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METRIC ENTROPY AND HORSESHOE FOR C^1 MAPS

YONG MOO CHUNG (広島大・工 鄭容武)

Department of Applied Mathematics, Graduate School of Engineering
Hiroshima University, Higashi-Hiroshima 739-8527, Japan

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ABSTRACT. We show that for a C^1 one-dimensional map there is a hyperbolic Cantor set in a neighborhood of the support of an invariant probability measure with positive metric entropy. Some results concerning the relation between entropy and expanding periodic orbits follow from this fact.

Misiurewicz and Szlenk [1, 10, 11] have proved that for any continuous map of an interval or the circle the growth rate of the number of periodic points is equal or greater than the topological entropy.

For C^1 Hölder maps of manifolds in any dimension, the author [2, 3] proved that the metric entropy of a hyperbolic measure is approximated by the topological entropy of a horseshoe. It is an extension of the result obtained by Katok [5, 7] for diffeomorphisms, and not requiring any conditions for critical orbits. Then in dimension one the topological entropy of the map is characterized by the number of expanding periodic points [2, 4]. In that proof he used Pesin theory [13, 14], a theory of nonuniformly hyperbolic dynamical systems, and the assumption of Hölder continuity of the derivative is crucial in that theory [15].

On the other hand, Katok and Mezhirov [8] proved that a large number of periodic orbits are expanding with exponent at least almost as large as entropy without assuming the regularity of the map beyond C^1.

The purpose of this paper is to show that the horseshoe result obtained for C^1 Hölder maps is valid without the Hölder continuous condition in dimension one.

Throughout this paper let M be a compact interval or the circle and $f : M \to M$ a C^1 map with finitely many critical points. We denote by $h(f)$ the topological entropy of f, and by $h_\mu(f)$ the metric entropy of μ for $\mu \in \mathcal{E}(f)$, where $\mathcal{E}(f)$ denotes the set of ergodic f-invariant Borel probability measures on M. Then it is well-known as the variational principle for entropy [17] that:

$$h(f) = \sup\{h_\mu(f) : \mu \in \mathcal{E}(f)\}.$$

The following is also known as the Ruelle entropy inequality [16]:

$$h_\mu(f) \le \lambda_\mu$$

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for any $\mu \in \mathcal{E}(f)$ with $h_\mu(f) > 0$, where λ_μ denotes the Lyapunov exponent of μ for f, that is

$$\lambda_\mu = \lambda_\mu(f) = \int \log |f'(x)| d\mu(x).$$

Our main result is the following:

Proposition. *Let μ be an f-invariant ergodic Borel probability measure and assume that the metric entropy $h_\mu(f)$ is positive. Then for any continuous functions $\xi_1, \ldots, \xi_l$ on M and a number $\epsilon > 0$ there are a positive integer m_0 and a Cantor set Λ of M with $f^{m_0}(\Lambda) = \Lambda$ such that:*

(1) $\Lambda \subset B_\epsilon(\mathrm{supp}(\mu))$;

(2) $f^{m_0}|_\Lambda : \Lambda \to \Lambda$ *is topologically conjugate to a one-sided fullshift and*

$$\frac{1}{m_0} h(f^{m_0}|_\Lambda) \geq h_\mu(f) - \epsilon;$$

(3) *for any $x \in \Lambda$ and $k = 1, \ldots, l$*

$$|(f^{m_0})'(x)| \geq e^{m_0(h_\mu(f) - \epsilon)} \quad \text{and} \quad \left|\frac{1}{m_0} \sum_{i=0}^{m_0-1} \xi_k(f^i(x)) - \int \xi_k d\mu\right| \leq \epsilon,$$

where $B_\epsilon(A)$ denotes the ϵ-neighborhood of a set A, and supp(μ) *the support of μ.*

We remark that if any critical point of the map does not belong to the support of the measure then the logarithm of the modulus of the derivative of the map is continuous on a neighborhood of the support, and hence the Cantor set stated in the proposition can be chosen so that on which the Lyapunov exponent is close to that of the measure.

For a periodic point p of f with period n, the Lyapunov exponent along its orbit is given by

$$\lambda(p) = \frac{1}{n} \log |(f^n)'(p)|.$$

From the proposition it follows immediately that:

Theorem 1. *Let μ be as above. Then for any $0 < \alpha < h_\mu(f)$ and $\epsilon > 0$,*

$$\begin{aligned} h_\mu(f) &\leq \lim_{\delta \to 0+} \limsup_{n \to \infty} \frac{1}{n} \log \sharp\{p \in B_\epsilon(\mathrm{supp}(\mu)) : f^n(p) = p, \\ &\qquad\qquad |(f^j)'(p)| \geq \delta e^{j\alpha} \quad \text{for all} \quad j \geq 0\} \\ &\leq \limsup_{n \to \infty} \frac{1}{n} \log \sharp\{p \in B_\epsilon(\mathrm{supp}(\mu)) : f^n(p) = p, \ \lambda(p) \geq \alpha\}, \end{aligned}$$

where $\sharp A$ denotes the cardinality of a set A.

Combining Theorem 1 with the variational principle we obtain:

Corollary 2. *If* $0 < \alpha < h(f)$ *then*

$$h(f) = \lim_{\delta \to 0+} \limsup_{n \to \infty} \frac{1}{n} \log \sharp\{p \in M : f^n(p) = p, \ \ |(f^j)'(f^i(p))| \geq \delta e^{j\alpha}$$

$$\text{for all} \ \ j \geq 0 \ \text{and} \ 0 \leq i \leq n-1\}$$

$$= \lim_{\delta \to 0+} \limsup_{n \to \infty} \frac{1}{n} \log \sharp\{p \in M : f^n(p) = p, \ \ \lambda(p) \geq \alpha,$$

$$|f'(f^i(p))| \geq \delta \ \ \text{for all} \ \ 0 \leq i \leq n-1\}.$$

Another consequence of the proposition is the following:

Theorem 3. *Let* μ *be as in the proposition. Then there is a sequence* p_j $(j = 1, 2, \ldots)$ *of periodic points of* f *such that*

$$\lim_{j \to \infty} p_j \in \operatorname{supp}(\mu), \quad \lim_{j \to \infty} \lambda(p_j) \geq h_\mu(f) \quad \text{and} \quad \lim_{j \to \infty} \frac{1}{n(p_j)} \sum_{i=0}^{n(p_j)-1} \delta_{f^i(p_j)} = \mu,$$

where δ_x *denotes the Dirac measure supported on a single point* x *and* $n(p)$ *the period of a periodic point* p.

Combining Theorem 3 with the ergodic decomposition, it is easy to see that any invariant probability measure of positive metric entropy is approximated by a measure of which support consists of finite number of expanding periodic orbits. Since the metric entropy is upper semi-continuous as a function of invariant probability measures [11] we get:

Collorary 4. *The metric entropy of generic* f*-invariant Borel probability measure is zero.*

It is checked that the corollary above is also valid for any continuous map of an interval or the circle from our proof.

From the hyperbolicity of the Cantor set stated in the proposition we have:

Theorem 5. *Let* $f : M \to M$ *and* μ *be as in the proposition. If a sequence* g_n $(n = 1, 2, \ldots)$ *of maps converges to* f *in* C^1 *topology, then there are* g_n*-invariant ergodic Borel probability measures* μ_n *supported on hyperbolic sets* Λ_n *of* g_n *such that*

$$\lim_{n \to \infty} \mu_n = \mu \quad \text{and} \quad \lim_{n \to \infty} h_{\mu_n}(g_n) = h_\mu(f).$$

Proof of Proposition

Replacing $f : M \to M$ to its N_0th iterate $f^{N_0} : M \to M$ and ξ_k to $(1/N_0) \cdot \sum_{i=0}^{N_0-1} \xi_k \circ f^i$ $(k = 1, 2, \ldots, l)$ for some large $N_0 \geq 1$ if necessary, we may assume that $h_\mu(f) \geq \log 3 + \epsilon$ without loss of generality. Take a finite partition $\mathcal{I}$ of M by intervals such that:

(1) $h_\mu(f, \mathcal{I}) \geq h_\mu(f) - \epsilon/8$, where $h_\mu(f, \mathcal{I})$ denotes the entropy of the partition $\mathcal{I}$;

(2) $\mathrm{crit}(f) \subset \cup_{I\in\mathcal{I}}\partial I$, where $\mathrm{crit}(f)$ denotes the set of critical points of f and ∂J the boundary of a set J;
(3) $\max_{I\in\mathcal{I}}|I| \leq \epsilon$, where $|J|$ denotes the diameter of a set J;
(4) $\max_{I\in\mathcal{I}}\varphi(\xi_k, I) \leq \epsilon/2$ for each $k = 1, 2, \ldots, l$, where $\varphi(\xi, J) = \sup_{x,y\in J}|\xi(x) - \xi(y)|$ for a function ξ and a set J.

Then f is monotone on each element of $\mathcal{I}$, and taking $\mathcal{I}$ to be fine enough we may assume that any element of $\vee_{i=0}^{n-1} f^{-i}\mathcal{I}$ is an interval. We denote by $\mathcal{I}_0$ the family of elements of $\mathcal{I}$ whose measures are positive. Then $I \subset B_\epsilon(\mathrm{supp}(\mu))$ holds for all $I \in \mathcal{I}_0$. Fix an integer $N_1 \geq 1$ and put

$$\begin{aligned}\Delta = \Delta_{N_1} = \{z \in M : \mu(\mathcal{I}_n(z)) \leq e^{-n(h_\mu(f,\mathcal{I})-\epsilon/8)},\\ |\frac{1}{n}\sum_{i=0}^{n-1}\xi_k(f^i(z)) - \int \xi_k d\mu| \leq \epsilon/2 \quad \text{for all } k = 1, 2, \ldots, l \text{ and } n \geq N_1\},\end{aligned}$$

where $\mathcal{I}_n(x)$ denotes the element of $\vee_{i=0}^{n-1} f^{-i}\mathcal{I}$ containing x. By the Birköff ergodic theorem and the Shannon-McMillan-Breiman theorem [12], taking large $N_1 \geq 1$ we may assume that $\mu(I \cap \Delta) > 0$ holds for all $I \in \mathcal{I}_0$. For $I \in \mathcal{I}_0$ and $n \geq N_1$ we put

$$\mathcal{J}(I;n) = \{J \in \vee_{i=0}^{n-1} f^{-i}\mathcal{I} : J \subset I,\ \mu(J \cap \Delta) > 0\}.$$

Then for each $J \in \mathcal{J}(I;n)$, taking $z \in J \cap \Delta$ we have

$$\begin{aligned}\mu(J) = \mu(\mathcal{I}_n(z)) &\leq e^{-n(h_\mu(f,\mathcal{I})-\epsilon/8)}\\ &\leq e^{-n(h_\mu(f)-\epsilon/4)},\end{aligned}$$

and if $x \in J$ then we have

$$\begin{aligned}|\frac{1}{n}\sum_{i=0}^{n-1}\xi_k(f^i(x)) - \int \xi_k d\mu| &\leq \frac{1}{n}\sum_{i=0}^{n-1}|\xi_k(f^i(x)) - \xi_k(f^i(z))|\\ &\quad + |\frac{1}{n}\sum_{i=0}^{n-1}\xi_k(f^i(z)) - \int \xi_k d\mu|\\ &\leq \epsilon/2 + \epsilon/2 \quad \leq \epsilon\end{aligned}$$

for all $k = 1, 2, \ldots, l$. For $I, I' \in \mathcal{I}_0$ and $n \geq N_1$ we put

$$\mathcal{J}(I, I';n) = \{J \in \mathcal{J}(I;n) : \mathrm{int} f^n(J) \supset \mathrm{cl} I'\},$$

where $\mathrm{int}A, \mathrm{cl}A$ denote the interior and the closure of a set A, respectively.

Lemma 6. *For any $I \in \mathcal{I}_0$ there are $n = n(I) \geq N_1$ and $I' = I'(I) \in \mathcal{I}_0$ such that*

$$\sharp\mathcal{J}(I, I';n) \geq e^{n(h_\mu(f)-\epsilon/2)}.$$

Proof. Put

$$\alpha_0 = \min\{\mu(I \cap \Delta) : I \in \mathcal{I}_0\} > 0.$$

Then for any $n \geq N_1$ and $I \in \mathcal{I}_0$ we have

$$\begin{aligned}\alpha_0 &\leq \mu(I \cap \Delta)\\ &\leq \sum_{J \in \mathcal{J}(I;n)} \mu(J)\\ &\leq \sharp\mathcal{J}(I;n) \cdot e^{-n(h_\mu(f)-\epsilon/4)},\end{aligned}$$

and hence

$$\begin{aligned}\sharp\mathcal{J}(I;n) &\geq \alpha_0 \cdot e^{n(h_\mu(f)-\epsilon/4)}\\ &\geq e^{n(h_\mu(f)-3\epsilon/8)} \quad > 3^n\end{aligned}$$

if $n \geq -(8/\epsilon)\log\alpha_0$. Then for each $I \in \mathcal{I}_0$ there is an integer $n = n(I) \geq N_1$ with $n \geq (8/\epsilon)\log\sharp\mathcal{I}_0$ such that

$$\sharp\mathcal{J}(I;n+1) \geq 3\sharp\mathcal{J}(I;n).$$

For each $J \in \mathcal{J}(I;n)$, since $f^n(J)$ is an (connected) interval, there are at most two elements of $\mathcal{I}_0$ that intersect $f^n(J)$ without whose closures are covered by it. Then

$$\begin{aligned}\sum_{I' \in \mathcal{I}_0} \sharp\mathcal{J}(I,I';n) &= \sum_{I' \in \mathcal{I}_0} \sharp\{J \in \mathcal{J}(I;n) : \mathrm{int} f^n(J) \supset \mathrm{cl} I'\}\\ &= \sum_{J \in \mathcal{J}(I;n)} \sharp\{I' \in \mathcal{I}_0 : \mathrm{int} f^n(J) \supset \mathrm{cl} I'\}\\ &\geq \sum_{J \in \mathcal{J}(I;n)} [\sharp\{I' \in \mathcal{I}_0 : f^n(J) \cap I' \neq \emptyset\} - 2]\\ &\geq \sharp\mathcal{J}(I;n+1) - 2\sharp\mathcal{J}(I;n)\\ &\geq \sharp\mathcal{J}(I;n)\\ &\geq e^{n(h_\mu(f)-3\epsilon/8)},\end{aligned}$$

and hence

$$\begin{aligned}\sharp\mathcal{J}(I,I';n) &\geq \frac{1}{\sharp\mathcal{I}_0} e^{n(h_\mu(f)-3\epsilon/8)}\\ &\geq e^{n(h_\mu(f)-\epsilon/2)}\end{aligned}$$

holds for some $I' = I'(I) \in \mathcal{I}_0$. □

By Lemma 6 we can choose a finite chain $I_0 = I_r, I_1, \ldots, I_{r-1} \in \mathcal{I}_0$ with $1 \leq r \leq \sharp\mathcal{I}_0$ and $n_0, n_1, \ldots, n_{r-1} \geq N_1$ such that

$$\sharp\mathcal{J}(I_s, I_{s+1}; n_s) \geq e^{n_s(h_\mu(f)-\epsilon/2)} \quad (s = 0, 1, \ldots, r-1).$$

Put $m(0) = 0, m(s) = \sum_{j=0}^{s-1} n_j$ for $s = 1, 2, \ldots, r$, and set

$$\begin{aligned}\mathcal{K} &= \{K_1, K_2, \ldots, K_t\}\\ &= \{K = K(J_0, J_1, \ldots, J_{r-1}) = \bigcap_{s=0}^{r-1} f^{-m(s)} \mathrm{cl} J_s\\ &\qquad J_s \in \mathcal{J}(I_s, I_{s+1}; n_s) \ \text{ for all } \ s = 0, 1, \ldots, r-1\}.\end{aligned}$$

Then for any $K \in \mathcal{K}$ if $x \in K$ then

$$
\begin{aligned}
|\frac{1}{N_2}\sum_{i=0}^{N_2-1}\xi_k(f^i(x)) - \int \xi_k d\mu| &\leq \frac{1}{N_2}\sum_{s=0}^{r-1}|\sum_{j=0}^{n_s-1}\xi_k(f^j(f^{m(s)}(x))) - n_s\int \xi_k d\mu| \\
&\leq \frac{1}{N_2}\sum_{s=0}^{r-1} n_s\epsilon \quad = \epsilon
\end{aligned}
$$

for all $k = 1, 2, \dots, l$, where $N_2 = m(r) = \sum_{j=0}^{r-1} n_j$. For $n \geq 1$ and $(a_0 \cdots a_{n-1}) \in \prod_{i=0}^{n-1}\{1, 2, \dots, t\}$ we denote

$$L(a_0 \cdots a_{n-1}) = \bigcap_{j=0}^{n-1} f^{-jN_2} K_{a_j}.$$

Then $L(a_0 \cdots a_{n-1})$ is a compact interval such that

$$f^{jN_2}L(a_0 \cdots a_{n-1}) = L(a_j \cdots a_{n-1}) \quad \text{for all} \quad 0 \leq j \leq n-1$$

and

$$f^{nN_2}L(a_0 \cdots a_{n-1}) \supset \mathrm{cl}I_0.$$

Let consider the product space $\Sigma = \prod_{i=0}^{\infty}\{1, 2, \dots, t\}$ and the fullshift $\sigma : \Sigma \to \Sigma$ in t-symbols. Then we have

$$
\begin{aligned}
h(\sigma) &= \log t = \log \sharp\mathcal{K} \\
&= \log \prod_{s=0}^{r-1} \sharp\mathcal{J}(I_s, I_{s+1}; n_s) \\
&\geq \log \prod_{s=0}^{r-1} e^{n_s(h_\mu(f) - \epsilon/2)} \\
&= \sum_{s=0}^{r-1} n_s(h_\mu(f) - \epsilon/2) \\
&= N_2(h_\mu(f) - \epsilon/2).
\end{aligned}
$$

Notice that it suffices to prove the proposition for f^{N_2} and $(1/N_2)\sum_{i=0}^{N_2-1}\xi_k \circ f^i$ instead of f and ξ_k $(k = 1, 2, \dots, l)$, respectively. Thus from now on we assume that $N_2 = 1$ without loss of generality. Then

$$|\xi_k(x) - \int \xi_k d\mu| \leq \epsilon$$

holds whenever $x \in K$ for all $K \in \mathcal{K}$. Let $q_0 = \sharp\mathrm{crit}(f) < \infty$ and fix an integer $l_0 \geq 1$. Then there are $(c_0^1 \cdots c_{l_0-1}^1), (c_0^2 \cdots c_{l_0-1}^2), \dots, (c_0^{q_0} \cdots c_{l_0-1}^{q_0}) \in \prod_{i=0}^{l_0-1}\{1, 2, \dots, t\}$

such that if $(a_0 \cdots a_{l_0-1}) \in \prod_{i=0}^{l_0-1}\{1,2,\dots,t\}$ satisfies $(a_0 \cdots a_{l_0-1}) \neq (c_0^p \cdots c_{l_0-1}^p)$ for all $p = 1,2,\dots,q_0$ then $L(a_0 \cdots a_{l_0-1}) \cap \mathrm{crit}(f) = \emptyset$. Put

$$X = X_{l_0} = \{(a_i) \in \Sigma : (a_j \cdots a_{j+l_0-1}) \neq (c_0^p \cdots c_{l_0-1}^p) \quad \text{for all} \quad j \geq 0, \ p = 1,2,\dots,q_0\}.$$

Then $\sigma' = \sigma|_X : X \to X$ is a subshift of finite type, and by [9] taking large l_0 we may assume that

$$\begin{aligned} h(\sigma') = h(\sigma|_X) &\geq h(\sigma) - \epsilon/4 \\ &\geq h_\mu(f) - 3\epsilon/4. \end{aligned}$$

Lemma 7. *There are an integer $k \geq 1$ and a subset Y of X with $\sigma^k(Y) = Y$ such that $\sigma^k|_Y : Y \to Y$ is a topological mixing subshift of finite type and $h(\sigma^k|_Y) = kh(\sigma')$.*

Proof. A subshift is of finite type if and only if it has the pseudo orbit tracing property. Then the nonwandering set of $\sigma' : X \to X$ is decomposed into finite number of invariant closed sets $Z_1, Z_2, \dots, Z_q$ such that $\sigma|_{Z_p} : Z_p \to Z_p$ is topologically transitive for each $p = 1,2,\dots,q$. Moreover for each $p = 1,2,\dots,q$ there is a subset Y_p of Z_p and an integer $m_p \geq 1$ such that

$$\sigma^i(Y_p) \cap \sigma^{i'}(Y_p) = \emptyset \quad \text{if} \quad 0 \leq i < i' \leq m_p - 1, \quad \sigma^{m_p}(Y_p) = Y_p,$$

$$Z_p = Y_p \cup \sigma(Y_p) \cup \cdots \cup \sigma^{m_p-1}(Y_p)$$

and $\sigma^{m_p}|_{Y_p} : Y_p \to Y_p$ is topologically mixing. We remark that $\sigma^{m_p}|_{Y_p} : Y_p \to Y_p$ is of finite type because it has the pseudo orbit tracing property. Then we have

$$\begin{aligned} h(\sigma') &= \max\{h(\sigma|_{Z_p}) : p = 1,2,\dots,q\} \\ &= h(\sigma|_{Z_r}) \\ &= h(\sigma^{m_r}|_{Y_r})/m_r \end{aligned}$$

for some r. Then $k = m_r$ and $Y = Y_r$ are what we want. □

Replacing $\sigma' : X \to X$ to $\sigma^k|_Y : Y \to Y$ we may assume that $\sigma' : X \to X$ is topologically mixing. For each integer $n \geq 1$ we say that a word $\underline{a} = (a_0 \cdots a_{n-1}) \in \prod_{i=0}^{n-1}\{1,2,\dots,t\}$ of length n is admissible in X if there is $\tilde{b} = (b_i) \in X$ such that $(b_0 \cdots b_{n-1}) = (a_0 \cdots a_{n-1})$. We denote by $W_n(X)$ the set of admissible words in X of length n. Then there is a constant $C_1 \geq 1$ such that

$$C_1^{-1} e^{nh(\sigma')} \leq \sharp W_n(X) \leq C_1 e^{nh(\sigma')}$$

for all $n \geq 1$. On the other hand, since $\sigma' : X \to X$ is a topologically mixing subshift of finite type, there exists an integer $k_0 \geq 1$ such that for any integer $n \geq 1$ and a pair $\underline{a}^1 = (a_0^1 \cdots a_{n-1}^1), \underline{a}^2 = (a_0^2 \cdots a_{n-1}^2) \in W_n(X)$ there is $B(\underline{a}_1, \underline{a}_2) = (b_0 \cdots b_{k_0-1}) \in W_{k_0}(X)$ with $(a_0^1 \cdots a_{n-1}^1 b_0 \cdots b_{k_0-1} a_0^2 \cdots a_{n-1}^2) \in W_{2n+k_0}(X)$. Notice that k_0 is independent of n. Moreover since $\cup_{(a_0 \cdots a_{l_0-1}) \in W_{l_0}(X)} L(a_0 \cdots a_{l_0-1})$

does not contain any critical point of f, there exists $\delta_0 > 0$ such that if $L(a_0 \cdots a_{l_0-1})$ with $(a_0 \cdots a_{l_0-1}) \in W_{l_0}(X)$ then $| f'(x) | \geq \delta_0$. For each int $n \geq 1$ we put

$$V_n = \{(a_0 \cdots a_{n-1}) \in W_n(X) : |L(a_0 \cdots a_{n-1})| \leq 2C_1|I_0|e^{-nh(\sigma')}, \\ \sum_{i=0}^{n-1} |f^i L(a_0 \cdots a_{n-1})| \leq \sqrt{n}\}.$$

Lemma 8. *$\sharp V_n \geq (4C_1)^{-1}e^{nh(\sigma')}$ holds for all large $n \geq 1$.*

Proof. Since $\sum_{(a_0 \cdots a_{n-1}) \in W_n(X)} |L(a_0 \cdots a_{n-1})| \leq |I_0|$ we have

$$(I) := \sharp\{(a_0 \cdots a_{n-1}) \in W_n(X) : |L(a_0 \cdots a_{n-1})| > 2C_1|I_0|e^{-nh(\sigma')}\} \\ \leq (2C_1)^{-1}e^{nh(\sigma')}.$$

On the other hand, since

$$\begin{aligned}
&\sum_{(a_0 \cdots a_{n-1}) \in W_n(X)} \sum_{j=0}^{n-1} |f^j L(a_0 \cdots a_{n-1})| \\
&= \sum_{j=0}^{n-1} \sum_{(a_0 \cdots a_{n-1}) \in W_n(X)} |f^j L(a_0 \cdots a_{n-1})| \\
&= \sum_{j=0}^{n-1} \sum_{(a_0 \cdots a_{n-1}) \in W_n(X)} |L(a_j \cdots a_{n-1})| \\
&\leq \sum_{j=0}^{n-1} \sum_{(a_j \cdots a_{n-1}) \in W_{n-j}(X)} \sharp W_j(X) \cdot |L(a_j \cdots a_{n-1})| \\
&\leq \sum_{j=0}^{n-1} \sharp W_j(X) \cdot |I_0| \\
&\leq \sum_{j=0}^{n-1} C_1 e^{jh(\sigma')}|I_0| \quad \leq C_2 e^{nh(\sigma')},
\end{aligned}$$

where $C_2 = C_1|I_0|e^{-h(\sigma')}/(1 - e^{-h(\sigma')})$, we have

$$(II) := \sharp\{(a_0 \cdots a_{n-1}) \in W_n(X) : \sum_{i=0}^{n-1} |f^i L(a_0 \cdots a_n - 1)| > \sqrt{n}\} \\ \leq (C_2/\sqrt{n})e^{nh(\sigma')}.$$

Then we obtain

$$\begin{aligned}
\sharp V_n &\geq \sharp W_n(X) - (I) - (II) \\
&\geq \{C_1^{-1} - (2C_1)^{-1} - (C_2/\sqrt{n})\}e^{nh(\sigma')} \\
&\geq (4C_1)^{-1}e^{nh(\sigma')}
\end{aligned}$$

for all $n \geq (4C_1C_2)^2$. $\square$

Lemma 9. *For any large integer $n \geq 1$ and $(a_0 \cdots a_{n-1}) \in V_n$ if $x \in L(a_0 \cdots a_{n-1})$ then*

$$|(f^{n-l_0})'(x)| \geq e^{n(h(\sigma')-\epsilon/8)}.$$

Proof. Take a small number $\beta > 0$ such that if $y, z \in M$ satisfy $|y - z| \leq \beta$ then $||f'(y)| - |f'(z)|| \leq \epsilon\delta_0/20$. For $(a_0 \cdots a_{n-1}) \in V_n$ and $i = 0, 1, \dots, n-1$ put $\beta_i = |f^i L(a_0 \cdots a_{n-1})|$ and $\eta_i = \varphi(|f'|, f^i L(a_0 \cdots a_{n-1}))$. Then $\sum_{i=0}^{n-1} \beta_i \leq \sqrt{n}$, and hence $\sharp\{i : \beta_i > \beta\} \leq \sqrt{n}/\beta$ holds. Thus we have

$$\begin{aligned}
\sum_{i=0}^{n-1} \eta_i &= \Big(\sum_{i:\beta_i > \beta} + \sum_{i:\beta_i \leq \beta} \Big) \eta_i \\
&\leq (\sqrt{n}/\beta) \cdot D + n \cdot (\epsilon\delta_0/20) \\
&\leq n\epsilon\delta_0/10
\end{aligned}$$

for all $n \geq (20D/\beta\delta_0\epsilon)^2$, where $D = \max_{x \in M} |f'(x)|$. On the other hand, since $|L(a_0 \cdots a_{n-1})| \leq 2C_1|I_0|e^{-nh(\sigma')}$ and $|f^n L(a_0 \cdots a_{n-1})| \geq |I_0|$, by the mean value theorem there is $y_0 \in L(a_0 \cdots a_{n-1})$ such that $|(f^n)'(y_0)| \geq (2C_1)^{-1} e^{nh(\sigma')}$, and hence

$$\begin{aligned}
|(f^{n-l_0})'(y_0)| &= |(f^n)'(y_0)| \cdot |(f^{l_0})'(f^{n-l_0}(y_0))|^{-1} \\
&\geq (2C_1)^{-1} D^{-l_0} e^{nh(\sigma')}.
\end{aligned}$$

Then for any $x \in L(a_0 \cdots a_{n-1})$, since $f^i(x) \in L(a_i \cdots a_{i+l_0-1})$ and $(a_i \cdots a_{i+l_0-1}) \in W_{l_0}(X)$ for all $i = 0, 1, \dots, n - l_0$, we have

$$\begin{aligned}
\log \frac{|(f^{n-l_0})'(y_0)|}{|(f^{n-l_0})'(x)|} &= \log |(f^{n-l_0})'(y_0)| - \log |(f^{n-l_0})'(x)| \\
&\leq \sum_{i=0}^{n-l_0-1} |\log |f'(f^i(y_0))| - \log |f'(f^i(x))|| \\
&\leq \delta_0{}^{-1} \sum_{i=0}^{n-l_0-1} ||f'(f^i(y_0))| - |f'(f^i(x))|| \\
&\leq \delta_0{}^{-1} \sum_{i=0}^{n-l_0-1} \eta_i \\
&\leq \delta_0{}^{-1} \sum_{i=0}^{n-1} \eta_i \\
&\leq n\epsilon/10,
\end{aligned}$$

and hence

$$\begin{aligned}
|(f^{n-l_0})'(x)| &\geq e^{-n\epsilon/10} |(f^{n-l_0})'(y_0)| \\
&\geq (2C_1)^{-1} D^{-l_0} e^{n(h(\sigma')-\epsilon/10)} \\
&\geq e^{n(h(\sigma')-\epsilon/8)}
\end{aligned}$$

for all $n \geq (40/\epsilon)\log(2C_1 D^{l_0})$. $\square$

Fix a large integer $n_0 \geq 1$ with $n_0 \geq (8/\epsilon)\cdot\max\{k_0 h(\sigma') + \log(4C_1), (l_0 + k_0)(h(\sigma') - \log\delta_0)\}$ and put $m_0 = n_0 + k_0$. Setting

$$\begin{aligned} Z = \{(a_i) \in X : \underline{a}^k = (a_{km_0}\cdots a_{km_0+n_0-1}) \in V_{n_0}, \\ (a_{km_0+n_0}\cdots a_{(k+1)m_0-1}) = B(\underline{a}^k, \underline{a}^{k+1}) \quad \text{for all} \quad k \geq 0\} \end{aligned}$$

we have $\sigma^{m_0}(Z) = Z$. Moreover $\sigma^{m_0}|_Z\colon Z \to Z$ is topologically conjugate to a fullshift in $\sharp V_{n_0}$-symbols. Now we define a compact set of M by

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{(a_0\cdots a_{n-1})\in W_n(Z)} L(a_0\cdots a_{n-1}).$$

Then $f^{m_0}(\Lambda) = \Lambda$ and $\Lambda \subset I_0 \subset B_\epsilon(\mathrm{supp}(\mu))$ hold. For any $x \in \Lambda$, taking $(a_0\cdots a_{m_0-1}) \in W_{m_0}(Z)$ with $x \in L(a_0\cdots a_{m_0-1})$, we have $f^i(x) \in K_{a_i}$ for all $i = 0, 1, \ldots, m_0 - 1$, and then

$$\begin{aligned} \left|\frac{1}{m_0}\sum_{i=0}^{m_0-1} \xi_k(f^i(x)) - \int \xi_k d\mu\right| &\leq \frac{1}{m_0}\sum_{i=0}^{m_0-1} \left|\xi_k(f^i(x)) - \int \xi_k d\mu\right| \\ &\leq \frac{1}{m_0}\sum_{i=0}^{m_0-1} \epsilon \quad = \epsilon \end{aligned}$$

for all $k = 1, 2, \cdots, l$. Since $(a_0\cdots a_{n_0-1}) \in V_{n_0}$, by Lemma 9 we have

$$\begin{aligned} |(f^{m_0})'(x)| &= |(f^{n_0+k_0})'(x)| \\ &= |(f^{l_0+k_0})'(f^{l_0}(x))| \cdot |(f^{n_0-l_0})'(x)| \\ &\geq {\delta_0}^{l_0+k_0} e^{n_0(h(\sigma')-\epsilon/8)} \\ &\geq {\delta_0}^{l_0+k_0} e^{-k_0 h(\sigma')} e^{m_0(h(\sigma')-\epsilon/8)} \\ &\geq e^{m_0(h(\sigma')-\epsilon/4)} \\ &\geq e^{m_0(h_\mu(f)-\epsilon)}. \end{aligned}$$

If $y, z \in L(a_0\cdots a_{km_0+l_0})$ with $(a_0\cdots a_{km_0+l_0}) \in W_{km_0+l_0}(Z)$ then

$$\begin{aligned} |y - z| &\leq e^{-km_0(h_\mu(f)-\epsilon)}|f^{km_0}(y) - f^{km_0}(z)| \\ &\leq e^{-km_0(h_\mu(f)-\epsilon)}|I_0|. \end{aligned}$$

Thus $\pi : \Lambda \to Z$ defind by $\pi(x) = (a_i)$ for $x \in \cap_{n=1}^{\infty} L(a_0\cdots a_{n-1})$ is a homeomorphism, and then Λ is a Cantor set. Further, it is obvious that $\pi \circ (f^{m_0}|_\Lambda) = (\sigma^{m_0}|_Z) \circ \pi$, and hence $f^{m_0}|_\Lambda\colon \Lambda \to \Lambda$ is topologically conjugate to a fullshift in

$\sharp V_{n_0}$-symbols. Moreover, by Lemma 8 we have

$$\begin{aligned}\frac{1}{m_0}h(f^{m_0}\,|_{\Lambda}) &= \frac{1}{m_0}\log \sharp V_{n_0} \\ &\geq \frac{1}{m_0}\log\{(4C_1)^{-1}e^{n_0 h(\sigma')}\} \\ &\geq \frac{1}{m_0}\log\{(4C_1)^{-1}e^{-k_0 h(\sigma')}e^{m_0 h(\sigma')}\} \\ &\geq h(\sigma') - \epsilon/4 \\ &\geq h_\mu(f) - \epsilon.\end{aligned}$$

This completes the proof of the proposition.

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