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The Borel Sum of Divergent Barnes Hypergeometric Series and its Application to a Partial Differential Equation

Kunio Ichinobe (市延 邦夫)
Graduate School of Mathematics, Nagoya University
(多元数理科学研究科, 名古屋大学)

1 Introduction and Results

We shall treat of the divergent Barnes hypergeomtric series ${}_qF_{p-1}$ $(p<q)$ which is defined by

$$(1.1)\qquad {}_qF_{p-1}(\boldsymbol{\alpha};\boldsymbol{\gamma};z)={}_qF_{p-1}\left(\begin{array}{c}\boldsymbol{\alpha}\\ \boldsymbol{\gamma}\end{array};z\right):=\sum_{n=0}^{\infty}\frac{(\boldsymbol{\alpha})_n}{(\boldsymbol{\gamma})_n}\frac{z^n}{n!},\quad z\in\mathbb{C},$$

where $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_q)\in\mathbb{C}^q$, $\boldsymbol{\gamma}=(\gamma_1,\cdots,\gamma_{p-1})\in\mathbb{C}^{p-1}$ and we use the following abbreviations

$$(\boldsymbol{\alpha})_n=\prod_{\ell=1}^{q}(\alpha_\ell)_n,\quad (\boldsymbol{\gamma})_n=\prod_{m=1}^{p-1}(\gamma_m)_n,$$

with $(c)_n=\Gamma(c+n)/\Gamma(c)$ $(c\in\mathbb{C})$ and Γ denotes the Gamma function.

Throughout this paper, we assume $\gamma_j\notin\mathbb{Z}_{\leq 0}$ for all j to make sense of this series and we also assume $\alpha_j\notin\mathbb{Z}_{\leq 0}$ for all j to avoid the trivial case where $\mathbb{Z}_{\leq 0}=\{0,-1,-2,\cdots\}$.

We are concerned with the Borel summability of this divergent series (1.1). In the previous papers [Ich] and [MI], we gave an explicit form of the Borel sum of this divergent series (1.1) and its analytic continuation around the origin, which were given by a linear conbination of ${}_pF_{q-1}$. The explicit formula of the Borel sum means the rediscovery of Barnes original one obained in [Bar], from the view point of Borel summability. In the proof of previous papers we employed the Barnes type integral representation for the generalized hypergeometric function which is Borel transform of the divergent series (1.1).

In this paper we shall give another proof by employing the Euler integral representation for the same function.

Before stating our results, we shall prepare some notations and definitions (cf. [Bal]).

For $d\in\mathbb{R}$, $\beta>0$ and ρ $(0<\rho\leq\infty)$, we define a sector $S=S(d,\beta,\rho)$ by

$$S(d,\beta,\rho):=\{z\in\mathbb{C};|d-\arg z|<\frac{\beta}{2},0<|z|<\rho\},$$

where d, β and ρ are called the direction, the opening angle and the radius of $S(d,\beta,\rho)$, respectively.

For $k > 0$, we define that $\hat{u}(z) = \sum_{n=0}^{\infty} u_n z^n$ belongs to $\mathbb{C}[[z]]_{1/k}$, which is called the formal power series of Gevrey order $1/k$, if there exist some positive constants C and K such that for any n we have

$$|u_n| \leq CK^n \Gamma\left(1 + \frac{n}{k}\right).$$

Let $k > 0$, $\hat{u}(z) = \sum_{n=0}^{\infty} u_n z^n \in \mathbb{C}[[z]]_{1/k}$ and $u(z) \in \mathcal{O}(S)$. Here $\mathcal{O}(S)$ denotes the set of holomorphic functions on a sector S. Then we define that

$$u(z) \cong_k \hat{u}(z) \quad \text{in } S,$$

if for any closed subsector S' of S, there exist some positive constants C and K such that for any N we have

$$\left| u(z) - \sum_{n=0}^{N-1} u_n z^n \right| \leq CK^N |z|^N \Gamma\left(1 + \frac{N}{k}\right), \quad z \in S'.$$

For $k > 0$, $d \in \mathbb{R}$ and $\hat{u}(z) \in \mathbb{C}[[z]]_{1/k}$, we define that $\hat{u}(z)$ is k-summable in d direction or Borel summable for short if there exist a sector $S = S(d, \beta, \rho)$ with $\beta > \pi/k$ and $u(z) \in \mathcal{O}(S)$ such that $u(z) \cong_k \hat{u}(z)$ holds in S.

Remark 1

(i) If $\beta \leq \pi/k$, then there are infinitely many u's satisfying $u(z) \cong_k \hat{u}(z)$ in $S(d, \beta, \rho)$ for any d and some $\rho > 0$.

(ii) If $\beta > \pi/k$, then a function $u(z)$ as mention above does not exist in general, but it is unique if it does exist. In this sense such a function u is called the Borel sum of $\hat{u}$ in d direction, or the Borel sum for short.

In what follows, we use the following abbreviations.

$$\boldsymbol{\alpha} + s = (\alpha_1 + s, \alpha_2 + s, \cdots, \alpha_q + s) \in \mathbb{C}^q, \quad \widehat{\boldsymbol{\alpha}_j} = (\alpha_1, \cdots, \alpha_{j-1}, \alpha_{j+1}, \cdots, \alpha_q) \in \mathbb{C}^{q-1},$$

$$\Gamma(\boldsymbol{\alpha}) = \prod_{\ell=1}^{q} \Gamma(\alpha_\ell), \quad \Gamma(\widehat{\boldsymbol{\alpha}_j}) = \prod_{\ell=1, \ell \neq j}^{q} \Gamma(\alpha_\ell).$$

Now we put $\hat{f}(z) = {}_qF_{p-1}(\boldsymbol{\alpha}; \boldsymbol{\gamma}; z) \in \mathbb{C}[[z]]_{q-p}$. Then our first result is stated as follows.

Theorem 1.1 (Borel sum)

Assume that $\alpha_i - \alpha_j \notin \mathbb{Z}$ $(i \neq j)$. Then $\hat{f}(z)$ is $1/(q-p)$-summable in any direction d such that $d \not\equiv 0 \pmod{2\pi}$ and its Borel sum $f(z)$ is given by

$$\begin{aligned} (1.2) \qquad f(z) &= \frac{C_{\boldsymbol{\alpha\gamma}}}{2\pi i} \int_I \frac{\Gamma(\boldsymbol{\alpha} + s)\Gamma(-s)}{\Gamma(\boldsymbol{\gamma} + s)} (-z)^s ds \\ &= C_{\boldsymbol{\alpha\gamma}} \sum_{j=1}^{q} C_{\boldsymbol{\alpha\gamma}}(j) \times (-z)^{-\alpha_j} {}_pF_{q-1}\left(\begin{matrix} \alpha_j, 1 + \alpha_j - \boldsymbol{\gamma} \\ 1 + \alpha_j - \widehat{\boldsymbol{\alpha}_j} \end{matrix} ; \frac{(-1)^{p-q}}{z} \right), \end{aligned}$$

where $z \in S(\pi, (q-p+2)\pi, \infty)$ *and*

$$C_{\alpha\gamma} = \frac{\Gamma(\gamma)}{\Gamma(\boldsymbol{\alpha})}, \quad C_{\alpha\gamma}(j) = \frac{\Gamma(\alpha_j)\Gamma(\widehat{\boldsymbol{\alpha}_j} - \alpha_j)}{\Gamma(\gamma - \alpha_j)}. \tag{1.3}$$

Here the path of integration I *runs from* $-i\infty$ *to* $+i\infty$ *on the imaginary axis in such a manner that the poles of* $\Gamma(\boldsymbol{\alpha}+s)$ *are on the left side of* I *and the poles of* $\Gamma(-s)$ *are on the right side of* I.

Next, our result for the analytic continuation of the Borel sum f is stated as follows.

Theorem 1.2 (Analytic Continuation of Borel sum)
Under the same assumptions as in Theorem 1.1, *we have*

$$\begin{aligned} & \frac{1}{2\pi i}\{f(z) - f(ze^{2\pi i})\} \\ = \ & \frac{C_{\alpha\gamma}}{2\pi i}\int_I \frac{\Gamma(\boldsymbol{\alpha}+s)}{\Gamma(\gamma+s)\Gamma(1+s)} z^s ds \\ = \ & C_{\alpha\gamma}\sum_{j=1}^{q} \frac{C_{\alpha\gamma}(j)}{\Gamma(\alpha_j)\Gamma(1-\alpha_j)} z^{-\alpha_j} {}_pF_{q-1}\left(\begin{matrix} \alpha_j, 1+\alpha_j-\gamma \\ 1+\alpha_j-\widehat{\boldsymbol{\alpha}_j} \end{matrix} ; \frac{(-1)^{p-q}}{z} \right), \end{aligned} \tag{1.4}$$

where $z \in S(0, (q-p)\pi, \infty)$ *and* $C_{\alpha\gamma}, C_{\alpha\gamma}(j)$ *and the path of integration* I *are the same ones as in Theorem* 1.1, *respectively.*

Remark 2 In the case $\alpha_i - \alpha_j \in \mathbb{Z}$ for some i and j, we can prove the similar results to Theorems, where the logarithmic terms appear (see [Ich]).

2 Proof of Theorem 1.1

In order to prove Theorems we use the following lemma for the Borel summability.

Lemma 2.1 *Let* $k > 0$, $d \in \mathbb{R}$ *and* $\hat{u}(z) = \sum_{n=0}^{\infty} u_n z^n \in \mathbb{C}[[z]]_{1/k}$. *Then the following three propositions are equivalent:*

(1) $\hat{u}(z)$ *is* k*-summable in* d *direction.*

(2) *Let* $g(\zeta)$ *be the formal* k*-Borel transformation of* $\hat{u}(z)$

$$g(\zeta) = (\hat{\mathcal{B}}_k \hat{u})(\zeta) := \sum_{n=0}^{\infty} \frac{u_n}{\Gamma(1+n/k)}\zeta^n, \tag{2.1}$$

which is holomorphic in a nighbourhood of $\zeta = 0$. *Then* $g(\zeta)$ *can be continued analytically in* $S(d, \varepsilon, \infty)$ *for some positive constant* ε *and satisfies a growth condition of exponential* *order at most k there, that is, there exist some positive numbers C and δ such that we have*

$$|g(\zeta)| \le C\exp\{\delta|\zeta|^k\}, \ \zeta \in S(d,\varepsilon,\infty). \tag{2.2}$$

In this case, the Borel sum $u(z)$ is obtained after an analytic continuation of the following Laplace integral

$$u(z) = (\mathcal{L}_k g)(z) := \frac{1}{z^k}\int_0^{\infty(d)} e^{-(\zeta/z)^k} g(\zeta)d(\zeta^k) \tag{2.3}$$

where $z \in S(d,\beta,\rho)$ with $\beta < \pi/k$ and $\rho > 0$ and the path of integration is taken from 0 to ∞ along the half line of argument d.

(3) *Let $j \ge 2$ and $k_1 > 0, \cdots, k_j > 0$ satisfy $1/k = 1/k_1 + \cdots + 1/k_j$. Let $h(\zeta)$ be the following iterated formal Borel transformations of $\hat{u}(z)$*

$$h(\zeta) = (\hat{\mathcal{B}}_{k_1} \circ \cdots \circ \hat{\mathcal{B}}_{k_j}\hat{u})(\zeta). \tag{2.4}$$

Then $h(\zeta)$ holds the same properties as $g(\zeta)$ above. In this case, the Borel sum $u(z)$ is obtained after an analytic continuation of the following iterated Laplace integrals

$$u(z) = (\mathcal{L}_{k_j} \circ \cdots \circ \mathcal{L}_{k_1} h)(z). \tag{2.5}$$

The equivalence of (i) and (ii) is given in [Bal] and the equivalence of (iii) with others is proved in [Miy].

Proof of Theorem 1.1. Let $h(\zeta)$ be the $(q-p)$ times iterated formal 1-Borel transformations of $\hat{f}(z)$

$$h(\zeta) = (\hat{\mathcal{B}}_1^{q-p}\hat{f})(\zeta) = {}_qF_{q-1}\left(\begin{matrix}\alpha \\ \gamma, 1, \cdots, 1\end{matrix}; \zeta\right). \tag{2.6}$$

This series is convergent in $|\zeta| < 1$. Then we can see that $h(\zeta) \in \mathcal{O}(\mathbb{C} \setminus [1,\infty))$ and $h(\zeta)$ has at most polynomial growth as $\zeta \to \infty$, because $h(\zeta)$ satisfies a Fuchsian equation with singular points $\{0,1,\infty\}$. Therefore $\hat{f}(z)$ is $1/(q-p)$-summable in any direction d such that $d \not\equiv 0 \pmod{2\pi}$ and the Borel sum $f(z)$ is given by the following iterated Laplace integrals and its analytic continuation

$$\begin{aligned} f(z) &= \frac{1}{z}\int_0^{\infty(d)} \exp\left(-\frac{s_1}{z}\right) ds_1 \frac{1}{s_1}\int_0^{\infty(d)} \exp\left(-\frac{s_2}{s_1}\right) ds_2 \cdots \\ &\cdots \times \frac{1}{s_{q-p-2}}\int_0^{\infty(d)} \exp\left(-\frac{s_{q-p-1}}{s_{q-p-2}}\right) ds_{q-p-1} \frac{1}{s_{q-p-1}}\int_0^{\infty(d)} \exp\left(-\frac{\zeta}{s_{q-p-1}}\right) h(\zeta)d\zeta, \end{aligned} \tag{2.7}$$

where $d = \arg\zeta = \arg s_j \not\equiv 0 \pmod{2\pi}$ and $|d - \arg z| < \pi/2$. By a change of variables

$$\frac{s_1}{z} = u_1, \quad \frac{s_2}{s_1} = u_2, \quad \cdots, \quad \frac{s_{q-p-1}}{s_{q-p-2}} = u_{q-p-1}, \quad \frac{\zeta}{s_{q-p-1}} = u_{q-p}, \tag{2.8}$$

$$(2.9)\qquad f(z) = \int_0^{\infty(a)} e^{-u_1}du_1 \int_0^{\infty(0)} e^{-u_2}du_2 \cdots \int_0^{\infty(0)} e^{-u_{q-p}} h(\boldsymbol{u}z)du_{q-p},$$

where $a = d - \arg z$ $(|a| < \pi/2)$ and $\boldsymbol{u} = u_1 \cdot u_2 \cdots u_{q-p}$.

To caluclate these integrals, we employ the Euler integral representation of $h(\zeta)$ which is given by

$$(2.10)\qquad h(\zeta) = C_0 \prod_{j=1}^{p-1} \int_0^1 t_j^{\alpha_j - 1}(1-t_j)^{\gamma_j - \alpha_j - 1} dt_j \prod_{j=p}^{q-2} \int_0^1 t_j^{\alpha_j - 1}(1-t_j)^{-\alpha_j} dt_j$$
$$\times \int_0^1 t_{q-1}{}^{\alpha_{q-1}-1}(1-t_{p-1})^{-\alpha_{q-1}}(1-\boldsymbol{t}\zeta)^{-\alpha_q} dt_{q-1},$$

where $\boldsymbol{t} = t_1 \cdot t_2 \cdots t_{q-1}$ and

$$(2.11)\qquad C_0 = \frac{\Gamma(\gamma)}{\prod_{j=1}^{p-1}\Gamma(\alpha_j)\Gamma(\gamma_j - \alpha_j)} \cdot \frac{\Gamma(1)}{\prod_{j=p}^{q-1}\Gamma(\alpha_j)\Gamma(1-\alpha_j)}.$$

Here in order to make sense of these integrals in (2.10), we assume the following integrability conditions

$$(2.12)\qquad \mathrm{Re}\ \gamma_j > \mathrm{Re}\ \alpha_j > 0\ (j = 1, \cdots, p-1), \quad 0 < \mathrm{Re}\ \alpha_j < 1\ (j = p, \cdots, q-1).$$

Moreover we assume

$$(2.13)\qquad 0 < \mathrm{Re}\ \alpha_q < 1.$$

We remark that we can remove such restrictions at the end of proof.

Then we obtain the following fundamental formula for the Borel sum

$$(2.14)\qquad f(z) = C_0 \int_0^{\infty(0)} e^{-u_2}du_2 \int_0^{\infty(0)} e^{-u_3}du_3 \cdots \int_0^{\infty(0)} e^{-u_{q-p}}du_{q-p}$$
$$\times \prod_{j=1}^{p-1} \int_0^1 t_j^{\alpha_j-1}(1-t_j)^{\gamma_j-\alpha_j-1}dt_j \prod_{j=p}^{q-1} \int_0^1 t_j^{\alpha_j-1}(1-t_j)^{1-\alpha_j-1}dt_j$$
$$\times \int_0^{\infty(a)} e^{-u_1}(1-\boldsymbol{tu}z)^{-\alpha_q}du_1.$$

In order to calculate the last integral, we give the following lemma.

Lemma 2.2 *Let* $0 < \mathrm{Re}\ \beta < 1$. *Then we have*

$$(2.15)\qquad \int_0^{\infty(a)} e^{-u}(1-zu)^{-\beta}du = \frac{\{\Gamma(\beta)\}^{-1}}{2\pi i}\int_{\tilde I}\Gamma(\beta+s)\Gamma(s+1)\Gamma(-s)(-z)^s ds,$$

where $a = (d - \arg z)$ $(0 < d < 2\pi)$ *with* $|a| < \pi/2$ *and the path of integration* $\tilde I$ *runs from* $\kappa - i\infty$ *to* $\kappa + i\infty$ *with* $-\mathrm{Re}\ \beta < \kappa < 0$.

Proof of Lemma 2.2. First, we notice

$$(2.16)\qquad (1-zu)^{-\beta} = \frac{\{\Gamma(\beta)\}^{-1}}{2\pi i}\int_{\widetilde{I}}\Gamma(\beta+s)\Gamma(-s)(-zu)^s ds.$$

Therefore by substituting this formula into the left side of (2.15) and exchanging the order of integration, we obtain the conclusion. □

By using Lemma 2.2 and exchanging the order of integrations in (2.14), we have

$$\begin{aligned}(2.17)\quad f(z) &= \frac{C_0}{2\pi i}\int_{\widetilde{I}_1}\Gamma(1+s)\Gamma(\alpha_q+s)\Gamma(-s)(-z)^s ds \\ &\quad\times\int_0^{\infty(0)} e^{-u_2}{u_2}^s du_2\int_0^{\infty(0)} e^{-u_3}{u_3}^s du_3\cdots\int_0^{\infty(0)} e^{-u_{q-p}}{u_{q-p}}^s du_{q-p} \\ &\quad\times\prod_{j=1}^{p-1}\int_0^1 {t_j}^{\alpha_j-1+s}(1-t_j)^{\gamma_j-\alpha_j-1}dt_j\prod_{j=p}^{q-1}\int_0^1 {t_j}^{\alpha_j-1+s}(1-t_j)^{-\alpha_j}dt_j \\ &= \frac{C_{\alpha\gamma}}{2\pi i}\int_{\widetilde{I}_1}\frac{\Gamma(\alpha+s)\Gamma(-s)}{\Gamma(\gamma+s)}(-z)^s ds,\end{aligned}$$

where $C_{\alpha\gamma}$ is the constant given by (1.3). Here the path of integration $\widetilde{I}_1$ runs from $\kappa_1 - i\infty$ to $\kappa_1 + i\infty$ with $-\min\{\alpha_j\} < \kappa_1 < 0$ and it is possible to take such κ_1. Finally, by changing the path of integration $\widetilde{I}_1$ into I in Theorem 1.1, we can remove the restriction (2.12) and (2.13) for the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$, so we obtain the desired first formula (1.2) of integral representation for the Borel sum. In addition, by residue theorem, we obtain the desired the second formula (cf. [Ich, Theorem 2.1]). □

3 Proof of Theorem 1.2

Proof of Theorem 1.2. From (2.14), we get the following formula

$$\begin{aligned}(3.1)\quad f(z)-f(ze^{2\pi i}) &= C_0\int_0^{\infty(0)} e^{-u_2}du_2\int_0^{\infty(0)} e^{-u_3}du_3\cdots\int_0^{\infty(0)} e^{-u_{q-p}}du_{q-p} \\ &\quad\times\prod_{j=1}^{p-1}\int_0^1 {t_j}^{\alpha_j-1}(1-t_j)^{\gamma_j-\alpha_j-1}dt_j\prod_{j=p}^{q-1}\int_0^1 {t_j}^{\alpha_j-1}(1-t_j)^{-\alpha_j}dt_j \\ &\quad\times\int_{C_+(\boldsymbol{t}\widehat{\boldsymbol{u}_1}z)} e^{-u_1}(1-\boldsymbol{tu}z)^{-\alpha_q}du_1.\end{aligned}$$

where the path of integration $C_+(X)$ with $X = \boldsymbol{t}\widehat{\boldsymbol{u}_1}z$ starts at ∞ on $\arg u_1 = -\arg X$, encircles the point $u_1 = X^{-1}$ in the positive direction and returns to the starting point.

In order to calculate the last integral, we give the following lemma.

Lemma 3.1 *Let* $0 < \mathrm{Re}\,\beta < 1$. *Then we have*

$$(3.2)\qquad \int_{C_+(z)} e^{-u}(1-zu)^{-\beta}du = \int_0^{\infty(a)} e^{-u}\left\{(1-zu)^{-\beta}-(1-zue^{2\pi i})^{-\beta}\right\}du = \frac{1}{\Gamma(\beta)}\int_{\widetilde{I}}\Gamma(\beta+s)z^s ds,$$

where a *and the path of integration* $\widetilde{I}$ *are the same ones as in Lemma* 2.2, *respectively.*

Proof of Lemma 3.1. Since we notice that $(1-zu)^{-\beta}$ is univalent in u on $|u| < |z|$, we can prove (3.2) in the same manner as in Lemma 2.2. □

By using Lemma 3.1 and exchanging the order of integrations in (3.1), we have

$$(3.3)\ f(z)-f(ze^{2\pi i}) = \frac{C_0}{\Gamma(\alpha_q)}\int_{\widetilde{I}_1}\Gamma(\alpha_q+s)z^s ds\prod_{j=2}^{q-p}\int_0^{\infty(0)} e^{-u_j}{u_j}^s du_j \times\prod_{j=1}^{p-1}\int_0^1 {t_j}^{\alpha_j-1+s}(1-t_j)^{\gamma_j-\alpha_j-1}dt_j\prod_{j=p}^{q-1}\int_0^1 {t_j}^{\alpha_j-1+s}(1-t_j)^{-\alpha_j}dt_j = C_{\alpha\gamma}\int_{\widetilde{I}_1}\frac{\Gamma(\alpha+s)}{\Gamma(\gamma+s)\Gamma(1+s)}z^s ds,$$

where the path of integration $\widetilde{I}_1$ is the same one as in (2.17). Therefore, by changing the path of integration, we can remove the restriction (2.12) and (2.13) for the parameters. This is the desired first formula (1.4) of integral representation and by using residue theorem we obtain the desired second formula (cf. [Ich, Theorem 2.2]). □

4 Appliction to a Partial Differential Equation

In this section, we shall give an application to a partial differential equation of Theorem 1.2. Let us consider the following Cauchy problem.

$$(4.1)\qquad \begin{cases} \partial_t^p u(t,x) = \partial_x^q u(t,x), \\ u(0,x) = \varphi(x), \quad \partial_t^j u(0,x) = 0\ (1 \le j \le p-1), \end{cases}$$

where $t, x \in \mathbb{C}$, $p < q$ and the Cauchy data $\varphi(x)$ is holomorphic in a neighbourhood at the origin.

This Cauchy problem has a unique formal power series solution in t-variable

$$(4.2)\qquad \hat{u}(t,x) = \sum_{n=0}^{\infty}\varphi^{(qn)}(x)\frac{t^{pn}}{(pn)!}.$$

Miyake [Miy] proved that the formal solution (4.2) of (4.1) is Borel summable in d direction in t-plane if and only if the Cauchy data $\varphi(x)$ satisfies the following two conditions:

(i) the Cauchy data $\varphi(x)$ can be continued analytically in q sectors

$$\Omega(p,q;d,\varepsilon)=\bigcup_{j=0}^{q-1} S\left(\frac{pd+2\pi ij}{q},\varepsilon,\infty\right) \tag{4.3}$$

for some $\varepsilon>0$.

(ii) the Cauchy data $\varphi(x)$ has the growth condition of exponential order at most $q/(q-p)$ in $\Omega(p,q;d,\varepsilon)$.

Under the above conditions, we get the integral representation of the Borel sum of (4.2) by using the kernel function as follows.

Theorem 4.1 (Integral representation of Borel sum)
Under the conditions (i) *and* (ii) *for the Cauchy data* $\varphi(x)$, *the Borel sum* $u(t,x)$ *of the formal solution* (4.2) *of the Cauchy problem* (4.1) *is given by*

$$u(t,x)=\int_0^{\infty(pd/q)}\Phi(x,\zeta)k(t,\zeta)d\zeta, \tag{4.4}$$

where

$$\Phi(x,\zeta)=\sum_{j=0}^{q-1}\varphi(x+\zeta\omega^j),\quad \omega=\exp(2\pi i/q), \tag{4.5}$$

and the kernel function $k(t,\zeta)$ *is given by*

$$k(t,\zeta) \;=\; \frac{D_{pq}}{\zeta}\sum_{j=0}^{q-1}D_{pq}(j)X^{-j/q}{}_pF_{q-1}\left(\begin{matrix}1+j/q-\boldsymbol{p}/p\\ 1+j/q-(\widehat{\boldsymbol{q}/q})_j\end{matrix};\frac{(-1)^{p-q}}{X}\right), \tag{4.6}$$

where

$$X=\frac{q^q}{p^p}\frac{t^p}{\zeta^q}, \tag{4.7}$$

and $\boldsymbol{p}=(1,2,\cdots,p)$, $\boldsymbol{q}=(1,2,\cdots,q)$, $D_{pq}=\dfrac{\Gamma(\boldsymbol{p}/p)}{\Gamma(\boldsymbol{q}/q)}$, $D_{pq}(j)=\dfrac{\Gamma((\widehat{\boldsymbol{q}/q})_j-j/q)}{\Gamma(\boldsymbol{p}/p-j/q)}$.

Proof of Theorem 4.1. Let $v(s,x)$ be the $(q-p)$ times iterated p-Borel transforms in t-variable of the formal solution (4.2)

$$v(s,x)=\left((\hat{\mathcal{B}}_p)^{q-p}\hat{u}(\cdot,x)\right)(s)=\sum_{n=0}^{\infty}\frac{\varphi^{(qn)}(x)}{(pn)!}\frac{s^{pn}}{(n!)^{q-p}}. \tag{4.8}$$

By Cauchy's integral formula, for sufficiently small $|s|$ and $|x|$ we have

$$\begin{aligned} v(s,x) &= \frac{1}{2\pi i}\oint_{|\zeta|=r}\frac{\varphi(x+\zeta)}{\zeta}\sum_{n=0}^{\infty}\frac{(qn)!}{(pn)!(n!)^{q-p}}\left(\frac{s^p}{\zeta^q}\right)^n d\zeta \\ &= \frac{1}{2\pi i}\oint_{|\zeta|=r}\frac{\varphi(x+\zeta)}{\zeta}h(s,\zeta)d\zeta, \end{aligned} \tag{4.9}$$

where $r > \sqrt[q]{(q^q/p^p)|s^p|}$ and $h(s,\zeta) = {}_qF_{q-1}\left(\boldsymbol{q}/q; \boldsymbol{p}/p, 1, \cdots, 1; Y\right)$ with $Y = q^q/p^p \cdot s^p/\zeta^q$. Here we notice that $h(s,\zeta)$ has q singular points in ζ-plane at q roots of $\zeta^q = (q^q/p^p)s^p$ for a fixed $s \neq 0$ with $\arg s = d$. We put $a = (q^q/p^p)^{1/q} s^{p/q}$ (the root with argument dp/q), and we denote by $[0,a]$ the segment joining the origin and a. Since we notice that $h(s,\zeta)$ is univalent in $\mathbb{C}_\zeta \setminus \bigcup_{j=0}^{q-1}[0, a\omega^j]$ (outside of q segments), we can deform the contour of integration for (4.9) as follows.

$$v(s,x) = \frac{1}{2\pi i}\int_0^{\infty(pd/q)} \frac{\Phi(x,\zeta)}{\zeta}\left\{h(s,\zeta) - h(s,\zeta\omega^{-1})\right\} d\zeta. \tag{4.10}$$

Hence the Borel sum $u(t,x)$ is given by the following iterated Laplace transforms of $v(s,x)$

$$\begin{aligned} u(t,x) &= \left((\mathcal{L}_p)^{q-p} v(\cdot,x)\right)(t) \\ &= \frac{1}{2\pi i}\int_0^{\infty(pd/q)} \frac{\Phi(x,\zeta)}{\zeta}\left\{(\mathcal{L}_p)^{q-p}\left(h(\cdot,\zeta) - h(\cdot,\zeta\omega^{-1})\right)\right\}(t) d\zeta. \end{aligned} \tag{4.11}$$

This observation shows that the kernel function $k(t,\zeta)$ is given by

$$k(t,\zeta) = \frac{1}{2\pi i}\frac{1}{\zeta}\left\{\left((\mathcal{L}_p)^{q-p}h\right)(\cdot,\zeta) - \left((\mathcal{L}_p)^{q-p}h\right)(\cdot,\zeta\omega^{-1})\right\}(t). \tag{4.12}$$

Now, we shall prove that the function $h(s,\zeta)/\zeta$ is an iterated formal Borel transforms of the formal solution of the following Cauchy problem for the adjoint equation

$$\begin{cases} \partial_t^p u(t,\zeta) = (-\partial_\zeta)^q u(t,\zeta), \\ u(0,\zeta) = 1/\zeta, \quad \partial_t^j u(0,\zeta) = 0 \ (1 \le j \le p-1). \end{cases} \tag{4.13}$$

This Cauchy problem (4.13) has a unique formal solution

$$\begin{aligned} \hat{e}(t,\zeta) &= \frac{1}{\zeta}\sum_{n=0}^{\infty}\frac{(qn)!}{(pn)!}\left(\frac{t^p}{\zeta^q}\right)^n \\ &= \frac{1}{\zeta}\, {}_qF_{p-1}\left(\frac{\boldsymbol{q}}{q}; \frac{\widehat{\boldsymbol{p}_p}}{p}; \frac{q^q}{p^p}\frac{t^p}{\zeta^q}\right) \underset{\text{put}}{=} \frac{1}{\zeta}\hat{f}(X), \end{aligned} \tag{4.14}$$

where X is given by (4.7). Let $g(Y) = \left((\hat{\mathcal{B}}_1)^{q-p}\hat{f}\right)(Y)$. Then we can see that

$$g(Y) = h(s,\zeta), \quad \left((\mathcal{L}_1)^{q-p}g\right)(X) = \left((\mathcal{L}_p)^{q-p}h(\cdot,\zeta)\right)(t). \tag{4.15}$$

Hence, by letting $f(X)$ be the Borel sum of $\hat{f}(X)$, we have $f(X) = \left((\mathcal{L}_1)^{q-p}g\right)(X)$ and the Borel sum $e(t,\zeta)$ of $\hat{e}(t,\zeta)$ is given by

$$e(t,\zeta) = \frac{1}{\zeta}f(X). \tag{4.16}$$

Thus we can see that the kernel function $k(t,\zeta)$ is given by

$$
\begin{aligned}
(4.17) \qquad k(t,\zeta) &= \frac{1}{2\pi i}\{e(t,\zeta)-e(t,\zeta e^{-\frac{2\pi i}{q}})\} \\
&= \frac{1}{2\pi i}\times\frac{1}{\zeta}\{f(X)-f(Xe^{2\pi i})\}, \quad X=q^q t^p/p^p\zeta^q.
\end{aligned}
$$

Therefore by using Theorem 1.2, we get the kernel function $k(t,\zeta)$ which is given by (4.6). This completes the proof of Theorem 4.1. □

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