{2m-2k-1} [k]_q [2m-2k]_q P_q(2m-k-2, k-2) e_{2m-2k-1} \\
& + \sum_{k=1}^{m} (-1)^{k-1} q^{\frac{k(k-1)}{2}} P_q(2m-k-1, k-1) e_{2m-2k+1} \\
= & \ e_{2m-1} + \sum_{k=2}^{m} (-1)^{k-1} q^{\frac{k(k-1)}{2}} \{P_q(2m-k-1, k-1) \\
& + q^{2m-2k+1} [k-1]_q [2m-2k+2]_q P_q(2m-k-1, k-3)\} e_{2m-2k+1}.
\end{aligned}
$$

This is equal to De_{2m} since we have

$$
\begin{aligned}
& P_q(2m-k-1, k-1) + q^{2m-2k+1} [k-1]_q [2m-2k+2]_q P_q(2m-k-1, k-3) \\
= & \ ([2m-2k+2]_q [2m-2k+1]_q + q^{2m-2k+1} [k-1]_q [2m-2k+2]_q) P_q(2m-k-1, k-3) \\
= & \ [2m-2k+2]_q [2m-k]_q P_q(2m-k-1, k-3) \\
= & \ P_q(2m-k, k-1).
\end{aligned}
$$

Similarly, we obtain $De_{2m+2} = -[2m+1]_q De_{2m} + ADe_{2m+1}$, which implies $[D, A]e_n = P^{(0)} e_n$ □

Since $\langle De_n, e_0\rangle_q$ is $(-1)^{m-1}q^{\frac{m(m-1)}{2}}[m-1]_q!$ when $n = 2m-1$, and 0 when $n = 2m$, we can formally compute D^*e_0 as

$$D^*e_0 = \sum_{m=1}^{\infty}(-1)^{m-1}q^{\frac{m(m-1)}{2}}\frac{[m-1]_q!}{[2m-1]_q!}e_{2m-1},$$

and the square of its Hilbert norm is

$$\|D^*e_0\|^2 = \sum_{m=1}^{\infty}|q|^{m(m-1)}\frac{([m-1]_q!)^2}{[2m-1]_q!}$$

which is finite for $-1 < q \le 1$ by the ratio test. This implies $e_0 \in \mathrm{dom}(D^*)$. Let us collect this in the following corollary.

Corollary 5.5. *For all $-1 < q < 1$, the vacuum vector $e_0 = \Omega$ is in the domain of the adjoint of the normalized dual operator, and the conjugate variable for the q-Gaussian is given by*

$$\xi = D^*e_0 = \sum_{m=1}^{\infty}(-1)^{m-1}q^{\frac{m(m-1)}{2}}\frac{[m-1]_q!}{[2m-1]_q!}e_{2m-1}.$$

Remark 5.6. We remark that the polynomial corresponding to e_n is the nth q-Hermite polynomial, $Q[e_n] = H_n(x|q)$. There is a relation between q-Hermite polynomials and the Chebyshev polynomials U_n of the second kind, which is the $q = 0$ version of q-Hermite. This can be stated as follows (for example, see Lemma 5.57 in [20]),

$$U_n(x\sqrt{1-q}) = \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k q^{\frac{(k+1)k}{2}}\binom{n-k}{k}_q \sqrt{1-q}^{\,n-2k}H_{n-2k}(x|q)$$

where

$$\binom{n-k}{k}_q = \frac{[n-k]_q!}{[k]_q![n-2k]_q!}.$$

In particular, by computing the coefficient of $H_0(x|q) = 1$, we have

$$\tau_q(U_{2n}(A\sqrt{1-q})) = (-1)^n q^{\frac{(n+1)n}{2}}$$

and $\tau_q(U_{2n-1}(A\sqrt{1-q})) = 0$ for any $n \in \mathbb{N}$.

In the one-variable case, there is actually a general formula for the conjugate variable ξ in terms of Chebyshev polynomials (provided this sum converges), namely (see Exercise 8.12 in [61])

$$\xi = \sum_{n=1}^{\infty}\tau(U_{n-1}(X))C_n(X)$$

where $C_n(X)$ is the Chebyshev polynomial of the first kind. If we apply this formula to $X = A\sqrt{1-q}$ and recall that a rescaling of the random variable by a factor α results in the rescaling of the conjugate variable by a factor $1/\alpha$, then we obtain from this general formula the following expression for the conjugate variable of the q-Gaussian:

$$\sqrt{1-q}\sum_{n=0}^{\infty}(-1)^n q^{\frac{(n+1)n}{2}}C_{2n+1}(A\sqrt{1-q}).$$

We can transform this formula into the sum of q-Hermite polynomials by using $C_1(X) = U_1(X) = X$, $C_n(X) = U_n(X) - U_{n-2}(X)$ for $n \geq 2$ and the relation from above between U_n and $H_n(x|q)$, resulting after some reformulations in

$$\sum_{m=0}^{\infty} (-1)^m q^{\frac{(m+1)m}{2}} (1-q)^{m+1} \sum_{n=m}^{\infty} q^{(n+1)(n-m)} (1+q^{n+1}) \binom{n+m+1}{n-m}_q e_{2m+1}.$$

By using the non-trivial identity

$$(1-q)^{m+1} \sum_{n=m}^{\infty} q^{(n+1)(n-m)} (1+q^{n+1}) \binom{n+m+1}{n-m}_q = \frac{[m]_q!}{[2m+1]_q!}$$

for all $m \in \mathbb{Z}_{\geq 0}$, we recover thus indeed our formula from Cor. 5.5.

5.2. **Multi-variable case.** A similar reduction gives us a formula for the multi-variable case. Let us consider the q-deformed Fock space $\mathcal{F}_q(H)$ of a d-dimensional Hilbert space H with orthonormal basis $e_1, \ldots, e_d$. Let $\{A_i\}_{i=1}^d$ be q-Gaussians with respect to $\{e_i\}_{i=1}^d$.

Then the equation $[D_i, A_j] = \delta_{ij} P^{(0)}$, together with $D_i e_\omega = 0$, allows us to determine D_i inductively.

Example 5.7. Since we have $e_{jw} = A_j e_w - l_j^* e_w$ for $j \in [d]$ and $w \in [d]^*$, we have by applying D_i and $[D_i, A_j] = \delta_{ij} P^{(0)}$,

$$D_i e_{jw} = D_i A_j e_w - D_i l_j^* e_w = A_j D_i e_w + \delta_{ij} P^{(0)} e_w - D_i l_j^* e_w.$$

This gives the recursion for $D_i e_w$ with $D_i e_\Omega = 0$. For example, we have for $i, j_1, j_2, j_3 \in [d]$,

$$\begin{aligned} D_i e_{j_1} &= A_{j_1} D_i e_\Omega + \delta_{ij_1} P^{(0)} e_\Omega = \delta_{ij_1} e_\Omega \\ D_i e_{j_2 j_1} &= A_{j_2} D_i e_{j_1} - \delta_{j_2 j_1} D_i e_\Omega = \delta_{ij_1} e_{j_2} \\ D_i e_{j_3 j_2 j_1} &= A_{j_3} D_i e_{j_2 j_1} - \delta_{j_3 j_2} D_i e_{j_1} - q \delta_{j_3 j_1} D_i e_{j_2} \\ &= \delta_{ij_1} A_{j_3} e_{j_2} - \delta_{j_3 j_2} \delta_{ij_1} e_\Omega - q \delta_{j_3 j_1} \delta_{ij_2} e_\Omega \\ &= \delta_{ij_1} e_{j_3 j_2} - q \delta_{ij_2} \delta_{j_3 j_1} e_\Omega. \end{aligned}$$

Similarly, we obtain the following formulas for $j_4, j_5, j_6 \in [d]$,

$$\begin{aligned} D_i e_{j_4 j_3 j_2 j_1} &= \delta_{ij_1} e_{j_4 j_3 j_2} - q^2 \delta_{ij_2} \delta_{j_4 j_1} e_{j_3} - q \delta_{ij_2} \delta_{j_3 j_1} e_{j_4} \\ D_i e_{j_5 j_4 j_3 j_2 j_1} &= \delta_{ij_1} e_{j_5 j_4 j_3 j_2} - \delta_{ij_2} (q^3 \delta_{j_5 j_1} e_{j_4 j_3} + q^2 \delta_{j_4 j_1} e_{j_5 j_3} + q \delta_{j_3 j_1} e_{j_5 j_4}) \\ &\quad + \delta_{ij_3} (q^4 \delta_{j_5 j_1} \delta_{j_4 j_2} + q^3 \delta_{j_5 j_2} \delta_{j_4 j_1}) e_\Omega \\ D_i e_{j_6 j_5 j_4 j_3 j_2 j_1} &= \delta_{ij_1} e_{j_6 j_5 j_4 j_3 j_2} \\ &\quad - \delta_{ij_2} (q^4 \delta_{j_6 j_1} e_{j_5 j_4 j_3} + q^3 \delta_{j_5 j_1} e_{j_6 j_4 j_3} + q^2 \delta_{j_4 j_1} e_{j_6 j_5 j_3} + q \delta_{j_3 j_1} e_{j_6 j_5 j_4}) \\ &\quad + \delta_{ij_3} (q^6 \delta_{j_6 j_1} \delta_{j_5 j_2} e_{j_4} + q^5 \delta_{j_6 j_1} \delta_{j_4 j_2} e_{j_5} + q^5 \delta_{j_6 j_2} \delta_{j_5 j_1} e_{j_4} \\ &\quad + q^4 \delta_{j_6 j_2} \delta_{j_4 j_1} e_{j_5} + q^4 \delta_{j_5 j_1} \delta_{j_4 j_2} e_{j_6} + q^3 \delta_{j_5 j_2} \delta_{j_4 j_1} e_{j_6}). \end{aligned}$$

Remark 5.8. As in Remark 5.3, we cannot define a linear operator D_i in the case $q = -1$. But now the case $q = 1$ also has to be excluded if $d \geq 2$. For example, we have $e_1 \otimes e_2 - e_2 \otimes e_1 = 0$ in $\mathcal{F}_1(H)$, but

$$D_1(e_1 \otimes e_2 - e_2 \otimes e_1) = -e_2 \neq 0.$$

For $-1 < q < 1$, on the other hand, we can define D_i since the operator $\bigoplus_{n=0}^{\infty} P^{\otimes n}$ is strictly positive (see Section 2) and $\{e_w\}_{w \in [d]^*}$ forms a linear basis of $\mathcal{F}_{\mathrm{alg}}(H) \subset \mathcal{F}_q(H)$.

From these examples, we can guess that the general formula for $D_i e_{j_n \cdots j_1}$ is characterized by partitions of $n+1$ vertices $n > n-1 > \cdots > 1 > i$ and counting their crossings. However, the usual definition of crossings does not work in this setting.

Example 5.9. In the examples above, we pick in $D_i e_{j_5 j_4 j_3 j_2 j_1}$ the term

$$q^4 \delta_{ij_3} \delta_{j_5 j_1} \delta_{j_4 j_2}.$$

This corresponds to the partition $\{(i,3),(1,5),(2,4)\}$. Since $(1,5)$ has no crossing with $(2,4)$, the number of crossings is 2, while the coefficient above is q^4.

This phenomenon also happens for the other terms

$$q^6 \delta_{ij_3} \delta_{j_6 j_1} \delta_{j_5 j_2} e_{j_4},\ q^5 \delta_{ij_3} \delta_{j_6 j_1} \delta_{j_4 j_2} e_{j_5},\ q^4 \delta_{ij_3} \delta_{j_5 j_1} \delta_{j_4 j_2} e_{j_6}$$

in $D_i e_{j_6 j_5 j_4 j_3 j_2 j_1}$.

We need to change the rules of counting crossings for the precise formula of $D_i e_w$. Here we list the rules of drawing partitions that are compatible with the formula. We remark that a similar method of counting crossings appears in Definition 3.13 in [3].

(i) Consider $n+1$ vertices $n > n-1 > \cdots > 1 > 0$.

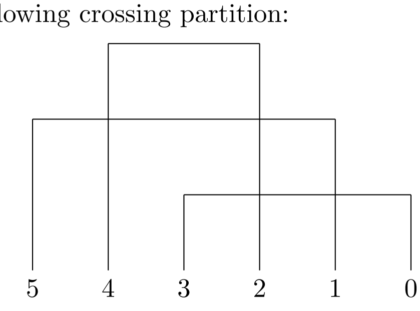
(ii) 0 must be connected to some $k \in \{1, \ldots, n\}$ with height 1.

(iii) $l \in \{1, \ldots, k-1\}$ must be coupled with one of $\{k+1, \ldots, n\}$ with height $l+1$.

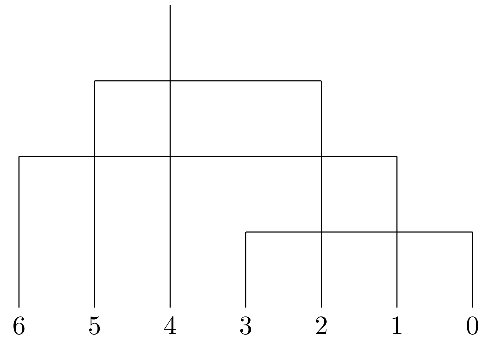
(iv) Vertices which are not coupled with $\{1, \ldots, k-1\}$ should be singletons and are drawn with straight lines to the top.

We define $B(n+1)$ as a set of partitions that satisfy the above rules. For $\pi \in B(n+1)$, we denote by $p(\pi)$ the set of parings in π and by $s(\pi)$ the singletons in π

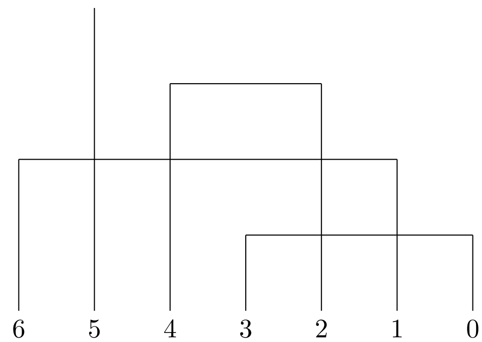
Example 5.10. Let us see what happens with the number of crossings if we follow the drawing rules from above. The term $q^4 \delta_{ij_3} \delta_{j_5 j_1} \delta_{j_4 j_2}$ from Example 5.9 is now represented by the following crossing partition:



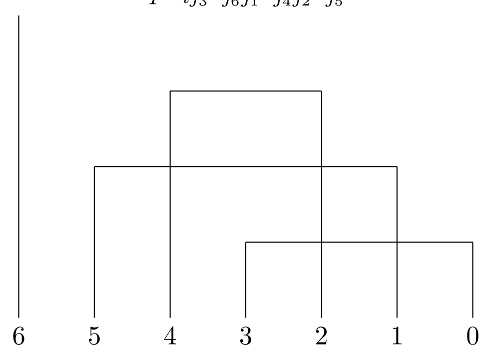
Note that the number of crossings in the picture above is now indeed 4, corresponding to the factor q^4. Similarly, the factors in the contributions $q^6 \delta_{ij_3} \delta_{j_6 j_1} \delta_{j_5 j_2} e_{j_4}$, $q^5 \delta_{ij_3} \delta_{j_6 j_1} \delta_{j_4 j_2} e_{j_5}$ and $q^4 \delta_{ij_3} \delta_{j_5 j_1} \delta_{j_4 j_2} e_{j_6}$ are accounted for correctly by the following partitions:



$q^6\delta_{ij_3}\delta_{j_6j_1}\delta_{j_5j_2}e_{j_4}$



$q^5\delta_{ij_3}\delta_{j_6j_1}\delta_{j_4j_2}e_{j_5}$



$q^4\delta_{ij_3}\delta_{j_5j_1}\delta_{j_4j_2}e_{j_6}$

We identify 0 with the index of a dual system and $k \in \{1, \dots, n\}$ with a letter $j_k \in [d]$ for a given word $j_n j_{n-1} \cdots j_1$. Then our examples from above motivate the following formula for D_i.

Proposition 5.11. *For $i, j_1, \dots, j_n \in [d]$, we define densely defined unbounded operators $D_1, \dots, D_d$, whose domains are the algebraic Fock space $\mathcal{F}_0(H)_{\mathrm{alg}}$, by linear extension of*

$$D_i e_\Omega = 0, \qquad D_i e_{j_n \cdots j_1} = \sum_{\pi \in B(n+1)} (-1)^{\pi(0)-1} q^{\mathrm{cross}(\pi)} \delta_{p(\pi)} e_{s(\pi)}$$

where $\mathrm{cross}(\pi)$ *is the number of crossings of π according to our drawing rules and where $\delta_{p(\pi)} = \prod_{(k,l)\in\pi} \delta_{j_k j_l}$ with $j_0 = i$ and $e_{s(\pi)} = e_{j_{k_s} \cdots j_{k_1}}$ for $s(\pi) = \{k_s > \cdots > k_1\}$.*

Then we have

$$[D_i, A_j] = \delta_{ij} P^{(0)}$$

on the domain $\mathcal{F}_{\text{alg}}(H)$.

Proof. We have to show that

$$[D_i, A_j] e_\Omega = \delta_{ij} P^{(0)} e_\Omega = \delta_{ij} e_\Omega$$

and

$$[D_i, A_j] e_{j_n \cdots j_1} = \delta_{ij} P^{(0)} e_{j_n \cdots j_1} = 0$$

for all $n > 0$ and $j_1, \ldots, j_n \in [d]$. The first formula is easy to check, so let us concentrate on the second one. We will there rename j to j_{n+1} and for better legibility we will also write sometimes $[j_n \cdots j_1]$ for $e_{j_n \cdots j_1}$.

Then we can compute on one hand

$$\begin{aligned} & A_{j_{n+1}} D_i e_{j_n \cdots j_1} \\ = & \sum_{\sigma \in B(n+1)} (-1)^{\sigma(0)-1} q^{\text{cross}(\sigma)} \delta_{p(\sigma)} A_{j_{n+1}} e_{s(\sigma)} \\ = & \sum_{\sigma \in B(n+1)} (-1)^{\sigma(0)-1} q^{\text{cross}(\sigma)} \delta_{p(\sigma)} e_{j_{n+1} s(\sigma)} \\ + & \sum_{\sigma \in B(n+1)} \sum_{k=1}^{|s(\sigma)|} (-1)^{\sigma(0)-1} q^{\text{cross}(\sigma)+|s(\sigma)|-k} \delta_{j_{n+1} j_{s(\sigma)_k}} \delta_{p(\sigma)} [s(\sigma)_{|s(\sigma)|} \cdots \check{s(\sigma)}_k \cdots s(\sigma)_1]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} D_i A_{j_{n+1}} e_{j_n \cdots j_1} &= D_i e_{j_{n+1} j_n \cdots j_1} + \sum_{l=1}^{n} \delta_{j_{n+1} j_l} q^{n-l} D_i [j_n \cdots \check{j}_l \cdots j_1] \\ &= D_i e_{j_{n+1} j_n \cdots j_1} \\ &+ \sum_{l=1}^{n} \sum_{\pi \in B(n)} (-1)^{\pi(0)-1} q^{\text{cross}(\pi)+n-l} \delta_{j_{n+1} j_l} \delta_{p(\pi)} e_{s(\pi)}, \end{aligned}$$

where $\check{j}_l$ means to omit j_l. Note that all partitions $\pi \in B(n)$ act on $n-1$ letters $j_n, \ldots, \check{j}_l, \ldots, j_1$ in the sum above.

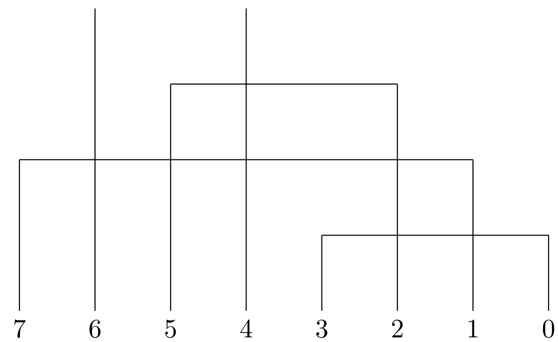
Let us first see that all terms in the last sum of $A_{j_{n+1}} D_i e_{j_n \cdots j_1}$ show also up as terms in $D_i A_{j_{n+1}} e_{j_n \cdots j_1}$.

To see this, let us consider the contribution corresponding to $\sigma \in B(n+1)$ and $k \in \{1, \ldots, |s(\sigma)|\}$. Since $s(\sigma)_k$ is a singleton, we can remove it and obtain a partition $\pi \in B(n)$. We also take l so that $l = s(\sigma)_k$. Then for these σ, π, k, l we have

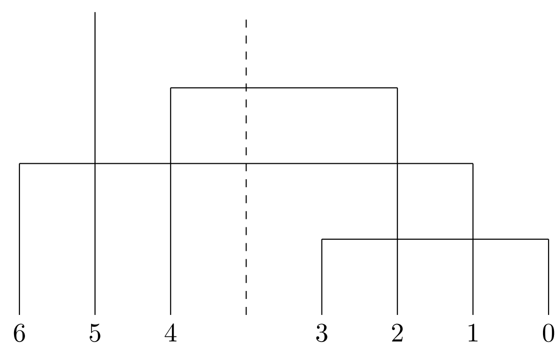
$$\delta_{j_{n+1} j_l} \delta_{p(\pi)} e_{s(\pi)} = \delta_{j_{n+1} j_{s(\sigma)_k}} \delta_{p(\sigma)} [s(\sigma)_{|s(\sigma)|} \cdots \check{s(\sigma)}_k \cdots s(\sigma)_1]$$

where π acts on $j_n, \ldots, \check{j}_l, \ldots, j_1$.

For example, if we take $\sigma \in B(8)$ represented by



and $k = 1$, then we take $l = 4$ and $\pi \in B(7)$ represented by



where we ignore the dashed line.

Note that we have also $\pi(0) = \sigma(0)$ since we removed $s(\sigma)_k$ which is on the left of $\sigma(0)$. By definition, the difference between σ and π is only the singleton $s(\sigma)_k$ and the difference between $\mathrm{cross}(\sigma)$ and $\mathrm{cross}(\pi)$ is the number of crossing points on the line of $s(\sigma)_k$. Recall that for $\pi \in B(n+1)$ the vertices that are not singletons must be coupled. Therefore the number of crossing points on the line of $s(\sigma)_k$ is equal to the number of vertices which are on the left of $s(\sigma)_k$ and not a singleton, which is equal to $(n - l) - (|s(\sigma)| - k)$; thus we have

$$\mathrm{cross}(\pi) + n - l = \mathrm{cross}(\sigma) + |s(\sigma)| - k.$$

This implies that the contribution corresponding to σ and k in the second sum of $A_{j_{n+1}} D_i e_{j_n \cdots j_1}$ shows also up as a contribution corresponding to π and l in $D_i A_{j_{n+1}} e_{j_n \cdots j_1}$

Next, we need to identify the remaining terms of the last sum. Note that the (π, l) which we can get under the above identification from (σ, k) can be identified with partitions π' of $n + 2$ vertices $n + 1 > n > \cdots > 1 > 0$ such that $n + 1$ is coupled with some $n + 1 > k > \pi'(0)$ and $\pi' \setminus (n + 1, k)$ belongs to $B(n)$ in an order preserving way. Thus the $l \in \{1, \ldots, n\}$ and $\pi \in B(n)$ such that $l > \pi(0)$ are exactly those terms corresponding to all possible (σ, k).

So we have

$$-[D_i, A_{j_{n+1}}]e_{j_n\cdots j_1} = \sum_{\sigma\in B(n+1)} (-1)^{\sigma(0)-1} q^{\mathrm{cross}(\sigma)} \delta_{p(\sigma)} e_{j_{n+1}s(\sigma)} - D_i e_{j_{n+1}j_n\cdots j_1}$$
$$+ \sum_{l=1}^{n} \sum_{\substack{\pi\in B(n)\\ \pi(0)\geq l}} (-1)^{\pi(0)} q^{\mathrm{cross}(\pi)+n-l} \delta_{j_{n+1}j_l} \delta_{p(\pi)} e_{s(\pi)}.$$

In order to see that this is actually equal to zero, we need now to understand the condition $l \leq \pi(0)$ in the last sum.

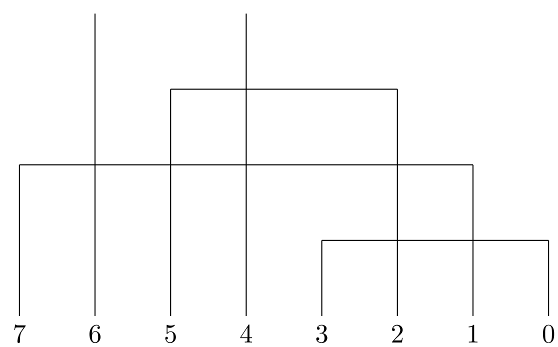
In this case, we can associate the term $\delta_{j_{n+1}j_l}\delta_{p(\pi)}e_{s(\pi)}$ to a partition π' in $B(n+2)$; π' is given by coupling $n+1$ with $l \leq \pi(0)$ and requiring that $\pi' \setminus (n+1, l)$ is equal to π. We have then

$$\pi(0) + 1 = \pi'(0),$$

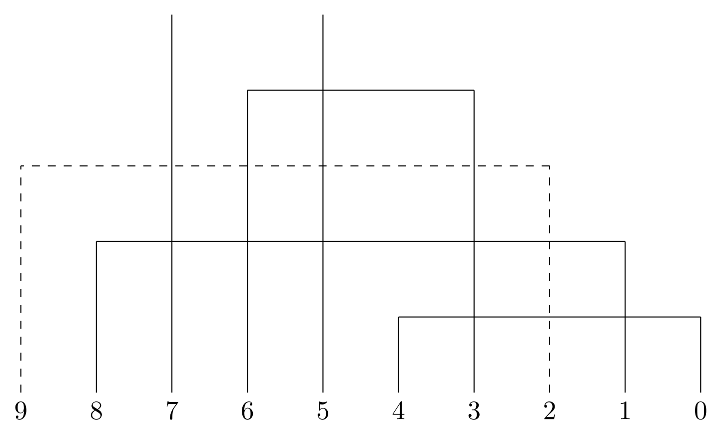
since $l \leq \pi(0)$ is inserted into π'. Moreover the difference between $\mathrm{cross}(\pi')$ and $\mathrm{cross}(\pi)$ is the number of crossing points on the pair $(n+1, l)$. Note that in our definitions, crossing points on $(n+1, l)$ consist of the coupling with $\pi'(0) > j > l$ (double count) and the coupling with $l > j \geq 0$ (single count) and also singletons of π (single count). Therefore, this difference is equal to $n - l$ and we have

$$\mathrm{cross}(\pi') = \mathrm{cross}(\pi) + n - l.$$

For example, take $\pi \in B(8)$ represented by



and take $l = 2$. Then $\pi' \in B(10)$ is represented by



Here we have

$$\pi'(0) = 4, \quad \pi(0) = 3: \qquad 3 + 1 = 4$$

and

$$\text{cross}(\pi') = 13, \quad \text{cross}(\pi) = 7 \quad n = 8, \quad l = 2: \qquad 13 = 7 + (8 - 2).$$

By combining these results we obtain

$$\begin{aligned}
-[D_i, A_{j_{n+1}}]e_{j_n\cdots j_1} + D_i e_{j_{n+1}j_n\cdots j_1} \\
&= \sum_{\sigma\in B(n+1)} (-1)^{\sigma(0)-1} q^{\text{cross}(\sigma)} \delta_{p(\sigma)} e_{j_{n+1}s(\sigma)} \\
&\qquad + \sum_{l=1}^{n} \sum_{\substack{\pi\in B(n) \\ \pi(0)\geq l}} (-1)^{\pi(0)} q^{\text{cross}(\pi)+n-l} \delta_{j_{n+1}j_l} \delta_{p(\pi)} e_{s(\pi)} \\
&= \sum_{\sigma\in B(n+1)} (-1)^{\sigma(0)-1} q^{\text{cross}(\sigma)} \delta_{p(\sigma)} e_{j_{n+1}s(\sigma)} \\
&\qquad + \sum_{\substack{\pi'\in B(n+2) \\ \pi'(n+1) \text{ is not a singleton}}} (-1)^{\pi'(0)-1} q^{\text{cross}(\pi')} \delta_{p(\pi')} e_{s(\pi')} \\
&= \sum_{\pi'\in B(n+2)} (-1)^{\pi'(0)-1} q^{\text{cross}(\pi')} \delta_{p(\pi')} e_{s(\pi')} \\
&= D_i e_{j_{n+1}j_n\cdots j_1},
\end{aligned}$$

and thus $[D_i, A_{j_{n+1}}]e_{j_n\cdots j_1} = 0$, which proves our assertion. □

In the following, we want to use this proposition to conclude that e_Ω lies in the domain of D_i^* and actually also derive a formula for $D_i^* e_\Omega$.

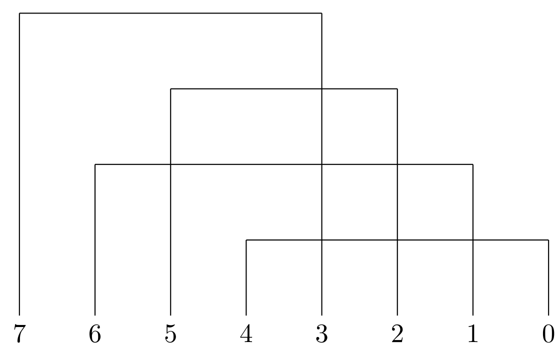
Theorem 5.12. *For any $-1 < q < 1$, there exists a normalized dual system and thus a conjugate system for q-Gaussians $A = (A_1, \dots, A_d)$.*

Proof. By Theorem 2.24, it suffices to see that $D_i^* e_\Omega$ exists in $\mathcal{F}_q(H)$. In order to see that e_Ω is in the domain of D_i^*, we have to show that the linear functional $\langle D_i \cdot\,, e_\Omega\rangle_q$ is bounded on the algebraic Fock space.

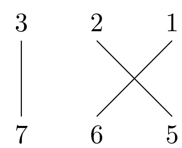
Let us take $\sum_{w\in[d]^*} \alpha_w e_w \in \mathcal{F}_{\text{alg}}(H)$. Note that we can compute $\langle D_i e_w, e_\Omega\rangle_q$ by counting summands without singletons by Proposition 5.11 and in that case the word length $|w|$ must be odd and $\pi \in B(2m)$ must connect 0 to m for $|w| = 2m-1$. Thus we have

$$\begin{aligned}
\langle D_i \sum_{w\in[d]^*} \alpha_w e_w, e_\Omega\rangle_q &= \sum_{m=1}^{\infty} \sum_{|w|=2m-1} \sum_{\substack{\pi\in B(2m) \\ \pi(0)=m}} \alpha_w (-1)^{m-1} q^{\text{cross}(\pi)} \delta_{p(\pi)w} \\
&= \sum_{m=1}^{\infty} \sum_{\substack{\pi\in B(2m) \\ \pi(0)=m}} \sum_{|w|=2m-1} \alpha_w (-1)^{m-1} q^{\text{cross}(\pi)} \delta_{p(\pi)w}.
\end{aligned}$$

Note that we write now $\delta_{p(\pi)w}$ for $\delta_{p(\pi)} = \prod_{(k,l)\in\pi} \delta_{j_k j_l}$ in order to make the dependency on $w = j_{2m-1}\cdots j_1$ explicit. For each $\pi \in B(2m)$, let us consider words w with $|w| = 2m-1$ such that $\delta_{p(\pi)w} = 1$. Such words can be represented by $\pi_p(w)iw$ where w is any word with $|w| = m-1$ and $\pi_p \in S_{m-1}$ is a permutation such that $\delta_{p(\pi)\pi_p(w)iw} = 1$. Note that there is a one-to-one correspondence between π and π_p. For example, the partition π in $B(8)$ represented by



induces the element π_p in S_3 represented by



The important observation is that we have

$$\mathrm{cross}(\pi) = \frac{m(m-1)}{2} + |\pi_p|$$

where $|\pi_p|$ is the number of inversions of π_p. Actually, when we take, for $1 \le k \le m-1$, the pair $(k, \pi(k)) \in \pi$, then this pair crosses with k pairs $(0, m)$, $(1, \pi(1))$, ..., $(k-1, \pi(k-1))$ on the right area $(m > m-1 > \cdots > 0)$, which implies the number of crossings in the right area is $\sum_{k=1}^{m-1} k = \frac{m(m-1)}{2}$. In addition, $(k, \pi(k)), (l, \pi(l))$ $(k < l \in \{1, \dots, m-1\})$ are crossing in the left area $(2m-1 > 2m-2 > \cdots > m)$ if and only if $\pi(l) < \pi(k)$, which implies the number of crossings in the left area is $|\pi_p|$.

Thus we can continue our calculation as follows:

$$\begin{aligned}
\langle D_i \sum_{w \in [d]^*} \alpha_w e_w, e_\Omega \rangle_q &= \sum_{m=1}^{\infty} \sum_{\substack{\pi \in B(2m) \\ \pi(0)=m}} \sum_{|w|=2m-1} \alpha_w (-1)^{m-1} q^{\mathrm{cross}(\pi)} \delta_{p(\pi)w} \\
&= \sum_{m=1}^{\infty} (-1)^{m-1} q^{\frac{m(m-1)}{2}} \sum_{\pi_p \in S_{m-1}} \sum_{|w|=m-1} \alpha_{\pi_p(w)iw} q^{|\pi_p|} \\
&= \sum_{m=1}^{\infty} (-1)^{m-1} q^{\frac{m(m-1)}{2}} \sum_{|w|=m-1} \sum_{\pi_p \in S_{m-1}} \sum_{|v|=m-1} \delta_{v\pi_p(w)} \alpha_{viw} q^{|\pi_p|} \\
&= \sum_{m=1}^{\infty} (-1)^{m-1} q^{\frac{m(m-1)}{2}} \sum_{|w|=m-1} \sum_{|v|=m-1} \sum_{\pi_p \in S_{m-1}} \langle e_v, e_{\pi_p(w)} \rangle \alpha_{viw} q^{|\pi_p|} \\
&= \sum_{m=1}^{\infty} (-1)^{m-1} q^{\frac{m(m-1)}{2}} \sum_{|w|=m-1} \langle \sum_{|v|=m-1} \alpha_{viw} e_v, e_w \rangle_q.
\end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned} |\langle D_i \sum_{w\in[d]^*} \alpha_w e_w, e_\Omega\rangle_q| &\leq \sum_{m=1}^{\infty} |q|^{\frac{m(m-1)}{2}} \sum_{|w|=m-1} |\langle \sum_{|v|=m-1} \alpha_{viw} e_v, e_w\rangle_q| \\ &\leq \sum_{m=1}^{\infty} |q|^{\frac{m(m-1)}{2}} \sum_{|w|=m-1} \| \sum_{|v|=m-1} \alpha_{viw} e_v\|_q \cdot \|e_w\|_q. \end{aligned}$$

Note that $\|e_w\|_q^2 \leq \sum_{\pi\in S_{m-1}} |q|^{|\pi|} = [m-1]_{|q|}!$.

On the other hand, we can write

$$\sum_{|v|=m-1} \alpha_{viw} e_v = R_{iw}^* \sum_{|v|=m-1} \alpha_{viw} e_{viw} = R_{iw}^* \sum_{|v|=2m-1} \alpha_v e_v$$

where $R_w^* = R_{w_1}^* \cdots R_{w_n}^*$ for $w = w_1 \cdots w_n$ is the free right annihilation operator of the word w.

By Lemma 5.1, the free right annihilation operators $R_1^*, \ldots, R_d^*$ are bounded and their operator norms are less than $C = \sqrt{w(q)}^{-1}$ where $w(q)$ is a positive constant which appears in [12]. Since R_{iw}^* is in our case a product of m such free right annihilation operators, we have $\|R_{iw}^*\| \leq C^m$ and thus

$$\| \sum_{|v|=m-1} \alpha_{viw} e_v\|_q \leq C^m \| \sum_{|v|=2m-1} \alpha_v e_v\|_q \leq C^m \| \sum_{w\in[d]^*} \alpha_w e_w\|_q.$$

So, finally, we have the following estimate:

$$\begin{aligned} |\langle D_i \sum_{w\in[d]^*} \alpha_w e_w, e_\Omega\rangle_q| &\leq \| \sum_{w\in[d]^*} \alpha_w e_w\|_q \sum_{m=1}^{\infty} |q|^{\frac{m(m-1)}{2}} \sum_{|w|=m-1} C^m \sqrt{[m-1]_{|q|}!} \\ &= \| \sum_{w\in[d]^*} \alpha_w e_w\|_q \sum_{m=1}^{\infty} |q|^{\frac{m(m-1)}{2}} d^{m-1} C^m \sqrt{[m-1]_{|q|}!} \end{aligned}$$

and, by the ratio test, we can check that

$$\sum_{m=1}^{\infty} |q|^{\frac{m(m-1)}{2}} d^{m-1} C^m \sqrt{[m-1]_{|q|}!} < \infty.$$

This implies that the linear functional $\langle D_i \cdot\, , e_\Omega\rangle_q$ is bounded and therefore $e_\Omega \in \mathrm{dom}(D_i^*)$. □

Corollary 5.13. *Let $(D_1, \ldots, D_d)$ be the normalized dual system of the q-Gaussian operators, as defined in Proposition 5.11. Then the corresponding conjugate system $(\xi_1, \ldots, \xi_d)$ is given by*

$$\xi_i = D_i^* e_\Omega = \sum_{w\in[d]^*} (-1)^{|w|} q^{\frac{(|w|+1)|w|}{2}} R_{w^*i} e_w$$

where R_i is the adjoint of free right annihilation operator $R_i = (R_i^*)^*$ and $R_{w^*i} = R_{w_n} \cdots R_{w_1} R_i$ for $w = w_1 \cdots w_n$. Moreover, the series for ξ_i is not only convergent with respect to the Hilbert space norm $\|\cdot\|_q$, but also with respect to the operator norm $\|\cdot\|$, if we identify operators in $\mathrm{W}^*(A)$ with elements in the Fock space. Thus $\xi_i = X_i e_\Omega$, where X_i is contained in the norm closure of non-commutative polynomials $\mathbb{C}\langle A\rangle$; i.e., in particular $X_i \in \mathrm{W}^*(A)$.

Proof. In the proof of Theorem 2.22, we have seen that

$$\begin{aligned}\langle D_i \sum_{v\in[d]^*} \alpha_v e_v, e_\Omega\rangle_q &= \sum_{m=1}^{\infty}(-1)^{m-1} q^{\frac{m(m-1)}{2}} \sum_{|w|=m-1} \langle \sum_{|v|=2m-1} R^*_{iw}\alpha_v e_v, e_w\rangle_q \\ &= \sum_{m=1}^{\infty}(-1)^{m-1} q^{\frac{m(m-1)}{2}} \langle \sum_{|v|=2m-1} \alpha_v e_v, \sum_{|w|=m-1} R_{w^*i} e_w\rangle_q.\end{aligned}$$

Note that

$$\langle \sum_{|v|=2m-1} \alpha_v e_v, \sum_{|w|=m'-1} R_{w^*i} e_w\rangle_q = 0, \qquad \text{if } m \neq m'$$

since R_{w^*i} maps e_w (with $|w| = m'-1$) to the subspace spanned by $\{e_v\}_{|v|=2m'-1}$. This also implies

$$\langle \sum_{|v|=2m} \alpha_v e_v, \sum_{|w|=m'-1} R_{w^*i} e_w\rangle_q = 0, \qquad \text{for any } m, m'.$$

Thus we obtain

$$\begin{aligned}&\sum_{m=1}^{\infty}(-1)^{m-1} q^{\frac{m(m-1)}{2}} \langle \sum_{|v|=2m-1} \alpha_v e_v, \sum_{|w|=m-1} R_{w^*i} e_w\rangle_q \\ &\qquad = \langle \sum_{v\in[d]^*} \alpha_v e_v\ , \sum_{m=1}^{\infty} \sum_{|w|=m-1} (-1)^{m-1} q^{\frac{m(m-1)}{2}} R_{w^*i} e_w\rangle_q.\end{aligned}$$

For the operator norm, we can estimate by the triangle inequality

$$\|D_i^* e_\Omega\| \leq \sum_{m=0}^{\infty} \sum_{|w|=m} |q|^{\frac{m(m+1)}{2}} \|R_{w^*i} e_w\|.$$

Now, we use Bożejko's Haagerup type inequality [13], which tells us for $|w| = m$

$$\|R_{w^*i} e_w\| \leq (2m+2) C_{|q|}^{\frac{3}{2}} \|R_{w^*i} e_w\|_q,$$

where $C_q^{-1} = \prod_{m=1}^{\infty}(1-q^m)$. Since $\|R_{w^*i}\| \leq \sqrt{w(q)}^{-(m+1)}$ and $\|e_w\|_q \leq \sqrt{[m]_{|q|}!}$ (see the proof of Theorem 2.22), we have

$$\begin{aligned}\sum_{m=0}^{\infty} \sum_{|w|=m} |q|^{\frac{m(m+1)}{2}} \|r^*_{iw} e_w\| &\leq \sum_{m=0}^{\infty} \sum_{|w|=m} |q|^{\frac{m(m+1)}{2}} (2m+2) C_{|q|}^{\frac{3}{2}} \sqrt{w(q)}^{-(m+1)} \sqrt{[m]_{|q|}!} \\ &= \sum_{m=0}^{\infty} d^m |q|^{\frac{m(m+1)}{2}} (2m+2) C_{|q|}^{\frac{3}{2}} \sqrt{w(q)}^{-(m+1)} \sqrt{[m]_{|q|}!},\end{aligned}$$

which is finite by the ratio test. Since e_w can be represented by non-commutative polynomials over q-Gaussians, $D_i^* e_\Omega$ belongs to the norm closure of $\mathbb{C}\langle A\rangle$. □

Remark 5.14. We can recover the results in Section 5.1 for the case of one variable from Proposition 5.11 and Corollary 5.13 by considering $d = 1$ and identifying e_{1^m} with e_m. In particular, we have then

$$R_1 e_m = \frac{1}{[m+1]_q} e_{m+1},$$

and

$$R_1^{m+1}e_m = \frac{1}{[m+1]_q[m+2]_q\cdots[2m+1]_q}e_{2m+1} = \frac{[m]_q!}{[2m+1]_q!}e_{2m+1}$$

and

$$\xi_1 = \sum_{m=0}^{\infty}(-1)^m q^{\frac{(m+1)m}{2}}R_1^{m+1}e_m = \sum_{m=0}^{\infty}(-1)^m q^{\frac{(m+1)m}{2}}\frac{[m]_q!}{[2m+1]_q!}e_{2m+1},$$

which recovers, by replacing m by $m-1$, the formula for $\xi = D^*e_\Omega$ in Corollary 5.5.

5.3. **Lipschitz conjugate.** Let us check that the conjugate system $(\xi_1,\dots,\xi_d)$ for the q-Gaussian variables are Lipschitz conjugate variables, namely, for each $i\in[d]$, $\xi_i = D_i^*e_\Omega \in \mathrm{dom}(\overline{\partial_j})$ and $\overline{\partial_j}\xi_i \in \mathrm{W}^*(A)\overline{\otimes}\mathrm{W}^*(A)$ for each $j\in[d]$. For this, we need to know $\partial_j e_w$ and this has a similar combinatorial formula to that of a normalized dual system. Again we have to consider a special set of partitions, consisting just of singletons and pairs, and draw them in a specific way to count their crossings.

(i) Consider $n+1$ vertices $n>\cdots>1>0$.

(ii) The vertex 0 must be coupled with some $k\in\{1,\dots,n\}$ with height 1.

(iii) Each $l\in\{1,\dots,k-1\}$ is a singleton or coupled with one of $\{k+1,\dots,n\}$ with height $l+1$.

(iv) Vertices which are not coupled with one of $\{1,\dots,k-1\}$ should be singletons and are drawn with straight lines to the top.

Let $C(n+1)$ be the set of partitions defined by the rules above. For each $\pi\in C(n+1)$, we define $s_l(\pi)$ and $s_r(\pi)$ as the set of singletons in the left area $n\ge k>\pi(0)$ and in the right area $\pi(0)>k\ge 1$, respectively. As before, we use the notation $\mathrm{cross}(\pi)$ for the number of crossings in the drawing according to these rules.

For each $w\in[d]^*$, we identify e_w with the non-commutative polynomial $Q[w]$ over q-Gaussians. We give now the combinatorial formula for $\partial_i e_{j_n\cdots j_1}$ identifying each index j_k with a vertex of k (where we put $j_0=i$).

Proposition 5.15. *For each $i\in[d]$, $n\in\mathbb{N}$ and $j_1,\dots,j_n\in[d]$, we have*

$$\partial_i e_{j_n\cdots j_1} = \sum_{\pi\in C(n+1)}(-1)^{|p(\pi)|-1}q^{\mathrm{cross}(\pi)-|s_r(\pi)|}\delta_{p(\pi)}e_{s_l(\pi)}\otimes e_{s_r(\pi)}$$

As before, we denote by $p(\pi)$ the set of parings in π and the factor $\delta_{p(\pi)}$ ensures that π has to pair the same indices.

Proof. We will prove the formula by induction over n. For $n=1$, it says that $\partial_i e_j = \delta_{ij}e_\Omega\otimes e_\Omega$, which is clearly true. Assume now that the formula is true for $n\ge 1$ and let us show it for $n+1$. By the definition of q-Gaussians, we have the recursion for non-commutative polynomials e_w

$$e_{j_{n+1}\cdots j_1} = A_{j_{n+1}}e_{j_n\cdots j_1} - \sum_{k=1}^{n}\delta_{j_{n+1}j_k}q^{n-k}[j_n\cdots\check{j}_k\cdots j_1],$$

which induces

$$\partial_i e_{j_{n+1}\cdots j_1} = \delta_{i j_{n+1}} 1 \otimes e_{j_n \cdots j_1} + (A_{j_{n+1}} \otimes 1) \cdot \partial_i e_{j_n \cdots j_1} - \sum_{k=1}^{n} \delta_{j_{n+1} j_k} q^{n-k} \partial_i [j_n \cdots \check{j}_k \cdots j_1].$$

By using the induction assumption, we can compute

$$\begin{aligned}
(A_{j_{n+1}} \otimes 1) \cdot \partial_i e_{j_n \cdots j_1} &= \sum_{\pi \in C(n+1)} (-1)^{|p(\pi)|-1} q^{\mathrm{cross}(\pi) - |s_r(\pi)|} \delta_{p(\pi)} A_{j_{n+1}} e_{s_l(\pi)} \otimes e_{s_r(\pi)} \\
&= \sum_{\pi \in C(n+1)} (-1)^{|p(\pi)|-1} q^{\mathrm{cross}(\pi) - |s_r(\pi)|} \delta_{p(\pi)} e_{j_{n+1} s_l(\pi)} \otimes e_{s_r(\pi)} \\
&\quad + \sum_{\pi \in C(n+1)} (-1)^{|p(\pi)|-1} \times \\
&\quad \times \Big\{ \sum_{m=1}^{|s_l(\pi)|} q^{\mathrm{cross}(\pi) - |s_r(\pi)| + |s_l(\pi)| - m} \delta_{p(\pi)} \delta_{j_{n+1} j_{s_l(\pi)_m}} [s_l(\pi)_{|s_l(\pi)|} \cdots s_l(\check{\pi})_m \cdots s_l(\pi)_1] \otimes e_{s_r(\pi)} \Big\}.
\end{aligned}$$

On the other hand, we can compute $\sum_{k=1}^{n} \delta_{j_{n+1} j_k} q^{n-k} \partial_i [j_n \cdots \check{j}_k \cdots j_1]$ as

$$\sum_{k=1}^{n} \sum_{\sigma \in C(n)} \delta_{j_{n+1} j_k} (-1)^{|p(\sigma)|-1} q^{\mathrm{cross}(\sigma) - |s_r(\sigma)| + n - k} \delta_{p(\sigma)} e_{s_l(\sigma)} \otimes e_{s_r(\sigma)}$$

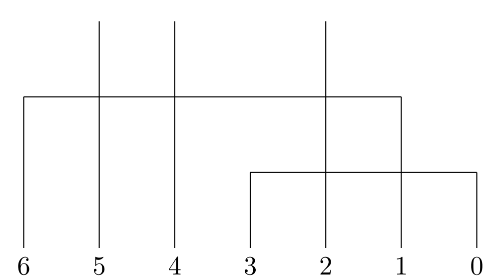
where σ acts on the word $j_n \cdots \check{j}_k \cdots j_1$. By the same argument as in Proposition 5.11, we can see that the last sum of $(A_{j_{n+1}} \otimes 1) \cdot \partial_i e_{j_n \cdots j_1}$ is canceled by $-\sum_{k=1}^{n} \delta_{j_{n+1} j_k} q^{n-k} \partial_i [j_n \cdots \check{j}_k \cdots j_1]$. Indeed, for each $\pi \in C(n+1)$ and $m \in \{1, \ldots, |s_l(\pi)|\}$, we take $k = s_l(\pi)_m$ and $\sigma = \pi \setminus s_l(\pi)_m \in C(n)$. Then $|p(\pi)| = |p(\sigma)|$. By counting the crossing points on the line $s_l(\pi)_m$, we have

$$\mathrm{cross}(\pi) - \mathrm{cross}(\sigma) = n - k - (|s_l(\pi)| - m)$$

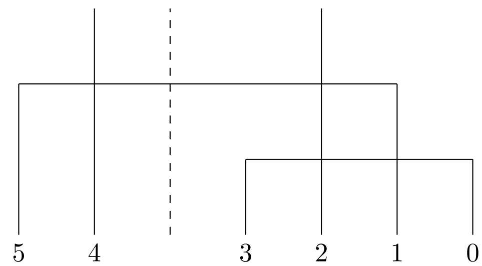
Since we have $|s_r(\pi)| = |s_r(\sigma)|$, we have

$$\mathrm{cross}(\pi) - |s_r(\pi)| + |s_l(\pi)| - m = \mathrm{cross}(\sigma) - |s_r(\sigma)| + n - k$$

For example, we take $\pi \in C(7)$ represented by



and take $m = 1$. Then, we take $k = 4$ and $\sigma \in C(6)$ represented by



In this case, we have

$$|p(\pi)| = |p(\sigma)| = 2, \qquad |s_l(\pi)| = 2, \qquad |s_r(\pi)| = |s_r(\sigma)| = 1$$

and thus

$$\mathrm{cross}(\pi) - |s_r(\pi)| + |s_l(\pi)| - m = 5-1+2-1 = 5 = 4-1+6-4 = \mathrm{cross}(\sigma) - |s_r(\sigma)| + n - k.$$

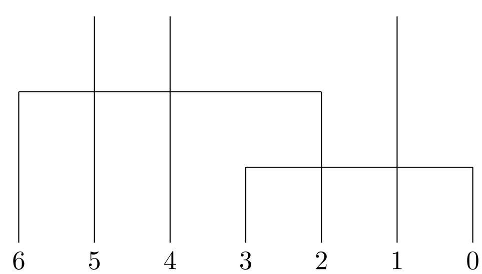
The remaining terms in $-\sum_{k=1}^{n} \delta_{j_{n+1} j_k} q^{n-k} \partial_i [j_n \cdots \check{j}_k \cdots j_1]$, which are characterized by $\sigma \in C(n)$ and $k \le \sigma(0)$, are corresponding to partitions $\sigma' \in C(n+2)$ which connect $n+1$ to k and satisfy $\sigma' \setminus (n+1, k) = \sigma$. Then $|p(\sigma')| = |p(\sigma)| + 1$ and by counting the crossing points on $(n+1, k)$ we have

$$\mathrm{cross}(\sigma') - \mathrm{cross}(\sigma) = n - k.$$

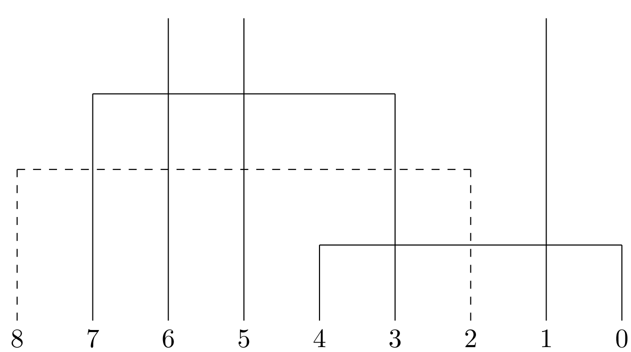
We also have $|s_r(\sigma')| = |s_r(\sigma)|$, and thus

$$\mathrm{cross}(\sigma') - |s_r(\sigma')| = \mathrm{cross}(\sigma) - |s_r(\sigma)| + n - k.$$

For example, consider $\sigma \in C(7)$ represented by



and take $k = 2$. Then we obtain $\sigma' \in C(9)$ represented by



In this case,

$$|p(\sigma')| = 3 = |p(\sigma)| + 1, \qquad \mathrm{cross}(\sigma') - \mathrm{cross}(\sigma) = 9 - 4 = 5 = 7 - 2.$$

The term $\delta_{ij_{n+1}} 1 \otimes e_{j_n \cdots j_1}$ is given by the partitions in $C(n+2)$ which connect 0 to $n+1$. The sum

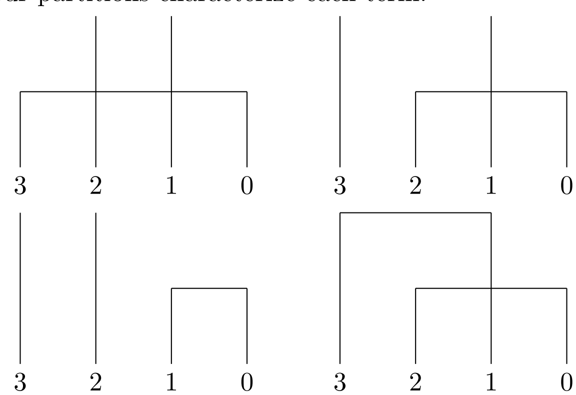
$$\sum_{\pi \in C(n+1)} (-1)^{|p(\pi)|-1} q^{\mathrm{cross}(\pi) - |s_r(\pi)|} \delta_{p(\pi)} e_{j_{n+1} s_l(\pi)} \otimes e_{s_r(\pi)}$$

is given by the partitions in $C(n+2)$ such that $n+1$ is a singleton. Therefore $\partial_i e_{j_{n+1} \cdots j_1}$ is given by the partitions in $C(n+2)$ and we have proved the claimed formula for $n+1$. □

Example 5.16. For $n = 3$, the proposition tells us

$$\partial_i e_{j_3 j_2 j_1} = \delta_{ij_3} 1 \otimes e_{j_2 j_1} + \delta_{ij_2} e_{j_3} \otimes e_{j_1} + \delta_{ij_1} e_{j_3 j_2} \otimes 1 - q \delta_{ij_2} \delta_{j_3 j_1} 1 \otimes 1.$$

The following four partitions characterize each term.



Corollary 5.17. *The conjugate system of q-Gaussians is Lipschitz conjugate for $-1 < q < 1$.*

Proof. Let $i, j \in [d]$. We will check that the following sum

$$\partial_j \xi_i = \sum_{w \in [d]^*} (-1)^{|w|} q^{\frac{(|w|+1)|w|}{2}} \partial_j R_{w^* i} e_w$$

converges in the operator norm of $B(\mathcal{F}_q(H)^{\otimes 2})$. Since $R_{w^* i} e_w$ is a linear span of $\{e_v\}_{|v|=2m+1}$ for each $|w| = m$, we can write $R_{w^* i} e_w = \sum_{|v|=2m+1} \alpha_v e_v$. (Note that α_v depends on w, but since the following estimates do not depend on w we will suppress this in the notation.) Note that $\alpha_v e_\Omega = R_v^* R_{w^* i} e_w$ and we can estimate

$$|\alpha_v| = \|R_v^* R_{w^* i} e_w\|_q \leq \sqrt{w(q)}^{-3m-2} \sqrt{[m]_{|q|}!}.$$

Therefore we can estimate, by the triangle inequality,

$$\|\partial_j R_{w^* i} e_w\| \leq \sum_{|v|=2m+1} \sqrt{w(q)}^{-3m-2} \sqrt{[m]_{|q|}!}\ \|\partial_j e_v\|.$$

By Proposition 5.15 and Bożejko's Haagerup type inequality [13], we obtain

$$\|\partial_j e_v\| \leq \sum_{\pi \in C(2m+2)} \|e_{s_l(\pi)}\| \cdot \|e_{s_r(\pi)}\| \leq C_{|q|}^3 (2m+1)^2 [2m]_{|q|}!\ |C(2m+2)|$$

where $|C(2m+2)|$ is the cardinality of $C(2m+2)$. Since we can regard $C(2m+2)$ as a subset of the symmetric group of degree $2m+2$, we obtain $|C(2m+2)| \le (2m+2)!$. Therefore we have

$$\begin{aligned}
\sum_{w\in[d]^*} \|(-1)^{|w|} q^{\frac{(|w|+1)|w|}{2}} \partial_j R_{w^*i} e_w\| &\le \sum_{m=0}^{\infty}\sum_{|w|=m} |q|^{\frac{(m+1)m}{2}} \|\partial_j r_{iw}^* e_w\| \\
&\le \sum_{m=0}^{\infty}\sum_{|w|=m} |q|^{\frac{(m+1)m}{2}} \sum_{|v|=2m+1} \sqrt{w(q)}^{-3m-2} \sqrt{[m]_{|q|}!}\; \|\partial_j e_v\| \\
&\le C_{|q|}^3 \sum_{m=0}^{\infty}\sum_{|w|=m} |q|^{\frac{(m+1)m}{2}} \times \\
&\quad \times \sum_{|v|=2m+1} \sqrt{w(q)}^{-3m-2} \sqrt{[m]_{|q|}!(2m+1)^2[2m]_{|q|}!(2m+2)!} \\
&= C \sum_{m=0}^{\infty} |q|^{\frac{(m+1)m}{2}} (2m+1)^2 (2m+2)! \left(\frac{d}{\sqrt{w(q)}}\right)^{3m} \sqrt{[m]_{|q|}!\; [2m]_{|q|}!}.
\end{aligned}$$

where

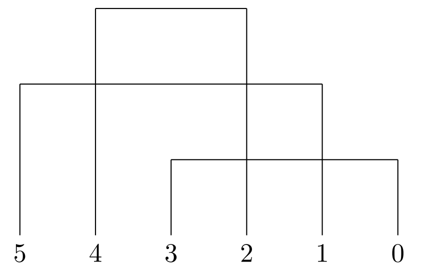
$$C = (dC_{|q|}^3)/w(q)$$

is a constant which is independent of m and the last sum is convergent by the ratio test. This implies $\xi_i \in \mathrm{dom}(\overline{\partial_j})$ and $\overline{\partial_j}\xi_i \in \mathrm{W}^*(A)\overline{\otimes}\mathrm{W}^*(A)$. □

Remark 5.18. It is likely that one can extend our results to more general deformations of Gaussian algebras; in particular, the case of mixed q_{ij}-Gaussians is quite straightforward. In this case the q-commutation relations $l_i^* l_j - q l_j l_i^* = \delta_{ij}$ are replaced by $l_i^* l_j - q_{i,j} l_j l_i^* = \delta_{ij}$, where the parameters $(q_{ij})_{1\le i,j\le d}$ just have to satisfy $-1 \le q_{ij} = q_{ji} \le 1$ and we are still looking on the von Neumann algebra generated by all $A_i := l_i + l_i^*$. As for the q-case there exists a representation of these operators as creation or annihilation operators on the Fock space [79, 10, 48]. The main difference is that in all formulas the factor q for a crossing has to be replaced by q_{ij}, where i and j are the indices of the two crossing strings (where one should note that we only get a non-vanishing contribution from a pairing if it pairs the same indices). To be more precise, the left creation operators are defined in the same way as in Section 2, but the left annihilation operators are defined by

$$l_i^* e_{j_n j_{n-1}\cdots j_1} = \sum_{k=1}^{n} \delta_{ij_k} q_{ij_n} q_{ij_{n-1}} \cdots q_{ij_{k+1}} e_{j_n\cdots \check{j_k}\cdots j_1}.$$

This induces the same combinatorial structure for $D_i e_{j_n j_{n-1}\cdots j_1}$ as in Proposition 5.11 as well as for $\partial_i e_{j_n j_{n-1}\cdots j_1}$ as in Proposition 5.15, if one replaces q by the appropriate q_{ij} for crossings according to our drawings in Section 5.2 and Section 5.3. For example, for the drawing



the corresponding term in $D_i e_{j_5 j_4 j_3 j_2 j_1}$ is given by

$$q_{j_1 j_2}^2 q_{i j_2} q_{i j_1} \delta_{i j_3} \delta_{j_1 j_5} \delta_{j_2 j_4} e_\Omega .$$

This is equal to $q_{j_5 j_4}^2 q_{i j_4} q_{i j_5} \delta_{i j_3} \delta_{j_1 j_5} \delta_{j_2 j_4} e_\Omega$ and $q_{j_2 j_1}^2 q_{j_2 i} q_{j_1 i} \delta_{i j_3} \delta_{j_1 j_5} \delta_{j_2 j_4} e_\Omega$ since δ_{ij} is the Kronecker's delta and $(q_{ij})_{1 \le i,j \le d}$ is symmetric. Therefore this term depends only on the crossings. For each $\pi \in B(n+1)$, we denote by $q^{\mathrm{cross}(\pi)} \delta_{p(\pi)}$ the coefficient as above. Then we have the same formula for a normalized dual system for mixed q_{ij}-Gaussians as in Proposition 5.11. Similarly, $\partial_i e_{j_n j_{n-1} \cdots j_1}$ is characterized by $C(n+1)$ in Section 5.3 and we count crossings as above, except for crossings of right singletons and the pair that includes 0.

We can also derive the conjugate system of mixed q_{ij}-Gaussians from this combinatorics. As in the proof of Theorem 2.22, we separate the crossings into two sets, the left area and the right area. Moreover, crossings in the left area correspond to the number of inversions of permutations which are induced by pair partitions, while crossings in the right area are independent of the choices of the pair partitions. By the same arguments as in the q-case we get the following formula for the conjugate system $(\xi_1, \ldots, \xi_d)$ for mixed q_{ij}-Gaussians $(A_1, \ldots, A_d)$:

$$\xi_i = \sum_{w \in [d]^*} (-1)^{|w|} q(w) R_{w^* i} e_w ,$$

where

$$q(w) = \prod_{\substack{1 \le k \le m \\ 0 \le l \le k-1}} q_{j_k j_l} \qquad \text{for} \qquad w = j_m \cdots j_1 .$$

Moreover, we can extend Lemma 5.1 to the q_{ij}-setting since Theorem 1 in [12] includes the q_{ij}-case. We also have Haagerup's inequality for the q_{ij}-setting, according to Theorem 26 in [51].

The factor $q(w)$ replaces now the factor $q^{m(m+1)/2}$, which was in the end responsible for the uniform convergence of all appearing power series expansions. As a consequence, if $\max_{i,j \in [d]} |q_{ij}| < 1$, then all our estimates work in the same way and we get thus that also the Lipschitz conjugate system for the mixed q_{ij}-Gaussians exists.

5.4. **Power series expansions of the conjugate variables and free Gibbs potential.** Finally, we also want to address estimates for the conjugate system in terms of non-commutative power series in the operators. This is relevant if we want to find a potential, such that our q-distribution is the corresponding free Gibbs state.

In order to write the conjugate system as such non-commutative power series, we need to represent $R_{w^* i} e_w$ ($i \in [d]$ and $w \in [d]^*$) as a non-commutative polynomial. In Theorem 3.1 of [37], one can find the concrete formula of $Q[w]$ (see Section 2 for the definition of $Q[w]$). Here, we present this formula, actually its extension for mixed q_{ij}-Gaussians, by using our combinatorics.

Let $D(n)$ be the set of partitions on n vertices $n > n-1 > \cdots > 1$ which consist of either singletons or pair partitions. We count crossings of $\pi \in D(n)$ by using hight as in Sections 5.2 and 5.3.

Proposition 5.19. *Consider the setting and the notations for the q_{ij}-Gaussians as in Remark 5.18. Then, for $j_1, \ldots, j_n \in [d]$, we have*

$$e_{j_n \cdots j_1} = \sum_{\pi \in D(n)} (-1)^{|p(\pi)|} q^{\mathrm{cross}(\pi)} \delta_{p(\pi)} A^{s(\pi)} e_\Omega$$

where $A^{s(\pi)} = A_{j_{k_s}} \cdots A_{j_{k_1}}$ for $s(\pi) = \{k_s > \cdots > k_1\}$.

Proof. We prove this formula by induction on n. For $n = 1$ it just says $e_j = (-1)^0 q^0 A_j e_\Omega$, which is clearly true. So assume we know it for $n \geq 1$ and let us prove it for $n+1$. We have

$$\begin{aligned} e_{j_{n+1} \cdots j_1} &= A_{j_{n+1}} e_{j_n \cdots j_1} - \sum_{k=1}^{n} \delta_{j_{n+1} j_k} q_{j_{n+1} j_n} \cdots q_{j_{n+1} j_{k+1}} e_{j_n \cdots \check{j_k} \cdots j_1} \\ &= \sum_{\pi \in D(n)} (-1)^{|p(\pi)|} q^{\mathrm{cross}(\pi)} \delta_{p(\pi)} A^{j_{n+1} s(\pi)} e_\Omega \\ &\quad - \sum_{k=1}^{n} \delta_{j_{n+1} j_k} \sum_{\sigma \in D(n-1)} q_{j_{n+1} j_n} \cdots q_{j_{n+1} j_{k+1}} (-1)^{|p(\sigma)|} q^{\mathrm{cross}(\sigma)} \delta_{p(\sigma)} A^{s(\sigma)} e_\Omega, \end{aligned}$$

where $\sigma \in D(n-1)$ acts on $j_n \cdots \check{j_k} \cdots j_1$. The first term corresponds to $\tilde{\pi} \in D(n+1)$ such that $\tilde{\pi}(n+1)$ is a singleton. For the second term we take, for each $k \in \{1, \ldots, n\}$ and $\sigma \in D(n-1)$, the $\tilde{\sigma} \in D(n+1)$ such that $n+1$ is connected to k and $\tilde{\sigma} \setminus (n+1, k) = \sigma$. Then we have

$$|p(\tilde{\sigma})| = |p(\sigma)| + 1 \qquad \text{and} \qquad q^{\mathrm{cross}(\tilde{\sigma})} \delta_{p(\tilde{\sigma})} = q_{j_{n+1} j_n} \cdots q_{j_{n+1} j_{k+1}} q^{\mathrm{cross}(\sigma)} \delta_{p(\sigma)} \delta_{j_{n+1} j_k}.$$

Thus the second term corresponds to such $\tilde{\sigma} \in D(n+1)$ and we have

$$\begin{aligned} e_{j_{n+1} \cdots j_1} &= \sum_{\substack{\tilde{\pi} \in D(n+1) \\ \tilde{\pi}(n+1) \text{ is a singleton}}} (-1)^{|p(\tilde{\pi})|} q^{\mathrm{cross}(\tilde{\pi})} \delta_{p(\tilde{\pi})} A^{s(\tilde{\pi})} e_\Omega \\ &\quad + \sum_{\substack{\tilde{\sigma} \in D(n+1) \\ \tilde{\sigma}(n+1) \text{ is not a singleton}}} (-1)^{|p(\tilde{\sigma})|} q^{\mathrm{cross}(\tilde{\sigma})} \delta_{p(\tilde{\sigma})} A^{s(\tilde{\sigma})} e_\Omega \\ &= \sum_{\pi \in D(n+1)} (-1)^{|p(\pi)|} q^{\mathrm{cross}(\pi)} \delta_{p(\pi)} A^{s(\pi)} e_\Omega. \end{aligned}$$

□

Using this we can rewrite our conjugate variables as non-commutative power series in $A_1, \ldots, A_d$. (In the following we will, for simplicity, again restrict to the q-case, though the q_{ij}-case can be treated in the same way.) The main point will be to see that we have good estimates for the operator norms of the summands in these series; this will be similar to the proof of Corollary 5.17. Let us fix $i \in [d]$. For each $w \in [d]^*$ with $|w| = m$, we write $R_{w^* i} e_w = \sum_{|v|=2m+1} \alpha_v e_v$ (as before we suppress in the notation for α_v the dependency on w). Recall that we have

$|\alpha_v| \leq \sqrt{w(q)}^{-3m-2}\sqrt{[m]_q!}$ for any $|v| = 2m+1$ (see the proof of Corollary 5.17). Moreover, by Proposition 5.19, we have

$$e_v = \sum_{\pi \in D(2m+1)} (-1)^{|p(\pi)|} q^{\mathrm{cross}(\pi)} \delta_{p(\pi)} A^{s(\pi)} e_\Omega.$$

Then we have

$$\begin{aligned} \xi_i &= \sum_{m=0}^{\infty} (-1)^m q^{\frac{m(m+1)}{2}} \sum_{|w|=m} R_{w^*i} e_w \\ &= \sum_{m=0}^{\infty} (-1)^m q^{\frac{m(m+1)}{2}} \sum_{|w|=m} \sum_{|v|=2m+1} \alpha_v e_v \\ &= \sum_{m=0}^{\infty} (-1)^m q^{\frac{m(m+1)}{2}} \sum_{|w|=m} \sum_{|v|=2m+1} \alpha_v \sum_{\pi \in D(2m+1)} (-1)^{|p(\pi)|} q^{\mathrm{cross}(\pi)} \delta_{p(\pi)} A^{s(\pi)} e_\Omega. \end{aligned}$$

This is our "concrete" realization for the ξ_i as a non-commutative power series in $A_1, \ldots, A_d$. We claim that these power series have an infinite radius of convergence. We set $A = \max_{i \in [d]} \|A_i\| > 1$. Then we can estimate the operator norm as follows:

$$\begin{aligned} \|\xi_i\| &\leq \sum_{m=0}^{\infty} |q|^{\frac{m(m+1)}{2}} \sum_{|w|=m} \sum_{|v|=2m+1} |\alpha_v| \sum_{\pi \in D(2m+1)} |q^{\mathrm{cross}(\pi)} \delta_{p(\pi)}| \|A^{s(\pi)}\| \\ &\leq \sum_{m=0}^{\infty} |q|^{\frac{m(m+1)}{2}} \sum_{|w|=m} \sum_{|v|=2m+1} |\alpha_v| (2m+1)! A^{2m+1} \\ &\leq \sum_{m=0}^{\infty} |q|^{\frac{m(m+1)}{2}} \sum_{|w|=m} \sum_{|v|=2m+1} \sqrt{w(q)}^{-3m-2} \sqrt{[m]_q!} (2m+1)! A^{2m+1} \\ &= \sum_{m=0}^{\infty} |q|^{\frac{m(m+1)}{2}} \left(\frac{d}{\sqrt{w(q)}} \right)^{3m+2} \sqrt{[m]_q!} (2m+1)! A^{2m+1} \end{aligned}$$

where we use $|D(2m+1)| \leq (2m+1)!$, since all partitions in $D(2m+1)$ have blocks of size either 1 or 2 and can thus be identified with permutations in the symmetric group of degree $2m+1$.

By the ratio test, this sum converges for any A and thus this implies that the conjugate system is a d-tuple of non-commutative power series which are uniformly convergent with a radius of convergence equal to ∞.

A free Gibbs potential (see Section 1.2 in [39]) for q-Gaussians is an operator $V \in \mathrm{W}^*(A_1, \ldots, A_d)$ which satisfies $\mathcal{D}_i V = \xi_i$ for all $i \in [d]$ where $\mathcal{D}_i$'s are the cyclic derivatives defined by $\mathcal{D}_i = m_{\mathrm{flip}} \circ \partial_i$ (m_{flip} is defined by $m_{\mathrm{flip}}(a \otimes b) = ba$). When we write the conjugate system as non-commutative power series $\xi_i = \sum_{w \in [d]^*} \alpha(w,i) A^w$, this potential V is formally given by (see the proof of Corollary 4.3 in [39])

$$V = \frac{1}{2} N^{-1} \left(\sum_{i=1}^{d} A_i \xi_i + \xi_i A_i \right) = \sum_{i=1}^{d} \sum_{w \in [d]^*} \frac{\alpha(w,i)}{2(1+|w|)} (A^{iw} + A^{wi})$$

where N is the number operator which maps A^w to $|w|A^w$. Estimates as above tell us the uniform convergence of the non-commutative power series on the right hand side, yielding the existence of a free Gibbs potential.

Proposition 5.20. *A free Gibbs potential exists for the q-Gaussians, for any $-1 < q < 1$.*

6. Strong convergence of q-Gaussians

This section is a part of the paper [65]. Here, we recall the main result of this section.

Theorem 6.1. *For any $-1 < q_0 < 1$, strong convergence of q-Gaussians $A^{(q)} = (A_1^{(q)}, \dots, A_d^{(q)})$ holds at q_0, i.e. for any non-commutative polynomial P,*

$$\lim_{q \to q_0} \|P(A^{(q)})\| = \|P(A^{(q_0)})\|.$$

The key fact for the proof is the Haagerup-type inequality of q-Gaussians proved by Bożejko [13]. Thanks to this inequality, we can apply Brannan's approach to show strong convergence from convergence in non-commutative distribution. This kind of argument also appears in Pisier's paper [71, Section 1 and Section 4].

Proof. Since we have

$$\|P(A^{(q)})\|_{2n} = \tau\left[(P^*P)^n(A^{(q)})\right]^{\frac{1}{2n}} \le \|P(A^{(q)})\|,$$

it is obvious from convergence in non-commutative distribution that

$$\|P(A^{(q_0)})\| \le \liminf_{q \to q_0} \|P(A^{(q)})\|.$$

For the other direction, we will apply Brannan's approach [16] with Bożejko's Haagerup-type inequality in Theorem 2.58. Let P be any non-commutative polynomial of degree m. Then we can write $P(A^{(q)}) = \sum_{k=0}^m \sum_{|w|=k} \alpha_w e_w^{(q)}$, and we have

$$\begin{aligned}
\|P(A^{(q)})\| &\le \sum_{k=0}^m \Big\| \sum_{|w|=k} \alpha_w e_w^{(q)} \Big\| \\
&\le \sum_{k=0}^m (k+1) C_{|q|}^{\frac{3}{2}} \left\| \sum_{|w|=k} \alpha_w e_w^{(q)} \right\|_2 \\
&\le (m+1) C_{|q|}^{\frac{3}{2}} \sum_{k=0}^m \left\| \sum_{|w|=k} \alpha_w e_w^{(q)} \right\|_2 \\
&\le (m+1)^{\frac{3}{2}} C_{|q|}^{\frac{3}{2}} \left\| \sum_{k=0}^m \sum_{|w|=k} \alpha_w e_w^{(q)} \right\|_2 \\
&= (m+1)^{\frac{3}{2}} C_{|q|}^{\frac{3}{2}} \|P(A^{(q)})\|_2.
\end{aligned}$$

where we use orthogonality of $e_w^{(q)}$ with respect to word length. Now, we apply this inequality to $(P^*P)^n$ which has a degree $2mn$. Then we have

$$\|(P^*P)^n(A^{(q)})\| = \|P(A^{(q)})\|^{2n} \leq (2mn+1)^{\frac{3}{2}} C_{|q|}^{\frac{3}{2}} \|(P^*P)^n(A^{(q)})\|_2.$$

By taking a limit $q \to q_0$ after taking the power of $\frac{1}{2n}$ in both sides, we have from the convergence in non-commutative distribution,

$$\limsup_{q \to q_0} \|P(A^{(q)})\| \leq (2mn+1)^{\frac{3}{4n}} C_{|q_0|}^{\frac{3}{4n}} \|(P^*P)^n(A^{(q_0)})\|_2^{\frac{1}{2n}} \underset{n\to\infty}{\to} \|P(A^{(q_0)})\|.$$

Thus we have strong convergence of q-Gaussians. □

Remark 6.2. When $q_0 = \pm 1$, our proof does not work. Actually, the 1-Gaussian is the standard Gaussian, which is unbounded. The (-1)-Gaussian is the discrete measure which takes ± 1 with probability $\frac{1}{2}$. For $-1 < q < 1$, it is known that the q-Gaussian has a density function which is supported on $[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}]$ (cf. Theorem 1.10 in [11]). Since $\lim_{q\to -1} \frac{2}{\sqrt{1-q}} = \sqrt{2}$, we don't have strong convergence at $q_0 = -1$.

As a corollary of the main theorem, we obtain the convergence of spectrums of a self-adjoint polynomial in the Hausdorff distance.

Corollary 6.3. *For any $-1 < q_0 < 1$ and any self-adjoint polynomial P, we have*

$$\lim_{q\to q_0} d_H(\sigma[P(A^{(q)})], \sigma[P(A^{(q_0)})]) = 0$$

where $\sigma[P(A^{(q)})]$ is the spectrum of $P(A^{(q)})$ and $d_H(\cdot,\cdot)$ is the Hausdorff distance.

Proof. Let $\epsilon > 0$ be given. Since $\sigma[P(A^{(q_0)})]$ is compact, we can take $\{x_k\}_{k=1}^m \subset \sigma[P(A^{(q_0)})]$ such that $\sigma[P(A^{(q_0)})] \subset \bigcup_{k=1}^m N_{\frac{\epsilon}{2}}(x_k)$ where $N_{\frac{\epsilon}{2}}(x_k)$ is the $\frac{\epsilon}{2}$-neighborhood of x_k. For each $k = 1, \ldots, m$, we take a continuous function f_k on $\mathbb{R}$ such that $0 \leq f_k \leq 1$ and $f_k(x_k) = 1$ and $f_k|_{N_{\frac{\epsilon}{2}}(x_k)^c} = 0$. Since $\|f_k(P(A^{(q_0)}))\| = 1$ for all k, we also have $\|f_k(P(A^{(q)}))\| > 0$ if $|q - q_0|$ is sufficiently small by Stone–Weierstrass theorem and strong convergence (actually, convergence in non-commutative distribution is enough for this claim). This implies $N_{\frac{\epsilon}{2}}(x_k) \cap \sigma[P(A^{(q)})] \neq \emptyset$ for each k and we have

$$\sigma[P(A^{(q_0)})] \subset \bigcup_{k=1}^m N_{\frac{\epsilon}{2}}(x_k) \subset \sigma[P(A^{(q)})] + (-\epsilon, \epsilon).$$

On the other hand, we take a continuous function g on $\mathbb{R}$ such that $0 \leq g \leq 1$ and $g = 0$ on $\sigma[P(A^{(q_0)})]$ and $g = 1$ on the complement of $\sigma[P(A^{(q_0)})] + (-\epsilon, \epsilon)$. Since $g(P(A^{(q_0)})) = 0$, we similarly have $\|g(P(A^{(q)}))\| < 1$ if $|q - q_0|$ is sufficiently small, which implies

$$\sigma[P(A^{(q)})] \subset \sigma[P(A^{(q_0)})] + (-\epsilon, \epsilon).$$

Therefore, we obtain $d_H(\sigma[P(A^{(q)})], \sigma[P(A^{(q_0)})]) < \epsilon$ if $|q - q_0|$ is sufficiently small. □

This kind of argument actually holds for any tuples of operators in C*-algebras with faithful states which satisfy "uniform RD property" and convergence in non-commutative $*$-distribution.

Proposition 6.4. *Let* $(\mathcal{A}_n, \phi_n)_{n\in\mathbb{N}}$ *and* $(\mathcal{A}_\infty, \phi_\infty)$ *be couples of a* C*-*algebra* $\mathcal{A}_n, \mathcal{A}_\infty$ *and a faithful state* ϕ_n, ϕ_∞. *Let* $X^{(n)} = (X_1^{(n)}, \ldots, X_d^{(n)})$ *be a* d*-tuple of operators (not necessarily self-adjoint) in* $\mathcal{A}_n$ *for each* $n \in \mathbb{N} \cup \{\infty\}$. *Assume* $(X^{(n)})_{n\in\mathbb{N}\cup\{\infty\}}$ *satisfy the following two properties*

- *"uniform RD property": there exist constants* $C, D > 0$ *such that for any* $n \in \mathbb{N}$ *and non-commutative* $*$*-polynomial* P,

$$\|P(X^{(n)})\| \le C(\mathrm{deg}P + 1)^D \|P(X^{(n)})\|_2.$$

  where $\mathrm{deg}P$ *is the degree of* P.

- *convergence in non-commutative* $*$*-distribution: for any non-commutative* $*$*-polynomial* P,

$$\lim_{n\to\infty} \phi_n[P(X^{(n)})] = \phi_\infty[P(X^{(\infty)})].$$

Then, $X^{(n)}$ *strongly converges to* $X^{(\infty)}$*; for any* $*$*-polynomial* P,

$$\lim_{n\to\infty} \|P(X^{(n)})\| = \|P(X^{(\infty)})\|.$$

We also have for any self-adjoint $*$*-polynomial* P,

$$\lim_{n\to\infty} d_H(\sigma[P(X^{(n)})], \sigma[P(X^{(\infty)})]) = 0.$$

Proof. As well as the proof of the main theorem, we can estimate $\|(P^*P)^k(X^{(n)})\|$ by using "uniform RD property". Since $\lim_{k\to\infty}[(2k \cdot \mathrm{deg}P + 1)^D]^{\frac{1}{2k}} = 1$, we obtain strong convergence from convergence in non-commutative $*$-distribution. By a similar argument in the proof of Corollary 6.3, we also obtain the convergence of spectrums in the Hausdorff distance. ☐

Remark 6.5. In [16], Brannan showed that the normalized standard generators of free orthogonal quantum groups O_N^+ satisfy both two assumptions where the degree function is replaced by a suitable length function. As a consequence, he proved these normalized generators strongly converge to a free semicircular system as $N \to \infty$. For more general free orthogonal quantum groups O_F^+, RD type estimate fails for all non-Kac, non-amenable free orthogonal quantum groups, see e.g. [17].

Remark 6.6. We conclude with a remark on the spectral radius $r[P(X^{(n)})]$ for a non-self-adjoint polynomial P. For the same reason that the infimum of continuous functions is upper semi-continuous, we can say by strong convergence (note that $r(T) = \inf_k \|T^k\|^{\frac{1}{k}}$ for a bounded operator T),

$$\limsup_{n\to\infty} r[P(X^{(n)})] \le r[P(X^{(\infty)})].$$

In particular, the spectral radius $r[P(A^{(q)})]$ is upper semi-continuous with respect to q. We were not able to show the lower semi-continuity. For this problem, a quantitative estimate for the difference between $\|P(A^{(q)})^k\|^{\frac{1}{k}}$ and $\|P(A^{(q')})^k\|^{\frac{1}{k}}$ with different $q, q' \in (-1, 1)$ should be helpful. By the Haagerup-type inequality, it is enough to see the difference between $\|P(A^{(q)})^k\|_2^{\frac{1}{k}}$ and $\|P(A^{(q')})^k\|_2^{\frac{1}{k}}$ for sufficiently large k. We leave it for future work.

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