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A remark on the 2D-Euler equation

In this paper we revisit the initial value problem for the 2D-Euler equation on a bounded domain. The main object is to streamline the proof of the global existence and uniqueness of a classical solution, given in the old paper [K], although there is nothing essentially new. In particular we use the vorticity $\zeta = \partial\wedge u$ (= curl(u)) as a basic ingredient of the theory. However, instead of assuming that the initial velocity a is $C^{1+\theta}$ as in [K], we simply assume that $\alpha = \partial\wedge a$ is C and construct a unique *weak* solution $u(t)$ in $\widehat{L}$, to be define below. Afterwards it is shown that if $a\in C^{1+\theta}$ then $u(t)\in C^{1+\theta}$. Almost all the necessary material is in [K]; the change is only in the order of their arrangement. Naturally we follow the notation of [K] as much as possible.

As in [K], we consider a bounded domain $\Omega\subset\mathbf{R}^{2}$; for simplicity we assume that Ω is smooth and simply connected, and that there is no external force. (The modification necessary for a multiply connected Ω will be commented on later.) Moreover, for notational convenience we assume that Ω is closed. (If necessary we use Ω° to denote the interior of Ω.)

We denote by $\|\ \ \|$ the $C(\Omega)$-norm, indiscriminately for scalar or vector valued functions. $\widehat{L}(\Omega;\mathbf{R}^{2})$ is the set of all vector valued functions on Ω such that

$$f\in W^{1,p}(\Omega;\mathbf{R}^{2})\quad\text{for}\quad 1<p<\infty,\quad\text{and}$$

$$|f(x)-f(y)|\leq\text{const.}\omega(|x-y|),\quad x,y\in\Omega,$$

where $\omega(s)=s(1+\log^{+}(1/s))$. The associated norm is denoted by $\|f\|_{\mathrm{ql}}$.

The initial value problem for the Euler equation is given by

$$\partial_{t}u+\partial.(uu)+\partial p=0,\quad\partial.u=0,\quad u(0)=a.\tag{1}$$

Here uu is a tensor with jk component $u_{j}u_{k}$; $\partial.(uu)$ is a vector with k component $\partial_{j}(u_{j}u_{k})$; $\partial.u=\text{div}(u)=\partial_{j}u_{j}$. (Summation convention is used throughout.)

Theorem I. Let $\partial\wedge a\in C(\Omega;\mathbf{R})$ and $T>0$. Then there is a unique weak solution $\{u,p\}$ to (1) such that

$$u\in C(I;\widehat{L}(\Omega;\mathbf{R}^{2})),\quad\partial p\in\qquad\qquad,\quad I=[0,T].\tag{2}$$

If in particular $\partial\wedge a\in C^{\theta}(\Omega;\mathbf{R})$ for some $\theta\in(0,1)$, then $\{u,p\}$ is a classical solution with the properties

$$u\in C(I;C^{1}(\Omega;\mathbf{R}^{2}))\cap B(I;C^{1+\theta}(\Omega;\mathbf{R}^{2})),\quad\partial_{t}u\in C(I;C(\Omega;\mathbf{R}^{2})),\quad\partial p\in$$

where B denotes the class of bounded functions.

For the proof we introduce the (scalar) vorticity

$$\zeta = \partial \wedge u = (\partial_1 u_2 - \partial_2 u_1). \tag{4}$$

As is well known ζ should satisfy the *vorticity equation*, which is a system consisting of (4) and

$$\partial_t \zeta + \partial.(u\zeta) = 0, \quad \zeta(0) = \alpha = \partial \wedge a. \tag{5}$$

Our plan is to start with a function φ in a certain subset S of $C(Q)$, where $Q = I \times \Omega$, and determine $u \in C(Q)$, which are q.L. in x, such that $\partial \wedge u = \varphi$. We then solve (4) for ζ, which is shown to be in a certain compact subset of S. Furthermore, we show that the map $\varphi \mapsto \zeta$ is continuous in $C(Q)$. A fixed point of the map, which exists by the Schauder fixed point theorem, gives a solution of the vorticity equation. u will then be shown to be the unique solution of (1) together with a certain gradient ∂p.

Lemma 1. For each $\varphi \in C(Q; \mathbf{R})$, there is a unique $u \in C(I; \widehat{L})$ such that

$$\partial.u(t) = 0 \quad \text{and} \quad \partial \wedge u(t) = \varphi(t) \text{ on } \Omega, \quad \|.u(t) = 0 \text{ on } b\Omega,$$

$$\|u(t)\|_L \leq c\|\varphi(t)\|, \quad t \in I, \tag{6}$$

where c is a constant depending only on Ω.

Proof. This follows immediately from [K,Lemma x.x]; note that $C(Q; \mathbf{R}) = C(I; C(\Omega))$.

Lemma 2. Let $u \in C(Q; \mathbf{R}^2)$ such that $u(t) \in \hat{L}(\Omega)$, $\partial.u(t) = 0$ on Ω and $\nu.u(t) = 0$ on $b\Omega$. Then the ordinary differential equation $dx/dt = u(t,x)$ is uniquely solvable for any initial time $s \in I$ and any initial condition $x(s) = y \in \Omega$, with the solution (characteristic function) $x = \Phi_{t,s}(y) \in \Omega$ existing for all $t \in I$. The map $\Phi : t, s, y \mapsto x$ is continuous in the three variables. For fixed t, s, it is a homeomorphism of Ω onto itself, satisfying the chain rule $\Phi_{t,s} \circ \Phi_{s,r} = \Phi_{t,r}$.

Proof. The existence of the solution for all t, s is due to the fact that $\partial.u = 0$ and $\nu.u = 0$ (see [K]). The uniqueness follows from the theorem of Osgood, since $1/\omega(r)$ is not integrable near $r = 0$. For the continuity properties, see e.g. [H].

Lemma 3. Let u_n, $n = 1, 2, ...$, be a sequence of functions satisfying the assumptions of Lemma 2, with the associated map Φ_n. Moreover, assume that $u_n \to u$ in $C(Q; \mathbf{R}^2)$. Then $\Phi_n \to \Phi$ in $C(Q; \mathbf{R}^2)$.

Proof. This is a continuous dependence theorem for the characteristic function. Usually it is stated as continuous dependence on a auxiliary continuous parameter μ (see e.g.[H]), but there is no difference in the proof when μ is replaced by a discrete parameter n.

Lemma 4 The homeomorphisms $\Phi_{t,s}$ are measure preserving.

Proof. Approximate u in $\widehat{L}$ by C^1 functions, for which Φ becomes C^1 in all three variables and the result is classical (see e.g.[H]). The required result follows on passing to the limit using Lemma 3.

Lemma 5 $\Phi_{t,s}(y)$ is uniformly Hölder continuous in the three variables for $t, s \in I$, $y \in \Omega$.

Proof. The result is due to the quasi-Lipashitzian property of u, see [K], Lemma x.x. The Hölder exponent may be very small when T is large.

Lemma 6 Let u be as in Lemma 2. Then the linearized vorticity equation (2) has a weak solution ζ given by

$$\zeta(t) = \alpha \circ \Phi_{0,t}, \quad t \in I. \tag{7}$$

Proof. This is well known for a classical solution if u and α were C^1. As it is, it requires a proof. Obviously (7) satisfies $\zeta(0) = \alpha$, since $\Phi_{0,0}$ is the identity on Ω. Thus it suffices to show that for any smooth scalar function χ on Q, one has

$$\partial_t < \zeta, \chi > = < \zeta u, \partial\chi > = < \zeta, u.\partial\chi >, \tag{8}$$

where $< \, , >$ denotes the scalar product on Ω for scalar or vector valued functions. In view of (7) and the measure preserving property of the map $\Phi_{t,s}$, (8) is equivalent to

$$\partial_t < \alpha, \chi \circ \Phi_{t,0} > = < \alpha, (u.\partial\chi) \circ \Phi_{t,0} >; \tag{9}$$

note that $\Phi_{t,0}$ is the inverse map of $\Phi_{0,t}$. Here the left member equals

$$< \alpha(x), \partial_t\chi(\Phi_{t,0}(x) > = < \alpha(x), \partial\chi(\Phi_{t,0}(x)).\partial_t\Phi_{t,0}(x) >$$

$$= < \alpha(x), \partial\chi(\Phi_{t,0}(x)).u(t, \Phi_{t,0}(x)) >$$

which is the right member of (9), q.e.d.

Remark. It appeares that Lemma 4 is nontrivial; it would be hard to prove it without the condition $\partial.u = 0$, which implies the measure preserving property.

Lemma 7 There is $u \in C(I; \hat{L}(\Omega; \mathbf{R}^2))$ such that $\zeta = \partial \wedge u$ is in $C(Q; \mathbf{R})$ and is a weak solution of the vorticity equation ( ).

Proof. Let $\alpha \in C(\Omega)$ be fixed. Let S be the ball in $C(Q)$ with center 0 and radius $\|\alpha\|$. For each $\varphi \in S$, construct u and then ζ according to Lemmas 2 and 5. Then it is obvious that $\|\zeta\| \leq \|\alpha\|$, hence $\zeta \in S$. Thus the map $F : \varphi \mapsto \zeta$ sends S into itself. F is continuous in the topology of $C(Q)$, as is seen from Lemmas 2,3. Moreover, the range of F is compact in $C(Q)$, since $\zeta(t,x) = \alpha(\Phi_{0,t}(x))$, where $\alpha \in C(\Omega)$ is fixed and $\Phi_{0,t}(x)$ is uniformly Hölder continuousin t, x by Lemma 5. It follows from Schauder's fixed point theorem that F has a fixed point ζ, which is a solution of the vorticity equation.