Summary of Thesis: **Theoretical Study of Invertible States Using Matrix Product States**

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Quantum mechanical phase degrees of freedom are known to have an interesting interplay with topology [1, 2]. A canonical example is the Dirac monopole where the presence of a magnetic monopole prevents quantum mechanical wave functions from being defined uniquely over the entire space. Instead, following the work by Wu and Yang [3], wave functions can be defined by introducing multiple patches, and at the intersection of two patches, wave functions from different patches are related by a transition function. The (large) gauge invariance results in the quantization of magnetic charges in units of the inverse of the fundamental charge. A magnetic monopole also arises in the context of the Berry phase, where a diabolic point of the Hamiltonian plays the role of the Dirac monopole of the Berry connection in a parameterized quantum system where the wave function $|\psi(x)\rangle$ depends smoothly on some adiabatic parameter(s) x taken from a parameter space X. The mathematical structure underlying these situations is a principle U(1) bundle over the parameter space X. Such bundles are characterized and classified by a topological invariant, the first Chern class taking its value in the second cohomology group of X, $\mathrm{H}^2(X;\mathbb{Z})$.

In recent years, it has been recognized that there are many-body systems where the regular notion of the Berry phase fails to capture topological properties. Specifically, a family of invertible many-body quantum states that depends on some parameter $x \in X$, which we shall call invertible states over X for short, has been discussed [4–12]. Such a family can be topologically non-trivial and can be considered as a generalization of the Thouless pump. It can also be considered as a generalization of regular gapped phases (SPT phases) which can be regarded as a special case where the parameter space is a single point. For example, it is known that there is a non-trivial family of $(d+1)$-dimensional systems with U(1) symmetry parameterized over S^d [5]. We however cannot use the ordinary Berry phase to detect its non-triviality in general. A cursory explanation is that the Berry connection and Berry curvature measure the non-triviality of $\mathrm{H}^2(X;\mathbb{Z})$, so for example when $d = 3$, they cannot be non-trivial on S^3. Even worse, if not introduced carefully, the Berry connection and curvature may be ill-defined in many-body quantum systems in the first place: For example, if we consider a chain of spins that are weakly interacting with each other and are each coupled to an adiabatically time-evolving magnetic field, the 1st Chern number diverges in the thermodynamic limit since each spin contributes independently.

Against this backdrop, Kapustin and Spodyneiko defined a quantity that generalizes Berry curvature for many-body Hamiltonians on a d-dimensional lattice[4, 13, 14]. This is called the higher Berry curvature. The formulation by Kapustin and Spodyneiko is applicable to a broad class of gapped lattice systems, making it promising for formulating various topological invariants of many-body systems. Indeed, Kapustin and Spodyneiko used this formulation for generalizing the Thouless pump and for applications to the thermal Hall effect[5, 15]. Analysis using field theory[16] also suggests its usefulness in characterizing quantum phase transition points. However, there are some issues with the Kapustin-Spodyneiko formulation:

- Problem 1: It is difficult to formalize higher Berry phases. The invariants constructed by Kapustin-Spodyneiko are integer-valued and can be seen as a generalization of Berry curvature. On the other hand, the existence of topological phases classified by higher Berry phases is also expected from field theory analysis. However, the Kapustin-Spodyneiko method cannot define higher Berry phases, making it insufficient as a topological invariant. This is due to the Hamiltonian-based formulation of the Kapustin-Spodyneiko method, which cannot handle gauge-dependent quantities like Berry connections.

- Problem 2: It is difficult to compute invariants using numerical calculations. To calculate the higher Berry curvature based on the Kapustin-Spodyneiko formulation, it is necessary to compute the Green's function for the many-body Hamiltonian, which is computationally expensive. For regular Berry phases, there are Hamiltonian-based and quantum state-based formulations. While theoretically equivalent, using the quantum state-based formulation is known to allow for low-cost computation of Berry phases [17], known as the Fukui-Hatsugai-Suzuki method.

The above issues are primarily due to the fact that Kapustin-Spodyneiko's method is a Hamiltonian-based formulation. Therefore, it is expected that providing a formulation of higher Berry phases based on quantum states will address Problems 1 and 2.

In order to reach these problems, it has been realized that a "higher" generalization of the Berry phase itself, which takes its value in the higher cohomology group, $\mathrm{H}^{d+2}(X;\mathbb{Z})$, is important [5, 11, 18]. Motivated by these developments, the purpose of this paper is to extend the ordinary Berry phase to $(1+1)$-dimensional quantum many-body systems and construct a topological invariant that takes its value in $\mathrm{H}^3(X;\mathbb{Z})$. In this paper, based on [11, 19, 20], we formulate the higher Berry phases for $1+1$-dimensional many-body systems using the matrix product states representations, and provide their physical and geometric understanding. See also [21].

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