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Geometric Conditions on Uniqueness Problem for Meromorphic Mappings

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Introduction.

This paper is a summary report of the author's recent research on the uniqueness problem of meromorphic mappings from the point of view of Nevanlinna theory. The study of the uniqueness problem of meromorphic mappings under condition on the preimages of divisors was first studied by G. Pólya and R. Nevanlinna ([21] and [17]). They proved the following famous five point theorem: Let f and g be nonconstant meromorphic functions on $\mathbb{C}$. If $f^{-1}(a_j) = g^{-1}(a_j)$ for distinct five points $a_1, \cdots, a_5$ in $\mathbb{P}_1(\mathbb{C})$, then f and g are identical (see also [18]). This may be called an absolute unicity theorem in as much as the condition concerns set equality. On the other hand, G. Pólya-R. Nevanlinna also have a relative unicity theorems. These theorems add the requirement that, for each inverse image in question, f and g take their value there with the same multiplicity. For example, the following four point theorem is well-known: Let f and g be as above. If $f^*a_j = g^*a_j$ as divisors for distinct four points $a_1, \cdots, a_4$ in $\mathbb{P}_1(\mathbb{C})$, then either $f \equiv g$ or $g = T(f)$ for an automorphism T of $\mathrm{Aut}(\mathbb{P}_1(\mathbb{C}))$ determined by $a_1, \cdots, a_4$. Until now, many researchers have studied unicity theorems for meromorphic functions on $\mathbb{C}$, as well there have been many contributions in the multidimensional case. Some of relevant papers listed in references. Among these, H. Fujimoto has proved a number of remarkable unicity theorems in relative case. For example, he proved the following brilliant theorem ([10, p. 1] and [11, p. 117]):

Theorem (Fujimoto). *Let f, $g : \mathbb{C}^m \to \mathbb{P}_n(\mathbb{C})$ be nonconstant meromorphic mappings with the same inverse images of q hyperplanes in general position.*

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(1) *If* $q = 3n+1$, *then there exists an automorphism* L *of* $\mathbb{P}_n(\mathbb{C})$ *such that* $f = L(g)$.
(2) *If* $q = 3n+2$ *and either* f *or* g *is linearly nondegenerate, then* $f \equiv g$.
(3) *If* $q = 2n+3$ *and either* f *or* g *is algebraically nondegenerate, then* $f \equiv g$.

His proofs are based on the Borel identity. On the other hand, S. J. Drouilhet [8] gave the first several variable extention of absolute unicity theorem as follows:

Theorem (Drouilhet). Let M be a projective algebraic manifold and A a smooth affine variety. Let $L \to M$ be an ample line bundle over M and $H \to \mathbb{P}_N(\mathbb{C})$ the hyperplane bundle over $\mathbb{P}_N(\mathbb{C})$. Let $\iota : M \to \mathbb{P}_N(\mathbb{C})$ be a nonconstant holomorphic mapping. Let $D \in |L|$ be hypersurfaces with normal crossings. Let $f,\ g : A \to M$ be transcendental meromorphic mappings. Suppose that $f^{-1}(D) = g^{-1}(D) = Z\ (\neq \emptyset)$ as point set. If $f = g$ on Z and $L \otimes K_M \otimes (-2\iota^* H)$ is ample, then $\iota \circ f \equiv \iota \circ g$.

In this paper, we deal with the absolute case. In [17] and thereafter methods used proving relative theorems have been essentially different from those in the absolute case. In the proof of absolute unicity theorems, we use a second main theorem for meromorphic mappings in an essentially computational way (cf. [1], [3], [4] and [8]). Note that the second main theorem for meromorphic mappings is established in a few cases. Hence, we deals with the case of dominant mappings. In what follows, we consider the following settings. Let $\pi : X \to \mathbb{C}^m$ be a finite analytic covering space and M a projective algebraic manifold. Let f_1, f_2 be dominant meromorphic mappings from X into M. Suppose that they have the same inverse images of given divisors on M. We first give conditions under which f_1, f_2 are algebraically related. We consider propagation of algebraic dependence of meromorphic mappings and their applications to uniqueness problem. Roughly speaking, our results say that if these mappings satisfy the same algebraic relation at all points of the set of the inverse images of divisors and if the given divisors are sufficiently ample, then they must satisfy this relationship identically. These results are considered as the propagation theorems of algebraic dependence. The propagation of dependence from a proper analytic subset to the whole space was first studied by L. Smiley [27] (cf. [29, p. 176]). There have been several studies on the propagation of dependence (cf. [9], [16] and [31]). So far, this problem has been studied under the conditions on the growth of meromorphic mappings. For example, W. Stoll [31] proved some interesting theorems on the propagation of dependence of meromorphic mappings $f : X \to M$ under a condition on the growth of mappings in different settings. In his results, at least one of the mappings f_j must grow quicker than the ramification divisor B of $\pi : X \to \mathbb{C}^m$. However, there can be only a few restricted cases where meromorphic mappings satisfying these conditions even if $\dim M = 1$ (cf. [20] and [22]). In this paper we first give criteria for the propagation of algebraic dependence of meromorphic mappings from X into M under the condition on the existence of meromorphic mappings separating the fibers of $\pi : X \to \mathbb{C}^m$. Thanks to the theory of algebroid reduction of meromorphic mappings, we can always find such a mapping. Thus it seems that our condition is more natural and essential than the above mentioned conditions. The theorem on algebroid reduction of meromorphic mappings and the ramification estimate due to J. Noguchi [19] are essentially important in the proofs of our results. In some of our criteria, we assume complicated conditions, but they have wider ranges of applicability. These criteria are actually corollaries of Lemmas 2.2 and 2.3, which are fundamental lemmas for our study. In §1, we recall some known facts in Nevanlinna theory of meromorphic mappings. In §2, we give those criteria. We consider the case where given divisors may determine distinct line bundles. In §§3–4, we will give their applications. We note that a certain kind of unicity theorems such as results in [3] and [8] may be considered as a special case of theorems on the propagation of dependence. In these theorems we can see that, for two meromorphic mappings $f,\ g : X \to M$ with the same inverse images of divisors as point sets (say Z) satisfying $f = g$ on Z, the algebraic relation $f = g$ on Z propagates to the whole space X. We give some unicity theorems from this point of view in §3. In §4, we study the uniqueness problem of holomorphic mappings into smooth elliptic curves. In particular, we give some conditions under which two holomorphic mappings are related by endomorphism of elliptic curves. For details, see [5] and [6].

§1. Preliminaries.

Let $\pi : X \to \mathbf{C}^m$ be a finite analytic (ramified) covering space over $\mathbf{C}^m$ and let s_0 be its sheet number, that is, X is a reduced irreducible normal complex space and $\pi : X \to \mathbf{C}^m$ is a proper surjective holomorphic mapping with discrete fibers. We denote by B the ramification divisor. Let $z = (z_1, \cdots, z_m)$ be the natural coordinate system in $\mathbf{C}^m$, and set

$$\|z\|^2 = \sum_{\nu=1}^{m} z_\nu \bar{z}_\nu, \quad X(r) = \pi^{-1}(\{z \in \mathbf{C}^m;\ \|z\| < r\}) \quad \text{and} \quad \alpha = \pi^* dd^c \|z\|^2,$$

where $d^c = (\sqrt{-1}/4\pi)(\bar{\partial} - \partial)$. For a (1,1)-current φ of order zero on X we set

$$N(r,\ \varphi) = \frac{1}{s_0} \int_1^r \langle \varphi \wedge \alpha^{m-1},\ \chi_{X(t)} \rangle \frac{dt}{t^{2m-1}},$$

where $\chi_{X(r)}$ denotes the characteristic function of $X(r)$. Let M be a compact complex manifold and let $L \to M$ be a line bundle over M. Denote by $|\cdot|$ a hermitian fiber metric in L and by ω its Chern form. Let $f : X \to M$ be a meromorphic mapping. We set

$$T_f(r,\ L) = N(r,\ f^*\omega),$$

and call it the characteristic function of f with respect to L. We also define $T_f(r,\ F)$ for $F \in \mathrm{Pic}(M) \otimes \mathbf{Q}$ in the following way. If ν is a positive integer with $\nu F \in \mathrm{Pic}(M)$, then we set

$$T_f(r,\ F) = \frac{1}{\nu} T_f(r,\ \nu F).$$

It is easy to see that $T_f(r,\ F)$ is well-defined. Let $|L|$ be the complete linear system determined by L. We have the following Nevanlinna's inequality for meromorphic mappings (cf. [19, p. 269]):

Theorem 1.1. *Let* $L \to M$ *be a line bundle over* M *and let* $f : X \to M$ *be a nonconstant meromorphic mapping. Then*

$$N(r, f^*D) \leq T_f(r, L) + O(1)$$

for $D \in |L|$ *with* $f(X) \not\subseteq \operatorname{Supp} D$, *where* $O(1)$ *stands for a bounded term as* $r \to +\infty$.

Let $f : X \to M$ be a meromorphic mapping, and let $D \in |L|$. Let E be an effective divisor on $\mathbb{C}^m$ such that $E = \sum_j \nu_j E'_j$ for distinct irreducible hypersurfaces E'_j in $\mathbb{C}^m$ and for nonnegative integers ν_j, and let k be a positive integer. We set

$$N_k(r, E) = \sum_j \min\{k, \nu_j\} N(r, E'_j).$$

A meromorphic mapping $f : X \to M$ is said to be *dominant* provided that $\operatorname{rank} f = \dim M$. The following second main theorem for dominant meromorphic mappings gives an essential computational way in the next section (cf. [19, Theorem 1]):

Theorem 1.2. *Let* M *be a projective algebraic manifold with* $m \geq \dim M$ *and let* $L \to M$ *be an ample line bundle. Suppose that* $D_1, \cdots, D_q$ *are divisors in* $|L|$ *such that* $D_1 + \cdots + D_q$ *has only simple normal crossings. Let* $f : X \to M$ *be a dominant meromorphic mapping. Then*

$$qT_f(r, L) + T_f(r, K_M) \leq \sum_{j=1}^{q} N_1(r, f^*D_j) + N(r, B) + S_f(r),$$

where $S_f(r) = O(\log T_f(r, L)) + o(\log r)$ *except on a Borel subset* $E \subseteq [1, +\infty)$ *with finite measure.*

In application of Theorem 1.2, it is essential to give the estimate for $N(r, B)$ by the characteristic function of f. In the case where $m = 1$ and $M = \mathbb{P}_1(\mathbb{C})$, the ramification theorem due to H. Selberg is well-known (cf. [24]). In the case of meromorphic mappings $f : X \to M$, we have a following ramification estimate proved by J. Noguchi.

Definition 1.3. Let Y be a compact complex manifold. We say that a meromorphic mapping $f : X \to Y$ *separates the fibers of* $\pi : X \to \mathbb{C}^m$, if there exists a point z in $\mathbb{C}^m - (\operatorname{Supp} \pi_* B \cup \pi(I(f)))$ such that $f(x) \neq f(y)$ for any distinct points $x, y \in \pi^{-1}(z)$. In this case, X is said to be the *proper existence domain of* f.

Assume that $f : X \to M$ separates the fibers of $\pi : X \to M$ and L is ample. Then there exist the least positive integer μ_0 and a pair of sections $\sigma_0, \sigma_1 \in H^0(M, \mu_0 L)$ such that a meromorphic function $f^*(\sigma_0/\sigma_1)$ separates the fibers of $\pi : X \to \mathbb{C}^m$. Then we have the following ramification estimate due to J. Noguchi ([19, p. 277]):

Theorem 1.4 (Noguchi). *Suppose that* $L \to M$ *is ample and* $f : X \to M$ *separates the fibers of* $\pi : X \to M$. *Let* μ_0 *be as above. Then*

$$N(r,\ B) \leq 2\mu_0(s_0 - 1)\ T_f(r,\ L) + O(1).$$

In the case where f does not separate fibers of $\pi : X \to M$, we cannot estimate the growth of the ramification divisor in general. However, we have the following reduction theorem proved by J. Noguchi ([19, p. 272]):

Theorem 1.5 (Noguchi). *Let* $f : X \to M$ *be a meromorphic mapping. Then there exist a finite analytic covering space* $\varpi : \underline{X} \to \mathbb{C}^m$, *a surjective proper holomorphic mapping* $\lambda : X \to \underline{X}$ *and a meromorphic mapping* $\underline{f} : \underline{X} \to M$ *separating the fibers of* $\varpi : \underline{X} \to \mathbb{C}^m$ *such that the following diagram*

$$\begin{tikzcd}
\mathbb{C}^m \arrow[d, "\mathrm{id}"'] & X \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "\lambda"'] & M \arrow[d, "\mathrm{id}"] \\
\mathbb{C}^m & \underline{X} \arrow[l, "\varpi"] \arrow[r, "\underline{f}"'] & M
\end{tikzcd}$$

is commutative. Furthermore, if f *is dominant, so is* $\underline{f}$.

From the above theorem, we can determine the proper domain of existence for an arbitrary meromorphic mappings $f : X \to M$. For the theory of algebroid reduction, see also [30].

Remark 1.6 We note that $\underline{X}$ is also a normal complex space. By making use of Theorem 1.4, we can easily obtain the following equalities (cf. [19, p. 273]):

$$T_f(r,\ L) = T_{\underline{f}}(r,\ L), \quad \text{and} \quad N(r,\ f^*D) = N(r,\ \underline{f}^*D).$$

Thus we also have

$$N_k(r,\ f^*D) = N_k(r,\ \underline{f}^*D)$$

for each positive integer k. Hence, by Theorems 1.2 and 1.4, we have the following: For an arbitrary dominant meromorphic mapping $f : X \to M$, the following inequality

$$qT_f(r,\ L) + T_f(r,\ K_M) \leq \sum_{j=1}^{q} N_1(r,\ f^*D_j) + N(r,\ \underline{B}) + S_f(r)$$

holds, where $\underline{B}$ is the ramification divisor of $\varpi : \underline{X} \to \mathbb{C}^m$. We also see that the following inequality holds:

$$N(r,\ \underline{B}) \leq 2\mu_0(s_0 - 1)\, T_f(r,\ L) + O(1).$$

Therefore we can apply Theorems 1.2 and 1.4 for an arbitrary meromorphic mapping $f : X \to M$. This observation is very useful and will be essentially used in the next

§2. Criteria for propagation of algebraic dependence.

We first give a definition of algebraic dependence of meromorphic mappings. Let M be a projective algebraic manifold and $L \to M$ an ample line bundle over M. Set $M^2 = M \times M$. For meromorphic mappings $f_1, f_2 : X \to M$, we define a meromorphic mapping $f_1 \times f_2 : X \to M^2$ by

$$(f_1 \times f_2)(z) = (f_1(z), f_2(z)), \quad z \in X - (I(f_1) \cup I(f_2)),$$

where $I(f_j)$ are the indeterminacy loci of f_j. A proper algebraic subset Σ of M^2 is said to be *decomposable* if there exist algebraic subsets Σ_1, Σ_2 such that $\Sigma = \Sigma_1 \times \Sigma_2$.

Definition 2.1. Let S be an analytic subset of X. Nonconstant meromorphic mappings $f_1, f_2 : X \to M$ are said to be *algebraically dependent* on S if there exists a proper algebraic subset Σ of M^2 such that $(f_1 \times f_2)(S) \subseteq \Sigma$ and Σ is not decomposable. In this case, we also say that f_1 and f_2 are *Σ-related* on S.

Let $D_1, \cdots, D_q$ be divisors in $|L|$ such that $D_1 + \cdots + D_q$ has only simple normal crossings. Let $S_1, \cdots, S_q$ be hypersurfaces in X such that $\dim S_i \cap S_j \leq m-2$ for any $i \neq j$. We define a hypersurface S in X by $S = S_1 \cup \cdots \cup S_q$. Let E be an effective divisor on X, and let k be a positive integer. If $E = \sum_j \nu_j E'_j$ for distinct irreducible hypersurfaces E'_j in X and for nonnegative integers ν_j, then we define the support of E with order at most k by

$$\mathrm{Supp}_k\, E = \bigcup_{0<\nu_j \leq k} E'_j.$$

Assume that $\mathrm{Supp}_{k_j} f_0^* D_j$ coincides with S_j for all j with $1 \leq j \leq q$, where k_j is a fixed positive integer. Let $\mathcal{F}$ be the set of all *dominant* meromorphic mappings $f : X \to M$ such that $\mathrm{Supp}_{k_j} f^* D_j$ is equal to S_j for each j with $1 \leq j \leq q$. Let F_1 and F_2 be big line bundles over M. We define a line bundles $\tilde{F}$ over M^2 by

$$\tilde{F} = \pi_1^* F_1 \otimes \pi_2^* F_2,$$

where $\pi_j : M^2 \to M$ are the natural projections on j-th factor. Let $\tilde{L}$ be a big line bundle over M^2. In the case of $\tilde{L} \neq \tilde{F}$, we assume that there exists a positive rational number $\tilde{\gamma}$ such that $\tilde{\gamma}\tilde{F} \otimes \tilde{L}^{-1}$ is big. If $\tilde{L} = \tilde{F}$, then we take $\tilde{\gamma} = 1$. Let $\mathcal{R}$ be the set of all hypersurfaces Σ in X such that $\Sigma = \mathrm{Supp}\, \tilde{D}$ for some $\tilde{D} \in |\tilde{L}|$ and Σ is not decomposable.

Assume that $f : X \to M$ separates the fibers of $\pi : X \to M$. Since L is ample, there exist a positive integer μ and a pair of sections $\sigma_0, \sigma_1 \in H^0(M, \mu L)$ such that a meromorphic function $f^*(\sigma_0/\sigma_1)$ separates the fibers of $\pi : X \to \mathbb{C}^m$ for all such mappings f. We denote by μ_0 the least positive integer among those μ's. We assume that there exists a line bundle, say F_0, in $\{F_1, F_2\}$ such that $F_0 \otimes F_j^{-1}$ is either big or trivial for $j = 1, 2$. Set $k_0 = \max_{1 \leq j \leq q} k_j$. We define $L_0 \in \mathrm{Pic}(M) \otimes \mathbb{Q}$ by

$$L_0 = \left(\sum_{j=1}^{q} \frac{k_j}{k_j + 1} - 2\mu_0(s_0 - 1) \right) L \otimes \left(-\frac{2\tilde{\gamma} k_0}{k_0 + 1} F_0 \right).$$

Then, by making use of Theorems 1.2 and 1.4, we have our basic result, from which we see that, if L_0 is sufficiently big, then the algebraic dependence on S propagates to the whole space X.

Lemma 2.2. *Let f_1 and f_2 be arbitrary mappings in $\mathcal{F}$ and $\Sigma \in \mathfrak{R}$. Suppose that f_1 and f_2 are Σ-related on S. If $L_0 \otimes K_M$ is big, then f_1 and f_2 are Σ-related on X.*

Now, let us consider a more general case. Let L_1 and L_2 be ample line bundles over M. Let $q_1, \cdots, q_l$ be positive integers and assume that $D_j = D_{j1} + \cdots + D_{jq_j} \in |q_j L|$ has only normal crossings, where $D_{jk} \in |L_j|$. Let Z be a hypersurface in X. Let $\mathcal{G}$ be a family of dominant meromorphic mappings $f : X \to M$ such that

$$\mathrm{Supp}_{k_j} f^* D_j = Z$$

for some j. In the case where $L_j = L$ for $j = 1, 2$, we define $G_0 \in \mathrm{Pic}(M) \otimes \mathbb{Q}$ by

$$G_0 = \left(\min_{j=1,\,2} \left\{ \frac{q_j k_j}{k_j + 1} \right\} - 2\mu_0(s_0 - 1) \right) L \otimes \left(-\frac{2\tilde{\gamma} k_0}{k_0 + 1} F_0 \right).$$

Then we have one more fundamental result for our study.

Lemma 2.3. *Let f_1 and f_2 be arbitrary mappings in $\mathcal{G}$ and $\Sigma \in \mathfrak{R}$. Suppose that f_1, f_2 are Σ-related on Z. If $G_0 \otimes K_M$ is big, then f_1, f_2 are Σ-related on X.*

Now, we will give criteria for the propagation of algebraic dependence of dominant meromorphic mappings, which is a corollary of Lemma 2.2. For $F \in \mathrm{Pic}(M) \otimes \mathbb{Q}$, we define $[F/L]$ by

$$[F/L] = \inf \left\{ \gamma \in \mathbb{Q};\ \gamma L \otimes F^{-1} \text{ is big} \right\}.$$

Set

$$p_0 = \sum_{j=1}^{q} \frac{k_j}{k_j + 1} - [K_M^{-1}/L] - 2\mu_0(s_0 - 1).$$

We also set

$$m_1 = q - [K_M^{-1}/L] - 2\mu_0(s_0 - 1) \qquad \text{and} \qquad m_2 = q - [K_M^{-1}/L]).$$

Then we have the following criterion for the propagation of algebraic dependence:

Corollary 2.4. *Let f_1, $f_2 \in \mathcal{F}$. Suppose that they are Σ-related on S. If m_j are positive and if*

$$p_0 - \frac{2\tilde{\gamma} k_0}{k_0 + 1} [F_1/L] + m_1 p_0 - \frac{2\tilde{\gamma} k_0}{k_0 + 1} [F_j/L] > 0,$$

then f_1, f_2 are Σ-related on X.

By making use of Lemma 2.3, we also have the following two criteria. Set

$$n_1 = q_1 - [K_M^{-1}/L_1] - 2\mu_0(s_0 - 1) \qquad \text{and} \qquad n_2 = q_2 - [K_M^{-1}/L_2].$$

We also set

$$p_j = \frac{q_j k_j}{1+k_j} - [K_M^{-1}/L_j] - 2\mu_0(s_0-1)$$

for $j = 1,\ 2$. Then we have the following criterion:

Corollary 2.5. *Let $f_1,\ f_2$ be arbitrary mappings in $\mathcal{G}$ and $\Sigma \in \mathfrak{R}$. Suppose that $f_1,\ f_2$ are Σ-related on Z. If all $n_j > 0$ and if*

$$p_1 - \frac{2\tilde{\gamma}k_0}{k_0+1}[F_1/L_1] + n_1 p_2 - \frac{2\tilde{\gamma}k_0}{k_0+1}[F_2/L_2] > 0,$$

then $f_1,\ f_2$ are Σ-related on X.

Set $e_0 = 2\mu_0(s_0-1)+1$. Then we also have the following:

Corollary 2.6. *Let $f_1,\ f_2$ be as in Corollary 2.5. If all $n_j > 0$ and if*

$$p_1 - \frac{2\tilde{\gamma}k_0}{k_0+1}[F_1/L_1] + n_1 p_2 - \frac{2\tilde{\gamma}e_0 k_0}{n_2(k_0+1)}[F_2/L_2] > 0,$$

then $f_1,\ f_2$ are Σ-related on X.

Remark 2.7. The case, where either all $k_j = 1$ or all $k_j = +\infty$, are especially important from the viewpoint of Nevanlinna theory. We now consider the case where $k_j = +\infty$ for some j. We first note that $\mathrm{Supp}\, f^*D = \mathrm{Supp}_{k_j} f^*D$ if $k_j = +\infty$. Set $k_j/(k_j+1) = 1$ and $1/(k_j+1) = 0$ for $k_j = +\infty$. Then it is easy to see that the proofs of Lemmas 2.2 and 2.3 also work in the case where $k_j = +\infty$ for some j. Hence the conclusions of the above propositions are still valid for the case where some of the $k_j = +\infty$. We also note that the proof of Lemma 2.2 also works in the case where some of the S_j are empty sets.

§3. Unicity theorems for meromorphic mappings.

In this section we give some unicity theorems as an application of criteria for dependence by taking line bundles F_j of special type. For the details of this direction, see [1], [3],[4], [8] and [28]. We keep the same notation as in §2. Let $\Phi : M \to \mathbb{P}_n(\mathbb{C})$ be a meromorphic mapping with $\mathrm{rank}\,\Phi = \dim\, M$. We denote by H the hyperplane bundle over $\mathbb{P}_n(\mathbb{C})$. Take $F_1 = F_2 = \Phi^* H$. We also take $\tilde{L} = \tilde{F}$. Then we see

$$L_0 = \left(\sum_{j=1}^{q} \frac{k_j}{k_j+1} - 2\mu_0(s_0-1)\right) L \otimes \left(-\frac{2k_0}{k_0+1}\Phi^* H\right).$$

We fix $f_0 \in \mathcal{F}$. Set

$$\Omega_0 = M - (\{w \in M - I(\Phi);\ \mathrm{rank}\, d\Phi(w) < \dim\, M\} \cup I(\Phi)),$$

where $I(\Phi)$ is the locus of indeterminacy of Φ. A set $\{D_j\}_{j=1}^q$ of divisors is said to be *generic with respect to* f_0 *and* Φ provided that

$$f_0(\mathbb{C}^m - I(f_0)) \cap \operatorname{Supp} D_j \cap \Omega_0 \neq \emptyset$$

for at least one $1 \leq j \leq q$, where $I(f_0)$ denotes the locus of indeterminacy of f_0. We assume that $\{D_j\}_{j=1}^q$ is generic with respect to f_0 and Φ in what follows. Let $\mathfrak{F}_1$ be the set of all mappings $f \in \mathfrak{F}$ such that $f = f_0$ on S. Then we have the following unicity theorems by Lemma 2.2 and by uniqueness of analytic continuation (cf. [3, Theorem 2.1]):

Theorem 3.1. *Suppose that* $L_0 \otimes K_M$ *is big. Then the family* $\mathfrak{F}_1$ *contains just one mapping* f_0.

We next consider the case $\dim M = 1$. Assume that M is a compact Riemann surface with genus g_0. In the case $g_0 = 0$, we have the following unicity theorem for meromorphic functions on X by Theorem 3.1, which is closely related to the uniqueness problem of algebroid functions (cf. [1, Theorem 3.3]).

Theorem 3.2. *Let* $f_1, f_2 : X \to \mathbb{P}_1(\mathbb{C})$ *be nonconstant holomorphic mappings. Let* $a_1, \cdots, a_d$ *be distinct points in* $\mathbb{P}_1(\mathbb{C})$. *The following hold.*

(1) *Suppose that* $\mathrm{Supp} f_1{}^* a_j = \mathrm{Supp} f_2{}^* a_j$ *for all* j. *If* $d \geq 2s_0 + 3$, *then* f_1 *and* f_2 *are identical on* X.

(2) *Suppose that* $\mathrm{Supp}_1 f_1{}^* a_j = \mathrm{Supp}_1 f_2{}^* a_j$ *for all* j. *If* $d \geq 4s_0 + 3$, *then* f_1 *and* f_2 *are identical on* X.

Note that the above theorem is sharp in the case $X = \mathbb{C}$.

Example 3.3. We consider the integral

$$z = \varphi(w) := \int_0^w (1 - t^4)^{-\frac{1}{2}} dt$$

on the unit disc in $\mathbb{C}$. Set $z_1 = \varphi(1)$, $z_2 = \varphi(\sqrt{-1})$, $z_3 = \varphi(-1)$ and $z_4 = \varphi(-\sqrt{-1})$. Then φ maps the unit disc onto the square $z_1 z_2 z_3 z_4$. By Schwarz's reflection principle, the inverse function of $z = \varphi(w)$ can be analytically continued over the complex plane $\mathbb{C}$, and the resulting function $w = f(z)$ is doubly periodic. Let $a_1 = 1$, $a_2 = \sqrt{-1}$, $a_3 = -1$, $a_4 = -\sqrt{-1}$, $a_5 = 0$ and $a_6 = \infty$. Set $f_1 = f$ and $f_2 = \sqrt{-1} f$. Then $\mathrm{Supp}_1 f_1{}^* a_j = \mathrm{Supp}_1 f_2{}^* a_j$ for all j, but $f_1 \not\equiv f_2$.

The uniqueness problem of holomorphic mappings into a compact Riemann surface with positive genus is not well studied (cf. [1], [9] and [23]). In the case of $g_0 = 1$, we will discuss the uniqueness for holomorphic mappings into smooth elliptic curves in §4. We now consider the case where $g_0 \geq 2$. Note that Riemann-Roch's theorem shows $\mu_0 \leq g_0 + 1$. In this case, by making use of Theorem 3.1, we have the following unicity theorem (cf. [1, Theorem 3.6)]):

Theorem 3.4. *Let $f_1, f_2 : X \to M$ be nonconstant holomorphic mappings. Let $a_1, \cdots, a_d$ be distinct points in M. The following hold.*

(1) *Suppose that* $\mathrm{Supp} f_1^* a_j = \mathrm{Supp} f_2^* a_j$ *for all j. If $d > \max\{4g_0, 2(g_0+1)(s_0-1)\}$, then f_1 and f_2 are identical on X.*

(2) *Suppose that* $\mathrm{Supp}_1 f_1^* a_j = \mathrm{Supp}_1 f_2^* a_j$ *for all j. If $d > \max\{4g_0, (2g_0+1)(2s_0+1) - 8g_0\}$, then f_1 and f_2 are identical on X.*

Note that under the condition of Theorems 3.2 and 3.4, at least one $\mathrm{Supp}_1 f_1^* a_j$ is not empty.

§4. Holomorphic mappings into smooth elliptic curves.

We finally consider the case where M is a smooth elliptic curve E. The uniqueness problem of holomorphic mappings into elliptic curves was first studied by E. M. Schmid [23] and Schmid obtained the following unicity theorem: Let $f, g : R \to E$ be nonconstant holomorphic mappings, where R is an open Riemann surface of a certain type. Then there exists a nonnegative integer d depending only on R such that, if $f^{-1}(a_j) = g^{-1}(a_j)$ for distinct $d+5$ points $a_1, \cdots, a_{d+5}$ in E, then f and g are identical. In the special case $R = \mathbb{C}$, we have $d = 0$.

So far, there have been only few studies on the uniqueness problem of holomorphic mappings $f : X \to E$ (cf. [9] and [23]). In this section, we consider the problem to determine the condition which yields $f = \varphi(g)$ for an endomorphism φ of the abelian group E. We first note the following fact: If $f : X \to E$ separates the fibers of $\pi : X \to \mathbb{C}^m$, then we can take $\mu_0 = 2$ (cf. [20, p. 286]). Let $L \in \mathrm{Pic}(E)$. Since $H^2(E, \mathbb{Z}) \cong \mathbb{Z}$, we identify the Chern class $c(L)$ of L with an integer. We now consider the infimum $[F/L]$ of the set of rational numbers γ such that $\gamma c(L) - c(F)$ is ample. We note that $[F/L] = [F/L']$ if $c(L) = c(L')$. Hence the conclusions of Lemma 2.3, Corollaries 2.5 and 2.6 are still valid provided that $D_j \in |q_j L_j|$ and all $c(L_j)$ are identical. We also note that $\tilde{\gamma}$ is not necessarily rational number in this section. It is well-known that

$$\mathrm{Pic}(E^2) \neq \pi_1^* \mathrm{Pic}(E) \oplus \pi_2^* \mathrm{Pic}(E).$$

We denote by $[p]$ the point bundle determined by $p \in E$. Let $F_1 = F_2 = [p]$. Let $f, g : X \to E$ be nonconstant holomorphic mappings. We denote by $\mathrm{End}(E)$ the ring of endomorphisms of E. If E has no complex multiplication, it is well-known that $\mathrm{End}(E) \cong \mathbb{Z}$. Hence $\varphi(x) = nx$ for some integer n.

We now seek conditions which yield $g = \varphi(f)$ for some $\varphi \in \mathrm{End}(E)$. Let $\varphi \in \mathrm{End}(E)$ and consider a curve

$$\tilde{S} = \{(x, y) \in E \times E;\ y = \varphi(x)\}$$

in $E \times E$. Let $\tilde{L}$ be the line bundle $[\tilde{S}]$ determined by $\tilde{S}$. In this section, $\tilde{\gamma}$ denotes the infimum of rational numbers such that $\gamma \tilde{F} \otimes [\tilde{S}]^{-1}$ is ample. Then we essentially use the following theorem proved by T. Katsura (see [6, §6]):

Theorem (Katsura). *Let $\tilde{\gamma}$ be as above. Then* $\tilde{\gamma} = \deg \varphi + 1$.

By the above theorem, we have the following corollary (cf. [26, p. 89]):

Corollary. *Let n be an integer. If $\varphi \in \mathrm{End}(E)$ is an endomorphism defined by $\varphi(x) = nx$, then $\tilde{\gamma} = n^2 + 1$.*

By making use of Lemma 2.3, we have the following:

Theorem 4.1. *Let f, g and φ be as above. Let $D_1 = \{a_1, \cdots, a_d\}$ be a set of d points and φ a endomorphism of E. Set $D_2 = \varphi(D_1)$. Assume that the number of points in D_2 is also d. Suppose that $\mathrm{Supp}_k\ f^*D_1 = \mathrm{Supp}_k\ g^*D_2$ for some k. If $d > 2(\deg \varphi + 1) + 8(s_0 - 1)(1 + k^{-1})$, then $g = \varphi(f)$.*

In the above theorem, we assume that the cardinality $\sharp D_2$ of the point set D_2 equals d. However, it may happen that $\sharp D_2 < d$. For example, if $\varphi(x) = nx$ $(n \in \mathbf{Z})$ and there exists at least one pair $(i,\ j)$ such that $a_i - a_j$ is n-torsion point, then $\sharp D_2 < d$. In this case, by making use of Corollary 2.6, we have the following:

Theorem 4.2. *Let $f,\ g : \mathbf{C}^m \to E$ be nonconstant holomorphic mappings. Let $D_1 = \{a_1, \cdots, a_d\}$ be a set of d points and $\varphi \in \mathrm{End}(E)$. Set $D_2 = \varphi(D_1)$. Assume that the number of points in D_2 is d'. Suppose that $\mathrm{Supp}_1\ f^*D_1 = \mathrm{Supp}_1\ g^*D_2$. If $dd' > (d + d')(\deg \varphi + 1)$, then $g = \varphi(f)$.*

Corollary 4.3. *Let f, g and X be as in Theorem 5.2. Let $D_1 = \{a_1, \cdots, a_d\}$ be a set of d points and set $D_2 = \{na_1, \cdots, na_d\}$ for some integer n. Assume that the number of points in D_2 is d'. Suppose that $\mathrm{Supp}_1\ f^*D_1 = \mathrm{Supp}_1\ g^*D_2$. If $dd' > (d + d')(n^2 + 1)$, then $g = nf$.* We do not know whether Theorem 5.2 is sharp or

not. However, if the condition $dd' > (d + d')(\deg \varphi + 1)$ is not satisfied, then it is not necessarily true that $g = \varphi(f)$.

Example 4.4. Let $\varphi \in \mathrm{End}(E)$ be an endomorphism defined by $\varphi(x) = 2x$. Define $f, g : \mathbf{C} \to E$ by $f(z) = \overline{\pi}(x)$ and $g(z) = -2\overline{\pi}(x)$, where $\overline{\pi} : \mathbf{C} \to E$ be the universal covering mapping. Let $D_1 = \{x \in E; 4x = 0\}$. Then $D_2 = \varphi(D_1) = 2D_1$. It is clear that $\mathrm{Supp}_1\ f^*D_1 = \mathrm{Supp}_1\ g^*D_2$. In this case, $d = 16$, $d' = 4$ and $\deg \varphi + 1 = 5$. Thus we have

$$dd' - (d + d')(\deg \varphi + 1) = -36 < 0$$

and $g \neq \varphi(f)$.

The following unicity theorem is a direct conclusion of Theorem 4.1:

Theorem 4.5. *Let $a_1, \cdots, a_d$ be distinct points in E. Let $f,\ g : X \to E$ be nonconstant holomorphic mappings. Suppose that $\mathrm{Supp}_k\ f^*a_j = \mathrm{Supp}_k\ g^*a_j$ for all j, where $1 \leq k \leq +\infty$. If $d > 8s_0 - 4 + 8k^{-1}(s_0 - 1)$, then f and g are identical.*

In the case of $X = \mathbf{C}^m$, we have the following:

Theorem 4.6. *Let* $a_1, \cdots, a_d$ *be distinct points in* E. *Let* $f,\ g : \mathbb{C}^m \to E$ *be nonconstant holomorphic mappings. Suppose that* $\mathrm{Supp}_1\, f^*a_j = \mathrm{Supp}_1\, g^*a_j$ *for all* j. *If* $d \geq 5$, *then* f *and* g *are identical.*

We give here the concluding remark. If we choose special points of E, we obtain an example which yields that Theorem 5.6 is sharp. Indeed, let $a_1, \cdots, a_4$ be two-torsion points in E and let $\wp$ be the Weierstrass $\wp$ function. If $f_1^*a_j = f_2^*a_j$ for $j = 1, \cdots, 4$, it is easy to see that $\wp \circ f_1 = \wp \circ f_2$ by Nevanlinna's four points theorem. Hence $f_1 = f_2$ or $f_1 = -f_2$. Since $p \mapsto -p\ (p \in E)$ is an automorphism of E, it is acceptable that f_1 and f_2 are essentially identical. In this example, it seems that the structure of the function field of E affects strongly the uniqueness problem for holomorphic mappings.

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