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最適化問題に対する Riccati 方程式
The Riccati equation for an optimization problem

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1 Introduction

Conjugate point is a global concept in the calculus of variations, and it is a key factor of optimality conditions. In variational problems, the variable is not a vector x in R^n but a function $x(t)$. Recently, we established a conjugate point theory for an elementary extremal problem:

$$(P_0) \qquad \text{Minimize} \quad f(x) \qquad x \in R^n.$$

In [3], we defined the Jacobi equation and (strict) conjugate points for (P_0), and we described necessary- and sufficient optimality conditions in terms of conjugate points. Furthermore, we extended the conjugate point theory to a constrained case in [4].

On the other hand, the Riccati equation also plays an important role in the classical conjugate point theory. The aims of this paper are to introduce the Riccati equation to (P_0), and to discuss optimality in terms of the solution for the Riccati equation.

In Section 2, we briefly review the classical conjugate point theory. In Section 3, we review the conjugate points for (P_0) presented in [3]. In Section 4, we define the Riccati equation for (P_0), and we describe a sufficient optimality conditions for (P_0) in terms of the Riccati equation. In Section 5, we clarify the algebraic meaning of the solution for the Riccati equation.

2 The classical conjugate point theory

Let $\bar{x}$ be a minimum for the simplest problem in the calculus of variations:

$$(SP) \qquad \begin{array}{ll} \text{Minimize} & F(x) := \int_0^T f(t, x(t), \dot{x}(t))dt \\ \text{subject to} & x(0) = A, \ x(T) = B \end{array}$$

Then it satisfies the Euler equation: $df_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t))/dt = f_x(t, \bar{x}(t), \dot{\bar{x}}(t))$, and the Legendre condition: $P(t) := f_{\dot{x}\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) \geq 0$.

Legendre attempted to prove that a sufficient condition for $F(x)$ have a weak minimum at $\bar{x}(t)$ is the strengthened Legendre condition: $f_{\dot{x}\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) > 0$ in addition to the Euler equation. Though Legendre did not get to the goal, his idea is fruitful, see [2, p. 104]. His approach was to first write the second variation

$$I(y) := \int_0^T \{P\dot{y}^2 + Ry^2\}dt \quad \text{for } y(0) = y(T) = 0$$

of $F(x)$ at $\bar{x}$ in the form

$$I(y) = \int_0^T \{P\dot{y}^2 + Ry^2 + \frac{d}{dt}(wy^2)\}dt = \int_0^T \{P\dot{y}^2 + 2y\dot{y}w + (R + \dot{w})y^2\}dt, \quad (1)$$

where $w(t)$ is an arbitrary piecewise smooth function. Next, he observed that the strengthened Legendre condition would be sufficient if it were possible to find a function $w(t)$ for which the integrand in (1) is a perfect square. However, this is not always possible, since $w(t)$ would have to satisfy the Riccati equation

$$P(R + \dot{w}) = w^2, \quad (2)$$

and although the Riccati equation may not have a solution on the whole interval $[0, T]$. Furthermore, by the change of variable

$$w = -\frac{\dot{y}}{y}P, \quad (3)$$

the Riccati equation is transformed into the Jacobi equation: $d(P\dot{y})/dt = Ry$, which implies that the Riccati equation has a solution $w(t)$ except zero points of a non-trivial solution $y(t)$ of the Jacobi equation.

In general, conjugate points are defined by the Jacobi equation:

$$\frac{d}{dt}\left\{\bar{f}_{\dot{x}x}y + \bar{f}_{\dot{x}\dot{x}}\dot{y}\right\} = \bar{f}_{xx}y + \bar{f}_{x\dot{x}}\dot{y}.$$

If there exists a non-trivial solution $y(t)$ of the Jacobi equation with $y(0) = 0$, then any zero point $c > 0$ of $y(t)$ is said to be conjugate to 0.

THEOREM 1 *(Jacobi) A sufficient condition for a feasible solution $\bar{x}(t)$ be a minimum is the combined condition of the Euler equation, the strengthened Legendre condition, and that there are no points conjugate to 0 on $[0, T]$. Conversely, if $\bar{x}$ is a minimum and the strengthend Legendre condition is satisfied, then there are no points conjugate to 0 on $[0, T)$.*

3 The Jacobi equation for (P_0)

In this section, we review the conjugate point theory for (P_0) presented in [3]. According to Sylvester's criterion, an $n \times n$-symmetric matrix $A = (a_{ij})$ is positive-definite if and only if its descending principal minors $|A_k|$ $(k = 1, \ldots, n)$ are positive, where $A_k := (a_{ij})$ $(1 \le i,\ j \le k)$.

The following lemma shows that the determinant of any square matrix is expanded with respect to the descending principal minors.

LEMMA 1 *([3]) For any $n \times n$-matrix $A = (a_{ij})$, its determinant is expanded as follows:*

$$|A| = \sum_{k=0}^{n-1} \sum_{\rho \in S(k+1,n)} \varepsilon(\rho) a_{k+1\rho(k+1)} a_{k+2\rho(k+2)} \cdots a_{n\rho(n)} |A_k| \quad (4)$$

where $|A_0| := 1$, $\varepsilon(\rho)$ denotes the sign of ρ, and $S(k+1, n)$ denotes the set of all permutations ρ on $\{k+1, \ldots, n\}$ satisfying that there is no $\ell > k$ such that ρ is closed on $\{\ell + 1, \ldots, n\}$.

DEFINITION 1 *([3]) For any $n \times n$-matrix $A = (a_{ij})$, we call the recursion relation on* $y_0, \dots, y_n$

$$y_k = \sum_{i=0}^{k-1} \sum_{\rho \in S(i+1,k)} \varepsilon(\rho) a_{i+1\rho(i+1)} a_{i+2\rho(i+2)} \cdots a_{k\rho(k)} y_i, \qquad k = 1, \dots, n \tag{5}$$

the Jacobi equation for A. We say that k is conjugate to 1 if the solution $\{y_i\}$ of the Jacobi equation with $y_0 > 0$ changes the sign from positive to non-positive at k. Namely,

$$y_0 > 0,\ y_1 > 0,\ \dots,\ y_{k-1} > 0,\ \textit{and}\ y_k \leq 0. \tag{6}$$

Concerning the reason why we call the recursion relation (5) the Jacobi equation, readers may refer to [3, p. 57].

THEOREM 2 *([3]) For any $n \times n$-symmetric matrix A, $A > 0$ if and only if there is no point conjugate to 1.*

THEOREM 3 *([3]) A sufficient condition for an extremal $\bar{x}$ to be a minimum for (P_0) is that there is no point conjugate to 1 for the Hesse matrix $f''(\bar{x})$.*

4 The Riccati equation for (P_0)

In this section, we define the Riccati equation for (P_0), and we clarify the relationship between the Riccati equation and the Jacobi equation. As we saw in Section 2, the key point to derive the (classical) Riccati equation was the perfect square. This idea works very well when we define the Riccati equation for (P_0), too.

Let us first consider the case where A is a tridiagonal matrix.

$$A = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n \end{pmatrix}. \tag{7}$$

Suppose that the quadratic form $x^T A x$ ($x \in R^n$) is expressed as a summation of n perfect squares

$$\begin{aligned} x^T A x &= a_1 x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2 + 2b_1 x_1 x_2 + 2b_2 x_2 x_3 + \cdots + 2b_{n-1} x_{n-1} x_n \\ &= \left(w_1 x_1 + \frac{b_1}{w_1} x_2 \right)^2 + \cdots + \left(w_{n-1} x_{n-1} + \frac{b_{n-1}}{w_{n-1}} x_n \right)^2 + w_n^2 x_n^2 \end{aligned}$$

for some $w_1, \dots, w_n \in R/\{0\}$. Then $\{w_k\}$ has to satisfy

$$w_1^2 = a_1, \quad w_k^2 = a_k - \frac{b_{k-1}^2}{w_{k-1}^2} \quad (k = 2, \dots n). \tag{8}$$

On the other hand, the Jacobi equation (5) for the tridiagonal matrix reduces to

$$y_k = a_k y_{k-1} - b_{k-1}^2 y_{k-2}. \tag{9}$$

Dividing (9) by y_{k-1}, we get

$$\frac{y_k}{y_{k-1}} = a_k - b_{k-1}^2 \frac{y_{k-2}}{y_{k-1}}. \tag{10}$$

Comparing (10) with (8), we obtain

$$w_k^2 = \frac{y_k}{y_{k-1}}. \tag{11}$$

Hence there is no nonzero $w_k \in R$ when $y_{k-1}y_k \leq 0$. This fact matches the classical result mentioned in Section 2.

For any matrix A, by dividing the Jacobi equation (5) by y_{k-1}, we obtain the definition of the Riccati equation.

DEFINITION 2 *For any $n \times n$-matrix $A = (a_{ij})$, we define the Riccati equation as follows.*

$$\begin{aligned} w_1^2 &= a_{11}, \\ w_k^2 &= a_{kk} + \sum_{\ell=1}^{k-1} \sum_{\rho \in S(\ell,k)} \frac{\varepsilon(\rho) a_{\ell\rho(\ell)} a_{\ell+1\rho(\ell+1)} \cdots a_{k\rho(k)}}{w_\ell^2 w_{\ell+1}^2 \cdots w_{k-1}^2}, \qquad k = 2, \ldots, n. \end{aligned}$$

The following theorem states the relationship between conjugate points and the Riccati equation.

THEOREM 4 *Let k be conjugate to 1. Then the Riccati equation has a solution $w_1, \ldots, w_{k-1} \in R/\{0\}$, but it has no solution $w_k \in R/\{0\}$. Conversely, if $w_1, \ldots, w_{k-1} \in R/\{0\}$ satisfy the Riccati equation, and if there is no $w_k \in R/\{0\}$ satisfying the Riccati equation, then k is conjugate to 1.*

THEOREM 5 *A sufficient condition for an extremal $\bar{x}$ to be a minimum for (P_0) is that the Riccati equation has non-zero real solution w_k $(k = 1, \ldots, n)$ for the Hesse matrix $A = f''(\bar{x})$.*

5 Perfect squares

As we have seen in the beginning of Section 4, when A is a positive-definite tridiagonal matrix, the quadratic form $x^T A x$ can be expressed as a summation of n perfect squares. The aim of this section is to prove that the above observation is true for an arbitrary positive-definite matrix A.

LEMMA 2 *Let A be an $n \times n$-symmetric matrix, and divide A as follows.*

$$A = \begin{pmatrix} \alpha & a^T \\ a & B \end{pmatrix}, \tag{12}$$

where $\alpha \in R$, $a \in R^{n-1}$, and B is an $(n-1) \times (n-1)$-symmetric matrix. Then $A > 0$ if and only if $\alpha > 0$ and $B - aa^T/\alpha > 0$.

THEOREM 6 *Let A be a positive-definite matrix of order n. Then the quadratic form x^TAx is expressed as a summation of n perfect squares by the following procedure.*

Step 1 Divide A and x as (12) and $x^T = (x_1,\ y^T) \in R \times R^{n-1}$, respectively.

Step 2 $x^TAx = \left(\sqrt{\alpha}x_1 + \dfrac{a^Ty}{\sqrt{\alpha}}\right)^2 + y^T\left(B - \dfrac{aa^T}{\alpha}\right)y$

Step 3 Choose $B - \dfrac{aa^T}{\alpha}$ as A, and go to Step 1.

Theorem 6 is rephrased as follows.

THEOREM 7 *Let $A = (a_{ij})$ be a positive-definite matrix of order n. Then the quadratic form x^TAx is expressed as a summation of n perfect squares*

$$x^TAx = \sum_{k=1}^{n} a_{kk}(k)\left\{x_k + \frac{\sum_{j=k+1}^{n} a_{kj}(k)x_j}{a_{kk}(k)}\right\}^2, \tag{13}$$

where $x^T = (x_1, \ldots, x_n)$ and $a_{ij}(k)$ is inductively defined by

$$a_{ij}(1) := a_{ij}, \quad 1 \leq i,\ j \leq n, \tag{14}$$

$$a_{ij}(k+1) := a_{ij}(k) - \frac{a_{ki}(k)a_{kj}(k)}{a_{kk}(k)}, \quad k+1 \leq i,\ j \leq n. \tag{15}$$

The following theorem provides an explicit representation of $a_{ij}(k)$, and it clarifies the algebraic meaning of $\{w_k\}$.

THEOREM 8 *Let $A = (a_{ij})$ be a positive-definite matrix of size n, let $a_{ij}(k)$ be the sequence defined by (14) and (15), and let $\{w_k\}$ be the solution of the Riccati equation. Then*

$$a_{ij}(k) = a_{ij} + \sum_{\ell=1}^{k-1} \sum_{\rho \in S_{ij}(\ell,k-1)} \frac{\varepsilon(\rho)a_{\ell\rho(\ell)}a_{\ell+1\rho(\ell+1)}\cdots a_{k-1\rho(k-1)}a_{i\rho(i)}}{w_\ell^2 w_{\ell+1}^2 \cdots w_{k-1}^2} \tag{16}$$

for any $k = 1, \ldots, n$ and $k \leq i,\ j \leq n$, where $S_{ij}(\ell, k-1)$ denote the set of all bijections $\rho : \{\ell, \ell+1, \ldots, k-1, i\} \to \{\ell, \ell+1, \ldots, k-1, j\}$ which satisfies that $\rho(\{\ell', \ell'+1, \ldots, k-1, i\}) \neq \{\ell', \ell'+1, \ldots, k-1, j\}$ for any $\ell < \ell' \leq k-1$. Furthermore,

$$w_k^2 = a_{kk}(k) \tag{17}$$

for any $k = 1, \ldots, n$.

The relation (17) guarantees that one may test the positive-definiteness of A by means of the procedure in Theorem 2 or equivalently (14) and (15).

References

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