Higher-Form Symmetry and Eigenstate Thermalization Hypothesis

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Abstract

The eigenstate thermalization hypothesis (ETH) is a successful framework providing criteria for thermalization in isolated quantum systems. The ETH for an operator ensures thermalization for the operator, i.e., its expectation value equilibrate to its thermal average. Although numerical and theoretical analyses support the ETH as a fundamental mechanism for explaining thermalization in diverse systems, it remains a challenge to analytically identify whether particular systems satisfy the ETH. In quantum many-body systems and quantum field theories (QFTs), phenomena that violate the ETH are expected to imply nontrivial thermalization processes, and are gathering increasing attention.

In this thesis, we elucidate how the existence of higher-form symmetries influences the dynamics of thermalization in isolated quantum systems. Under reasonable assumptions, we analytically show that a p-form symmetry in a $(d+1)$-dimensional QFT leads to the breakdown of the ETH for many nontrivial $(d-p)$-dimensional observables. In the case of discrete higher-form (i.e., $p \geq 1$) symmetry, this indicates the absence of thermalization for observables that are non-local but much smaller than the entire system size even though the system has no local conserved quantities. We provide numerical evidence for this argument for the (2+1)-dimensional $\mathbb{Z}_2$ lattice gauge theory. While local observables such as the plaquette operator thermalize even for mixed symmetry sectors, the non-local observable such as the one exciting a magnetic dipole instead relaxes to the generalized Gibbs ensemble that takes account of the $\mathbb{Z}_2$ 1-form symmetry.

The assumptions of the ETH-violation above include the mixing of symmetry sectors within a given energy shell. This condition is rather challenging to verify because it requires information on the eigenstates in the middle of the spectrum. In the subsequent chapter, we further reconsider this assumption from the viewpoint of the projective phase to alleviate this difficulty. In the case of $\mathbb{Z}_N$ symmetries, we can circumvent the difficulty by considering $\mathbb{Z}_N \times \mathbb{Z}_N$-symmetric theories with a mixed 't Hooft anomaly, and then perturbing the Hamiltonian while preserving one of the $\mathbb{Z}_N$ symmetries of interest. Additionally, we carry out numerical analyses for $(1+1)$-dimensional spin chains and the $(2+1)$-dimensional $\mathbb{Z}_2$ lattice gauge theory to demonstrate this scenario.

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Chapter 1

Introduction

1.1 Thermalization through unitary time evolution

Conceptual and practical foundation of thermalization in many-body system has been a subject of great interest and tackled from various aspects over a century. Thermalization process, which appears irreversible, frequently emerges even in systems microscopically governed by deterministic and reversible laws. For quantum systems, these setups are characterized by the unitary time evolution of pure states. It is somewhat counterintuitive for generic pure states to evolve into thermal states since the density matrix remains that of pure states, but pure states can indeed thermalize in the modern understanding. Such processes takes place in high energy physics, regarding the unitary time evolution of closed universe or highly excited dynamics of quantum field theories; in nuclear physics, where phenomena such as heavy-ion collisions and quark-gluon plasma become of concern; and needless to say, in isolated quantum many-body systems including spin chains in the context of condensed matter physics.

Attempts to formulate thermalization in isolated quantum systems were initiated by J. von Neumann [1], proving the so-called quantum ergodic theorem, which states that a majority of systems undergo thermalization in terms of macroscopic variables. Although his work can be regarded as a pioneering achievement for thermalization in isolated quantum systems, its claim is rather restrictive in the following sense: Firstly, the quantum ergodic theorem asserts thermalization for majority of Hamiltonians obtained by unitary transformations to a reference Hamiltonian, and it does not give any implications to individual systems. In particular, most of Hamiltonians constructed in this manner are unphysical ones that include highly non-local interactions. As a second point, von Neumann's work does not consider thermalization of microscopic quantities. The recent theoretical

and experimental studies support the fact that almost all pure states with subextensive energy fluctuations is *locally* indistinguishable from the microcanonical ensemble in sufficiently large systems. It implies that even local observables rather than macroscopic ones may exhibit thermalization. In this scenario, quantum entanglement plays an essential role and there are no classical counterpart for the local thermalization.

To deal with these difficulties, we first have to characterize "thermal equilibrium" as a realized final state since thermalization is the process where a generic initial state equilibrate to a thermal stationary state. The notion of thermal equilibrium is closely linked with that of *typicality*. Roughly speaking, typicality in the context of statistical mechanics means that the vast majority of "typical" states cannot be distinguished by physical observables in many-body systems, and thereby defining thermal equilibrium. Corresponding to the arguments in the last paragraph, we can introduce two types of typicality: *thermal*(/macroscopic) typicality whose characterizing observables should be macroscopic; and *canonical*(/microscopic) typicality where even microscopic quntities such as the entanglement entropy cannot distinguish the typical states. Based on thermal and canonical typicality, we can define the macroscopic thermal equilibrium (MATE) and the microscopic thermal equilibrium (MITE), respectively. MATE is a natural analog of Boltzmann's idea in classical systems, but for quantum systems, MITE offers a more fundamental characterization particularly in relatively small, say, ten-site spin systems.

One of the most celebrated criteria for thermalization in isolated quantum systems is the eigenstate thermalization hypothesis (ETH) [2–4] (for reviews, see e.g., [5]). The statement of the ETH is phrased as "all of the energy eigenstates can be regarded as thermal". As we will discuss in Chapter 2, the ETH provides a sufficient condition of microsocpic thermalization for arbitrary initial conditions in an energy shell. Supported by many numerical theoretical analyses, the ETH is considered as a basic mechanism to explain thermalization in various systems [4, 6–33], while providing a rigorous proof of the ETH remains a significant challenge. Indeed, some exotic counterexamples have been identified such as integrable systems, quantum many-body scars, Hilbert space fragmentation, and so on. These special but physically intriguing systems are attracting considerable attention, highlighting the need for systematic ways to diagnose the applicability of the ETH.

In quantum field theories, we have to appropriately regularize the degrees of freedom to examine the validity of the ETH, because the ETH is mainly formulated for systems with discrete spectra. For conformal field theories, this procedure can be carried out by truncating heavy states, which make

the ETH tractable in holographic setups [34–41]. We can also consider perturbations that explicitly break the conformal symmetry, and the ETH is verified for such perturbed theories [42–45]. Another approach is lattice regularization with truncation of the local degrees of freedom. Lattice quantum field theories accommodate rich structures such as quantum scars relevant to the ETH [46–49], and is the main focus of this thesis.

1.2 Symmetry and thermalization

In the field of statistical mechanics, symmetries play a significant role. When a system has local conserved quantities resulting from symmetries, the statistical ensemble is supposed to include these quantities appropriately. This observation has drawn much attention in recent research on how isolated quantum many-body systems reach thermal equilibrium [50–56]. While non-integrable systems without any symmetries are typically expected to relax locally to the canonical ensemble, the existence of symmetries can influence the dynamics of the systems in some cases [57–63]. For example, integrable systems, which have many symmetries, do not achieve thermalization in the standard way but rather approach the generalized Gibbs ensemble (GGE) [64–73], which incorporates the (quasi-)local conserved quantities.

With this in mind, the existence of symmetries are also expected to affect the validity of the ETH since it gives sufficient conditions of thermalization. Indeed, local conserved quantities originated from, e.g., continuous global symmetry or integrability, violate the ETH with respect to the entire Hilbert space, and in this case, the ETH can be recovered after fixing the values of conserved quantities [58, 62, 68]. In contrast, the effect of non-local conserved quantities arising from e.g., discrete symmetries, is more subtle. This is because they do not typically break the (diagonal) ETH, and thermalization of local observables is not influenced even when we consider mixed symmetry sectors [58, 74, 75]. This fact is indeed surprising since the existence of conserved quantities can be sometimes related to non-ergodicity [76–94].

Recently, the notion of *higher-form symmetry* has emerged in the context of quantum field theory, offering a framework for analyzing the phase structures [95–108] (for recent reviews, see Ref. [109–115] including applications to condensed matter physics). Higher-form symmetry is regarded as a generalization of conventional global symmetries, and characterized by topological symmetry operators, whose correlation functions remain invariant under their continuous deformations.

This generalization involves the dimensionality of both charged objects and symmetry operators: in a $(d+1)$-dimensional spacetime, p-form symmetries incorporates p-dimensional charged objects, with symmetry operators being $(d-p)$-dimensional. In the case of $p \geq 1$, we refer to that p-form symmetry as a higher-form symmetry. This is in contrast to conventional global symmetries, which are identified as 0-form symmetries.

Despite extensive research on the static aspects of higher-form symmetry, its dynamic implications remain largely unexplored. Higher-form symmetries are intrinsically present in gauge theories, such as Yang-Mills theories, which have attracted considerable attention in both condensed matter and atomic-molecular-optical physics contexts [116–133], as well as in high energy physics. While there are a few studies that discuss thermalization dynamics in specific models with these generalized symmetries [134–138], general implications of higher-form symmetry have seldom been fully uncovered.

As mentioned above, in many cases, it is believed that systems with local conserved quantities such as integrable models satisfy the ETH only if all of the symmetry sectors for the conserved quantities are resolved, while the ETH does not hold with the mixed symmetry sector. The result of our work [139] indeed provides a general proof to the violation of the ETH in the presence of higher-form symmetry. We stress that it is also applicable even to the case of non-local conserved quantities since discrete symmetries typically give rise to them.

1.3 Organization of the thesis

The remainder of this thesis is organized as follows. Chapter 2 provides the review of thermalization in isolated quantum systems. We first introduce the notion of typicality and characterize thermal equilibrium. The ETH is then implemented as sufficient conditions of the thermal equilibrium. In Chapter 3, we discuss the relation between the ETH and higher-form symmetry based on our work [139]. We analytically show that the ETH for the $(d-p)$-dimensional operator are broken in presence of p-dimensional quantum field theories. As a consequence of the ETH-violation, we also propose the generalized Gibbs ensemble (GGE) for discrete p-form symmetries as a realized thermal equilibrium. In Chapter 4, based on our work [140], we revisit the precise condition of the ETH-violation from the perspective of projective phases or 't Hooft anomaly of p-form symmetry. From this viewpoint, we can derive the ETH-violation without any direct reference to the eigenstates in the middle of spectrum of our interest. Chapter 5 is devoted to the conclusion and discussion. We there summarize the main results of this thesis and discuss outlook to possible future directions.

Chapter 2

Thermalization in isolated quantum systems

In this chapter, we review the notion of thermalization in isolated quantum systems, and introduce the eigenstate thermalization hypothesis (ETH), a widely recognized criterion that give sufficient conditions for thermalization. For more detailed reviews, see [55].

2.1 Thermal equilibrium

In order to discuss the thermalization and its sufficient conditions, we first consider how *thermal equilibrium* is characterized. In the following, we focuses on lattice systems, such as quantum spins, bosons, or fermions. To adopt quantum field theories, we thus should lattice-regularize the space manifold. If you need the finite Hilbert space, it can be obtained by further truncating the local Hilbert space. Here, the notion of locality can be naturally introduced since we consider a Hilbert space of the form

$$\mathcal{H}_{\mathcal{M}} := \bigotimes_{i\in\mathcal{M}} \mathcal{H}_i, \tag{2.1.1}$$

where $\mathcal{H}_i$ are local Hilbert spaces, and $\mathcal{M}$ denotes the lattice-regularized d-dimensional space manifold. Each microstate of the system is identified by a state vector in the Hilbert space $\mathcal{H}_{\mathcal{M}}$. The Hamiltonian of the system is denoted by H, with its eigenstates and eigenvalues represented as $|E_\alpha\rangle$ and E_α, i.e, $H|E_\alpha\rangle = E_\alpha|E_\alpha\rangle$. We also define a subspace that is constrained to have an almost definite macroscopic energy value E, denoted by

$$\mathcal{H}_{\mathcal{M},E} := \mathrm{span}\{|E_\alpha\rangle \in \mathcal{H}_{\mathcal{M}} \mid E_\alpha \in [E, E+\Delta E]\}\ (\subset \mathcal{H}_{\mathcal{M}}), \tag{2.1.2}$$

with ΔE being an energy range small on a macroscopic scale but large microscopically. We refer to $\mathcal{H}_{\mathcal{M},E}$ as the energy shell of the quantum system. The dimension of $\mathcal{H}_{\mathcal{M},E}$ is denoted by

$$D_{\mathcal{M},E} := \dim(\mathcal{H}_{\mathcal{M},E}). \tag{2.1.3}$$

In the following discussion, we deal with a set of operators

$$\mathscr{O} := \{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n\}, \tag{2.1.4}$$

where $\mathcal{O}_i$ are operators acting on the Hilbert space $\mathcal{H}_M$. The support of each operator $\mathrm{Supp}(\mathcal{O}_i)$ is defined as the minimal set $X_i \subset \mathcal{M}$ such that the operator is expressed as $\mathcal{O}_i = \mathcal{O}_{X_i} \otimes \mathbf{1}_{X_i^{\mathrm{c}}}$. Here, $\mathbf{1}_{X_i^{\mathrm{c}}}$ is the identity operator acting on $\otimes_{i \in X_i^{\mathrm{c}}} \mathcal{H}_i$, where X_i^{c} is the complement of X_i. The support for the set $\mathscr{O}$

$$\mathrm{Supp}\,\mathscr{O} := \bigcup_{\mathcal{O}_i \in \mathscr{O}} \mathrm{Supp}\,\mathcal{O}_i\,. \tag{2.1.5}$$

If an operator $\mathcal{O}_i$ satisfies the scaling $\mathrm{Vol}(\mathrm{Supp}\,\mathcal{O}_i)/\,\mathrm{Vol}(\mathcal{M}) \to 0$ as $\mathrm{Vol}(\mathcal{M}) \to \infty$, the operator $\mathcal{O}_i$ is said to be local, where $\mathrm{Vol}(\mathcal{M})$ denotes the volume of the manifold $\mathcal{M}$.

In many cases, quantum many-body Hamiltonians H of our interest in the context of thermalization are represented as a sum of local operators, which are referred to as local Hamiltonians. Along this line, we also introduce *local* conserved quantities. Among conserved quantities Q with $[H, Q] = 0$, Q that can be expressed as a sum of local operators, is called a local conserved quantity, and otherwise we call it a non-local conserved quantity. In the thermodynamic limit $\mathrm{Vol}(\mathcal{M}) \to \infty$, local conserved quantities are expected to behave as extensive variable since they take the form $Q \simeq \int_{\mathcal{M}} dx\, \mathcal{O}(x)$ in this limit.

2.1.1 Macroscopic/microscopic thermal equilibrium

As an analogy to classical many-body systems, we can consider the following as a characterization of thermal equilibrium [1,141,142]: given a set of macroscopic observables $\mathscr{M} = \{M_1, M_2, \ldots, M_j\}$, these operators $(M_1, \ldots, M_j)$ take particular value $(M_{1,\mathrm{thermal}}, \ldots, M_{j,\mathrm{thermal}})$ in the thermodynamic limit, and such states specify the thermal equilibrium. We shall elaborate more on what "macroscopic observable" means. Dividing the space manifold as $\mathcal{M} = \cup_k \Lambda_k$ such that $\Lambda_k \cap \Lambda_{k'} = \phi$ for any $k \neq k'$, we say $M_i \in \mathscr{M}$ is a macroscopic observable iff the support of M_i is one of Λ_k or their unions (including $\mathcal{M}$).

This characterization is referred to as *macroscopic thermal equilibrium* (MATE), and it is supported by the supposition called *thermodynamic typicality*, described as

$$^{\exists}\gamma > 0, \qquad 1 - \frac{D_{\mathscr{M}_{\rm eq}}}{D_{\mathcal{M},E}} < \exp(-\gamma\,\mathrm{Vol}(\mathcal{M})). \tag{2.1.6}$$

where $D_{\mathcal{M},E}$ is the dimension of the Hilbert space $\mathcal{H}_{\mathcal{M},E}$, and $D_{\mathscr{M}_{\rm eq}}$ is the one of its subspace with $(M_1,\ldots,M_j) \simeq (M_{1,\rm thermal},\ldots,M_{j,\rm thermal})$.

There is another notion of thermal equilibrium intrinsic to quantum systems (i.e., with no analogs in classical many-body systems) called *microscopic thermal equilibrium* (MITE). The notion of MITE is motivated by introduction of *canonical typicality* [143–145]. In order to explain canonical typicality, we divide the space $\mathcal{M}$ into a small subset X and its complement $X^{\rm c}$, where $X^{\rm c}$ is regarded as the "bath" with respect to X. The claim of canonical typicality is as follows: For almost all state $|\psi\rangle \in \mathcal{H}_{E,\mathcal{M}}$, there exists a density matrix $\rho_{\rm eq}$ such that

$$\mathrm{tr}_{X^{\rm c}}\,|\psi\rangle\langle\psi| \simeq \mathrm{tr}_{X^{\rm c}}\,\rho_{\rm eq}, \tag{2.1.7}$$

where $\mathrm{tr}_{X^{\rm c}}$ denotes the partial trace with respect to $X^{\rm c}$. We can elaborate on this relation utilizing the inequality [144]

$$P_{|\psi\rangle}\Big[\big|\big|\,\mathrm{tr}_{X^{\rm c}}\,|\psi\rangle\langle\psi| - \mathrm{tr}_{X^{\rm c}}\,\rho_{\rm eq}\big|\big|_1 \geq \eta\Big] \leq \eta', \tag{2.1.8}$$

$$\eta = \epsilon + \sqrt{\frac{D_X}{D_{\rm env}}}, \qquad \eta' = 4\exp\left(-\frac{\epsilon^2}{18\pi^3}D_{\mathcal{M},E}\right), \qquad \epsilon > 0, \tag{2.1.9}$$

$$D_X := \dim\left(\bigotimes_{i\in X}\mathcal{H}_i\right), \tag{2.1.10}$$

where $P_{|\psi\rangle}[\text{condition}]$ is the probability such that the condition holds for uniformly distributed $|\psi\rangle \in \mathcal{H}_{\mathcal{M},E}$, and $||\cdot||_1$ denotes the trace norm

$$||\rho||_1 := \mathrm{tr}_X\,\sqrt{\rho^\dagger\rho}\,. \tag{2.1.11}$$

The constant $D_{\rm env}$ is the effective dimension of the Hilbert space supported by $X^{\rm c}$[1]. For $\mathrm{Vol}(\mathcal{M}) \gg \mathrm{Vol}(X)$, the effective dimension behaves as $D_{\rm env} \simeq D_{\mathcal{M},E}/D_X$. One can show the relation (2.1.8) by utilizing Levy's lemma for general quantum systems [144].

[1]The precise definition of $D_{\rm env}$ is given by

$$D_{\rm env} := 1\Big/\,\mathrm{tr}_X\left(\mathrm{tr}_{X^{\rm c}}\,\frac{\mathbf{1}_{\mathcal{M},E}}{D_{\mathcal{M},E}}\right)^2, \tag{2.1.12}$$

where $\mathbf{1}_{\mathcal{M},E}$ is the identity matrix whose entries have nonzero value for the components of $\mathcal{H}_{\mathcal{M},E}$. If we consider the situation where ΔE is large enough, i.e., $\mathcal{H}_{\mathcal{M},E} = \mathcal{H}_{\mathcal{M}}$, $D_{\rm env}$ is equal to $D_{\mathcal{M},E}/D_X$.

The most significant case is when the volumes satisfy $\mathrm{Vol}(\mathcal{M}) \gg \mathrm{Vol}(X)$. For example, this relation holds if X has lower dimensional support compared to the total space $\mathcal{M}$, and this is exactly the case in Chapter 3 and Chapter 4. We define the "system size" ℓ with the dimension of length, and then the dimensions of the Hilbert spaces are given by $D_{\mathcal{M},E} = e^{O(\ell^d)}$ and $D_X = e^{O(\ell^{d-p})}$ for a $(d-p)$-dimensional X. We then consider the situation $D_{\mathcal{M},E} \gg D_X \gg 1$, and set ϵ suitably, say $\epsilon = D_{\mathcal{M},E}^{-1/3}$. Recalling the relation (2.1.9), we have

$$\eta \simeq D_{\mathcal{M},E}^{-1/3} + \sqrt{\frac{D_X^2}{D_{\mathcal{M},E}}} = e^{-O\left(\ell^{\min\{d,p\}}\right)}, \qquad \eta' = 4\exp\left(-\frac{1}{18\pi^3}e^{O(\ell^d)}\right). \tag{2.1.13}$$

In the $\ell \to \infty$ limit, both of the constants η and η' tends to zero as long as $p \geq 1$. This result shows that the canonical typicality (2.1.7) holds in the thermodynamic limit $\ell \to \infty$ with sufficiently large bath (in the context of (2.1.13), $p \geq 1$). We stress that the canonical typicality does NOT require the locality of X.

Remarkably, the canonical typicality (2.1.7) is valid regardless of the existence of conserved quantities. The standard lore states that if a system has a conserved quantities, the relevant thermal ensemble should take account of it (e.g., introducing a suitable chemical potential). Indeed, the density matrix is not necessarily the one of the canonical ensemble, and one can expect that the resulting thermal equilibrium is a grand canonical-like ensemble in presence of conserved quantities.

We lastly comment on the relation between MITE and MATE. In particular, MITE and MATE are neither sufficient nor necessary condition to each other, and MITE makes sense even in small systems while MATE does not. For a system in MATE, the quantum fluctuations of macroscopic observables is small. In contrast, MITE claims small fluctuations only for observables $\mathcal{O}$ with a small support X, and the ones of macroscopic observables are not necessarily small since they are sum of $\mathcal{O}$. In this sense, MATE can be considered stronger than MITE. On the other hand, MITE indicates the system can not be distinguished even microscopically, although this property does not necessarily hold for MATE, and this fact make us feel that MITE is stronger than MATE.

The inequality (2.1.8) implies that the notion of MITE is applicable if $D_{\mathcal{M},E} \gg D_X^2$ and $D_{\mathcal{M},E} \gg 1$. For example, this condition can be satisfied setting $d = 2$, $p = 1$, and $\ell = 4$ since the dimension of the Hilbert spaces scales as $D_{\mathcal{M},E} = e^{O(\ell^d)}$ and $D_X = e^{2O(\ell^{d-p})}$. The notion of MATE is meaningful only for large systems such that $\gamma\,\mathrm{Vol}(\mathcal{M}) \gg 1$ so that the fluctuation of macroscopic observables should be small.

2.1.2 Thermal equilibrium for operator sets

In the spirit of MITE, even microscopic subsystems realize thermal equilibriums characterized by "canonically typical" states. This fact is intrinsic to quantum systems and will be the main focus in the remainder of this thesis. The notion of MITE motivate us to consider thermal average even for local operators and non-extensive observables with a support X. When a state $|\psi\rangle$ is in MITE (2.1.7), we can see

$$\langle\psi|\mathcal{O}|\psi\rangle \simeq \mathrm{tr}(\mathcal{O}\rho_{\mathrm{eq}}) =: \langle\mathcal{O}\rangle_{\mathrm{eq}} \tag{2.1.14}$$

for an operator $\mathcal{O}$ with the support X. This relation suggests that a single pure state can reproduce the result of the microcanonical ensemble. In [145–147], the following relation is indeed shown:

$$P_{|\psi\rangle}\Big[\,\langle\psi|\mathcal{O}|\psi\rangle - \langle\mathcal{O}\rangle_{\mathrm{eq}} \geq \epsilon\Big] \leq \frac{||\mathcal{O}||^2}{\epsilon^2 D_{\mathcal{M},E}}\,, \tag{2.1.15}$$

where $||\cdot||$ denotes the operator norm

$$||\mathcal{O}|| := \sup_{\psi\in\mathcal{H}_{\mathcal{M}}:\langle\psi|\psi\rangle=1} \sqrt{\langle\psi|\mathcal{O}^\dagger\mathcal{O}|\psi\rangle}\,. \tag{2.1.16}$$

With the discussion above in mind, we now sort out the concepts of MITE and thermal equilibrium for operators. We here say a pure state $|\psi\rangle \in \mathcal{H}_{\mathcal{M},E}$ is in MITE for X iff

$$\Big|\Big|\,\mathrm{tr}_{X^{\mathrm{c}}}\,|\psi\rangle\langle\psi| - \mathrm{tr}_{X^{\mathrm{c}}}\,\rho_{\mathrm{eq}}\Big|\Big|_1 \to 0\,, \qquad \text{as } \mathrm{Vol}(\mathcal{M}) \to \infty. \tag{2.1.17}$$

On the other hand, we define the thermal equilibrium with respect to an operator set $\mathscr{O}$ (such that $\mathrm{Supp}\,\mathscr{O} = X$) by

$$\max_{\mathcal{O}_i\in\mathscr{O}} \left|\frac{\langle\psi|\mathcal{O}_i|\psi\rangle - \mathrm{tr}_{\mathcal{M}}(\rho_{\mathrm{eq}}\mathcal{O}_i)}{\Delta_{\mathcal{O}_i}}\right| \to 0\,, \qquad \text{as } \mathrm{Vol}(\mathcal{M}) \to \infty, \tag{2.1.18}$$

where $\Delta_{\mathcal{O}_i}$ is the spectral width of the operator $\mathcal{O}_i$ defined by

$$\Delta_{\mathcal{O}_i} := \max_k \lambda_k - \min_k \lambda_k\,, \qquad \mathcal{O}_i = \sum_k \lambda_k\,|\lambda_k\rangle\langle\lambda_k| \quad (\text{:spectral decomposition}). \tag{2.1.19}$$

Note here that the operators $\mathcal{O}_i \in \mathscr{O}$ are hermitian operators.

In the following, we show that

$$\begin{aligned} &|\psi\rangle \in \mathcal{H}_{\mathcal{M},E} \text{ is in thermal equilibrium with respect to } \tilde{\mathscr{O}} := \{^\forall\mathcal{O} \mid \mathrm{Supp}\,\mathcal{O} = X\} \\ \iff\quad &|\psi\rangle \in \mathcal{H}_{\mathcal{M},E} \text{ is in MITE for } X. \end{aligned} \tag{2.1.20}$$

To this end, we consider the quantity

$$\begin{aligned}
&\max_{\mathcal{O}_i\in\tilde{\mathscr{O}}}\left|\frac{\langle\psi|\mathcal{O}_i|\psi\rangle-\mathrm{tr}_{\mathcal{M}}(\rho_{\mathrm{eq}}\mathcal{O}_i)}{\Delta_{\mathcal{O}_i}}\right| \\
&=\max_{\mathcal{O}_i\in\tilde{\mathscr{O}}}\left|\mathrm{tr}_{\mathcal{M}}\left(\frac{\mathcal{O}_i}{\Delta_{\mathcal{O}_i}}(\rho-\rho_{\mathrm{eq}})\right)\right| \qquad\qquad \big(\rho:=|\psi\rangle\langle\psi|\,\big) \\
&=\max_{\mathcal{O}_i\in\tilde{\mathscr{O}}}\left|\mathrm{tr}_X\left(\frac{\mathcal{O}_i}{\Delta_{\mathcal{O}_i}}\Big(\mathrm{tr}_{X^{\mathrm{c}}}(\rho)-\mathrm{tr}_{X^{\mathrm{c}}}(\rho_{\mathrm{eq}})\Big)\right)\right| \\
&=\Big|\Big|\,\mathrm{tr}_{X^{\mathrm{c}}}(\rho)-\mathrm{tr}_{X^{\mathrm{c}}}(\rho_{\mathrm{eq}})\Big|\Big|_1\Big/2\,. \qquad (2.1.21)
\end{aligned}$$

In the last equality, we utilized the relation

$$\max_{\mathcal{A}}\mathrm{tr}\left(\frac{\mathcal{A}}{\Delta_{\mathcal{A}}}\mathcal{B}\right)=\frac{||\mathcal{B}||_1}{2}\,. \qquad (2.1.22)$$

This can be easily shown because for a Hermitian matrix $\mathcal{B}=\sum_k b_k\,|b_k\rangle\langle b_k|$, the $||\cdot||_1$ norm just reduces to $||\mathcal{B}||_1=\sum_k|b_k|$. In the relation (2.1.21), the vanishing left-hand as $\mathrm{Vol}(\mathcal{M})\to\infty$ leads to the vanishing right-hand side, and vice versa. Thus, we can immediately realize that the left-hand side of (2.1.20) is equivalent to the right-hand side.

After all, we conclude that even pure states can be regarded as thermal equilibrium in the sense of (2.1.17) and (2.1.18). Because thermal equilibrium for operator sets (2.1.18) makes sense, we can further delve into a question "what is the natural choice for $\mathscr{O}$?" Obviously, the most strong characterization for a given X is realized when you set $\mathscr{O}=\tilde{\mathscr{O}}:=\{{}^{\forall}\mathcal{O}\mid\mathrm{Supp}\,\mathcal{O}=X\}$. However, it is also plausible that some operator $\mathcal{O}_1$ such that $\mathrm{Supp}(\mathcal{O}_1)=X$ is in thermal equilibrium while another operator $\mathcal{O}_2$ with the same support is not. This subtleties actually one of major interests in the following chapters.

2.1.3 Entanglement entropy in thermal equilibrium

We now consider the entanglement entropy for a pure state $|\psi\rangle$, which is defined by

$$S(|\psi\rangle)=S_{\mathrm{vN}}\big(\mathrm{tr}_{X^{\mathrm{c}}}|\psi\rangle\langle\psi|\,\big):=-\mathrm{tr}_X\left[\big(\mathrm{tr}_{X^{\mathrm{c}}}|\psi\rangle\langle\psi|\,\big)\log\big(\mathrm{tr}_{X^{\mathrm{c}}}|\psi\rangle\langle\psi|\,\big)\right]. \qquad (2.1.23)$$

If the state $|\psi\rangle$ in MITE (2.1.7), we can naively expect

$$S(|\psi\rangle)\simeq S_{\mathrm{vN}}\big(\mathrm{tr}_{X^{\mathrm{c}}}\rho_{\mathrm{mc}}\big)=-\mathrm{tr}_X\left[\big(\mathrm{tr}_{X^{\mathrm{c}}}\rho_{\mathrm{mc}}\big)\log\big(\mathrm{tr}_{X^{\mathrm{c}}}\rho_{\mathrm{mc}}\big)\right]. \qquad (2.1.24)$$

Here, we assumed the equilibrium is given by the microcanonical ensemble, so that the system is supposed to have no extra structures such as symmetries. We note that although the relation (2.1.24)

seems plausible, it is quite nontrivial whether this relation holds true precisely up to subextensive quantity.

Once we suppose the expression (2.1.24), the equivalence of thermal ensemble leads to

$$S(|\psi\rangle) \simeq S_{\mathrm{vN}}\big(\mathrm{tr}_{X^{\mathrm{c}}}\,\rho_{\mathrm{mc}}\big) \simeq S_{\mathrm{vN}}\big(\rho_{\mathrm{can},X}\big), \tag{2.1.25}$$

if the subsystem X satisfies $\mathrm{Vol}(\mathcal{M}) \gg \mathrm{Vol}(X) \gg 1$. The matrix $\rho_{\mathrm{can},X}$ denote the density matrix of the canonical ensemble:

$$\rho_{\mathrm{can},X} := \frac{1}{\mathrm{tr}_X\, e^{-\beta H_X}} e^{-\beta H_X}, \tag{2.1.26}$$

where H_X is the Hamiltonian of the subsystem X[2], and the temperature β can be defined by $\beta := \partial S_{\mathrm{mc}}(E, \mathrm{Vol}(\mathcal{M}))/\partial E$. After all, we can reasonably believe the entanglement entropy for $|\psi\rangle$ with respect to X tends to the thermal entropy in the thermodynamic limit if the relation (2.1.24) holds. This fact is often used as a criterion to determine whether a given state $|\psi\rangle$ is a thermal state in some literature.

2.2 Sufficient conditions for thermalization

The concept of typicality in thermal equilibrium suggests that thermalization is the process where a system transitions from an atypical non-equilibrium initial state to a typical state representing thermal equilibrium. This typicality-based reasoning, however, falls short in explaining why certain systems undergo thermalization while others do not. Notably, some many-body systems do not achieve thermalization despite the typicality of thermal equilibrium being applicable. Therefore, to gain a deeper understanding, it is essential to establish specific criteria that can determine the occurrence or lack of thermalization in individual systems.

2.2.1 Equilibration and thermalization

The most naive way to define thermalization with respect to $\mathscr{O}$ for an initial $|\psi\rangle$ is as follows: ${}^{\exists}t_{\mathrm{relax}}$, ${}^{\forall}t > t_{\mathrm{relax}}$, ${}^{\forall}\mathcal{O}_i \in \mathscr{O}$ such that $\langle\psi(t)|\mathcal{O}_j|\psi(t)\rangle \simeq \mathrm{tr}(\mathcal{O}_i\rho_{\mathrm{eq}})$, but this characterization clearly fails due to the quantum recurrence theorem [148, 149]. In isolated quantum systems with unitary time evolution, we can not avoid the quantum recurrence theorem stating

$${}^{\forall}\epsilon > 0, \quad {}^{\exists}t_n,\ (n = 1, 2, \dots), \qquad \big|\big|\,|\psi(t_n)\rangle - |\psi(0)\rangle\,\big|\big| \leq \epsilon, \tag{2.2.1}$$

[2]We here assume the Hamiltonian for the subsystem X can be defined by cutting the boundary interactions appropriately, since the interactions included in the Hamiltonian are local.

where $||\cdot||$ denotes the standard Euclidean norm

$$\left|\left|\,|\psi\rangle\,\right|\right| := \sqrt{\langle\psi|\psi\rangle}\,. \tag{2.2.2}$$

In light of 2.2.1, the "relaxation time" in this sense does not exists, and thus we need more refined formulation of thermalization.

A natural refinement is requiring $\langle\psi(t)|\mathcal{O}_i|\psi(t)\rangle \simeq \mathrm{tr}(\mathcal{O}_i\rho_{\mathrm{eq}})$ for *almost* all time $t > 0$. To make this statement more precise, we repeatedly utilize the time average

$$\overline{\langle\mathcal{O}\rangle} := \lim_{T\to\infty}\int_0^T dt\,\langle\psi(t)|\mathcal{O}|\psi(t)\rangle\,. \tag{2.2.3}$$

The key ideas are classified into the following two ingredients:

Equilibration

We first require the fluctuation of observables with respect to time average to be small for almost all time. This statement can be phrased as

$$\max_{\mathcal{O}_i\in\mathscr{O}}\frac{\sigma_{\mathrm{time}}(\mathcal{O}_i)}{\Delta_{\mathcal{O}_i}} \to 0 \qquad \text{as } \mathrm{Vol}(\mathcal{M})\to\infty, \tag{2.2.4}$$

$$\sigma^2_{\mathrm{time}}(\mathcal{O}) := \overline{\left(\langle\psi(t)|\mathcal{O}|\psi(t)\rangle - \overline{\langle\mathcal{O}\rangle}\right)^2}\,. \tag{2.2.5}$$

Thermalization

If a system equilibrate, the time average of the observable should tend to the thermal average[3], i.e.,

$$\max_{\mathcal{O}_i\in\mathscr{O}}\frac{\left|\overline{\langle\psi(t)|\mathcal{O}_i|\psi(t)\rangle} - \langle\mathcal{O}_i\rangle_{\mathrm{mc}}(E)\right|}{\Delta_{\mathcal{O}_i}} \to 0 \qquad \text{as } \mathrm{Vol}(\mathcal{M})\to\infty, \tag{2.2.6}$$

$$\langle\mathcal{O}\rangle_{\mathrm{mc}}(E) := \mathrm{tr}(\mathcal{O}\rho_{\mathrm{mc}}(E))\,, \qquad \rho_{\mathrm{mc}}(E) := \sum_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M},E}}\frac{|E_\alpha\rangle\langle E_\alpha|}{D_{\mathcal{M},E}}\,. \tag{2.2.7}$$

For a given initial state $|\psi\rangle\in\mathcal{H}_{\mathcal{M},E}$, we say $|\psi\rangle$ is *thermalized* with respect to $\mathscr{O}$ iff it satisfies (2.2.4) and (2.2.6) under the unitary time evolution $|\psi(t)\rangle = e^{-iHt}\,|\psi(0)\rangle$. The notion of equilibration and thermalization is schematically depicted in Fig. 2.1.

[3]We here adopt the microcanonical ensemble since the realized thermal ensemble is expected to be the microcanonical ensemble without any special reason such as symmetries, Hilbert space fragmentation, and so on.

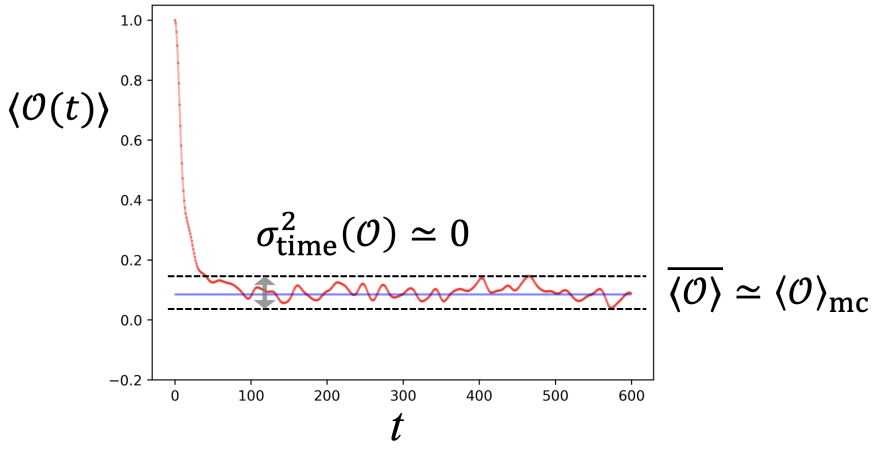

Figure 2.1: Schematics of equilibration and thermalization. Equilibration (2.2.4) indicates the fluctuation with respect to time average is small, and thermalization (2.2.6) ensures the equilibrated value of $\langle \mathcal{O} \rangle$ tends to its micro-canonical average.

Recalling Chebyshev's inequality[4], we obtain the expression

$$P_{t\in[0,\infty)}\left[\frac{\left|\langle\psi(t)|\mathcal{O}_i|\psi(t)\rangle - \overline{\langle\mathcal{O}_i\rangle}\right|}{\Delta_{\mathcal{O}_i}} \geq \epsilon\right] \leq \frac{\sigma^2_{\text{time}}(\mathcal{O}_i)}{\epsilon^2\Delta^2_{\mathcal{O}_i}} \tag{2.2.9}$$

$$\begin{aligned}\Rightarrow \quad & P_{t\in[0,\infty)}\left[\max_{\mathcal{O}_i\in\mathscr{O}}\left(\frac{\left|\langle\psi(t)|\mathcal{O}_i|\psi(t)\rangle - \overline{\langle\mathcal{O}_i\rangle}\right|}{\Delta_{\mathcal{O}_i}}\right) \geq \epsilon\right] \\ & = P_{t\in[0,\infty)}\left[\bigvee_{\mathcal{O}_i\in\mathscr{O}}\left(\frac{\left|\langle\psi(t)|\mathcal{O}_i|\psi(t)\rangle - \overline{\langle\mathcal{O}_i\rangle}\right|}{\Delta_{\mathcal{O}_i}} \geq \epsilon\right)\right] \\ & \leq \sum_{\mathcal{O}_i\in\mathscr{O}}\frac{\sigma^2_{\text{time}}(\mathcal{O}_i)}{\epsilon^2\Delta^2_{\mathcal{O}_i}} \leq |\mathscr{O}|\max_{\mathcal{O}_i\in\mathscr{O}}\frac{\sigma^2_{\text{time}}(\mathcal{O}_i)}{\epsilon^2\Delta^2_{\mathcal{O}_i}}\,,\end{aligned} \tag{2.2.10}$$

where $P_{t\in[0,\infty)}$ is the probability for uniformly distributed $t \in [0,\infty)$[5]. We then notice that the (2.2.4) leads to

$$P_{t\in[0,\infty]}\left[\max_{\mathcal{O}_i\in\mathscr{O}}\left(\frac{\left|\langle\psi(t)|\mathcal{O}_i|\psi(t)\rangle - \overline{\langle\mathcal{O}_i\rangle}\right|}{\Delta_{\mathcal{O}_i}}\right) \geq \epsilon\right] \to 0 \tag{2.2.12}$$

[4]Chebyshev's inequality is given by

$$P_X\Big[|X - \bar{X}| \geq \epsilon\Big] \leq \frac{\sigma_X^2}{\epsilon^2}, \tag{2.2.8}$$

where $\bar{X}$ and σ_X^2 is the expectation value and the variance with respect to probability distribution of X.

[5]More precisely $P_{t\in[0,\infty)}$ should be defined by

$$P_{t\in[0,\infty)} := \lim_{T\to+\infty} P_{t\in[0,T]}\,. \tag{2.2.11}$$

for an appropriately chosen ϵ such that

$$\frac{1}{\epsilon^2}\max_{\mathcal{O}_i\in\mathscr{O}}\frac{\sigma^2_{\rm time}(\mathcal{O}_i)}{\Delta^2_{\mathcal{O}_i}}\to 0 \qquad \text{as } \mathrm{Vol}(\mathcal{M})\to\infty. \tag{2.2.13}$$

Note that we here fixed the number of elements of $\mathscr{O}$. This result indeed matches the intuitive characterization of equilbration, where $\langle\psi(t)|\mathcal{O}_i|\psi(t)\rangle \simeq \overline{\langle\mathcal{O}_i\rangle} \simeq \mathrm{tr}(\mathcal{O}_i\rho_{\rm eq})$ for almost all time.

2.2.2 Eigenstate thermalization hypothesis

We now consider when equilibration (2.2.4) and thermalization (2.2.6) undergo for a given initial state. In the following, we work on the unitary time evolution $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$ under the Hamiltonian H with $H|E_\alpha\rangle = E_\alpha|E_\alpha\rangle$, suppose that the energy has no degeneracy: $E_\alpha = E_\beta \Rightarrow \alpha=\beta$; and no resonance: $E_\alpha - E_\beta = E_\gamma - E_\delta \neq 0 \Rightarrow \alpha=\gamma,\ \beta=\delta$ for simplicity. The non-resonance condition means that there is no degeneracy of energy gaps. Given an initial pure state

$$|\psi(0)\rangle = \sum_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M},E}} c_\alpha|E_\alpha\rangle, \qquad c_\alpha = \langle E_\alpha|\psi(0)\rangle, \tag{2.2.14}$$

the expectation value of an operator $\mathcal{O}$ reads

$$\langle\mathcal{O}(t)\rangle := \overline{\langle\psi(t)|\mathcal{O}|\psi(t)\rangle} = \sum_{\alpha,\beta} c^*_\alpha c_\beta e^{i(E_\alpha-E_\beta)t}\langle E_\alpha|\mathcal{O}|E_\beta\rangle, \tag{2.2.15}$$

$$\Rightarrow \qquad \overline{\langle\mathcal{O}(t)\rangle} = \sum_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M},E}} |c_\alpha|^2\langle E_\alpha|\mathcal{O}|E_\alpha\rangle \tag{2.2.16}$$

because of non-degeneracy condition. The variance with respect to time average can be also obtained as

$$\begin{aligned}
\sigma^2_{\rm time}(\mathcal{O}) &= \overline{\big(\langle\mathcal{O}(t)\rangle - \overline{\langle\mathcal{O}\rangle}\big)^2} = \overline{\left(\sum_{\substack{\alpha\neq\beta\\ |E_\alpha\rangle,|E_\beta\rangle\in\mathcal{H}_{\mathcal{M},E}} c^*_\alpha c_\beta e^{i(E_\alpha-E_\beta)t}\langle E_\alpha|\mathcal{O}|E_\beta\rangle\right)^2} \\
&= \sum_{\alpha\neq\beta,\,\gamma\neq\delta} c^*_\alpha c_\beta c^*_\gamma c_\delta \overline{e^{i(E_\alpha-E_\beta+E_\gamma-E_\delta)}}\langle E_\alpha|\mathcal{O}|E_\beta\rangle\langle E_\gamma|\mathcal{O}|E_\delta\rangle \\
&= \sum_{\alpha\neq\beta,\,\gamma\neq\delta} c^*_\alpha c_\beta c^*_\gamma c_\delta\,\delta_{\alpha\delta}\delta_{\beta\gamma}\langle E_\alpha|\mathcal{O}|E_\beta\rangle\langle E_\gamma|\mathcal{O}|E_\delta\rangle \\
&= \sum_{\substack{\alpha\neq\beta\\ |E_\alpha\rangle,|E_\beta\rangle\in\mathcal{H}_{\mathcal{M},E}}} |c_\alpha|^2|c_\beta|^2\left|\langle E_\alpha|\mathcal{O}|E_\beta\rangle\right|^2.
\end{aligned} \tag{2.2.17}$$

In the third line of (2.2.17), we utilized the non-resonance condition. We notice that the expectation (2.2.16) and the variance (2.2.17) only involve the diagonal part and the off-diagonal part of the matrix elements, respectively. Comparing these expressions with (2.2.4) and (2.2.6), one can present

sufficient conditions for thermalization and equilibration, which is commonly known as the *eigenstate thermalization hypothesis* (ETH).

Diagonal ETH

A sufficient for thermalization (2.2.6) with respect to $\mathscr{O}$ is given by[6]

$$^{\forall}\mathcal{O}_i \in \mathscr{O}, \quad \max_{|E_\alpha\rangle \in \mathcal{H}_\mathcal{M}} \frac{\left| \langle E_\alpha|\mathcal{O}_i|E_\alpha\rangle - \langle \mathcal{O}_i\rangle_{\mathrm{mc}}(E_\alpha) \right|}{\Delta_{\mathcal{O}_i}} \to 0 \qquad \text{as } \mathrm{Vol}(\mathcal{M}) \to \infty. \tag{2.2.18}$$

Off-diagonal ETH

The off-diagonal part provides a sufficient condition for equilibration (2.2.4) as

$$^{\forall}\mathcal{O}_i \in \mathscr{O}, \quad \max_{\substack{|E_\alpha\rangle, |E_\beta\rangle \in \mathcal{H}_\mathcal{M} \\ \alpha \neq \beta}} \frac{\left| \langle E_\alpha|\mathcal{O}_i|E_\beta\rangle \right|}{\Delta_{\mathcal{O}_i}} \to 0 \qquad \text{as } \mathrm{Vol}(\mathcal{M}) \to \infty. \tag{2.2.19}$$

The diagonal-ETH can be phrased as "all energy eigenstates are thermal" in light of the definition of thermal equilibrium (2.1.18). (See Fig. 2.2.) Unless otherwise noted, the term "ETH" will be used to refer solely to the diagonal ETH throughout this thesis. Compared to the weak ETH, which we will mention later in this section, the diagonal ETH is called the *strong ETH* as well.

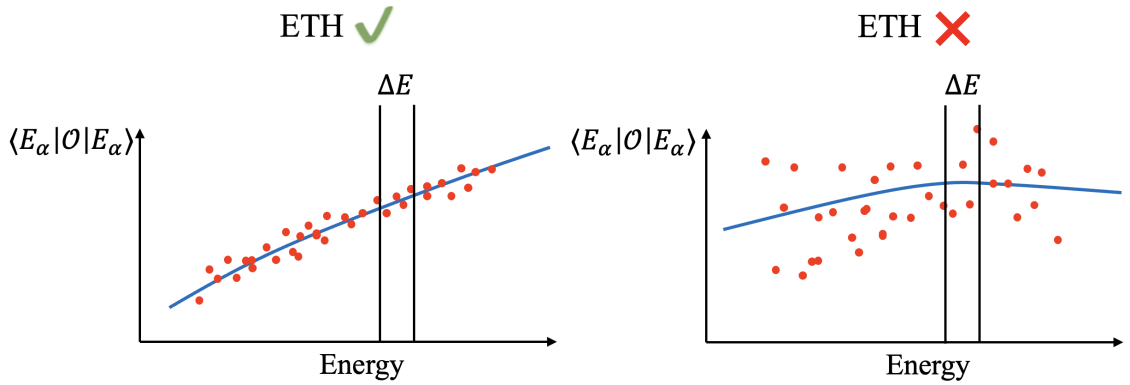

Figure 2.2: Schematic illustrations of the diagonal ETH. The red points denote the plots of energy eigenstates, and the blue line is the microcanonical average. In the left figure, the variance of the expectation values are small and tend to zero in the thermodynamic limit, which indicates the diagonal ETH hold for $\mathcal{O}$. In the right figure, the diagonal ETH is not satisfied since the variance remains finite after taking the thermodynamic limit.

Let us confirm the diagonal ETH (2.2.18) and the off-diagonal ETH (2.2.19) indeed suffice to thermalization (2.2.6) and equilibration (2.2.4) for generic systems. Noting the normalization

[6] Although we here take eigenvectors $|E_\alpha\rangle$ from the entire Hilbert space $\mathcal{H}_\mathcal{M}$, the prime interest in the context of thermalization is states in the middle of spectrum. It is thus sufficient to formulate the ETH (2.2.18) for eigenstates $|E_\alpha\rangle$ with sufficiently excited states such that $E_\alpha \in [E_1, E_2]$ for some E_1, E_2. This argument is applied to the off-diagonal ETH (2.2.19).

condition $\sum_\alpha |c_\alpha|^2$, we now evaluate the left hand side of (2.2.6).

$$\begin{aligned}
&\frac{\left|\overline{\langle\psi(t)|\mathcal{O}_i|\psi(t)\rangle} - \langle\mathcal{O}_i\rangle_{\mathrm{mc}}(E)\right|}{\Delta_{\mathcal{O}_i}} \\
&= \frac{\left|\sum_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M},E}} |c_\alpha|^2 \langle E_\alpha|\mathcal{O}|E_\alpha\rangle - \langle\mathcal{O}_i\rangle_{\mathrm{mc}}(E)\right|}{\Delta_{\mathcal{O}_i}} \\
&= \frac{\left|\sum_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M},E}} |c_\alpha|^2 \big(\langle E_\alpha|\mathcal{O}|E_\alpha\rangle - \langle\mathcal{O}_i\rangle_{\mathrm{mc}}(E_\alpha) + \langle\mathcal{O}_i\rangle_{\mathrm{mc}}(E_\alpha)\big) - \langle\mathcal{O}_i\rangle_{\mathrm{mc}}(E)\right|}{\Delta_{\mathcal{O}_i}} \\
&\le \frac{\sum_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M},E}} |c_\alpha|^2 \left|\langle E_\alpha|\mathcal{O}|E_\alpha\rangle - \langle\mathcal{O}_i\rangle_{\mathrm{mc}}(E_\alpha)\right|}{\Delta_{\mathcal{O}_i}} + \frac{\sum_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M},E}} |c_\alpha|^2 \left|\langle\mathcal{O}_i\rangle_{\mathrm{mc}}(E_\alpha) - \langle\mathcal{O}_i\rangle_{\mathrm{mc}}(E)\right|}{\Delta_{\mathcal{O}_i}} \\
&\le \frac{\max_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M},E}} \left|\langle E_\alpha|\mathcal{O}|E_\alpha\rangle - \langle\mathcal{O}_i\rangle_{\mathrm{mc}}(E_\alpha)\right|}{\Delta_{\mathcal{O}_i}} + \frac{\max_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M},E}} \left|\langle\mathcal{O}_i\rangle_{\mathrm{mc}}(E_\alpha) - \langle\mathcal{O}_i\rangle_{\mathrm{mc}}(E)\right|}{\Delta_{\mathcal{O}_i}}.
\end{aligned} \tag{2.2.20}$$

The second term in the last line should vanish as $\mathrm{Vol}(\mathcal{M}) \to \infty$ if we assume that $\langle\mathcal{O}_i\rangle_{\mathrm{mc}}(E)$ is a continuous function of $E/\mathrm{Vol}(\mathcal{M})$ and the energy shell scales as $o(\mathrm{Vol}(\mathcal{M}))$. This supposition seems reasonable for generic systems, and it is indeed encapsulated into Srednicki's ansatz [150] to be discussed later in this section. The first term of (2.2.20) converges to zero as well because of the diagonal ETH (2.2.18). Since the discussion here holds for arbitrary $\mathcal{O}_i \in \mathscr{O}$, we can conclude that the initial state $|\psi(0)\rangle$ is thermalized in the sense of (2.2.6). We then check the equilibration argument as

$$\begin{aligned}
\frac{\sigma^2_{\mathrm{time}}(\mathcal{O}_i)}{\Delta^2_{\mathcal{O}_i}} &= \sum_{\substack{\alpha\neq\beta \\ |E_\alpha\rangle,|E_\beta\rangle\in\mathcal{H}_{\mathcal{M},E}}} |c_\alpha|^2|c_\beta|^2 \frac{|\langle E_\alpha|\mathcal{O}_i|E_\beta\rangle|^2}{\Delta^2_{\mathcal{O}_i}} \\
&\le \left(\max_{\substack{\alpha\neq\beta \\ |E_\alpha\rangle,|E_\beta\rangle\in\mathcal{H}_{\mathcal{M},E}}} \frac{|\langle E_\alpha|\mathcal{O}_i|E_\beta\rangle|^2}{\Delta^2_{\mathcal{O}_i}}\right) \sum_{\substack{\alpha\neq\beta \\ |E_\alpha\rangle,|E_\beta\rangle\in\mathcal{H}_{\mathcal{M},E}}} |c_\alpha|^2|c_\beta|^2 \\
&\le \max_{\substack{\alpha\neq\beta \\ |E_\alpha\rangle,|E_\beta\rangle\in\mathcal{H}_{\mathcal{M},E}}} \frac{|\langle E_\alpha|\mathcal{O}_i|E_\beta\rangle|^2}{\Delta^2_{\mathcal{O}_i}}
\end{aligned} \tag{2.2.21}$$

Since the left hand side tends to zero as $\mathrm{Vol}(\mathcal{M}) \to \infty$, the left hand side also vanishes in the thermodynamic limit for arbitrary $\mathcal{O}_i \in \mathscr{O}$. The off-diagonal ETH (2.2.19) leads to the vanishing right hand side, and we can see the initial state $|\psi\rangle$ is equilibrated as desired.

We note that equilibration can be achieved without imposing the off-diagonal ETH if only the initial state is "random" enough, which is quantified as

$$D_{\mathrm{eff}} := \Big(\sum_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M},E}} |c_\alpha|^4\Big)^{-1} \to \infty \qquad \text{as } \mathrm{Vol}(\mathcal{M}) \to \infty. \tag{2.2.22}$$

The effective dimension D_{eff} is regarded as a measure of delocalization, which equals to 1 if $c_\alpha = \delta_{\alpha\tilde{\beta}}$ with some $\tilde{\beta}$, or $D_{\text{eff}} = e^{O(\text{Vol}(\mathcal{M}))}$ for uniformly random c_α. If we suppose (2.2.22), the variance reads

$$\begin{aligned}
\sigma^2_{\text{time}}(\mathcal{O}_i) &= \sum_{\substack{\alpha\neq\beta \\ |E_\alpha\rangle,|E_\beta\rangle\in\mathcal{H}_{\mathcal{M},E}}} |c_\alpha|^2|c_\beta|^2|\left\langle E_\alpha|\mathcal{O}_i|E_\beta\right\rangle|^2 \\
&\leq \frac{1}{2}\sum_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M},E}} |c_\alpha|^4 \left\langle E_\alpha|\mathcal{O}_i^2|E_\alpha\right\rangle + \frac{1}{2}\sum_{|E_\beta\rangle\in\mathcal{H}_{\mathcal{M},E}} |c_\beta|^4 \left\langle E_\beta|\mathcal{O}_i^2|E_\beta\right\rangle \\
&= \sum_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M},E}} |c_\alpha|^4 \left\langle E_\alpha|\mathcal{O}_i^2|E_\alpha\right\rangle \\
&\leq \sum_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M},E}} |c_\alpha|^4 \sup_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M},E}} \left\langle E_\alpha|\mathcal{O}_i^2|E_\alpha\right\rangle \\
&\leq \sum_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M},E}} |c_\alpha|^4 ||\mathcal{O}_i||^2 \qquad \left(\because \sup_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M},E}} \left\langle E_\alpha|\mathcal{O}_i^2|E_\alpha\right\rangle \leq \sup_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M}}} \left\langle E_\alpha|\mathcal{O}_i^2|E_\alpha\right\rangle = ||\mathcal{O}_i||^2\right) \\
&= \frac{||\mathcal{O}_i||^2}{D^2_{\text{eff}}} \leq \frac{\Delta^2_{\mathcal{O}_i}}{D^2_{\text{eff}}}\,.
\end{aligned} \tag{2.2.23}$$

In the second line, we utilized the inequality about the arithmetic and geometric mean

$$|c_\alpha|^2|c_\beta|^2 \leq \frac{1}{2}\left(|c_\alpha|^4 + |c_\beta|^4\right). \tag{2.2.24}$$

The inequality in the last line of (2.2.23) holds iff the maximum and minimum value of the eigenvalues of $\mathcal{O}_i$ are nonnegative and nonpositve, respectively. Without loss of generality, we can always shift the operator $\mathcal{O}_i$ by a constant so that this condition is satisfied. Comparing the relation (2.2.23) with the delocalization condition (2.2.22), we have $\sigma^2_{\text{time}}(\mathcal{O}_i) \to 0$ in the thermodynamic limit $\text{Vol}(\mathcal{M}) \to \infty$.

In the discussion so far, we focused on the (strong) ETH. We next sort out the relevant terminologies associated with the ETH. In formulating the ETH (2.2.18), we dealt with all of the energy eigenstates in the energy shell $\mathcal{H}_{\mathcal{M},E}$, although it is too strict to prove in general systems. A more tractable framework called the *weak ETH* was proposed in [7]. The weak ETH is expressed as

$$P_{|E_\alpha\rangle\in\mathcal{H}_{\mathcal{M},E}}\left[\max_{\mathcal{O}_i\in\bar{\mathscr{O}}} \frac{\left|\left\langle E_\alpha|\mathcal{O}_i|E_\alpha\right\rangle - \langle\mathcal{O}_i\rangle_{\text{mc}}\right|}{\Delta_{\mathcal{O}_i}} \leq \epsilon\right] \to 0 \qquad \text{as } \text{Vol}(\mathcal{M}) \to \infty \tag{2.2.25}$$

for sufficently small $\epsilon > 0$, i.e., $\epsilon = o(\text{Vol}(\mathcal{M}))$. It can be paraphrased as the total number of "rare" states are much smaller than the dimension of the energy shell. In contrast to the strong ETH, the weak ETH does not indicate thermalization for all of initial states in the energy shell. In particular,

the weak ETH has been demonstrated even for integrable systems, where thermalization is often absent. For systems where only the weak ETH is satisfied, we can deduce thermalization for initial states without substantial overlaps with rare states. Therefore, in such systems, we have to tackle with a highly nontrivial problem: determining whether physically a realistic initial state, such as one prepared by a quantum quench, has a substantial overlap with rare states or not.

Srednicki's ansatz

Along the line of the postulate such that a generic many-body Hamiltonian behaves similar to a random matrix, Srenicki [150] conjectured that matrix elements in the energy basis take the form

$$\langle E_\alpha|\mathcal{O}|E_\beta\rangle = \mathcal{O}(\bar{E})\delta_{\alpha\beta} + e^{-S_{\rm mc}(\bar{E})/2} f_{\mathcal{O}}(\bar{E},\omega) R_{\alpha\beta}, \tag{2.2.26}$$

where $\mathcal{O}(\bar{E})$ and $f_{\mathcal{O}}(\bar{E},\omega)$ are smooth functions of $\bar{E} := (E_\alpha + E_\beta)/2$ and $\omega := E_\alpha - E_\beta$. Complex numerical factors $R_{\alpha\beta}$ are treated as independent random variables with zero means and unit variances. The microcanonical entropy is denoted by $S_{\rm mc}(\bar{E}) := \log D_{\mathcal{M},E}$, and it scales as $S_{\rm mc}(\bar{E}) = O(\mathrm{Vol}(\mathcal{M}))$.

One can notice that Srednicki's ansatz (2.2.26) is a stronger assumption than the diagonal ETH (2.2.18) and the off-diagonal ETH (2.2.19). Since off-diagonal elements only appear in the second term of (2.2.26), they are exponentially dumping as $e^{-O(\mathrm{Vol}(\mathcal{M}))}$ in the thermodynamic limit, which suffices to the off-diagonal ETH. In this limit, the remaining matrix elements are the first term of (2.2.26), and thus we obtain

$$\mathcal{O}(\bar{E}) \simeq \langle \mathcal{O} \rangle_{\rm mc}(\bar{E}) \tag{2.2.27}$$

with a sufficiently narrow energy shell since $\mathcal{O}(\bar{E})$ is assumed to be a smooth function.

The small off-diagonal part of the matrix elements immediately indicates small fluctuations with respect to time, i.e., $\sigma_{\rm time}(\mathcal{O}) = e^{-O(\mathrm{Vol}(\mathcal{O}))} = O(D_{\mathcal{M},E}^{-1/2})$ since we have the relation (2.2.17). In addition, Srednicki's ansatz allows us to evaluate the accuracy of the microcanonical ensemble for finite systems [151].

Chapter 3

Violation of the ETH in QFTs with higher-form symmetry

In this chapter, we discuss how the existence of higher-form symmetry affects thermalization based on our work [139]. We analytically show that nontrivial observables break the ETH when higher-form symmetries are present under certain assumptions (Fig. 3.1). In the case of discrete symmetry groups, the breakdown of the ETH is caused by non-local conserved quantities. For a p-form symmetry, such ETH-breaking operators become $(d-p)$-dimensional, which are non-local but have a much smaller size than the entire system for $p \geq 1$. We demonstrate this statement for the two-dimensional $\mathbb{Z}_2$ lattice gauge theory with $\mathbb{Z}_2$ 1-form symmetry[1]. Furthermore, while local observables relax to the canonical ensemble, the non-local operator exciting a magnetic dipole instead relaxes to the GGE that considers the higher-form symmetry. Our results indicate that symmetries cause nontrivial thermalization processes revealed by non-local observables, which go beyond conventional statistical mechanics.

3.1 Higher-form symmetry

We consider quantum field theories in a $(d+1)$-dimensional spacetime, $\mathcal{M} \times \mathbb{R}$, where $\mathcal{M}$ is a connected d-dimensional space manifold. Let G be an abelian group, and the system is supposed to have a G p-form symmetry, i.e., there exists a $(d-p)$-dimensional topological operator $U_\alpha(\mathcal{C})$ ($\alpha \in G$), where $\mathcal{C} \subset \mathcal{M} \times \mathbb{R}$ denotes a $(d-p)$-dimensional closed surface. Here, an operator $U_\alpha(\mathcal{C})$

[1]Note that several studies [46, 152–155] investigated thermalization and its breakdown for gauge theories. For example, Refs. [156–158] considered disorder-free localization with mixed super-selection sectors defined from the gauge symmetries. However, our scope is distinct since we focus on the single physical sector satisfying the Gauss law [159].

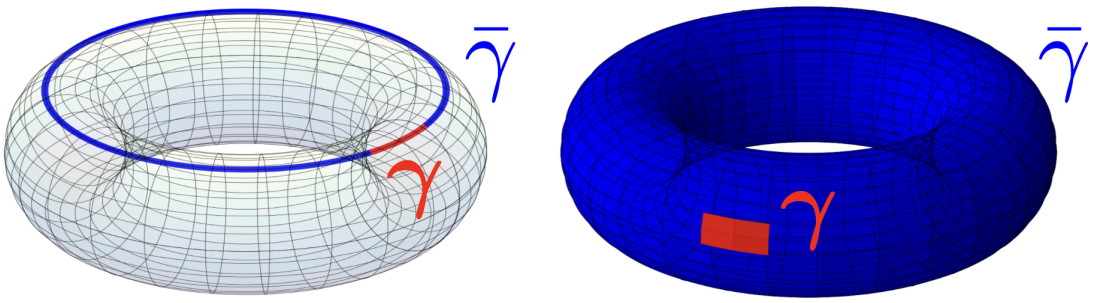

Figure 3.1: Schematics of p-form symmetry and its influence on the eigenstate thermalization hypothesis (ETH) [139]. The figure is for $\mathcal{M} = T^2$ ($d = 2$). The blue and red regions respectively denote the areas where $U(\bar{\gamma})$ and $U(\gamma)$ nontrivially act, and the symmetry operator is given by $U(\tilde{C}) = U(\bar{\gamma})U(\gamma)$. If $U(\gamma)$ satisfies the ETH with a non-vanishing thermal average, the ETH for $U(\bar{\gamma})$ is broken. (Left) The case with higher-form symmetry. The ETH-breaking operator $U(\bar{\gamma})$ is $(d-p)$-dimensional ($p = 1$ is shown) and has a support much smaller than the size of the "bath" $\mathcal{M}\backslash\bar{\gamma}$. (Right) The case with the conventional symmetry ($p = 0$), where the support of $U(\bar{\gamma})$ is comparable with the size of $\mathcal{M}$.

is said to be topological if and only if the transition amplitude satisfies

$$\langle f|\mathcal{T}(U_\alpha(\mathcal{C})e^{-i\int_{t_i}^{t_f}Hdt})|i\rangle = \langle f|\mathcal{T}(U_\alpha(\mathcal{C}')e^{-i\int_{t_i}^{t_f}Hdt})|i\rangle \tag{3.1.1}$$

for arbitrary initial and final states $|i\rangle$ and $|f\rangle$, and homotopically equivalent closed surfaces $\mathcal{C}' \subset \mathcal{M}\times\mathbb{R}$, where $\mathcal{T}$ denotes the time-ordering. For p-dimensional closed manifold $\check{\mathcal{C}} \subset \mathcal{M}\times\mathbb{R}$, a charged operator under the p-form symmetry then satisfies

$$\langle f|\mathcal{T}(W(\check{\mathcal{C}})U_\alpha(\mathcal{C})e^{-i\int_{t_i}^{t_f}Hdt}\dots)|i\rangle = e^{iq_\alpha \,\mathrm{link}(\check{\mathcal{C}},\mathcal{C})}\langle f|\mathcal{T}(W(\check{\mathcal{C}})e^{-i\int_{t_i}^{t_f}Hdt}\dots)|i\rangle, \tag{3.1.2}$$

where $q_\alpha \in \mathbb{R}$ is the charge of the operator $W(\check{C})$, and $\mathrm{link}(\check{\mathcal{C}},\mathcal{C})$ denotes the linking number of the closed surfaces $\check{\mathcal{C}}$ and $\mathcal{C}$. (See Fig. 3.2.)

We specifically consider the symmetry operator $U_\alpha(C)$ lying in the space for each fixed time, i.e., $C \subset \mathcal{M}$. Then, the topological property of $U_\alpha(C)$ leads to $[U_\alpha(C), e^{-iH\delta t}] = 0$ for an infinitesimal time slice δt, and thus $[H, U_\alpha(C)] = 0$. Importantly, $U_\alpha(C)$ has a $(d-p)$-dimensional support C, which is non-local but much smaller than the entire d-dimensional system for $p \geq 1$. This contrasts with conventional 0-form symmetries, whose symmetry operator is d-dimensional, i.e., its support is comparable to the entire system (Fig. 3.1). While the following discussion holds both for continuous and discrete symmetries, we especially focus on discrete symmetry, which typically entails non-local conserved quantities alone.

Under this setting, the Hamiltonian is block-diagonalized by $U_\alpha(C)$. Then, the ETH for $U_\alpha(C)$ trivially breaks down for the entire Hilbert space within the energy shell, where symmetry sectors are

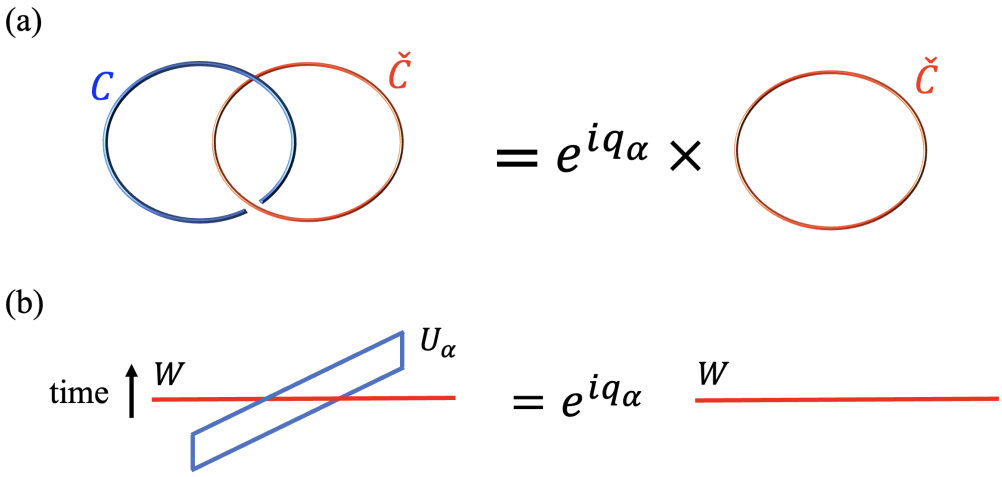

Figure 3.2: The properties of the topological operator. The blue and red line denote the support of the operators $U_\alpha(\mathcal{C})$ and $W(\check{\mathcal{C}})$, respectively. (a) When we unlink the topological operator from the charged operator, the correlation function is multiplied by e^{iq_α} with the charge q_α of $W(\check{\mathcal{C}})$. (b) The charged operator and symmetry operator lying to the spacial direction. In this configuration, the relation (3.1.2) just reduces to $U^\dagger W U = e^{iq_\alpha} W$, which is referred to as a space-like symmetry [160].

mixed. However, this does not necessarily indicate the breakdown of the ETH for other nontrivial observables since $U_\alpha(C)$ does not necessarily provide a local conserved quantity. Indeed, there are several evidences [58, 74, 75] that the ETH holds for local observables despite the existence of the discrete symmetry, especially the 0-form symmetry. In that case, eigenstate expectation values of those observables can be the same even for different symmetry sectors. For example, the transverse-field Ising model $H_{\mathrm{TFIM}} = \sum_{\mathbf{R},\mathbf{R}'\in\mathcal{M}} J_{\mathbf{R},\mathbf{R}'}\sigma^3(\mathbf{R})\sigma^3(\mathbf{R}') + \sum_{\mathbf{R}\in\mathcal{M}} g_{\mathbf{R}}\sigma^1(\mathbf{R})$ has a $\mathbb{Z}_2$ 0-form symmetry $U(C=\mathcal{M}) = \prod_{\mathbf{R}\in\mathcal{M}}\sigma^1(\mathbf{R})$, where $\sigma^{1,2,3}(\mathbf{R})$ denote the Pauli matrices acting on the vertices $\mathbf{R}$. While we have two symmetry sectors with $U = \pm 1$, they will not lead to distinct eigenstate expectation values for typical local observables, say $\sigma^1(\mathbf{R})$, in the thermodynamic limit.

3.2 Breakdown of the ETH for nontrivial operators

We now state our main result: higher-form symmetry of a non-degenerate Hamiltonian leads to the breakdown of the ETH even for many *nontrivial* $(d-p)$-dimensional operators. For this purpose, we require the following reasonable assumptions: i) the operator $U_\alpha(\tilde{C})$ can be decomposed as $U_\alpha(\tilde{C}) = U_\alpha(\gamma)U_\alpha(\bar{\gamma})$, where we have introduced a $(d-p)$-dimensional submanifold $\gamma\,(\subset\tilde{C})$ and its complement $\bar{\gamma} := \tilde{C}\backslash\gamma$, both of which have boundaries. ii) For at least one nontrivial closed surface, say $\tilde{C}\,(\subset\mathcal{M})$, the energy shell contains eigenstates in different symmetry sectors defined by $U_\alpha(\tilde{C})$.

iii) The microcanonical average $\langle U_\alpha(\gamma)\rangle_{\rm mc}^{\Delta E}(E)$ defined from the energy shell $[E, E+\Delta E]$ takes a nonzero value in the thermodynamic limit.

Under the above assumptions, we show that either $U_\alpha(\gamma)$ or $U_\alpha(\bar{\gamma})$ necessarily breaks the ETH[2] within the energy shell $[E, E+\Delta E]$ (see Appendix A for a proof). Our result indicates that, while the discrete symmetry and topology may not affect the thermalization of conventional local observables, their effect significantly emerges in the dynamics of $(d-p)$-dimensional non-local objects. We stress that, while such non-local observables go beyond conventional statistical mechanics, they have actively been studied since they play an essential physical role in gauge theory [110,161]. Furthermore, non-local operators have become accessible in state-of-the-art experiments using artificial quantum systems [162–164].

Let us point out the notable aspect of our results for higher-form symmetry with $p \geq 1$, although our results provide a hitherto unknown consequence even for the conventional symmetry $p = 0$. For $p = 0$, the ETH-breaking operators (say, $U_\alpha(\bar{\gamma})$) are d-dimensional, and the volume of their support, $V_{\bar{\gamma}}$, is comparable with the volume of the "bath" $V_{\mathcal{M}\setminus\bar{\gamma}}$ for large system-size limit, i.e., $V_{\bar{\gamma}}/V_{\mathcal{M}\setminus\bar{\gamma}} \to$ finite (see Fig. 3.1, right). For the example of $H_{\rm TFIM}$, $\prod_{\mathbf{R}\in\mathcal{M}\setminus\gamma}\sigma^1(\mathbf{R})$ breaks the ETH if $\prod_{\mathbf{R}\in\gamma}\sigma^1(\mathbf{R})$ satisfies it. Thus, the breakdown of the ETH might also be attributed to the smallness of the bath. In contrast, for higher-form symmetry with $p \geq 1$, we have $V_{\bar{\gamma}}/V_{\mathcal{M}\setminus\bar{\gamma}} \to 0$ in the thermodynamic limit (Fig. 3.1, left). Thus, the higher-form symmetry hinders thermalization even when the bath is regarded as much larger than the support of the observable of our interest.

Our main claim is generalized to an operator $U(\bar{\gamma})A(g)^\dagger$, where $A(g)$ is an operator defined on an arbitrary region $g\,(\subset\mathcal{M})$ satisfying $g\cap\bar{\gamma}=\phi$. That is, $A(g)U(\gamma)$ or $A(g)^\dagger U(\bar{\gamma})$ violates the ETH if we impose an assumption iii)' $\langle A(g)U(\gamma)\rangle_{\rm mc}^{\Delta E}(E)\neq 0$ instead of iii). This generalization indicates that for a fixed γ, we have many ETH-violating operators corresponding to the choice of g and $A(g)$. Note that, while $g\subseteq\gamma$ for 0-form symmetries, g may not be included in (or have even larger dimension than) γ for higher-form symmetries.

Finally, the symmetry can, in turn, *ensure* the ETH[3] for certain operators. Indeed, the so-called charged operators W, for which $U_\alpha(C)WU_\alpha^{-1}(C)=e^{i\alpha w}W$ holds with some charge w, satisfy

[2]Note that $U_\alpha(\gamma)$ and $U_\alpha(\bar{\gamma})$ are in general non-Hermitian. However, the breakdown of the ETH for these operators leads to that of certain Hermitian operators as well. Indeed, if $U_\alpha(\bar{\gamma})$ breaks the ETH, we can show that either of the Hermitian operators $U_\alpha(\bar{\gamma})+U_\alpha^\dagger(\bar{\gamma})$ or $i(U_\alpha(\bar{\gamma})-U_\alpha^\dagger(\bar{\gamma}))$ breaks the ETH.

[3]In this thesis, we only consider the diagonal ETH and do not discuss the off-diagonal ETH [165], which states that the off-diagonal matrix elements with respect to the energy eigenstates vanish in the thermodynamics limit.

$\langle E_n|W|E_n\rangle = 0$ for all n when $e^{i\alpha w} \neq 1$. Then, W satisfies the ETH, and the long-time average of $\langle W(t)\rangle$ becomes zero. For H_{TFIM} with 0-form symmetry, $W = \sigma^3(\mathbf{R})$ satisfies this condition. For the higher-form symmetry, the Wilson line in the lattice gauge theory discussed later satisfies this condition.

3.3 $\mathbb{Z}_2$ lattice gauge theory

We demonstrate the general discussion above using the (2+1)-dimensional $\mathbb{Z}_2$ lattice gauge theory on a square lattice forming a 2-torus $\mathcal{M} = T^2$. The Hamiltonian $H_{\mathbb{Z}_2}$ is given by [159, 166, 167]

$$H_{\mathbb{Z}_2} = -\sum_{\boldsymbol{r}} J_{\boldsymbol{r},xy}\sigma^3_{\boldsymbol{r},x}\sigma^3_{\boldsymbol{r}+\boldsymbol{e}_x,y}\sigma^3_{\boldsymbol{r}+\boldsymbol{e}_y,x}\sigma^3_{\boldsymbol{r},y} - \sum_{\boldsymbol{r},j}\sigma^1_{\boldsymbol{r},j}, \tag{3.3.1}$$

where $\sigma^{1,2,3}_{\boldsymbol{r},j}$ denote the Pauli matrices acting on the link $(\boldsymbol{r}, j)$, which is specified by the coordinate of vertices $\boldsymbol{r}$ and the direction $j = x, y$ (For the relation with the path integral formulation, see Appendix D. This system has a $\mathbb{Z}_2$ 1-form symmetry, and the spatial symmetry operators are characterized by $H_1(T^2, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ [168]. Indeed, the system has two independent symmetry operators corresponding to the x-cycle and y-cycle.

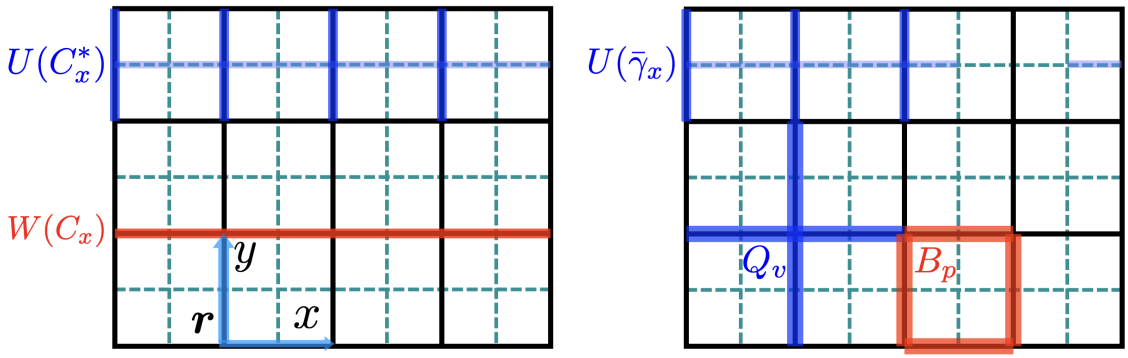

Figure 3.3: Schematic diagram of our lattice model ($N_x = 4$ and $N_y = 3$ are shown) under the periodic boundary condition [140]. The solid and green lines denote the lattice and the dual lattice, respectively. We illustrate examples of the operators $Q_v, U(C_x^*), W(C_x), U(\bar{\gamma}_x)$, and B_p, where the Pauli matrices σ^3 and σ^1 respectively act on the red and blue links.

To remove the residual gauge redundancies after the temporal gauge-fixing [159], we project the entire Hilbert space onto the physical one. Here, spatial gauge transformation is generated by the local operator

$$Q_v := \prod_{b:\text{ spatial link}, b\ni v} \sigma^1_b, \tag{3.3.2}$$

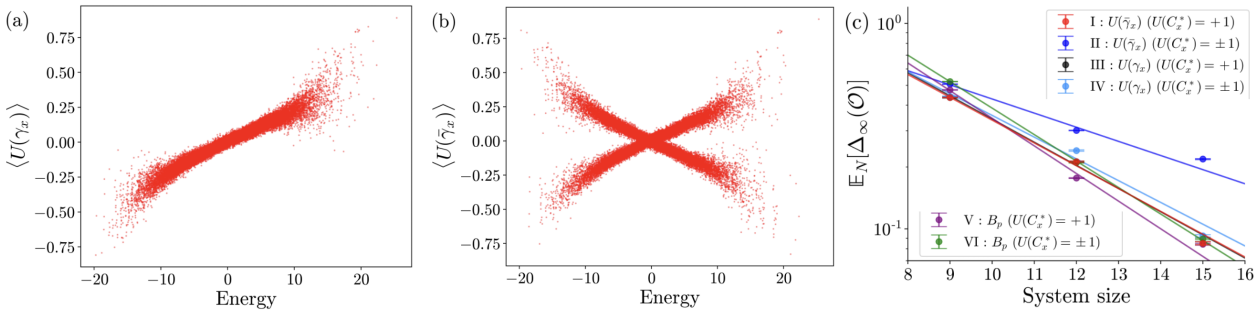

Figure 3.4: Expectation values for energy eigenstates in the 5×3 lattice [139]. (a) The local observable $U(\gamma_x)$ satisfies the ETH. (b) The observable with a one-dimensional support $U(\bar{\gamma}_x)$ violates the ETH. The expectation values are separated into two sectors classified by the value of 1-form symmetry, i.e., $U(C_x^*) = \pm 1$. (c) System-size dependence of the ETH measure Δ_∞. The decay for $U(\bar{\gamma}_x)$ with the total symmetry sectors (Case II) is much slower than the other cases, which indicates that the ETH is hindered. The fitting parameters are shown in Appendix B.

where v denotes the vertex. The operator Q_v satisfies $Q_v^2 = 1$, $[H_{\mathbb{Z}_2}, Q_v] = 0$. Then, the physical Hilbert space is given by [159] span $\left\{|\psi\rangle \;\middle|\; Q_v|\psi\rangle = +|\psi\rangle,\ {}^\forall v :\ \text{vertices}\right\}$, where the constraint can be regarded as the $\mathbb{Z}_2$ analog of the Gauss law. After this projection, the expectation value of a non-gauge invariant operator for physical states $|\psi\rangle$ always vanishes.

We next define the 't Hooft and Wilson operators on the spatial directions as [169–171]

$$U(C^*) := \prod_{b^* \in C^*} \sigma_{b^*}^1 = U^{-1}(C^*), \tag{3.3.3}$$

$$W(C) := \prod_{b \in C} \sigma_b^3 = W^{-1}(C). \tag{3.3.4}$$

Here, C and C^* are closed loops on the lattice and dual lattice, respectively (see Fig. 3.3). Both $U(C^*)$ and $W(C)$ commute with the operator Q_v and thus are gauge invariant.

The 't Hooft operator $U(C^*)$ serves as the $\mathbb{Z}_2$ 1-form symmetry operator of this model, satisfying $[H_{\mathbb{Z}_2}, U(C^*)] = 0$. This operator is topological since it satisfies $U(C_1^*)|\psi\rangle = U(C_2^*)|\psi\rangle$ if C_1^* and C_2^* are homotopically equivalent. It follows that $U(C^*)|\psi\rangle = |\psi\rangle$ if the dual closed loop C^* is topologically trivial, i.e., it can be continuously deformed to a point.

The 't Hooft operator $U(C^*)$ measures the "electric" charge of the Wilson operator. We define closed loops on the lattice winding around the x-/y-cycle by C_x and C_y (and similarly the loops on the dual lattice by C_x^* and C_y^*). Then, the operators W and U satisfy $U(C_i^*)W(C_j)U^{-1}(C_i^*) = (-1)^{\delta_{ij}+1}W(C_j)$, which is operator-realization of the electric $\mathbb{Z}_2$ 1-form symmetry [118].

3.3.1 ETH breaking by $\mathbb{Z}_2$ 1-form symmetry

Let us demonstrate the violation of the ETH for the $\mathbb{Z}_2$ lattice gauge theory. We take the coupling constants $J_{\mathbf{r},xy}$ in (3.3.1) to be weakly random (i.e., $J_{\boldsymbol{r},xy}$ is uniformly chosen from $[0.7, 0.8]$) to avoid unwanted degeneracy and integrability. We consider the $N_x \times N_y$ 2-torus and define x/y-cycles for the lattice and dual lattice as $C_{x/y}$ and $C^*_{x/y}$, respectively (Fig. 3.3).

We calculate the eigenstate expectation values of local operators $U(\gamma_x)$, B_p, and nonlocal one-dimensional observable $U(\bar{\gamma}_x)$. Here, γ_x is just one link included in C^*_x, and $\bar{\gamma}_x := C^*_x \backslash \gamma_x$. The operators $U(\gamma_x)$ and $U(\bar{\gamma}_x)$ represent a magnetic dipole excitation residing at the endpoints of γ_x [159]. The plaquette operator B_p is defined by

$$B_p := \prod_{b \in \text{plaquette } p} \sigma^3_b. \tag{3.3.5}$$

Figure 3.4(a) shows that the local observable $U(\gamma_x)$ satisfies the ETH. In contrast, Fig. 3.4(b) demonstrates that the non-local one-dimensional observable $U(\bar{\gamma}_x)$ has two branches of the eigenstate-expectation values, indicating the breakdown of the ETH owing to the general mechanism explained above. The ETH is recovered when we consider eigenstates within the symmetry sector for $U(C^*_x) = 1$ or -1, even without separating the sector for $U(C^*_y)$. The result for B_p is given in Appendix B.

To test the ETH more quantitatively, we perform the finite-size scaling analysis. We define the deviation measure for an observable $\mathcal{O}$ [25] by

$$\Delta_\infty(\mathcal{O}) := \max_{n, E_n \in [E, E+\delta E]} \left| \langle E_n | \mathcal{O} | E_n \rangle - \langle \mathcal{O} \rangle^{\delta E}_{\text{mc}}(E_n) \right|, \tag{3.3.6}$$

where $E_n = \langle E_n | H_{\mathbb{Z}_2} | E_n \rangle$ is an energy eigenvalue. The strong ETH corresponds to $\Delta_\infty \to 0$ in the thermodynamic limit. Furthermore, Δ_∞ is expected to decay exponentially $\sim e^{-s(E)N/2}$ for a fully chaotic system, where $s(E)$ is the entropy density at energy E [25, 54]. Figure 3(c) shows the system-size dependence of the disorder-averaged measure $\mathbb{E}[\Delta_\infty(\mathcal{O})]$, which is fitted with a function e^{-aN+b}. First, the local observables B_p and $U(\gamma_x)$ (irrespective of whether we resolve the symmetry sector) and the non-local observable $U(\bar{\gamma}_x)$ after resolving the symmetry sector exhibit sufficiently fast exponential decay with a relatively similar rate. This indicates the ETH for these observables. In contrast, $U(\bar{\gamma}_x)$ for the total symmetry sector decays much slower than the other cases, though it keeps decreasing due to the finite-size effect. Combining the general argument and the ETH for $U(\gamma_x)$, we conclude that the ETH for $U(\bar{\gamma}_x)$ breaks down due to the higher-form symmetry.

Note that the Wilson line always satisfies $\langle E_n|W(C_{x/y})|E_n\rangle = 0$ for all n, because of the general discussion for the charged operator discussed previously. Consequently, the long-time average of the Wilson operator always vanishes.

3.3.2 GGE with $\mathbb{Z}_2$ 1-form symmetry

We next argue that the stationary value of an observable that is non-local in the x-direction but local in the y-direction (e.g., $U(\bar{\gamma}_x)$) is described by the GGE that takes account of the $\mathbb{Z}_2$ 1-form symmetry. That is, we have $\langle \mathcal{O}\rangle_{\rm GGE} = {\rm Tr}[\mathcal{O}\,\rho_{\rm GGE}(\beta, \lambda_x, \mu_x)]$ with

$$\rho_{\rm GGE} = \frac{1}{Z_{\rm GGE}} e^{-\beta H_{\mathbb{Z}_2} - \lambda_x U(C_x^*) - \mu_x U(C_x^*) H_{\mathbb{Z}_2}}, \tag{3.3.7}$$

where "chemical potentials" λ_x and μ_x are uniquely determined from the initial values of the conserved quantities $H_{\mathbb{Z}_2}, U(C_x^*)$, and $U(C_x^*)H_{\mathbb{Z}_2}$, and $Z_{\rm GGE}$ is the normalization constant[4]. Our GGE is justified as the stationary state if we assume the restricted ETH for each $U(C_x^*)$-symmetry sector for the observable $\mathcal{O}$ (See Appendix C).

Figure 3.5 shows time evolutions of $\mathcal{O} = U(\gamma_x)$ and $U(\bar{\gamma}_x)$. For $U(\gamma_x)$, the stationary value is well described by the canonical ensemble $\rho_{\rm can} = Z_{\rm can}^{-1} e^{-\beta_{\rm can} H_{\mathbb{Z}_2}}$, where ${\rm Tr}[U(\gamma_x)\rho(t)] \simeq {\rm Tr}[U(\gamma_x)\rho_{\rm can}]$ holds for most of the time. In contrast, the canonical ensemble fails for $U(\bar{\gamma}_x)$. Instead, ${\rm Tr}[U(\bar{\gamma}_x)\rho(t)] \simeq {\rm Tr}[U(\bar{\gamma}_x)\rho_{\rm GGE}]$ holds for most of the time. We stress that $\rho_{\rm GGE}$ works well even though we do not consider the effect of $U(C_y^*)$, probably because $U(\bar{\gamma}_x)$ is local in the y-direction.

Note that the GGE suitable for a general finite Abelian group G is obtained by assuming the restricted ETH for each symmetry sector as $\rho_{\rm GGE}^G = e^{-\beta^G H - \sum_j \lambda_j^G P_j - \sum_j \mu_j^G P_j H}$, where P_i are the projections to each symmetry sector. For symmetry sectors defined by $U(C_x^*)$ with $G = \mathbb{Z}_2$, this ensemble is indeed equivalent to (3.3.7) after redefinitions of the chemical potentials.

3.4 Summary of Chapter 3

We analytically show that the existence of a p-form symmetry leads to the ETH-violation of many $(d-p)$-dimensional observables in the form of $U_\alpha(\bar{\gamma})$ under certain assumptions. A significant feature

[4]To adopt operators extended to the y-direction as well, we should include at most seven chemical potentials in total as

$$\tilde{\rho}_{\rm GGE} = \frac{1}{\tilde{Z}_{\rm GGE}} \exp\left(-\beta H_{\mathbb{Z}_2} - \sum_{i=x,y} \lambda_i U(C_i^*) - \sum_{i=x,y} \mu_i U(C_i^*) H_{\mathbb{Z}_2} - \alpha U(C_x^*)U(C_y^*) - \alpha' U(C_x^*)U(C_y^*)H_{\mathbb{Z}_2}\right). \tag{3.3.8}$$

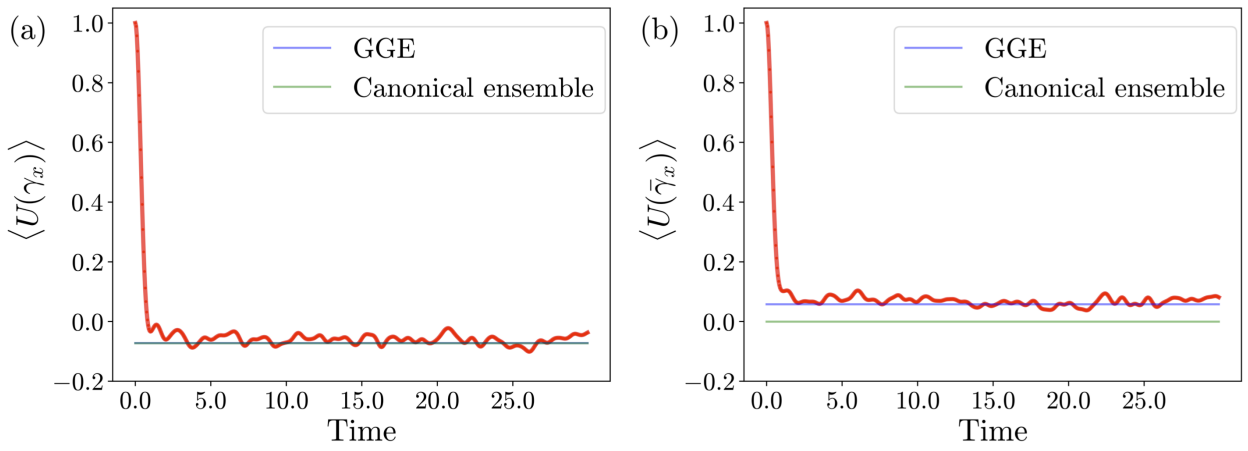

Figure 3.5: Time evolution of the expectation values of $U(\gamma_x)$ and $U(\bar{\gamma}_x)$ for the 4×3 lattice [139]. The blue and green lines indicate the prediction of the GGE in Eq. (3.3.7) and the standard canonical ensemble, respectively. For the local observable $U(\gamma_x)$ (left), the stationary state is described by the canonical ensemble, which is almost overlapping with the GGE result. In contrast, the stationary value of $U(\bar{\gamma}_x)$ (right) differs from the canonical ensemble and is described by the GGE. The initial states are random superpositions of eigenstates of $U(\gamma_x)$ (left) or $U(\bar{\gamma}_x)$ (right) with the eigenvalue $+1$, whose energy expectations lie within $[-5.0, -3.0]$.

of this statement is that the ETH-violating observable has a non-local but lower-dimensional support rather than the whole d-dimensional space manifold for $p \geq 1$. This implies that such objects can be described by the suitable GGE instead of the canonical ensemble. We use the $\mathbb{Z}_2$ lattice gauge theory to demonstrate the above statements. The discussion on the breakdown of the ETH can be applied to systems with p-form symmetries, e.g., various quantum field theories such as $SU(N)$ Yang-Mills theory with center symmetries.

Our results indicate that symmetries cause nontrivial thermalization dynamics for non-local observables, which go beyond conventional statistical mechanics. We stress that this ETH-violation stably holds even under local perturbations to the Hamiltonians because the higher-form symmetry is robust against them.

Chapter 4

Effects of projective phase on the ETH

In this chapter, we reconsider the ETH-violation from the viewpoint of projective phase and 't Hooft anomaly for higher-form symmetries based on our work [140]. As discussed in Chapter 3 a $(d+1)$-dimensional system with p-form symmetry is shown to accommodate many $(d-p)$-dimensional ETH-violating observables other than the symmetry operator itself under some reasonable assumptions. The assumptions consist of i) the endability of the symmetry operator, ii) mixture of symmetry sectors in a given energy shell, and iii) nonvanishing microcanonical average of the operator of our interest. The outcome of these conditions is applicable to general nondegenerate Hamiltonians with p-form symmetry even when the system exhibits a discrete symmetry with a nonlocal conserved quantity.

Higher-form symmetries often appear accompanied by the 't Hooft anomaly. The 't Hooft anomaly is defined as an obstruction to promote global symmetry to local gauge symmetry, and known to constrain the infrared theories of systems with conventional symmetries [172–177] or generalized symmetries [95–97, 99, 178]. One of the significant consequence of the 't Hooft anomaly for discrete symmetries is degeneracies of the ground state. In the operator formalism, the 't Hooft anomaly is realized as projective representation on the Hilbert space. From this point of view, we see that the degeneracy exist not only for the ground states but for all the energy eigenstates [179].

The purpose of this chapter is to reconsider the sufficient conditions for the ETH-breakdown by p-form symmetry. Specifically, the condition ii) above involves detailed information about the eigenstates in the middle of the energy spectrum, and thus it is rather challenging to verify this condition without explicit numerical calculations in general. The main idea here is to utilize the

mixed 't Hooft anomaly involving the $\mathbb{Z}_N$ p-form symmetry under consideration. In this paper, we employ a $G = \mathbb{Z}_N \times \mathbb{Z}_N$ symmetry with a mixed 't Hooft anomaly, and perturb the Hamiltonian to explicitly break one of the $\mathbb{Z}_N$ symmetries. By choosing the perturbation parameter λ appropriately, we obtain a theory with the $\mathbb{Z}_N$ p-form symmetry, which satisfies the condition ii). Since the broken symmetry is expected not to affect the thermal ensemble in the thermodynamic limit, the system just reduces to have the $(d-p)$-dimensional ETH-violating operators eventually. It is remarkable that along this construction, we do not need any direct reference to the details of the energy eigenstates in the middle of the spectrum.

In the subsequent sections of this chapter, we begin by revisiting the ETH violation induced by p-form symmetry, in the context of 't Hooft anomaly, in Section 4.1. Section 4.2 focuses on the degeneracies resulting from a $\mathbb{Z}_N \times \mathbb{Z}_N$ mixed 't Hooft anomaly and introduces a symmetry-breaking perturbation that fulfills the conditions for ETH violation. This is followed by a numerical analysis in Section 4.3 to demonstrate our argument, featuring models such as $(1+1)$-dimensional $\mathbb{Z}_2$-symmetric/$\mathbb{Z}_3$-symmetric spin chains and a $(2+1)$-dimensional $\mathbb{Z}_2$ lattice gauge theory. Finally, Section 4.4 is dedicated to summarizing the key findings of the chapter.

4.1 Conditions for the ETH-violation by p-form symmetry

In this section, we briefly review how the ETH is broken due to the p-form symmetry, and sort out its sufficient conditions in [139]. We consider a $(d+1)$-dimensional manifold $\mathcal{M} \times \mathbb{R}$, where $\mathcal{M}$ is a d-dimensional space manifold. Let the system have a G p-form symmetry with $(d-p)$-dimensional topological symmetry operator, where G is an Abelian group. Throughout this paper, symmetry operators extend to the spatial directions, and they are represented as unitary operators $U_\alpha(\tilde{C})$ with the support $\tilde{C} \subset \mathcal{M}$.

The main claim in [139] states that higher-form symmetry of a non-degenerate Hamiltonian leads to the breakdown of the ETH for nontrivial $(d-p)$-dimensional operators. To show this statement, we assume the following:

i) The symmetry operator $U_\alpha(\tilde{C})$ can be decomposed as $U_\alpha(\tilde{C}) = U_\alpha(\gamma)U_\alpha(\bar{\gamma})$ for an arbitrary $(d-p)$-dimensional submanifold $\gamma(\subset \tilde{C})$ and the complement $\bar{\gamma} := \tilde{C}\backslash\gamma$ (see Fig. 4.1 (a)). This implies the operator with boundaries $U(\gamma)$ and $U(\bar{\gamma})$ are well-defined (not-null) operators.

ii) An energy shell $[E, E+\delta E]$ contains eigenstates in different symmetry sectors defined by

$U_\alpha(\tilde{C})$, i.e., for at least one nontrivial closed surface, say $\tilde{C}\,(\subset \mathcal{M})$, there exist energy eigenstates $|E_n\rangle$, $|E_m\rangle$ with $E_n, E_m \in [E, E+\delta E]$ such that $\langle E_n|U_\alpha(\tilde{C})|E_n\rangle \neq \langle E_m|U_\alpha(\tilde{C})|E_m\rangle$.

iii) Given an energy shell $[E, E+\delta E]$, the microcanonical average $\langle U_\alpha(\gamma)\rangle_{\rm mc}^{\delta E}(E)$ takes a nonzero value in the thermodynamic limit.

It follows that either $U_\alpha(\gamma)$ or $U_\alpha(\bar{\gamma})$ necessarily breaks the ETH within the energy shell $[E, E+\Delta E]$ under the above condition It can be shown as follows (the proof for more general case can be found in Appendix A). We first consider a $(d-p)$-dimensional surface γ with boundary, which satisfies the property iii). In the case where $U_\alpha(\gamma)$ does not satisfy the ETH, our claim holds in the first place; we thus suppose $U_\alpha(\gamma)$ satisfies the ETH, i.e.,

$$\langle E_n|U_\alpha(\gamma)|E_n\rangle \simeq \langle E_m|U_\alpha(\gamma)|E_m\rangle \simeq \langle U_\alpha(\gamma)\rangle_{\rm mc}^{\Delta E}(E). \tag{4.1.1}$$

The Hamiltonian H is assumed to have no degeneracy, and thus its eigenstates $|E_n\rangle$, $|E_m\rangle$ are eigenstates of $U_\alpha(\tilde{C})$ as well. Since the group G is Abelian, the eigenvalues are expressed as

$$U_\alpha(\tilde{C})|E_n\rangle = e^{i\alpha q_n}|E_n\rangle, \qquad U_\alpha(\tilde{C})|E_m\rangle = e^{i\alpha q_m}|E_m\rangle, \tag{4.1.2}$$

where $q_n, q_m \in \mathbb{R}$. The assumption ii) now indicates that $|E_n\rangle$ and $|E_m\rangle$ belong to different sectors, i.e., $e^{i\alpha q_n} \neq e^{i\alpha q_m}$. The definition of $\bar{\gamma}$ leads to

$$\langle E_n|U_\alpha^{-1}(\bar{\gamma})|E_n\rangle = \langle E_n|U_\alpha(\gamma)U_\alpha(\tilde{C})^{-1}|E_n\rangle = e^{-i\alpha q_n}\langle E_n|U_\alpha(\gamma)|E_n\rangle \tag{4.1.3}$$

and $\langle E_m|U_\alpha^{-1}(\bar{\gamma})|E_m\rangle = e^{-i\alpha q_m}\langle E_m|U_\alpha(\gamma)|E_m\rangle$. Recalling iii) and the supposition of the ETH, i.e.,

$$\langle E_n|U_\alpha(\gamma)|E_n\rangle \simeq \langle E_m|U_\alpha(\gamma)|E_m\rangle \simeq \langle U_\alpha(\gamma)\rangle_{\rm mc}^{\Delta E}(E) \neq 0, \tag{4.1.4}$$

we obtain the relation

$$\langle E_n|U_\alpha^{-1}(\bar{\gamma})|E_n\rangle \neq \langle E_m|U_\alpha^{-1}(\bar{\gamma})|E_m\rangle \quad \Rightarrow \quad \langle E_n|U_\alpha(\bar{\gamma})|E_n\rangle \neq \langle E_m|U_\alpha(\bar{\gamma})|E_m\rangle. \tag{4.1.5}$$

After all, we see that $U_\alpha(\bar{\gamma})$ violates the ETH, and the claim has been proven.

Note here that symmetries do not automatically lead to the degeneracy of the spectrum and the mixing of the symmetry sectors. As a simple example, we can consider a Hamiltonian $H = \mathrm{diag}(1,1,-1,-2)$ and a charge $Q = \mathrm{diag}(1,1,1,-1)$. The operators H and Q commute each other, but there are energy eigenstates without degeneracy, and the eigenstates with $H=1$ do not exhibit the mixing of the symmetry sectors. However, in the presence of symmetries with the 't

Hooft anomaly, we can always obtain the degeneracies and the mixture of the symmetry sectors as discussed in Section 4.2, and this is a key to the following discussion.

Let us comment on the volume of the "bath" for the ETH-breaking observables (say, $U_\alpha(\bar{\gamma})$). Let $V_{\bar{\gamma}}$ and $V_{\mathcal{M}\backslash\bar{\gamma}}$ denote the volume of $\bar{\gamma}$ and $\mathcal{M}\backslash\bar{\gamma}$, respectively. For the operator $U_\alpha(\bar{\gamma})$ with the $(d-p)$-dimensional support, the d-dimensional complements $\mathcal{M}\backslash\bar{\gamma}$ can be regarded as the bath. If we consider a 0-form symmetry, the ratio $V_{\bar{\gamma}}/V_{\mathcal{M}\backslash\bar{\gamma}}$ remains finite in the thermodynamic limit $V := V_{\bar{\gamma}} \cup V_{\mathcal{M}\backslash\bar{\gamma}} \to \infty$, and thus the breakdown of the ETH may be attributed to the smallness of the bath. In contrast, for higher-form symmetry with $p \geq 1$, the volumes scales as $V_{\bar{\gamma}}/V_{\mathcal{M}\backslash\bar{\gamma}} \to 0$ in the thermodynamic limit since $\bar{\gamma}$ and $\mathcal{M}\backslash\bar{\gamma}$ are $(d-p)$-dimensional and d-dimensional, respectively. Thus, the higher-form symmetry hinders thermalization even when the support of the observable is much smaller than its bath.

Figure 4.1 (b) shows the numerical result for the $(2+1)$-dimensional $\mathbb{Z}_2$ gauge theory, which is described in detail in Section 4.3.2. The 1-dimensional operator $U(\bar{\gamma}) := U_1(\bar{\gamma})$ violates the ETH while the ETH for the local operator $U(\gamma)$ holds. Although the conditions i), ii) and iii) are indeed satisfied in this case, it is generally challenging to confirm whether the conditions, especially ii), are satisfied without explicit numerical calculation. In the following sections, we show that the condition ii) is satisfied when the system has a mixed 't Hooft anomaly between the symmetry of interest and an auxiliary symmetry which is to be broken by perturbations for the Hamiltonian.

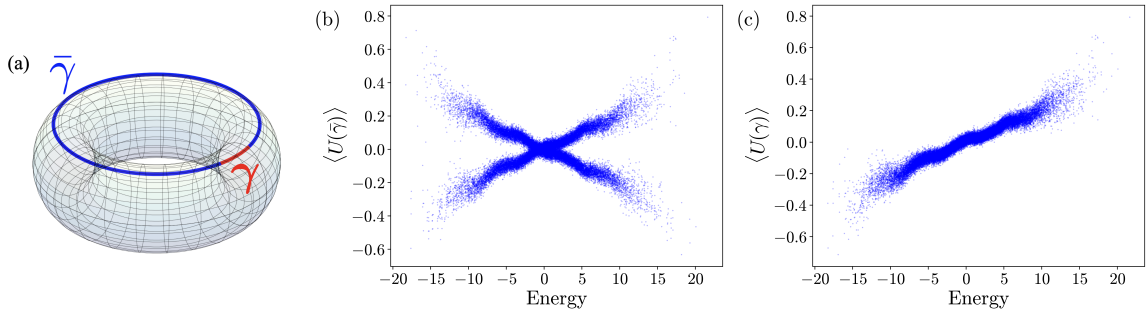

Figure 4.1: (a) Schematics of γ and $\bar{\gamma}$ for $\mathcal{M} = T^2$. The union of γ and $\bar{\gamma}$ constitutes a closed manifold $\tilde{C}$ [140]. (b)(c) The expectation values of the operator $U(\bar{\gamma})$ and $U(\gamma)$ with respect to the energy eigenstates for the $\mathbb{Z}_2$ gauge theory [140]. The ETH for $U(\bar{\gamma})$ can be seen violated because there are deviations for a fixed energy E, while the ETH for $U(\gamma)$ holds.

We comment on a related framework referred to as the subsystem eigenstate thermalization hypothesis (ETH) [180]. The subsystem ETH claims energy eigenstates can be regarded as the microscopic thermal equilibrium (MITE) [141, 142], which claims the reduced density tends to the

microcanonical density matrix in the thermodynamic limit. This condition is stronger than the ETH for operators with a given small support. Thus, the violation of the ETH for $U(\bar{\gamma})$ also indicates the breakdown of the subsystem ETH with respect to the conventional microcanonical ensemble.

4.2 Projective representation for Abelian group

In this section, we show that the condition ii) can be always satisfied if a $\mathbb{Z}_N \times \mathbb{Z}_N$ symmetry with a mixed 't Hooft anomaly is explicitly broken by perturbations. Before delving into such perturbations, we first discuss the degeneracy due to projective representations.

4.2.1 Degeneracy by projective representation

Let G and be an Abelian group and the theory have a G symmetry. In the operator formalism, 't Hooft anomalies are interpreted as projective representation of the symmetry, given by

$$\begin{aligned} U_{g_1}U_{g_2} &= e^{i\phi(g_1,g_2)}U_{g_1g_2}, \qquad g_1,g_2 \in G. \\ U_{g_1}U_{g_2} &= e^{i\phi(g_1,g_2)-i\phi(g_2,g_1)}U_{g_2}U_{g_1}, \end{aligned} \tag{4.2.1}$$

where U_{g_1}, U_{g_2} are unitary operators, and $\phi : G \times G \to \mathbb{R}$ is the projective phase. Since the theory has the G symmetry, the Hamiltonian H commutes with the unitary operators $[H, U_g] = 0$, $^\forall g \in G$. The non-vanishing projective phase with $\exp(i(\phi(g_1,g_2) - \phi(g_2,g_1))) \neq 1$ immediately leads to the degeneracy of arbitrary eigenstates of the Hamiltonian. This is because if you have a simultaneous eigenstate s.t. $H|E\rangle = E|E\rangle$ and $U_{g_1}|E\rangle = e^{i\alpha}|E\rangle, \alpha \in \mathbb{R}$, we obtain

$$\langle E|U_{g_2}|E\rangle = \langle E|U_{g_1}^\dagger U_{g_2}U_{g_1}|E\rangle = e^{-(i\phi(g_1,g_2)-i\phi(g_2,g_1))}\langle E|U_{g_2}|E\rangle \quad \Rightarrow \quad \langle E|U_{g_2}|E\rangle = 0. \tag{4.2.2}$$

Since $|E\rangle$ and $U_{g_2}|E\rangle$ are orthogonal to each other, they are degenerate energy eigenstates with the eigenvalue E. Note that $|E\rangle$ and $U_{g_2}|E\rangle$ belong to different symmetry sector of U_{g_1}:

$$U_{g_1}(U_{g_2}|E\rangle) = e^{i\phi(g_1,g_2)-i\phi(g_2,g_1)}e^{i\alpha}(U_{g_2}|E\rangle), \tag{4.2.3}$$

where α is the charge for the state $|E\rangle$.

We define simultaneous eigenstates of the Hamiltonian H and U_{g_1} by

$$H|E,\alpha\rangle = E|E,\alpha\rangle, \quad H|E,\beta\rangle = E|E,\beta\rangle, \tag{4.2.4}$$

$$U_{g_1}|E,\alpha\rangle = e^{i\alpha}|E,\alpha\rangle, \quad U_{g_1}|E,\beta\rangle = e^{i\beta}|E,\beta\rangle, \tag{4.2.5}$$

The charged operator under the symmetry G can be also introduce as

$$U_{g_1}^\dagger W_q U_{g_1} = e^{iq_{g_1}} W_q, \quad U_{g_2}^\dagger W_q U_{g_2} = e^{iq_{g_2}} W_q. \tag{4.2.6}$$

Now we consider the matrix elements of the charged operator W_q. The diagonal part satisfies

$$\begin{aligned} &\langle E,\alpha|\, W_q\, |E,\alpha\rangle = \langle E,\alpha|\, U_{g_1}^\dagger W_q U_{g_1}\, |E,\alpha\rangle\, e^{-iq_{g_1}} = \langle E,\alpha|\, W_q\, |E,\alpha\rangle\, e^{-iq_{g_1}}, \\ \Rightarrow \quad &\langle E,\alpha|\, W_q\, |E,\alpha\rangle = 0, \end{aligned} \tag{4.2.7}$$

if the operator W_q is nontrivially charged under the action of g_1, i.e., $e^{-iq_{g_1}} \neq 1$. On the other hand, the operator with trivial charge under g_1, but charged under g_2, i.e., $e^{-iq_{g_2}} \neq 1$ can have the nonvanishing expectation value in this basis while the off-diagonal part with $\alpha \neq \beta$ necessarily vanishes:

$$\begin{aligned} &\langle E,\alpha|\, W_q\, |E,\beta\rangle = \langle E,\alpha|\, U_{g_1}^\dagger W_q U_{g_1}\, |E,\beta\rangle = e^{i(\beta-\alpha)}\, \langle E,\alpha|\, W_q\, |E,\beta\rangle, \\ \Rightarrow \quad &\langle E,\alpha|\, W_q\, |E,\beta\rangle = 0. \end{aligned} \tag{4.2.8}$$

We stress that all of the properties discussed here can be applied not only to the ground states but also to arbitrary energy eigenstate, although topological robustness of degeneracy does not hold for general eigenstates since the gaps are exponentially small.

4.2.2 Symmetry violating perturbation

We now discuss a consequence of weak breaking of symmetries with 't Hooft anomaly. To this end, let G_1 and G_2 be Abelian groups and the group $G = G_1 \times G_2$ projectively acts on the Hilbert space of the theory. The corresponding unitary operator is given by U_{g_1} and $\tilde{U}_{g_2}$ for G_1 and G_2, respectively. We consider a situation such that each of the symmetry does not have a 't Hooft anomaly, but they have nontrivial projective phases between them: $U_{g_1}\tilde{U}_{g_2} = e^{i\phi(g_1,g_2)}\tilde{U}_{g_2}U_{g_1}$. We perturb the Hamiltonian by adding a term $\sum_{\text{site}} W_q$, where W_q is a charged operator under G_2 but trivially transforms under G_1, i.e.,

$$U_{g_1}^\dagger W_q U_{g_1} = W_q, \quad \tilde{U}_{g_2}^\dagger W_q \tilde{U}_{g_2} = e^{iq_{g_2}} W_q, \qquad g_1 \in G_1,\ g_2 \in G_2. \tag{4.2.9}$$

Here, we assume the the operator W_q is a local operator.

In the following, we focus on the system with a discrete spectrum realized by appropriate regularizations. For brevity, we specify $G_1 = G_2 = \mathbb{Z}_N$ as the symmetry groups and thus the charge

of W_q is simply given by $q_m = qm$ for $m \in G_2 = \mathbb{Z}_N$. As explained in the previous subsection, arbitrary energy eigenstates are degenerate, and then we work with the simultaneous eigenbasis of the unperturbed Hamiltonian H and U_{g_1} ($^\forall g_1 \in G_1 = \mathbb{Z}_N$), i.e., $H|E,\alpha\rangle = E|E,\alpha\rangle$, $U_{g_1}|E,\alpha\rangle = e^{i\alpha}|E,\alpha\rangle$. Note that the number of the degeneracy of each energy eigenstate is at least N as long as $\exp(i(\phi(g_1,g_2)-\phi(g_2,g_1))) \neq 1$ with $^\forall g_1 \in G_1$, $^\forall g_2 \in G_2$, since $\tilde{U}_m|E,\alpha\rangle$ ($m \in G_2 = \mathbb{Z}_N$) are eigenstates of U_{g_1} with different eigenvalues, and $\langle E,\alpha|\tilde{U}_m^\dagger \tilde{U}_n|E,\alpha\rangle = \langle E,\alpha|\tilde{U}_{n-m}|E,\alpha\rangle = 0$ for $n \neq m$. Although extra degeneracies are possible in general, such accidental degeneracies can be removed by deforming the Hamiltonian while preserving the $\mathbb{Z}_N \times \mathbb{Z}_N$ symmetry. We thus assume that all of the energy eigenstates are N-fold degenerate in the following discussion. The degenerate subspace $\mathcal{H}(E) := \mathrm{span}\{|E,\alpha\rangle \,|\, \alpha = 0,1,\ldots,N-1\}$ is also expressed as $\mathcal{H}(E) = \mathrm{span}\{\tilde{U}_{g_2}|E,\alpha\rangle\,, ^\forall g_2 \in G_2\}$.

The perturbed Hamiltonian is defined by

$$\tilde{H}(\lambda) := H + \lambda H_1, \qquad H_1 := \sum_{j:\,\mathrm{site}} \frac{W_q(j) + W_q(j)^\dagger}{2}\,, \tag{4.2.10}$$

where λ is the perturbation parameter. After this perturbation, the system with $\tilde{H}(\lambda)$ only exhibits G_1 symmetry since the operator $W_q(j)$ has a trivial charge under G_1. The perturbation part H_1 is diagonalized by $|E,\alpha\rangle$ basis in the subspace $\mathcal{H}(E)$ since the off-diagonal part $\langle E,\alpha|W_q|E,\beta\rangle$ ($\alpha \neq \beta$) always vanishes as in (4.2.8). In order to estimate the energy modification, we utilize the Hellmann-Feynman theorem for degenerate spectra [181–186]. Once the operator $d\tilde{H}(\lambda)/d\lambda = H_1$ is diagonalized in the subspace $\mathcal{H}(E)$, we can obtain

$$\frac{dE(\alpha;\lambda)}{d\lambda} = \langle E,\alpha;\lambda|H_1|E,\alpha;\lambda\rangle = \sum_{j:\,\mathrm{site}} \mathrm{Re}\,\langle E,\alpha;\lambda|W_q(j)|E,\alpha;\lambda\rangle\,, \tag{4.2.11}$$

where $E(\alpha;\lambda)$ is the energy eigenvalue for the eigenstate that depends on the parameter λ: $E(\alpha;\lambda)|E,\alpha;\lambda\rangle = \tilde{H}(\lambda)|E,\alpha;\lambda\rangle$. In the first order, the perturbed energy reads

$$E(\alpha;\lambda) = E + \lambda \sum_{j:\,\mathrm{site}} \mathrm{Re}\,\langle E,\alpha|W_q(j)|E,\alpha\rangle + \mathcal{O}(\lambda^2). \tag{4.2.12}$$

Significantly, for all elements of $\mathcal{H}(E)$, the expectation $\langle E,\alpha;\lambda|W_q|E,\alpha;\lambda\rangle$ have different values because of the relation $\langle E,\alpha;\lambda|\tilde{U}_m^\dagger W_q \tilde{U}_m|E,\alpha;\lambda\rangle = e^{iqm}\langle E,\alpha;\lambda|W_q|E,\alpha;\lambda\rangle$. Except for the case $\langle E,\alpha;\lambda|\tilde{U}_m^\dagger W_q \tilde{U}_m|E,\alpha;\lambda\rangle = \langle E,\alpha;\lambda|W_q|E,\alpha;\lambda\rangle^*$, we can see that the perturbed energies (4.2.12) are split for $|E,\alpha;\lambda\rangle$ and $\tilde{U}_m|E,\alpha;\lambda\rangle$. Even if the energies are still degenerate in the first order

perturbations, the degeneracies are lifted by higher order perturbations due to the mixing with other energy eigenstates.

Note here that the standard perturbation theory for higher order breaks down in the large system-size limit $V \to \infty$. Since the separations of the energies tend to be exponentially small in the limit, they become small enough compared to the perturbation, i.e., $E - E' \simeq \lambda \langle E, \alpha | H_1 | E, \alpha \rangle$, where E' is an energy eigenvalues of another eigenstate. The higher-order perturbation is no longer valid unless $\lambda \langle E, \alpha | H_1 | E, \alpha \rangle \simeq \mathcal{O}(e^{-V})$, and thus we have to resort to the Hellmann-Feynman theorem as in (4.2.11).

After all, we obtain the energy eigenstates with the energy separations of order λ (see Fig. 4.2). The key point is that these eigenstates in $\mathcal{H}(E)$ have distinct charges of U_m. In the thermodynamics limit $V \to \infty$, we should take the width of the energy shell $\delta E = \mathcal{O}(V^{1/2})$, and we suppose the energy splitting also scales as $\mathcal{O}(V^{1/2})$, i.e.,

$$\lambda \left(\sum_{j:\,\text{site}} \mathrm{Re} \langle E, \alpha | W_q(j) | E, \alpha \rangle - \sum_{j:\,\text{site}} \mathrm{Re} \langle E, \beta | W_q(j) | E, \beta \rangle \right) \simeq \mathcal{O}(V^{1/2}), \qquad \alpha \neq \beta. \tag{4.2.13}$$

This relation is expected to be naturally realized, but we can also force (4.2.13) by e.g., setting $\lambda \simeq \mathcal{O}(V^{-1/2})$ since W_q is a local operator. Another approach is to implement a weak randomness such that $\lambda H_1 = \lambda \sum_j r_j W_q(j)$, where r_j is a uniformly chosen constants from $[-r, r]$, $0 < r \ll 1$. Assuming the expectation value $\langle W_q(j) \rangle$ is almost uniform, we obtain the variation $\langle H_1 \rangle \simeq \mathcal{O}(V^{1/2})$. Under the supposition (4.2.13), we notice that given an energy window with the width δE, we can arrange eigenstates with different symmetry sectors with respect to G_1 within it by tuning the parameter λ. This exactly indicates the condition ii) is satisfied.

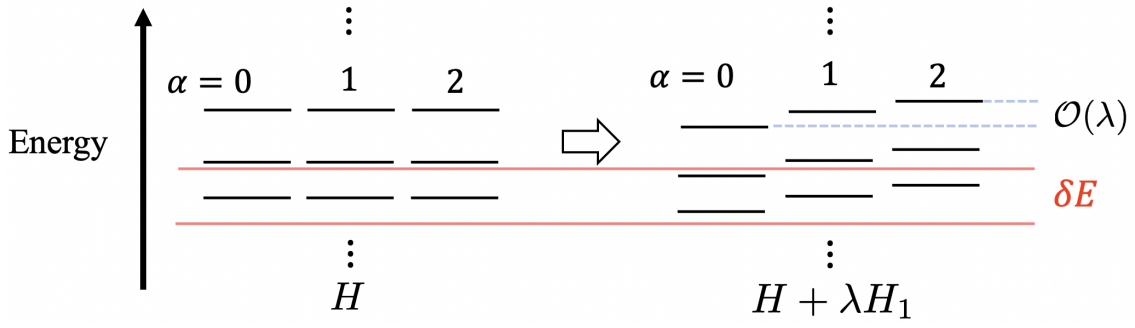

Figure 4.2: Schematics for the spectrum [140]. α denotes the charge for U_m s.t. $U_m |E, \alpha\rangle = e^{i\alpha} |E, \alpha\rangle$. After the perturbation with sufficiently small λ, the degeneracies are resolved so that the condition ii) is satisfied.

4.3 Demonstration for lattice models

In this section, we demonstrate the statement discussed in Section 4.2 by numerically calculating the energy spectra for concrete examples.

4.3.1 $(1+1)$-dimensional spin chains

As the first example, we consider a $(1+1)$-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spin chain and its $\mathbb{Z}_3 \times \mathbb{Z}_3$-symmetric generalization. Both of the models only exhibit 0-form symmetries, and thus the ETH-violating operators based on the mechanism in [139] are 1-dimensional. Even though one can not tell whether ETH violation is caused by the smallness of the baths for such operators that have the same dimensionality with the space, it is instructive to illustrate that the discussion in Section 4.2 indeed holds for those models.

A projective representation of $\mathbb{Z}_N \times \mathbb{Z}_N$ is realized on the N-dimensional Hilbert space spanned by $|g\rangle$, $g = 0, 1, \ldots, N-1$. The generators of each $\mathbb{Z}_N$ are represented by "clock" operators Z and "shift" operators X, which satisfy the relations [187]

$$ZX = e^{\frac{2\pi i}{N}} XZ. \tag{4.3.1}$$

The operators act on the Hilbert space as

$$Z\,|g\rangle = e^{2\pi i \frac{g}{N}}\,|g\rangle\,, \qquad X\,|g\rangle = |g+1 \mod N\rangle\,. \tag{4.3.2}$$

In the matrix form, they can be explicitly expressed as

$$Z = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & e^{2\pi i \frac{1}{N}} & 0 & \cdots & 0 \\ 0 & 0 & e^{2\pi i \frac{2}{N}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{2\pi i \frac{N-1}{N}} \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{4.3.3}$$

For $N = 2$, these are just the standard Pauli matrices.

In order to accommodate a spin chain with L sites, we introduce the tensor product $|g\rangle \otimes \cdots \otimes |g\rangle$, and the operators acting only the j-th site $Z_j := \mathbf{1} \otimes \cdots \otimes Z \otimes \cdots \otimes \mathbf{1}$ and $X_j := \mathbf{1} \otimes \cdots \otimes X \otimes \cdots \otimes \mathbf{1}$. The symmetry operators are then given by

$$U_1 := \prod_{j=1}^{L} Z_j, \qquad \tilde{U}_1 := \prod_{j=1}^{L} X_j. \tag{4.3.4}$$

We note that the projective phase between U_i and $\tilde{U}_j$ can be trivial for the case $\gcd(N, L) \neq 1$, and thus we suppose $\gcd(N, L) = 1$ so that the discussion in the previous section always holds.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spin chain

In the $N = 2$ case, a $\mathbb{Z}_2 \times \mathbb{Z}_2$-invariant $(1+1)$-dimensional spin chain is given by

$$H_{N=2} = \sum_{j=1}^{L} \left(J_j^x X_j X_{j+1} + J_j^y Y_j Y_{j+1} + J_j^z Z_j Z_{j+1} \right) , \tag{4.3.5}$$

where $Y_j := iX_jZ_j$. This Hamiltonian is nothing but the one of the XYZ Heisenberg spin chain. To remove unwanted spacetime symmetry, we introduce a weak randomness to the couplings J_j^x, J_j^y and J_j^z. There is still an extra symmetry that flips the sign of one of the Pauli matrices, e.g., $Y_j \mapsto -Y_j$, $^\forall j$. We thus work with a deformed Hamiltonian

$$H_{N=2} = \sum_{j=1}^{L} \left(J_j^x X_j X_{j+1} + J_j^y Y_j Y_{j+1} + J_j^z Z_j Z_{j+1} \right) + \alpha \sum_{j=1}^{L} X_j Y_{j+1} Z_{j+2} \, , \tag{4.3.6}$$

which is indeed $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric. Here we consider the periodic boundary condition with $j \sim j+L$, which indicates the topology of the space is S^1.

To weakly break the $\mathbb{Z}_2$ generated by $\tilde{U}_1$, we perturb the Hamiltonian as

$$\tilde{H}_{N=2} := H_{N=2} + \lambda \sum_{j=1}^{L} Z_j \, . \tag{4.3.7}$$

The numerical results are shown in Fig. 4.3. Since the surviving symmetry operator is given by $U_1 := \prod_{j=1}^{L} Z_j$, one of the ETH-violating operator in this case is given by $U_1(\bar{1}) := \prod_{j=2}^{L} Z_j$. After the perturbation, the double degeneracy is completely broken, and these eigenstates leads to mixed symmetry sector in a energy shell.

$\mathbb{Z}_3 \times \mathbb{Z}_3$-symmetric spin chain

In the $N = 3$ case, the Hamiltonian for a $\mathbb{Z}_3 \times \mathbb{Z}_3$-symmetric spin chain is given by

$$H_{N=3} := \sum_{j=1}^{L} \left(J_j^w W_j W_{j+1}^\dagger + J_j^x X_j X_{j+1}^\dagger + J_j^y Y_j Y_{j+1}^\dagger + J_j^z Z_j Z_{j+1}^\dagger \right) + (\text{h.c.}), \tag{4.3.8}$$

where $W_j := Z_j^\dagger X_j$ and $Y_j := Z_j X_j$. Again, we take the periodic boundary condition $j \sim j + L$. If the couplings J_j^w, J_j^w, J_j^w and J_j^w are weakly random and not real, the theory has no relevant symmetries other than $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry represented by U_m and $\tilde{U}_m$ $(m = 1, 2)$. Under these symmetry action, the local operators transform as

$$\begin{aligned} &U_m^\dagger W_j U_m = e^{\frac{4}{3}\pi m i} W_j, \quad U_m^\dagger X_j U_m = e^{\frac{4}{3}\pi m i} X_j, \quad U_m^\dagger Y_j U_m = e^{\frac{4}{3}\pi m i} Y_j, \quad U_m^\dagger Z_j U_m = Z_j, \\ &\tilde{U}_m^\dagger W_j \tilde{U}_m = e^{\frac{4}{3}\pi m i} W_j, \quad \tilde{U}_m^\dagger X_j \tilde{U}_m = X_j, \quad \tilde{U}_m^\dagger Y_j \tilde{U}_m = e^{\frac{2}{3}\pi m i} Y_j, \quad \tilde{U}_m^\dagger Z_j \tilde{U}_m = e^{\frac{2}{3}\pi m i} Z_j, \end{aligned} \tag{4.3.9}$$

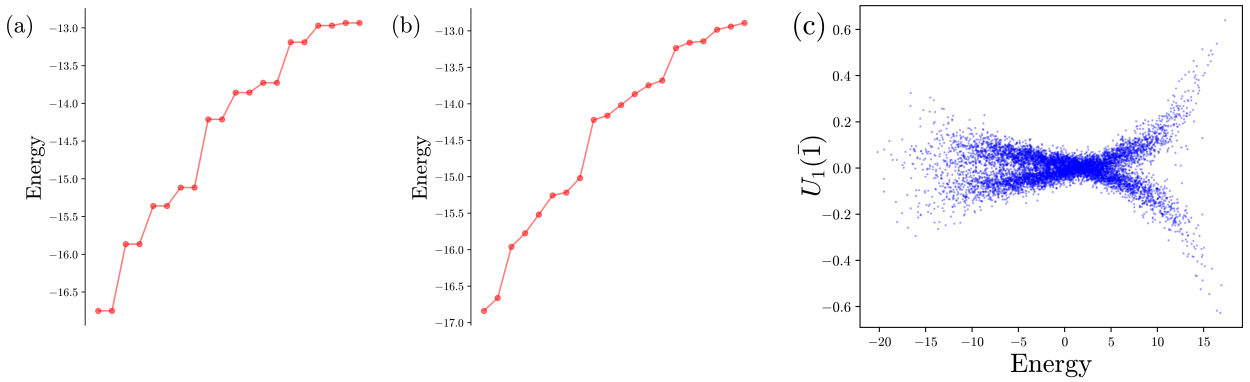

Figure 4.3: (a)(b) Part of energy spectra for $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry spin chain for $L = 11$, $\lambda = 0.1$ [140]. The coupling constants are uniformly distributed in $J_j^x \in [0.9, 1.0]$, $J_j^y \in [0.7, 0.8]$, $J_j^z \in [0.6, 0.7]$, and the parameter is given by $\alpha = 0.9$. The degeneracy in the original Hamiltonian (4.3.6) (a) is resolved by the perturbation (4.3.7) (b). (c) The expectation value of $U_1(\bar{1})$ for $L = 13$, $\lambda = 0.4$ [140]. The expectations are separated into two sectors, and thus the ETH for $U_1(\bar{1})$ is not satisfied.

Since the local operator Z_j is not charged under U_i, a desired perturbation can be performed as

$$\tilde{H}_{N=3} := H_{N=3} + \lambda \sum_{j=1}^{L} Z_j, \tag{4.3.10}$$

and then the Hamiltonian is invariant only under the action of U_i. As shown in Fig. 4.4, the triple degeneracies of the energy spectrum are resolved by the perturbation, and the operator $U(\bar{1}) = \prod_{j=2}^{L} Z_j$ does not satisfy the ETH.

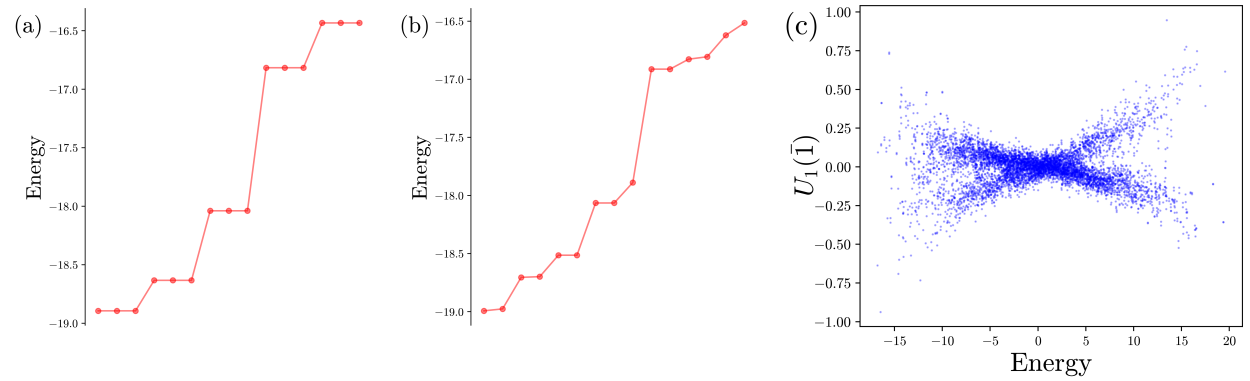

Figure 4.4: (a)(b) Part of energy spectra for $\mathbb{Z}_3 \times \mathbb{Z}_3$-symmetry spin chain for $L = 7$, $\lambda = 0.1$ [140]. The coupling constants are uniformly distributed in $J_j^w \in [1.0 + 0.2i, 1.1 + 0.2i]$, $J_j^x \in [0.9, 1.0]$, $J_j^y \in [0.1, 0.2]$, $J_j^z \in [0.2, 0.3]$. The degeneracy in the original Hamiltonian (4.3.8) (a) is resolved by the perturbation (4.3.10) (b). (c) The expectation value of $U_1(\bar{1})$ for $L = 8$, $\lambda = 0.4$ [140]. The expectations are separated into two sectors, and thus the ETH for $U_1(\bar{1})$ is not satisfied.

4.3.2 $(2+1)$-dimensional $\mathbb{Z}_2$ gauge theory

Here we consider the (2+1)-dimensional $\mathbb{Z}_2$ lattice gauge theory defined on a $L_x \times L_y$ square lattice with the periodic boundary conditions. In this model, the argument in Section 4.2 can be applied to the projective representation of $\mathbb{Z}_2$ electric one-form symmetry and "time reversal" symmetry.

The Hamiltonian is given by [159, 166, 167]

$$H_{\mathbb{Z}_2} = -\sum_{\boldsymbol{r}} \lambda_{\boldsymbol{r},xy} \sigma^3_{\boldsymbol{r},x} \sigma^3_{\boldsymbol{r}+\boldsymbol{e}_x,y} \sigma^3_{\boldsymbol{r}+\boldsymbol{e}_y,x} \sigma^3_{\boldsymbol{r},y} - \lambda \sum_{\boldsymbol{r},j} \sigma^1_{\boldsymbol{r},j}, \tag{4.3.11}$$

where $\sigma^{1,2,3}_{\boldsymbol{r},j}$ denote the Pauli matrices acting on the link variable $(\boldsymbol{r}, j)$, specified by the coordinate of vertices $\boldsymbol{r}$ and the direction $j = x, y$. The coupling constants $\lambda_{\boldsymbol{r},xy}$ and λ are real numbers. Note here that we take the different definition of the coupling constants from the one in Section 3.3, to apply the discussion in this chapter. Along the line of Section 4.2, we can regard the term $\lambda \sum_{\boldsymbol{r},j} \sigma^1_{\boldsymbol{r},j}$, as a perturbation term. We can observe that for $\lambda = 0$, the Hamiltonian $H_{\mathbb{Z}_2}$ is invariant under the "time reversal" symmetry[1] represented by

$$\tilde{U} := \sum_{\boldsymbol{r},j} \sigma^2_{\boldsymbol{r},j}. \tag{4.3.12}$$

This theory also enjoys the electric $\mathbb{Z}_2$ 1-form symmetry, and the spatial symmetry operators can be characterized by $H_1(T^2, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ [168]. The generators of $\mathbb{Z}_2$ correspond two independent symmetry operators corresponding to the x-cycle and y-cycle.

Though the total Hilbert space of the system for the $L_x \times L_y$ lattice is $(2^{2L_xL_y})$-dimensional, we have to project it onto the physical Hilbert space. This is because there exist residual gauge redundancies, after the temporal gauge-fixing, which is analogous to the gauge $A_0(\boldsymbol{r}) = 0$ for the Maxwell theory. Spatial gauge transformation is generated by the local operator

$$Q_v := \prod_{b:\ \text{spatial link}, b \ni v} \sigma^1_b, \qquad v : \text{vertex}, \tag{4.3.13}$$

which satisfies $Q_v^2 = 1$ and $[H_{\mathbb{Z}_2}, Q_v] = 0$. The physical Hilbert space is then obtained as

$$\text{span}\left\{ |\psi\rangle \;\middle|\; Q_v|\psi\rangle = +|\psi\rangle,\ {}^\forall v :\ \text{vertices} \right\}. \tag{4.3.14}$$

[1] This operation is not the time reversal symmetry in the usual sense, because it does not accompany the complex conjugation. However, the complex conjugation does not affect the Hamiltonian $H_{\mathbb{Z}_2}$, and we just referred to this symmetry as "time reversal."

This constraint can be regarded as the $\mathbb{Z}_2$ analog of the Gauss law $\nabla \cdot E|\psi\rangle_{\rm phys} = 0$ since we can write $Q_v = (\sigma^1_{\boldsymbol{r}-\boldsymbol{e}_x,x})^{-1}\sigma^1_{\boldsymbol{r},x}(\sigma^1_{\boldsymbol{r}-\boldsymbol{e}_y,y})^{-1}\sigma^1_{\boldsymbol{r},y}$. After this projection, the expectation value of non-gauge invariant operators with respect t physical states $|\psi\rangle$ always vanishe.

The Wilson and 't Hooft operators on the spatial directions are defined as [169–171]

$$W(C) = \prod_{b\in C}\sigma^3_b, \qquad U(C^*) = \prod_{b^*\in C^*}\sigma^1_{b^*}, \tag{4.3.15}$$

where, C and C^* are closed loops on the lattice and dual lattice, respectively (see Fig. 3.3). Both $U(C^*)$ and $W(C)$ are gauge invariant operators since they commute with Q_v.

The 't Hooft operator $U(C^*)$ satisfies $[H_{\mathbb{Z}_2}, U(C^*)] = 0$, and serves as the $\mathbb{Z}_2$ 1-form symmetry operator. This operator is topological since continuous deformations of the path C^* do not change the action of $U(C^*)$ on the physical states, i.e., $U(C^*_1)|\psi\rangle = U(C^*_2)|\psi\rangle$ if C^*_1 and C^*_2 are homotopically equivalent. If a dual closed loop C^* is topologically trivial, it follows that $U(C^*)|\psi\rangle = |\psi\rangle$. The "electric" charge of the Wilson operator is measured by the 't Hooft operator $U(C^*)$. Defining closed loops winding around the x-/y-cycle by C_x and C_y (and similarly the loops on the dual lattice by C^*_x and C^*_y), we see that the operators W and U satisfy

$$\begin{aligned} &U(C^*_y)W(C_x)U^{-1}(C^*_y) = -W(C_x), \quad U(C^*_x)W(C_y)U^{-1}(C^*_x) = -W(C_y), \\ &U(C^*_x)W(C_x)U^{-1}(C^*_x) = +W(C_x), \quad U(C^*_y)W(C_y)U^{-1}(C^*_y) = +W(C_y), \end{aligned} \tag{4.3.16}$$

which is indeed operator-realization of the electric $\mathbb{Z}_2$ 1-form symmetry [118]. (See also Appendix D.)

After these setups, we can explicitly observe that the symmetry operators satisfy $U(C^*_x)\tilde{U} = -\tilde{U}U(C^*_x)$ and $U(C^*_y)\tilde{U} = -\tilde{U}U(C^*_y)$, and then the perturbation $H_1 = \sum_{\boldsymbol{r},j}\sigma^1_{\boldsymbol{r},j}$ lift the degeneracy. It suffices to show the breakdown of the ETH as shown in Fig. 4.1, where the numerical calculation are performed for the 5×3 lattice, and the coupling constants $\lambda_{\boldsymbol{r},xy}$ are uniformly chosen from $[1.0, 1.1]$ and the parameter is set to $\lambda = 0.6$. On the other hand, the ETH holds for other operators such as the plaquette operator $B_p := \prod_{i\in p:\text{plaquette}}\sigma^3_i$ and a double insertion of the Wilson operators $W(C_1)W(C_2)$ (Fig. 4.5). We stress that the nonlocality of the operator does not immediately lead to the breakdown of the ETH since the non-local operator $W(C_1)W(C_2)$ satisfy the ETH.

4.4 Summary of Chapter 4

In this chapter, we have shown that the the one of the sufficient conditions for the ETH-violation is satisfied if we consider perturbations which breaks the symmetry with 't Hooft anomaly. Following

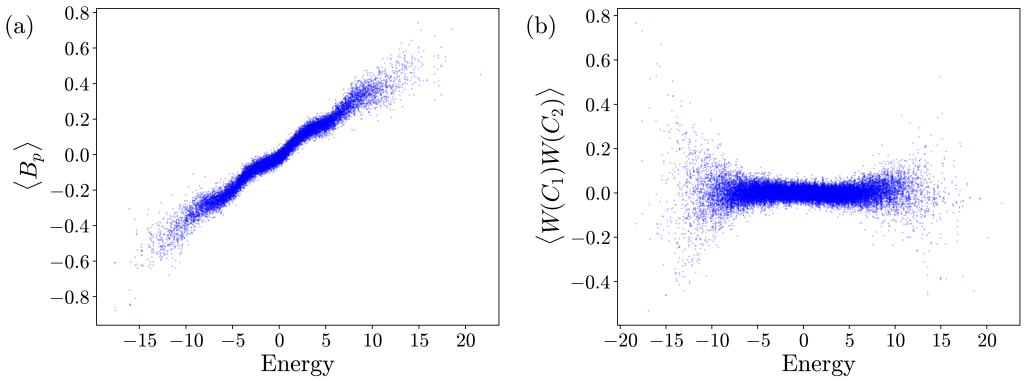

Figure 4.5: (a)(b) The expectation values of the plaquette operator $B_p := \prod_{i \in p:\text{plaquette}} \sigma_i^3$ and a double insertion of the Wilson operators $W(C_1)W(C_2)$ with respect to the energy eigenstates for the $\mathbb{Z}_2$ gauge theory [140]. The ETH for both of the operators are satisfied while the operator $W(C_1)W(C_2)$ is a 1-dimensional non-local operator.

this treatment, we just have to suppose the following: 1) the unperturbed Hamiltonian exhibits a $\mathbb{Z}_N \times \mathbb{Z}_N$ (p-form, in general) symmetry with a mixed 't Hooft anomaly; 2) the symmetry operator corresponding to one of the $\mathbb{Z}_N$ symmetry can be divided as $U_m(\tilde{C}) = U_m(\gamma)U_m(\bar{\gamma})$ with open manifolds γ and $\bar{\gamma}$; 3) $\langle U_m(\gamma) \rangle_{\text{mc}}^{\delta E} \neq 0$. Under these assumptions, the ETH for the $(d-p)$-dimensional operator $U_m(\gamma)$ or $U_m(\bar{\gamma})$ is always violated after perturbing the Hamiltonian by $\lambda \sum_j W_q(j)$ with the scaling (4.2.13). Although the conditions ii) in Section 4.1 (and in [139]) require information about each eigenstate in the middle of the spectrum a priori, the conditions 1), 2) and 3) lead to the same conclusion with the tractable condition to such eigenstates.

We also performed numerical calculations for $(1+1)$-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric/$\mathbb{Z}_3 \times \mathbb{Z}_3$-symmetric spin chains, and $(2+1)$-dimensional $\mathbb{Z}_2$ gauge theory. All of the models have indeed the 1-dimensional ETH-violating operators, and their mechanisms are boiled down to the general discussion above. We can thus conclude that our treatment indeed results in the ETH-violation for these concrete examples.

Chapter 5

Conclusion and discussion

In this thesis, we have discussed thermalization in lattice regularized quantum field theories, especially regarding higher-form symmetry and its 't Hooft anomaly. Chapter 2) was devoted to review of thermalization in isolated quantum systems and the eigenstate thermalization hypothesis (ETH), a successful framework to give sufficient conditions for thermalization. Based on our work [139], we discussed the influence of higher-form symmetry, which is a generalized concept of the conventional global symmetries, on the ETH in Chapter 3. Chapter 4 was devoted to reconsideration of the sufficient condition for ETH-violation by p-form symmetries, based on our work [140]

In Chapter 2, we first provided an overview of the recent understanding of thermal equilibrium in isolated quantum many-body systems. The notion of thermal equilibrium has been conventionally accompanied with the principle of typicality in classical and quantum many-body systems. In quantum systems, there are two widely recognized characterization of typicality: thermodynamic typicality and canonical typicality. Based on these concepts of typicality, we can regard "typical states" as in thermal equilibrium. Corresponding to thermodynamic/canonical typicality, we can formulate the notion of thermal equilibrium called MATE and MITE, respectively. The notion of MITE further motivate us to see thermal equilibrium at the level of operators, which is one of the main theme of this thesis. The ETH was introduced as a sufficient condition of thermalization in the sense of equilibrium for operator sets. The criterion being most actively studied is the (diagonal/strong) ETH, which states all eigenstates in a energy shell are thermal. In the subsequent chapters, we studied how certain operators violate the diagonal ETH while other operators do not.

In Chapter 3, we considered $(d+1)$-dimensional quantum field theories with p-form symmetry, and discussed its consequences on thermalization. Our analytical study reveals that the presence of a p-form symmetry results in the violation of the (diagonal) ETH for numerous observables of

$(d-p)$-dimensions, specifically in the form of $U_\alpha(\bar{\gamma})$, under a set of reasonable assumptions. A notable aspect of this observation is that for $p \geq 1$, the observables that violate the ETH have non-local yet lower-dimensional support, as opposed to acting on the entire d-dimensional space manifold $\mathcal{M}$. This fact indicates that the ETH-violation is not attributed by the smallness of the bath. We next proposed a generalized Gibbs ensemble (GGE) that describe the thermal ensemble in these systems. In presence of p-form symmetry, we should suitably include chemical potentials for the projectors onto each symmetry sector. To illustrate these concepts, we numerically studied the $\mathbb{Z}_2$ lattice gauge theory.

In Chapter 4, we followed up the discussion in Chapter 3, and reformulated the assumption regarding the mixing of symmetry sectors, in terms of 't Hooft anomaly. To discuss it in detail, we focused on the situation with $\mathbb{Z}_N \times \mathbb{Z}_N$. The main results of this chapter is as follows: we can make the theory satisfy the assumption of the ETH-violation in Chapter 3, by appropriately perturbing the Hamiltonian preserving one of the $\mathbb{Z}_N$ symmetry. Consequently, we no longer need detailed information about the eigenstates in the middle of the spectrum, and we can provide more tractable conditions for the ETH-violation. Along the line of the general arguments, we also numerically demonstrated how the energy spectra are deformed, and the ETH-violation is justified.

As an outlook, our analysis in Chapter 3 regarding the breakdown of the ETH has implications for more general systems with p-form symmetries. This includes a range of quantum field theories, notably the $SU(N)$ Yang-Mills theory, which is known to exhibit the center symmetries. The influence of higher-form symmetry on the entanglement structures of certain subsystems merits investigation. This includes the dynamics of entanglement entropy, a non-local quantity. Since entanglement entropy plays an essential role in holography, its exploration could illuminate black-hole dynamics, particularly in the context of gauge/gravity duality.

As for the formulation in Chapter 4, the application to other groups would be possible since essential procedure to mix the symmetry sectors should be common to the case of $\mathbb{Z}_N \times \mathbb{Z}_N$. It enables us to ensure the breakdown of the ETH for broader class of quantum field theories with higher-form symmetry including lattice gauge theories. In [41], it is shown that the local ETH suffices to the subsystem ETH for 0-dimensional subsystems. On the other hand, considering the prevalence of 0-form symmetries in general 2-dimensional conformal field theories, the results in Chapter 4 suggest an ETH violation for 1-dimensional operators. Studying the implications of this for the subsystem ETH in 1-dimensional subsystems in conformal field theories would be a

compelling direction for future research. Additionally, examining the impact of mixed 't Hooft anomalies on thermalization processes presents an intriguing area of study. We hope that our results will contribute to a deeper understanding of non-equilibrium dynamics in various quantum field theories and quantum many-body systems.

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Appendix A

Proof of the breakdown of the ETH under p-form symmetries

As discussed in the main text, higher-form symmetry of a non-degenerate Hamiltonian leads to the breakdown of the eigenstate thermalization hypothesis (ETH) even for many nontrivial operators. We here show this fact. The case $A(g) = 1$ in the following corresponds to the main claim in the main text, and the ETH-violating operators are $(d-p)$-dimensional in this case.

As stated in the main text, we require the following reasonable assumptions: i) the topological operator $U_\alpha(\tilde{C})$ can have boundaries, i.e., $U_\alpha(\gamma)$ with an arbitrary $(d-p)$-dimensional submanifold γ ($\subset \tilde{C}$) is a well-defined (not-null) operator. One can then decompose the operator as $U_\alpha(\tilde{C}) = U_\alpha(\gamma)U_\alpha(\bar{\gamma})$, where we have introduced the complement of γ as $\bar{\gamma} := \tilde{C}\backslash\gamma$. ii) For at least one nontrivial closed surface, say $\tilde{C}$ ($\subset \mathcal{M}$), there are energy eigenstates $|E_n\rangle$, $|E_m\rangle$ with $E_n, E_m \in [E, E+\Delta E]$ such that $\langle E_n|U_\alpha(\tilde{C})|E_n\rangle \neq \langle E_m|U_\alpha(\tilde{C})|E_m\rangle$. In other words, the energy shell of our interest contains eigenstates in different symmetry sectors defined by $U_\alpha(\tilde{C})$. iii)' The microcanonical average $\langle A(g)U_\alpha(\gamma)\rangle_{\rm mc}^{\Delta E}(E)$ of the operator $A(g)U_\alpha(\gamma)$ defined from the energy shell $[E, E+\Delta E]$ takes a nonzero value in the thermodynamic limit. The operator $A(g)$ here is defined on an arbitrary region g ($\subset \mathcal{M}$) that satisfies $g \cup \bar{\gamma} = \phi$ (empty region). If $A(g)$ is the identity operator, the assumption iii)' just reduces to iii).

With these assumptions, we can show that either $A(g)U_\alpha(\gamma)$ or $U_\alpha(\bar{\gamma})A(g)^\dagger$ necessarily breaks the ETH within the energy shell $[E, E+\Delta E]$. To see this, we consider a $(d-p)$-dimensional surface γ with boundary, which satisfies the property iii)'. If the operator $A(g)U_\alpha(\gamma)$ does not satisfy the ETH, our claim holds; we thus consider the case where $A(g)U_\alpha(\gamma)$ satisfies the ETH, i.e.,

$$\langle E_n|A(g)U_\alpha(\gamma)|E_n\rangle \simeq \langle E_m|A(g)U_\alpha(\gamma)|E_m\rangle \simeq \langle A(g)U_\alpha(\gamma)\rangle_{\rm mc}^{\Delta E}(E). \tag{A.0.1}$$

Since H is assumed to have no degeneracy, its eigenstates $|E_n\rangle$, $|E_m\rangle$ are also eigenstates of $U_\alpha(\tilde{C})$. Since the group G is abelian, the eigenvalues are expressed as

$$U_\alpha(\tilde{C})|E_n\rangle = e^{i\alpha q_n}|E_n\rangle \tag{A.0.2}$$

and

$$U_\alpha(\tilde{C})|E_m\rangle = e^{i\alpha q_m}|E_m\rangle, \tag{A.0.3}$$

where $q_n, q_m \in \mathbb{R}$. From the assumption ii), we can assume that $|E_n\rangle$ and $|E_m\rangle$ belong to different sectors, i.e., $e^{i\alpha q_n} \neq e^{i\alpha q_m}$.

Now, the definition of $\bar{\gamma}$ indicates

$$\langle E_n|A(g)U_\alpha^{-1}(\bar{\gamma})|E_n\rangle = \langle E_n|A(g)U_\alpha(\gamma)U_\alpha(\tilde{C})^{-1}|E_n\rangle = e^{-i\alpha q_n}\langle E_n|A(g)U_\alpha(\gamma)|E_n\rangle \tag{A.0.4}$$

and $\langle E_m|A(g)U_\alpha^{-1}(\bar{\gamma})|E_m\rangle = e^{-i\alpha q_m}\langle E_m|A(g)U_\alpha(\gamma)|E_m\rangle$. Recalling the assumption of the ETH and iii)', i.e.,

$$\langle E_n|A(g)U_\alpha(\gamma)|E_n\rangle \simeq \langle E_m|A(g)U_\alpha(\gamma)|E_m\rangle \simeq \langle A(g)U_\alpha(\gamma)\rangle_{\mathrm{mc}}^{\Delta E}(E) \neq 0, \tag{A.0.5}$$

we obtain the relation $\langle E_n|A(g)U_\alpha^{-1}(\bar{\gamma})|E_n\rangle \neq \langle E_m|A(g)U_\alpha^{-1}(\bar{\gamma})|E_m\rangle$. Finally, taking the complex conjugate, we have

$$\langle E_n|U_\alpha(\bar{\gamma})A(g)^\dagger|E_n\rangle \neq \langle E_m|U_\alpha(\bar{\gamma})A(g)^\dagger|E_m\rangle. \tag{A.0.6}$$

Thus, $U_\alpha(\bar{\gamma})A(g)^\dagger$ breaks the ETH, and our claim has been proven.

Appendix B

Numerical details of the ETH for the $\mathbb{Z}_2$ gauge theory

B.1 Fitting parameters for the finite-size scaling analysis

Figure 3(c) in the main text shows the finite-size scaling of the deviation measure

$$\Delta_\infty(\mathcal{O}) := \max_{n, E_n \in [E, E+\delta E]} \left| \langle E_n|\mathcal{O}|E_n\rangle - \langle \mathcal{O}\rangle_{\rm mc}^{\delta E}(E_n) \right|. \tag{B.1.1}$$

The measure $\mathbb{E}[\Delta_\infty(\mathcal{O})]$ averaged over disorder is fitted with a function e^{-aN+b}. Table S-1 shows the fitting parameters for each case. The decay for $U(\bar{\gamma}_x)$ with the total symmetry sectors (Case II) is much slower than the other cases, which indicates that the ETH is hindered only for this case. We use the square lattices $(N_x, N_y) = (3,3)$, $(4,3)$, $(5,3)$ with 10^4, 10^3, and 10^2 samples, respectively. The energy window is set as $[E, E+\delta E] = [-1.3\,N_x, -1.0\,N_x]$.

Table B.1: Observables, symmetry sectors, and the fitting parameters a and b for six different cases.

Case	Observable	Symmetry sector	a	b
I	$U(\bar{\gamma}_x)$	$U(C_x^*) = 1$	0.256 ± 0.017	1.47 ± 0.16
II		$U(C_x^*) = \pm 1$	0.158 ± 0.020	0.74 ± 0.20
III	$U(\gamma_x)$	$U(C_x^*) = 1$	0.260 ± 0.021	1.52 ± 0.23
IV		$U(C_x^*) = \pm 1$	0.244 ± 0.036	1.41 ± 0.40
V	B_p	$U(C_x^*) = 1$	0.310 ± 0.024	2.03 ± 0.24
VI		$U(C_x^*) = \pm 1$	0.297 ± 0.005	2.02 ± 0.05

B.2 ETH for the plaquette operator

Here, we show the detail of the ETH for a plaquette operator B_p along the same line as those for $U(\gamma_x)$ and $U(\bar{\gamma}_x)$ in the main text. Though this operator can be regarded as the minimal Wilson

operator, it acts nontrivially on the links around one plaquette p. In this sense, the operator B_p can be regarded as a local observable.

The expectation values versus eigenenergies are shown in Fig. B.1(a). Along with Fig. 3(c) in the main text, we conclude that the operator B_p satisfies the ETH even without resolving symmetry sectors. Figure B.1(b) shows the time evolution of the expectation value $\langle B_p(t)\rangle$. The stationary state is well described by the ensemble average of the generalized Gibbs ensemble (GGE) and that of the canonical ensemble, where both ensembles provide almost the same prediction.

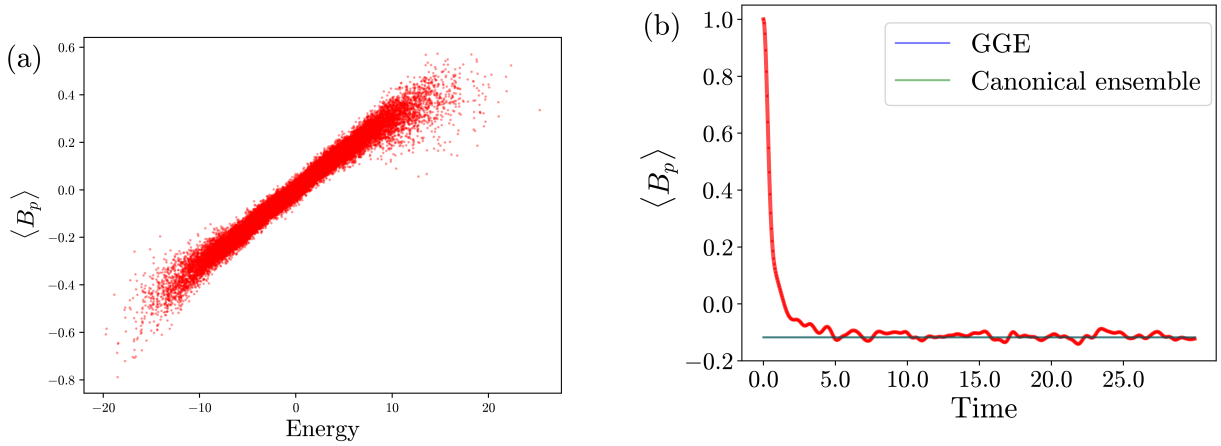

Figure B.1: (a) Expectation values of the local operator B_p with respect to energy eigenstates for the 5×3 lattice [139]. We see that B_p satisfies the ETH (also see Fig. 3(c) in the main text), even without resolving the symmetry sectors associated with $U(C_x^*)$ and $U(C_y^*)$.
(b) Time evolution of the expectation value of B_p for the 4×3 lattice [139]. The initial state is a random superposition of the eigenstates of B_p with the eigenvalue $+1$, whose energy expectations lie within an energy window $E \in [-5.0, -3.0]$. The prediction for the GGE and that for canonical ensemble make no difference (i.e., two results are almost overlapped), and the stationary value of $\langle B_p \rangle$ is described by them.

Appendix C

Justification of the GGE

C.1 Justification for the case with $H_{\mathbb{Z}_2}$

In the main text, we have introduced the GGE that takes into account discrete 1-form symmetry, $U(C_x^*)$. Here we show that this GGE can be justified if we assume the ETH restricted for each symmetry sector with respect to $U(C_x^*)$. Our GGE that takes account of the $\mathbb{Z}_2$ 1-form symmetry is given by

$$\begin{aligned} \langle \mathcal{O} \rangle_{\mathrm{GGE}} &= \mathrm{Tr}(\mathcal{O}\rho_{\mathrm{GGE}}), \\ \rho_{\mathrm{GGE}}(\beta, \lambda_x, \mu_x) &:= \frac{1}{Z_{\mathrm{GGE}}(\beta, \lambda_x, \mu_x)} e^{-\beta H_{\mathbb{Z}_2} - \lambda_x U(C_x^*) - \mu_x U(C_x^*) H_{\mathbb{Z}_2}}, \\ Z_{\mathrm{GGE}}(\beta, \lambda_x, \mu_x) &:= \mathrm{Tr}[e^{-\beta H_{\mathbb{Z}_2} - \lambda_x U(C_x^*) - \mu_x U(C_x^*) H_{\mathbb{Z}_2}}], \end{aligned} \tag{C.1.1}$$

where β, λ_x, μ_x are determined from the initial values of the conserved quantities $H_{\mathbb{Z}_2}, U(C_x^*)$, and $U(C_x^*)H_{\mathbb{Z}_2}$ (see Eq. (C.1.12)).

To see that the GGE describes the stationary state, we first obtain the expression for the stationary state, i.e., the long-time average of the expectation value of an observable $\mathcal{O}$ (we assume that the temporal fluctuation around the long-time average is negligible in the thermodynamic limit). For this purpose, we define the projection operator onto each symmetry sector by

$$P_{\pm}^x := \frac{1}{2}\big(1 \pm U(C_x^*)\big), \tag{C.1.2}$$

Once we assume the ETH with respect to each symmetry sector, the expectation values $\langle P_+^x \rangle$, $\langle P_+^x H_{\mathbb{Z}_2} P_+^x \rangle$ and $\langle P_-^x H_{\mathbb{Z}_2} P_-^x \rangle$ suffice to specify the stationary state. To see this, we consider the time evolution from an initial state $|\psi\rangle = \sum_n c_n |E_n\rangle$ with the energy eigenstates $|E_n\rangle$. Then, if we assume no degeneracy for the energy eigenvalues, the long-time average $\overline{\langle \psi | \mathcal{O} | \psi \rangle} := \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \, \langle \psi(t) | \mathcal{O} | \psi(t) \rangle$

is given by

$$\begin{aligned}\overline{\langle\psi|\mathcal{O}|\psi\rangle} &= \sum_n |c_n|^2 \langle E_n|\mathcal{O}|E_n\rangle \\ &= \sum_{n_+:U(C_x^*)=+1 \text{ sector}} |c_{n_+}|^2 \langle E_{n_+}^+|\mathcal{O}|E_{n_+}^+\rangle + \sum_{n_-:U(C_x^*)=-1 \text{ sector}} |c_{n_-}|^2 \langle E_{n_-}^-|\mathcal{O}|E_{n_-}^-\rangle \\ &\simeq \langle P_+^x\rangle \langle P_+^x\mathcal{O}P_+^x\rangle_{\rm mc}^{\delta E_+}(\mathcal{E}_+) + \langle P_-^x\rangle \langle P_-^x\mathcal{O}P_-^x\rangle_{\rm mc}^{\delta E_-}(\mathcal{E}_-), \end{aligned} \tag{C.1.3}$$

where $|E_{n_+}^+\rangle$ and $|E_{n_-}^-\rangle$ denote the eigenstates with $U(C_x^*)=+1$ and $U(C_x^*)=-1$, respectively. Here, we have used the ETH for each symmetry sector

$$\langle E_{n_\pm}^\pm|\mathcal{O}|E_{n_\pm}^\pm\rangle \simeq \langle P_\pm^x\mathcal{O}P_\pm^x\rangle_{\rm mc}^{\delta E_\pm}(E_{n_\pm}^\pm) \tag{C.1.4}$$

and assumed that $\frac{|c_{n_\pm}|^2}{\sum_{n_\pm}|c_{n_\pm}|^2}$ is localized around the mean energy for each symmetry sector

$$\mathcal{E}_\pm = \frac{\sum_{n_\pm}|c_{n_\pm}|^2 E_{n_\pm}}{\sum_{n_\pm}|c_{n_\pm}|^2} = \frac{\langle P_\pm^x H_{\mathbb{Z}_2} P_\pm^x\rangle}{\langle P_\pm^x\rangle}, \tag{C.1.5}$$

where $\sum_{n_\pm}$ stands for $\sum_{n_\pm:U(C_x^*)=\pm1 \text{ sector}}$ in the following. Note that the precise values of $\delta E_\pm$ are not important, and we will not consider them in the following.

Next, we consider the GGE. For this purpose, we define canonical ensembles for each sector

$$\langle\mathcal{O}\rangle_{\rm can,\pm} := \frac{1}{Z_\pm}\sum_{n_\pm}\mathcal{O}_{n_\pm n_\pm} e^{-(\beta\pm\mu_x)E_{n_\pm}^\pm \mp\lambda_x}, \qquad Z_\pm := \sum_{n_\pm} e^{-(\beta\pm\mu_x)E_{n_\pm}^\pm\mp\lambda_x}. \tag{C.1.6}$$

In terms of these ensembles, the GGE (C.1.1) is expressed as

$$\begin{aligned}\mathrm{Tr}(\mathcal{O}\rho_{\rm GGE}) &= \frac{1}{Z_{\rm GGE}}\left(\sum_{n_+} e^{-\beta E_{n_+}-\lambda_x-\mu_x E_{n_+}}\mathcal{O}_{n_+n_+} + \sum_{n_-} e^{-\beta E_{n_-}+\lambda_x+\mu_x E_{n_-}}\mathcal{O}_{n_-n_-}\right) \\ &= \sum_{s=\pm}\frac{Z_s}{Z_{\rm GGE}}\langle\mathcal{O}\rangle_{{\rm can},s} = \sum_{s=\pm}\langle P_s^x\rangle_{\rm GGE}\langle\mathcal{O}\rangle_{{\rm can},s}\,. \end{aligned} \tag{C.1.7}$$

As for the standard canonical ensemble, by appropriately choosing $\beta\pm\mu_x$, we can set

$$\mathcal{E}_\pm = \langle H\rangle_{\rm can,\pm} = \frac{\langle P_\pm^x H_{\mathbb{Z}_2} P_\pm^x\rangle_{\rm GGE}}{\langle P_\pm^x\rangle_{\rm GGE}}. \tag{C.1.8}$$

Under this condition, we assume that the distributions $\langle E_{n_\pm}^\pm|\rho_\pm|E_{n_\pm}^\pm\rangle = \exp\Big(-(\beta\pm\mu_x)E_{n_\pm}^\pm \mp\lambda_x\Big)/Z_\pm$ are sufficiently localized around $\mathcal{E}_\pm$. Then, using the ETH for each symmetry sector, we can write the operator averages as

$$\langle\mathcal{O}\rangle_{\rm can,\pm} \simeq \frac{1}{Z_\pm}\sum_{n_\pm} e^{-(\beta\pm\mu_x)E_{n_\pm}^\pm\mp\lambda_x}\langle P_\pm^x\mathcal{O}P_\pm^x\rangle_{\rm mc}(\mathcal{E}_\pm) = \langle P_\pm^x\mathcal{O}P_\pm^x\rangle_{\rm mc}(\mathcal{E}_\pm). \tag{C.1.9}$$

Choosing λ_x appropriately, we can set

$$\langle P^x_+\rangle_{\rm GGE} = \langle P^x_+\rangle\,, \qquad \big(\Rightarrow \quad \langle P^x_-\rangle_{\rm GGE} = 1-\langle P^x_+\rangle_{\rm GGE} = 1-\langle P^x_+\rangle = \langle P^x_-\rangle\big). \tag{C.1.10}$$

In summary, by setting β, μ_x, and λ_x such that Eqs. (C.1.8) and (C.1.10) hold true, we can show that the long-time average of the observable $\mathcal{O}$ (C.1.3) can be identified with the GGE prediction:

$$\overline{\langle\psi|\mathcal{O}|\psi\rangle} \simeq \sum_{\pm} \langle P^x_\pm\rangle_{\rm GGE}\,\langle\mathcal{O}\rangle_{{\rm can},\pm} = {\rm Tr}(\mathcal{O}\rho_{\rm GGE}). \tag{C.1.11}$$

Note that the conditions Eqs. (C.1.8) and (C.1.10) are equivalent to the following condition for the conserved quantities in the GGE,

$$\langle H_{\mathbb{Z}_2}\rangle_{\rm GGE} = \langle H_{\mathbb{Z}_2}\rangle\,,\ \langle U(C^*_x)\rangle_{\rm GGE} = \langle U(C^*_x)\rangle\,,\ \langle U(C^*_x)H_{\mathbb{Z}_2}\rangle_{\rm GGE} = \langle U(C^*_x)H_{\mathbb{Z}_2}\rangle\,, \tag{C.1.12}$$

as expected.

The above discussion has focused on observables that are non-local in the x-direction but local in the y-direction. To adopt operators extended to the y-direction as well, we should include at most seven chemical potentials in total as $\tilde{\rho}_{\rm GGE} = \tilde{Z}^{-1}_{\rm GGE}\exp\big(-\beta H_{\mathbb{Z}_2} - \sum_{i=x,y}\lambda_i U(C^*_i) - \sum_{i=x,y}\mu_i U(C^*_i)H_{\mathbb{Z}_2} - \alpha U(C^*_x)U(C^*_y) - \alpha' U(C^*_x)U(C^*_y)H_{\mathbb{Z}_2}\big)$.

C.2 Justification for the general case

In the case of a general finite abelian group G, we propose that the following density matrix serves as the GGE:

$$\rho^G_{\rm GGE}(\beta^G,\{\lambda^G_j\},\{\gamma^G_j\}) = \frac{1}{Z^G_{\rm GGE}(\beta^G,\{\lambda^G_j\},\{\gamma^G_j\})}e^{-\beta^G H-\sum_{j=1}^{N-1}\lambda^G_j P_j-\sum_{j=1}^{N-1}\mu^G_j P_j H}, \tag{C.2.1}$$

where P_j $(j=1,\dots,N)$ are the projections to each symmetry sector, and the number of the sectors is given by $N = |H_{d-p}(\mathcal{M},G)|$. Note here that the summation over the chemical potentials is performed over $1\le j\le N-1$ because operators P_N and $P_N H$ are not independent of the other conserved quantities.

Assuming the ETH for each symmetry sector,

$$\langle E^j_{n_j}|\mathcal{O}|E^j_{n_j}\rangle = \langle P_j\mathcal{O}P_j\rangle^{\delta E_j}_{\rm mc}(E^j_{n_j}) \quad (1\le j\le N), \tag{C.2.2}$$

we can obtain the stationary state as in the $\mathbb{Z}_2$ case (C.1.3):

$$\overline{\langle\psi|\mathcal{O}|\psi\rangle} = \sum_n |c_n|^2 \langle E_n|\mathcal{O}|E_n\rangle = \sum_{j=1}^{N}\sum_{n_j} |c_{n_j}|^2 \langle E^j_{n_j}|\mathcal{O}|E^j_{n_j}\rangle$$
$$\simeq \sum_{j=1}^{N} \langle P_j\rangle \langle P_j\mathcal{O}P_j\rangle^{\delta E_j}_{\rm mc}(\mathcal{E}_j), \tag{C.2.3}$$

where $|E^j_{n_j}\rangle$ are energy eigenstates in each symmetry sector and

$$\mathcal{E}_j = \frac{\langle P_jHP_j\rangle}{\langle P_j\rangle}. \tag{C.2.4}$$

Here, $\sum_{n_j}$ means that the sum is taken only over eigenstates that belong to the symmetry sector j. We have also assumed the localization of the energy distribution in each sector around $\mathcal{E}_j$. In the following, we omit δE_j, whose details are not important in the thermodynamic limit.

Along the same lines as the $\mathbb{Z}_2$ case, the GGE (C.2.1) can be written as

$$\langle\mathcal{O}\rangle^G_{\rm GGE} = \frac{1}{Z^G_{\rm GGE}}\left(\sum_{j=1}^{N-1}\sum_{n_j} e^{-(\beta^G+\mu^G_j)E_{n_j}-\lambda_j}\mathcal{O}_{n_jn_j} + \sum_{n_N} e^{-\beta^G E_{n_N}}\mathcal{O}_{n_Nn_N}\right)$$
$$= \sum_{j=1}^{N}\frac{Z_j}{Z^G_{\rm GGE}}\langle\mathcal{O}\rangle_{{\rm can},j} = \sum_{j=1}^{N}\langle P_j\rangle^G_{\rm GGE}\langle\mathcal{O}\rangle_{{\rm can},j}\,, \tag{C.2.5}$$

where canonical ensembles restricted to each sector are defined by

$$\langle\mathcal{O}\rangle_{{\rm can},j} := \frac{1}{Z_j}\sum_{n_j}\mathcal{O}_{n_jn_j}e^{-(\beta^G+\mu^G_j)E_{n_j}-\lambda^G_j}, \qquad Z_j := \sum_{n_j} e^{-(\beta^G+\mu^G_j)E_{n_j}-\lambda^G_j}, \qquad (j=1,\dots,N-1),$$
$$\langle\mathcal{O}\rangle_{{\rm can},N} := \frac{1}{Z_N}\sum_{n_N}\mathcal{O}_{n_Nn_N}e^{-\beta^G E_{n_N}}, \qquad Z_N := \sum_{n_N} e^{-\beta^G E_{n_N}}. \tag{C.2.6}$$

Here, we can choose β^G and μ^G_j $(j=1,\cdots,N-1)$ such that

$$\mathcal{E}_j = \langle H\rangle_{{\rm can},j} = \frac{\langle P_jHP_j\rangle^G_{\rm GGE}}{\langle P_j\rangle^G_{\rm GGE}} \tag{C.2.7}$$

holds for $1\le j\le N$. Now, the assumption that the distributions $\langle E_{n_j}|\rho_j|E_{n_j}\rangle := \exp\Big(-(\beta^G+\mu^G_j)E_{n_j}-\lambda^G_j\Big)/Z_j$ and $\langle E_{n_N}|\rho_N|E_{n_N}\rangle := \exp\big(-\beta^G E_{n_N}\big)$ are localized around $\mathcal{E}_j$ again leads to the following:

$$\langle\mathcal{O}\rangle_{{\rm can},j} = \langle P_j\mathcal{O}P_j\rangle_{\rm mc}(\mathcal{E}_j), \qquad (j=1,\dots,N). \tag{C.2.8}$$

We can determine λ^G_j $(j=1,\dots,N-1)$ uniquely such that the following holds true:

$$\langle P_j\rangle^G_{\rm GGE} = \langle P_j\rangle\,, \qquad (j=1,\dots,N-1), \tag{C.2.9}$$

$$\Rightarrow \quad \langle P_N \rangle^G_{\mathrm{GGE}} = \frac{Z_N}{Z_{\mathrm{GGE}}} = 1 - \sum_{j=1}^{N-1} \frac{Z_j}{Z_N} = 1 - \sum_{j=1}^{N-1} \langle P_j \rangle = \langle P_N \rangle \,. \tag{C.2.10}$$

In summary, by choosing β^G, λ_j^G, and μ_j^G $(j = 1, \cdots, N-1)$ such that Eqs. (C.2.7) and (C.2.9) hold true, we can show that the time average (C.2.3) is given by

$$\overline{\langle \psi | \mathcal{O} | \psi \rangle} = \sum_{j=1}^{N} \langle P_j \rangle^G_{\mathrm{GGE}} \langle \mathcal{O} \rangle_{\mathrm{can},j} = \langle \mathcal{O} \rangle^G_{\mathrm{GGE}} \,. \tag{C.2.11}$$

Finally, the conditions Eqs. (C.2.7) and (C.2.9) are equivalent to the following condition for the conserved quantities in the GGE,

$$\langle H \rangle^G_{\mathrm{GGE}} = \langle H \rangle \,,\ \langle P_j \rangle^G_{\mathrm{GGE}} = \langle P_j \rangle \,,\ \langle P_j H \rangle^G_{\mathrm{GGE}} = \langle P_j H \rangle \ \ (1 \le j \le N-1), \tag{C.2.12}$$

where we have used $P_j H P_j = P_j H$ and $\langle H \rangle = \sum_{j=1}^N \langle P_j H P_j \rangle$.

Recalling the relation (C.1.2), we can recover the GGE (C.1.1) by redefining β^G, λ_j^G, and μ_j^G for the $G = \mathbb{Z}_2$ case. In addition, similar reasoning justifies the GGE $\tilde{\rho}_{\mathrm{GGE}}$ for $G = \mathbb{Z}_2$ discussed in the main text, where we consider the non-local observables both for the x- and y-directions.

Appendix D

Hamiltonian formulation of the $\mathbb{Z}_2$ lattice gauge theory

D.1 $\mathbb{Z}_2$ gauge theory

We briefly review the Hamiltonian formalism of the $\mathbb{Z}_2$ lattice gauge theory here. Before introducing the Hamiltonian, we start with the path integral formulation in order to give an overview of basic concepts. Unless otherwise noted, lattices are on the $(2+1)$-dimensional Euclidean spacetime with coordinates x^0, x^1, x^2. The partition function of a gauge theory with a gauge group G is defined by

$$\mathcal{Z} = \int \left(\prod_{\boldsymbol{r},\mu} dU_\mu(\boldsymbol{r}) \right) \exp\left[\frac{1}{2g} \sum_{\boldsymbol{r},\mu\nu} \left(\mathrm{tr}\big(U_\mu(\boldsymbol{r}) U_\nu(\boldsymbol{r}+\boldsymbol{e}_\mu) U_\mu^{-1}(\boldsymbol{r}+\boldsymbol{e}_\nu) U_\nu^{-1}(\boldsymbol{r})\big) + (\text{c.c.}) \right) \right], \quad \text{(D.1.1)}$$

where the dynamical variable variables $U(\boldsymbol{r})$ reside on each link of the lattice, and labelled by its origin. The integral $\int dU_\mu(\boldsymbol{r})$ is with respect to the Haar measure of the group G. In the case of $G = \mathbb{Z}_2$, the dynamical variable U_μ is just denoted by

$$U_\mu = \tau_\mu \in \{\pm 1\}, \quad \text{(D.1.2)}$$

and the partition function reads

$$\mathcal{Z}_{\mathbb{Z}_2} = \sum_{\{\tau_\mu\}=\pm 1} \exp\left[\sum_{p \in \text{plaquettes}} \frac{1}{g_p} G_p \right], \quad \text{(D.1.3)}$$

$$G_p := \tau_\mu(\boldsymbol{r}) \tau_\nu(\boldsymbol{r}+\boldsymbol{e}_\mu) \tau_\mu^{-1}(\boldsymbol{r}+\boldsymbol{e}_\nu) \tau_\nu^{-1}(\boldsymbol{r}), \qquad (\mu\nu \in p). \quad \text{(D.1.4)}$$

We can see that the field strength G_p is invariant under gauge transformations

$$\begin{aligned} &\tau_\mu(\boldsymbol{r}_0) \mapsto -\tau_\mu(\boldsymbol{r}_0), \quad \tau_\mu(\boldsymbol{r}_0 - \boldsymbol{e}_\mu) \mapsto -\tau_\mu(\boldsymbol{r}_0 - \boldsymbol{e}_\mu), \qquad {}^\forall\mu, \\ &\tau_\mu(\boldsymbol{r}) \mapsto \tau_\mu(\boldsymbol{r}), \qquad \text{for } \boldsymbol{r} \neq \boldsymbol{r}_0. \end{aligned} \quad \text{(D.1.5)}$$

Sets of transformation points $\boldsymbol{r}_0$ form two-dimensional surface defects defined as

$$\langle \widetilde{U}(S^*)\rangle = \frac{1}{\mathcal{Z}_{\mathbb{Z}_2}} \mathcal{Z}_{\mathbb{Z}_2}\big(\tau_\mu \mapsto -\tau_\mu \text{ for } \tau_\mu \in S^*\big)\,. \tag{D.1.6}$$

A two-dimensional surface S^* is defined on the dual surface. The field strength itself is invariant under the transformation, $\langle \widetilde{U}(S^*)\rangle = 1$ holds for any topology of the spacetime. A gauge invariant non-local operator can be defined by

$$W(C) := \prod_{(\boldsymbol{r},\mu)\in C} \tau_\mu(\boldsymbol{r})\,, \qquad C:\ \text{closed loop}, \tag{D.1.7}$$

which is referred to as the Wilson operator. In the three-dimensional spacetime, there exists a topological disorder defect called 't Hooft operator, whose expectation value can be evaluated as

$$\langle U(C^*)\rangle = \frac{1}{\mathcal{Z}_{\mathbb{Z}_2}} \sum_{\{\tau_\mu\}=\pm 1} \left[\left(\prod_{p\notin C^*} \exp\left[\frac{1}{g_p} G_p\right]\right)\left(\prod_{p\in C^*} \exp\left[\frac{1}{g_p}(-G_p)\right]\right)\right], \tag{D.1.8}$$

where a closed loop C^* is defined on the dual lattice. An 't Hooft operator $U(C^*)$ is topological in the sense that it is invariant under continuous deformation, and thus $U(C^*) = 1$ holds as long as the closed loop C^* is topologically trivial. Though $U(C^*)$ looks like a disorder in the original $\mathbb{Z}_2$ gauge theory, it can be explicitly represented by the dual dynamical variable. For details, see appendix D.2. One can easily check that a Wilson operator and an 't Hooft operator satisfy the following relation:

$$\begin{aligned}
\langle W(C)U(C^*)\rangle &:= \frac{1}{\mathcal{Z}_{\mathbb{Z}_2}} \sum_{\{\tau_\mu\}=\pm 1} \left[\left(\prod_{p\notin C^*} \exp\left[\frac{1}{g_p} G_p\right]\right)\left(\prod_{p\in C^*} \exp\left[\frac{1}{g_p}(-G_p)\right]\right) \prod_{(\boldsymbol{r},\mu)\in C} \tau_\mu(\boldsymbol{r})\right] \\
&= (-1)^{\mathrm{link}(C,C^*)} \frac{1}{\mathcal{Z}_{\mathbb{Z}_2}} \sum_{\{\tau_\mu\}=\pm 1} \left[\left(\prod_{p\in \text{plaquettes}} \exp\left[\frac{1}{g_p} G_p\right]\right) \prod_{(\boldsymbol{r},\mu)\in C} \tau_\mu(\boldsymbol{r})\right] \\
&= (-1)^{\mathrm{link}(C,C^*)} \langle W(C)\rangle\,,
\end{aligned} \tag{D.1.9}$$

where $\mathrm{link}(C, C^*)$ denotes the linking number of the closed loops C and C^*. The expression (D.1.9) means that the 't Hooft operator $U(C^*)$ measures the "electric" charge of the Wilson operator. This is thus the realization of the electric $\mathbb{Z}_2$ 1-form symmetry. 't Hooft operators can also have endpoints, i.e., they are defined on an open path $\tilde{\gamma}$ with two endpoints s_1^*, s_2^* as

$$\langle U(\tilde{\gamma})\rangle = \frac{1}{\mathcal{Z}_{\mathbb{Z}_2}} \sum_{\{\tau_\mu\}=\pm 1} \left[\left(\prod_{p\notin \tilde{\gamma}} \exp\left[\frac{1}{g_p} G_p\right]\right)\left(\prod_{p\in \tilde{\gamma}} \exp\left[\frac{1}{g_p}(-G_p)\right]\right)\right]. \tag{D.1.10}$$

Since the path $\tilde{\gamma}$ can be continuously deformed without varying the expectation value, this operator is regarded as a dynamical "monopole" and "anti-monopole" residing at the dual sites s_1^* and s_2^*.

We now move on to the Hamilton formulation. To this end, we take a continuum limit in the time direction. In this limit, the partition function (D.1.3) admits the following expression:

$$[\mathcal{Z}_{\mathbb{Z}_2}]_{\text{temporal gauge:}\tau_0(\boldsymbol{r})=+1} = \mathcal{Z}_{\mathbb{Z}_2}/\text{const.} \longrightarrow \operatorname{tr} e^{-\beta H_{\mathbb{Z}_2}} . \tag{D.1.11}$$

The Hamiltonian is given by

$$H_{\mathbb{Z}_2} = -\sum_{\boldsymbol{r},j} \sigma_j^1(\boldsymbol{r}) - \sum_{\boldsymbol{r},jk} \lambda_{\boldsymbol{r},jk} \sigma_j^3(\boldsymbol{r}) \sigma_k^3(\boldsymbol{r}+\boldsymbol{e}_j) \sigma_j^3(\boldsymbol{r}+\boldsymbol{e}_k) \sigma_j^3(\boldsymbol{r}) , \tag{D.1.12}$$

where $\sigma_j^{1,2,3}(\boldsymbol{r})$ denotes the Pauli matrices acting on the link $(\boldsymbol{r}, j)$. Though the total Hilbert space of the system for the $N_x \times N_y$ lattice is $(2^{2N_xN_y})$-dimensional, we have to project it onto the *physical* Hilbert space. This is because there exist residual gauge redundancies, after we impose the temporal gauge condition $\tau_0(\boldsymbol{r}) = 1$, which is analogous to the gauge $A_0(\boldsymbol{r}) = 0$ for the Maxwell theory.

The spatial gauge transformation is generated by the local operator

$$Q_v := \prod_{\substack{b \in \text{spatial links} \\ b \ni v}} \sigma_b^1(\boldsymbol{r}_b) , \qquad\qquad v : \text{ vertex} . \tag{D.1.13}$$

The operator Q_v obviously satisfies $Q_v^2 = 1$ and commutes with $H_{\mathbb{Z}_2}$. The spatial gauge redundancy should be removed, and thus the physical Hilbert space is given by

$$\text{span}\left\{ |\psi\rangle_{\text{phys}} \;\middle|\; Q_v |\psi\rangle_{\text{phys}} = +|\psi\rangle_{\text{phys}} , \; {}^\forall v : \text{ vertices} \right\} = \text{span}\left\{ \prod_{v \in \text{vertices}} \left(\frac{1+Q_v}{2} \right) |\psi\rangle \right\} . \tag{D.1.14}$$

Note here that the constraint for the physical states can be regarded as the $\mathbb{Z}_2$ analog of the Gauss law $\nabla \cdot E |\psi\rangle_{\text{phys}} = 0$ since we can write $Q_v = (\sigma_x^1(\boldsymbol{r}-\boldsymbol{e}_x))^{-1} \sigma_x^1(\boldsymbol{r}+\boldsymbol{e}_x) (\sigma_y^1(\boldsymbol{r}-\boldsymbol{e}_y))^{-1} \sigma_y^1(\boldsymbol{r}+\boldsymbol{e}_y)$. After this projection, the expectation value of a non-gauge invariant operator with respect to physical states $|\psi\rangle_{\text{phys}}$ always vanishes. States in the physical space (D.1.14) will be denoted just by $|\psi\rangle$ in the following discussion.

The next task is to represent the Wilson and 't Hooft line as operators acting on the Hilbert space. The Wilson line and the 't Hooft line lying on the spatial directions are given by

$$W(C) = \prod_{b \in C} \sigma_b^3 = W^{-1}(C) , \tag{D.1.15}$$

$$U(C^*) = \prod_{b^* \in C^*} \sigma^1_{b^*} = U^{-1}(C^*)\,, \tag{D.1.16}$$

where C and C^* are closed loops lying on the lattice and dual lattice, respectively. Both of them commutes with the operator Q_v , and they are indeed gauge invariant. Corresponding to the topological property of the 't Hooft operator (D.1.8), continuous deformations of the path C^* does not change $U(C^*)$ acting on physical states, i.e.,

$$U(C^*_1)|\psi\rangle = U(C^*_2)|\psi\rangle\,, \tag{D.1.17}$$

if C'_1 and C'_2 are homotopically equivalent. It immediately follows that $U(C')|\psi\rangle = |\psi\rangle$ if the dual closed loop C' is trivial. In addition to (D.1.17), the operator U is topological in the time direction in the sense that

$$[H_{\mathbb{Z}_2}, U(C^*)] = 0\,. \tag{D.1.18}$$

In the following, we take a 2-torus as the space manifold with periodicity N_x and N_y in the 1,2 direction, respectively. We define closed loops winding around the x-/y-cycle by C_x and C_y , and the loops on the dual lattice are denoted by C^*_x and C^*_y in the same manner. The operators W and U then satisfy the relation

$$\begin{aligned} &U(C^*_y)W(C_x)U^{-1}(C^*_y) = -W(C_x)\,, \qquad U(C^*_x)W(C_y)U^{-1}(C^*_x) = -W(C_y)\,, \\ &U(C^*_x)W(C_x)U^{-1}(C^*_x) = +W(C_x)\,, \qquad U(C^*_y)W(C_y)U^{-1}(C^*_y) = +W(C_y)\,, \end{aligned} \tag{D.1.19}$$

which are indeed operator-formulation of the electric $\mathbb{Z}_2$ 1-form symmetry (D.1.9).

Since the 2-torus has a homology group $H_1(T^2;\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, we have four sectors characterized by the charge of the $\mathbb{Z}_2$ 1-form symmetry. Though simultaneous diagonalization of $U(C^*_x)$, $U(C^*_y)$, and $H_{\mathbb{Z}_2}$ leads to four separated dynamics similarly to standard global symmetries, a significant feature is their dimensionality. For standard *0-form* symmetries, the symmetry operator U is a codimension-1 operator comparison to the spacetime dimension (2+1-dimension in this case). On the other hand, 1-form symmetries incorporate a codimension-2 symmetry operator.

We remark that there also exists a standard $\mathbb{Z}_2$ 0-form symmetry which flips all of link variables. The symmetry operator is a 2-dimensional object defined by

$$\widehat{U}_{\mathbb{Z}_2} = \prod_{b \in \text{links}} \sigma^1_b\,. \tag{D.1.20}$$

Though this operator is not relevant to the $\mathbb{Z}_2$ 1-form symmetry in general, it acts on the physical states equivalently to $U(C^*)$ for the lattice gauge theories:

$$\widehat{U}_{\mathbb{Z}_2}|\psi\rangle = U(C_x^*)^{N_x} U(C_y^*)^{N_y}|\psi\rangle\,, \tag{D.1.21}$$

and thus the global $\mathbb{Z}_2$ 0-form symmetry has no effect on the four symmetry sectors.

D.2 Dual variable representation of the $\mathbb{Z}_n$ gauge theory

The three-dimensional $\mathbb{Z}_n$ gauge theory is dual to the three-dimensional $\mathbb{Z}_n$ Ising spin model [170] if the spacetime manifold $\mathcal{M}$ has a trivial homology $H_1(\mathcal{M},\mathbb{Z}) = 1$. This can be explicitly shown by expressing the theory in terms of dual variables. We will repeatedly utilize the identity

$$\frac{1}{n}\sum_{l=0}^{n-1}\exp\left(-2\pi i\frac{ml}{n}\right) = \delta_{m,0}(\mathrm{mod}\, n) := \begin{cases} 1 & n = 0\ (\mathrm{mod}\, n) \\ 0 & n \neq 0\ (\mathrm{mod}\, n) \end{cases}\,. \tag{D.2.1}$$

The link variable of the the $\mathbb{Z}_n$ gauge theory is given by

$$U_\mu(\boldsymbol{r}) \in \left\{\exp\left(2\pi i\frac{k}{n}\right),\ k \in \{0,1,\ldots,n-1\}\right\}\,, \tag{D.2.2}$$

and then the partition function reads

$$\mathcal{Z}_{\mathbb{Z}_n} = \sum_{\{U_\mu(\boldsymbol{r})\}}\left(\prod_{p\in\text{plaquettes}}\exp[F_p(G_p)]\right), \qquad F_p(x) := \frac{1}{2g_p}\big(x + x^*\big) = \frac{1}{2g_p}\big(x + x^{-1}\big)\,, \tag{D.2.3}$$

$$G_p := U_\mu(\boldsymbol{r})U_\nu(\boldsymbol{r}+\boldsymbol{e}_\mu)U_\mu^{-1}(\boldsymbol{r}+\boldsymbol{e}_\nu)U_\nu^{-1}(\boldsymbol{r})\,. \tag{D.2.4}$$

Noting the identity (D.2.1), the summand is expressed as

$$\exp[F_p(G_p)] = \sum_{l_p=0}^{n-1} I_{l_p} G_p^{l_p}\,, \tag{D.2.5}$$

$$I_{\ell_p} := \frac{1}{n}\sum_{n_p=0}^{n-1}\exp\left(-2\pi i\frac{n_p l_p}{n}\right)\exp\left[F_p\left(e^{2\pi i\frac{n_p}{n}}\right)\right], \tag{D.2.6}$$

since the summation with respect to n_p leads to the delta function. In order to obtain the partition function, we perform the summation over all of the configuration of $\{U_\mu(\boldsymbol{r})\}$. This summation again give rise to the delta function, and thus we obtain

$$\mathcal{Z}_{\mathbb{Z}_n} = \sum_{\{l_p\}=0}\left(\prod_{p\in\text{plaquettes}} I_{l_p}\right), \tag{D.2.7}$$

where the notation $\{l_p\} = 0$ means summing over configurations which satisfies

$$\sum_{p \ni b} l_p = 0 \pmod n , \qquad {}^{\forall} b : \text{link} . \tag{D.2.8}$$

Here we introduce the dual variable

$$\xi_p := \exp\left(2\pi i \frac{l_p}{n} \right) , \qquad \text{with the constraint: } \prod_{p \ni b} \xi_p = 1 . \tag{D.2.9}$$

The constraint can be solved by denoting the dynamical variable living each dual site:

$$\xi_p = \zeta_{s_1^*} \zeta_{s_2^*}^{-1} , \tag{D.2.10}$$

where the dual link p has a starting point s_1^* and an ending point s_2^*. After all, the expression (D.2.7) takes the form

$$\mathcal{Z}_{\mathbb{Z}_n} = \sum_{\{\zeta_{s^*}\}} \left(\prod_{p \in \text{plaquettes}} \sum_{n_p=0}^{n-1} \xi_p^{-n_p} \exp\left[F_p(e^{2\pi i \frac{n_p}{n}}) \right] \right) \tag{D.2.11}$$

$$= \sum_{\{\zeta_{s^*}\}} \prod_{p \in \text{plaquettes}} \exp\left[\tilde{F}_p(\xi_p) \right] . \tag{D.2.12}$$

The dual action can be explicitly obtained by

$$\tilde{F}_p(\xi_p) = \log\left(\sum_{k=0}^{n-1} \xi_p^{-k} \exp\left[F_p(e^{2\pi i \frac{k}{n}}) \right] \right) . \tag{D.2.13}$$

In the case of the $\mathbb{Z}_2$ gauge theory $(n = 2)$, the action is just $F_p(G_p) = G_p/g_p$, and thus $\tilde{F}_p$ is given by

$$\tilde{F}_p(\xi_p) = \log\left(\sum_{k=0}^{1} \xi_p^{-k} \exp\left[\frac{(-1)^k}{g_p} \right] \right) = \log\left(1 + \xi_p e^{-1/g_p} \right) , \tag{D.2.14}$$

which implies the dual theory is the standard $\mathbb{Z}_2$ Ising model with the nearest neighbor, whose Hamiltonian reads

$$\begin{aligned} \tilde{H}_I(\{\zeta_{s^*}\}) &= \sum_{s_1^* s_2^* \in \text{dual link}} \left(g_{s^*} \zeta_{s_1^*} \zeta_{s_2^*} + \alpha_{s^*} \right) , \\ g_{s^*} &:= \frac{1}{2} \log\left(\frac{1 + e^{-1/g_p}}{1 - e^{-1/g_p}} \right) , \quad \alpha_{s^*} := \frac{1}{2} \log\left((1 - e^{-1/g_p})(1 + e^{-1/g_p}) \right) . \end{aligned} \tag{D.2.15}$$

The relation between the coupling constants g_p and g_{s^*} can be rewritten as

$$1 = \sinh(2 g_{s^*}) \sinh\left(\frac{1}{g_p} \right) . \tag{D.2.16}$$

Since the Wilson operator is given by $W(C) := \prod_{b\in C} U_{\mu_b}(\boldsymbol{r}_b)$ for a closed loop C, we have

$$\begin{aligned}
\langle W(C)\rangle &= \frac{1}{\mathcal{Z}_{\mathbb{Z}_n}} \sum_{\{U_\mu(\boldsymbol{r})\}} \left[\left(\prod_{p\in\text{plaquettes}} \exp[F_p(G_p)] \right) \prod_{b\in C} U_{\mu_b}(\boldsymbol{r}_b) \right] \\
&= \frac{1}{\mathcal{Z}_{\mathbb{Z}_n}} \sum_{\{U_\mu(\boldsymbol{r})\}} \left[\left(\prod_{p\in\text{plaquettes}} \exp[F_p(G_p)] \right) \prod_{p\in S} G_p \right] \\
&= \frac{1}{\mathcal{Z}_{\mathbb{Z}_n}} \sum_{\{\zeta_{s^*}\}} \left(\prod_{p\in\text{plaquettes}} \sum_{n_p=0}^{n-1} \xi_p^{-n_p} \exp\Big[F_p(e^{2\pi i \frac{n_p}{n}})\Big] \prod_{p\in S} e^{2\pi i \frac{n_p}{n}} \right) \\
&= \frac{1}{\mathcal{Z}_{\mathbb{Z}_n}} \sum_{\{\zeta_{s^*}\}} \left[\left(\prod_{p\notin S} \exp\Big[\tilde{F}_p(\xi_p)\Big] \right) \left(\prod_{p\in S} \exp\Big[\tilde{F}_p(\xi_p e^{-2\pi i/n})\Big] \right) \right] .
\end{aligned} \tag{D.2.17}$$

The 't Hooft operator $U(\tilde{\gamma})$ is just a string-like operator which consists of ξ_p:

$$U(\tilde{\gamma}) := \prod_{p\in\tilde{\gamma}} \xi_p = \zeta_{s_1^*} \zeta_{s_2^*}^{-1} , \tag{D.2.18}$$

where s_1^* and s_2^* are the end points of the path $\tilde{\gamma}$. The expectation value of the 't Hooft operator can be then written in terms of the original variables as

$$\begin{aligned}
\langle U(\tilde{\gamma})\rangle &= \frac{1}{\mathcal{Z}_{\mathbb{Z}_n}} \sum_{\{\zeta_{s^*}\}} \prod_{p\in\text{plaquettes}} \zeta_{s_1^*} \zeta_{s_2^*}^{-1} \exp\Big[\tilde{F}_p(\xi_p)\Big] \\
&= \frac{1}{\mathcal{Z}_{\mathbb{Z}_n}} \sum_{\{\zeta_{s^*}\}} \prod_{p\in\text{plaquettes}} \sum_{n_p=0}^{n-1} \xi_p^{-n_p} \exp\Big[F_p(e^{2\pi i \frac{n_p}{n}})\Big] \prod_{p\in\tilde{\gamma}} \xi_p \\
&= \frac{1}{\mathcal{Z}_{\mathbb{Z}_n}} \sum_{\{U_\mu(\boldsymbol{r})\}} \left[\left(\prod_{p\notin\tilde{\gamma}} \exp[F_p(G_p)] \right) \left(\prod_{p\in\tilde{\gamma}} \exp\Big[F_p(G_p e^{2\pi i/n})\Big] \right) \right] .
\end{aligned} \tag{D.2.19}$$

Note here that in the summation over $\{\xi_p\}$ in the discussion above, the dynamical variable ξ_p is subject to the constraint $\prod_{p\ni b} \xi_p = 1$ for all links b.

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