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Fidelity in a quantum teleportation model

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1 Introduction

A model of *Quantum Teleportation* has been first given by Bennett et al.[1, 2], in which Alice perfectly sends an unknown state to Bob using the EPR entangled state. In their model, every state is perfectly teleported. The key to make a perfect teleportation scheme is to use a *maximally* entangled state (EPR state) over Alice and Bob. It is known, however, that preparation of such a maximally entangled states is difficult to realize. Therefore it is important to consider schemes with partially (not maximally) entangled states. As having been pointed out[3], with such an incompletely entangled state one can not obtain a perfect teleortation scheme. In [4, 5], protocols employing a partially entangled state constructed by beam splitting technique were introduced to provide the examples for both perfect and nonperfect teleportation.The scheme introduced in [4, 5] generalized that of Bennett et al.. In the protocol in nonperfect realistic teleportation, Alice and Bob make *tests* on their own systems and give up the experiments if the tests are not passed. If the tests are fortunately passed, the obtained state by Bob is shown to be perfectly same with the original one first possessed by Alice. We calculated the probability to complete successful teleportation, which approaches unity as the mean energy of the entangled state goes to infinity even in the nonperfect model.
We, in the present paper, do not employ the protocol with *tests* [4, 5] but original *naive* protocol given in [3] with beam splittings. For fixing the notations, let us review what the naive scheme is (See [3, 12]).

Step 0: A girl named Alice has an unknown quantum state ρ on an N-dimensional subspace $\mathcal{L}$ of a Hilbert space $\mathcal{H}_1$ and she was asked to teleport it to a boy named Bob.

Step 1: For this purpose, we need two other Hilbert spaces $\mathcal{H}_2$ and $\mathcal{H}_3$, $\mathcal{H}_2$ is attached to Alice and $\mathcal{H}_3$ is attached to Bob. Prearrange a so-called entangled state σ on $\mathcal{H}_2 \otimes \mathcal{H}_3$ having certain correlations and prepare an ensemble of the combined system in the state $\rho \otimes \sigma$ on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$.

Step 2: One then fixes a family of mutually orthogonal projections $(F_{nm})_{n,m=1}^{N}$ on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ corresponding to an observable $F := \sum_{n,m} z_{n,m} F_{nm}$. To complete the set of projections, we define another projection $F_0 := \mathbf{1} - \sum_{nm} F_{nm}$. Alice performs a measurement of the observable F, involving only the $\mathcal{H}_1 \otimes \mathcal{H}_2$ part of the system in the state $\rho \otimes \sigma$. Possible outcomes are $\{z_{nm}\}$'s and 0. When Alice obtains z_{nm}, according to the von Neumann rule, after Alice's measurement, the state becomes

$$\rho_{nm}^{(123)} := \frac{(F_{nm} \otimes \mathbf{1})\rho \otimes \sigma(F_{nm} \otimes \mathbf{1})}{\mathrm{tr}_{123}(F_{nm} \otimes \mathbf{1})\rho \otimes \sigma(F_{nm} \otimes \mathbf{1})}$$

where tr_{123} is the full trace on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$. On the other hand, when Alice obtains 0, the state becomes

$$\rho_0^{(123)} := \frac{(F_0 \otimes \mathbf{1})\rho \otimes \sigma(F_0 \otimes \mathbf{1})}{\mathrm{tr}_{123}(F_0 \otimes \mathbf{1})\rho \otimes \sigma(F_0 \otimes \mathbf{1})}.$$

Step 3: Bob is informed which outcome was obtained by Alice. This is equivalent to transmit the information that the eigenvalue z_{nm} or 0 was detected. This information is transmitted from Alice to Bob without disturbance and by means of classical tools.

Step 4: Having been informed an outcome of Alice's measurement, Bob performs a corresponding unitary operation onto his system. That is, if the outcome was z_{nm}, Bob operates a unitary operator W_{nm} and change the state into

$$(\mathbf{1} \otimes \mathbf{1} \otimes W_{nm})\rho_{nm}^{(123)}(\mathbf{1} \otimes \mathbf{1} \otimes W_{nm}^*) = \frac{(F_{nm} \otimes W_{nm})\rho \otimes \sigma(F_{nm} \otimes W_{nm}^*)}{\mathrm{tr}_{123}(F_{nm} \otimes \mathbf{1})\rho \otimes \sigma(F_{nm} \otimes \mathbf{1}}).$$

If the outcome was 0, Bob operates a unitary operator W_0 and the state becomes

$$(\mathbf{1}\otimes\mathbf{1}\otimes W_0)\rho_0^{(123)}(\mathbf{1}\otimes\mathbf{1}\otimes W_0^*) = \frac{(F_0\otimes W_0)\rho\otimes\sigma(F_0\otimes W_0^*)}{\mathrm{tr}_{123}(F_0\otimes\mathbf{1})\rho\otimes\sigma(F_0\otimes\mathbf{1})}.$$

Step 5: Making only partial measurements on the third part on the system means that Bob will control a state given by the partial trace on $\mathcal{H}_1\otimes\mathcal{H}_2$. Thus the state obtained by Bob is

$$\begin{aligned}\Gamma_{nm}^*(\rho) &= \mathrm{tr}_{12}\,(\mathbf{1}\otimes\mathbf{1}\otimes W_{nm})\rho_{nm}^{(123)}(\mathbf{1}\otimes\mathbf{1}\otimes W_{nm}^*)\\ &= \mathrm{tr}_{12}\frac{(F_{nm}\otimes W_{nm})\rho\otimes\sigma(F_{nm}\otimes W_{nm}^*)}{\mathrm{tr}_{123}(F_{nm}\otimes\mathbf{1})\rho\otimes\sigma(F_{nm}\otimes\mathbf{1})}\end{aligned}$$

in case when the outcome is $z_{nm}$and

$$\begin{aligned}\Gamma_0^*(\rho) &= \mathrm{tr}_{12}\,(\mathbf{1}\otimes\mathbf{1}\otimes W_0)\rho_0^{(123)}(\mathbf{1}\otimes\mathbf{1}\otimes W_0^*)\\ &= \mathrm{tr}_{12}\frac{(F_0\otimes W_0)\rho\otimes\sigma(F_0\otimes W_0^*)}{\mathrm{tr}_{123}(F_0\otimes\mathbf{1})\rho\otimes\sigma(F_0\otimes\mathbf{1})}.\end{aligned}$$

if the outcome was 0. Thus the whole teleportation scheme given by the family (F_{nm}) and the entangled state σ can be characterized by the family Γ_{nm} and Γ_0 of channels from the set of states on $\mathcal{H}_1$ into the set of states on $\mathcal{H}_3$ and the family $\{p_{nm}(\rho)\}$ and $p_0(\rho)$ given by

$$p_{nm}(\rho) := \mathrm{tr}_{123}(F_{nm}\otimes\mathbf{1})\rho\otimes\sigma(F_{nm}\otimes\mathbf{1})$$

$$p_0(\rho) := \mathrm{tr}_{123}(F_0\otimes\mathbf{1})\rho\otimes\sigma(F_0\otimes\mathbf{1})$$

of the probabilities that Alice's measurement according to the observable F will show the value z_{nm} and 0.

Once knowing the result of Alice's measurement, the channel becomes nonlinear because of the probabilties $p_{nm}(\rho)$ and $p_0(\rho)$ which appear in the denominator. We, however, do not know the result of Alice's measurement before the experiment. Therefore it is also important to consider an expected state which is obtained by mixing all possible states with multiplying their probabilities to occur. That is, the teleportation scheme can be written by a linear channel (completely positive map)

$$\Xi^*(\rho) = \sum_{nm} \Xi^*_{nm}(\rho) + \Xi^*_0(\rho), \tag{1}$$

where

$$\begin{aligned} \Xi^*_{nm}(\rho) &:= p_{nm}(\rho)\Gamma^*_{nm}(\rho) = \mathrm{tr}_{1,2}(F_{nm} \otimes W_{nm})\rho(F_{nm} \otimes W_{nm})^* \\ \Xi^*_0(\rho) &:= p_0(\rho)\Gamma^*_0(\rho) = \mathrm{tr}_{1,2}(F_0 \otimes W_0)\rho(F_0 \otimes W_0)^*. \end{aligned}$$

We investigate how close the obtained state $\Xi^*(\rho)$ to the original state ρ. In the next section, we review some mathematical notions which are used to construct rigorously a teleportation scheme by beam splittings. In section 3 we introduce a naive teleportation scheme and in section 4 we discuss how perfect the protocol is by use of a quantity, fidelity.

2 Basic Notions and Notations

First we collect some basic facts concerning the (symmetric) Fock space. We will introduce the Fock space in a way adapted to the language of counting measures. For details we refer to [6, 7, 8, 9, 10] and other papers cited in [8].

Let G be an arbitrary complete separable metric space. Further, let μ be a locally finite diffuse measure on G, i.e. $\mu(B) < +\infty$ for bounded measurable subsets of G and $\mu(\{x\}) = 0$ for all singletons $x \in G$. In order to describe the teleportation of states on a finite dimensional Hilbert space through the k–dimensional space $\mathbf{R}^k$, especially we are concerned with the case

$$\begin{aligned} G &= \mathbf{R}^k \times \{1, \dots, N\} \\ \mu &= l \times \# \end{aligned}$$

where l is the k–dimensional Lebesgue measure and $\#$ denotes the counting measure on $\{1, \dots, N\}$.

Now by $M = M(G)$ we denote the set of all finite counting measures on G. Since $\varphi \in M$ can be written in the form $\varphi = \sum_{j=1}^{n} \delta_{x_j}$ for some $n = 0, 1, 2, \dots$ and $x_j \in G$ (where δ_x denotes the Dirac measures corresponding to $x \in G$) the elements of M can be interpreted as finite (symmetric) point configurations in G. We equip M with its canonical σ–algebra $\mathfrak{W}$ (cf. [6], [7]) and we consider the measure F by setting

$$F(Y) := \mathcal{X}_Y(O) + \sum_{n \geq 1} \frac{1}{n!} \int_{G^n} \mathcal{X}_Y \left(\sum_{j=1}^{n} \delta_{x_j} \right) \mu^n(d[x_1, \dots, x_n])(Y \in \mathfrak{W})$$

Hereby, $\mathcal{X}_Y$ denotes the indicator function of a set Y and O represents the empty configuration, i. e., $O(G) = 0$. Observe that F is a σ–finite measure.

Since μ was assumed to be diffuse one easily checks that F is concentrated on the set of a simple configurations (i.e., without multiple points)

$$\hat{M} := \{\varphi \in M | \varphi(\{x\}) \leq 1 \text{ for all } x \in G\}$$

DEFINITION 2.1 $\mathcal{M} = \mathcal{M}(G) := L^2(M, \mathfrak{W}, F)$ *is called the (symmetric) Fock space over G.*

In [6] it was proved that $\mathcal{M}$ and the Boson Fock space $\Gamma(L^2(G))$ in the usual definition are isomorphic.
For each $\Phi \in \mathcal{M}$ with $\Phi \neq 0$ we denote by $|\Phi >$ the corresponding normalized vector

$$|\Phi > := \frac{\Phi}{||\Phi||}$$

Further, $|\Phi >< \Phi|$ denotes the corresponding one–dimensional projection, describing the pure state given by the normalized vector $|\Phi >$. Now, for each $n \geq 1$ let $\mathcal{M}^{\otimes n}$ be the n–fold tensor product of the Hilbert space $\mathcal{M}$. Obviously, $\mathcal{M}^{\otimes n}$ can be identified with $L^2(M^n, F^n)$.

DEFINITION 2.2 *For a given function* $g : G \to \mathbb{C}$ *the function*

$\exp(g) : M \to \mathbb{C}$ defined by

$$\exp(g)(\varphi) := \begin{cases} 1 & \text{if} \quad \varphi = 0 \\ \prod_{x \in G, \varphi(\{x\})>0} g(x) & \quad \textit{otherwise} \end{cases}$$

is called exponential vector generated by g.

Observe that $\exp(g) \in \mathcal{M}$ if and only if $g \in L^2(G)$ and one has in this case
$||\exp(g)||^2 = e^{||g||^2}$ and $|\exp(g)> = e^{-\frac{1}{2}||g||^2}\exp(g)$. The projection $|\exp(g)><\exp(g)|$ is called the coherent state corresponding to $g \in L^2(G)$. In the special case $g \equiv 0$ we get the vacuum state

$$|\exp(0)> = \mathcal{X}_{\{0\}} .$$

The linear span of the exponential vectors of $\mathcal{M}$ is dense in $\mathcal{M}$, so that bounded operators and certain unbounded operators can be characterized by their actions on exponential vectors.

DEFINITION 2.3 *Let T be a linear operator on $L^2(G)$ with $||T|| \leq 1$. Then the operator $\Gamma(T)$ called second quantization of T is the (uniquely determined) bounded operator on $\mathcal{M}$ fulfilling*

$$\Gamma(T)\exp(g) = \exp(Tg) \quad (g \in L^2(G))$$

Clearly, it holds

$$\begin{aligned} \Gamma(T_1)\Gamma(T_2) &= \Gamma(T_1T_2) \\ \Gamma(T^*) &= \Gamma(T^*) \end{aligned} \tag{2}$$

It follows that $\Gamma(T)$ is an unitary operator on $\mathcal{M}$ if T is an unitary operator on $L^2(G)$.

LEMMA 2.4 *Let K_1, K_2 be linear operators on $L^2(G)$ with property*

$$K_1^*K_1 + K_2^*K_2 = \mathbf{1} . \tag{3}$$

Then there exists exactly one isometry ν_{K_1,K_2} from $\mathcal{M}$ to $\mathcal{M}^{\otimes 2} = \mathcal{M} \otimes \mathcal{M}$ with

$$\nu_{K_1,K_2}\exp(g) = \exp(K_1g) \otimes \exp(K_2g) \quad (g \in L^2(G)) \tag{4}$$

Further it holds

$$\nu_{K_1,K_2} = (\Gamma(K_1) \otimes \Gamma(K_2))D \tag{5}$$

(at least on dom(D) *but one has the unique extension).*
The adjoint $\nu^*_{K_1,K_2}$ *of* ν_{K_1,K_2} *is characterized by*

$$\nu^*_{K_1,K_2}(\exp(h) \otimes \exp(g)) = \exp(K_1^* h + K_2^* g) \quad (g, h \in L^2(G)). \tag{6}$$

REMARK 2.5 *From* K_1, K_2 *we get a transition expectation* $\xi_{K_1K_2} : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$, *using* ν_{K_1,K_2} *and the lifting* $\xi^*_{K_1K_2}$ *may be interpreted as a certain splitting (cf. [9]).*

Here we explain fundamental scheme of beam splitting [8]. We define an isometric operator $V_{\alpha,\beta}$ for coherent vectors such that

$$V_{\alpha,\beta}|\exp(g)\rangle = |\exp(\alpha g)\rangle \otimes |\exp(\beta g)\rangle$$

with $|\alpha|^2 + |\beta|^2 = 1$. This beam splitting is a useful mathematical expression for optical communication and quantum measurements [9].

EXAMPLE 2.6 $(\alpha = \beta = 1/\sqrt{2}$ *above)* *Let* $K_1 = K_2$ *be the following operator of multiplication on* $L^2(G)$

$$K_1 g = \frac{1}{\sqrt{2}} g = K_2 g \quad (g \in L^2(G))$$

We put

$$\nu := \nu_{K_1,K_2}$$

and obtain

$$\nu \exp(g) = \exp\left(\frac{1}{\sqrt{2}} g\right) \otimes \exp\left(\frac{1}{\sqrt{2}} g\right) \quad (g \in L^2(G))$$

EXAMPLE 2.7 *Let* $L^2(G) = \mathcal{H}_1 \oplus \mathcal{H}_2$ *be the orthogonal sum of the subspaces* $\mathcal{H}_1, \mathcal{H}_2$. K_1 *and* K_2 *denote the corresponding projections.*

We will use Example 2.6 in order to describe a teleportation model where Bob performs his experiments on the same ensemble of the systems like Alice.

Further we will use a special case of Example 2.7 in order to describe a teleportation model where Bob and Alice are spatially separated (cf. section 5).

REMARK 2.8 *The property (3) implies*

$$\|K_1 g\|^2 + \|K_2 g\|^2 = \|g\|^2 \quad (g \in L^2(G)) \tag{7}$$

REMARK 2.9 *Let U, V be unitary operators on $L^2(G)$. If operators K_1, K_2 satisfy (3), then the pair $\hat{K}_1 = UK_1$, $\hat{K}_2 = VK_2$ fulfill (3).*

3 A naive teleportation scheme

In this section we define a naive version of the teleportation scheme by beam splitting [4, 5]. We fix an ONS $\{g_1, \dots, g_N\} \subseteq L^2(G)$, operators K_1, K_2 on $L^2(G)$ with (3), an unitary operator T on $L^2(G)$, and $d > 0$. We assume

$$TK_1 g_k = K_2 g_k \quad (k = 1, \dots, N), \tag{8}$$

$$\langle K_1 g_k, K_1 g_j \rangle = 0 \quad (k \neq j;\ k, j = 1 \dots, N), \tag{9}$$

Using (7) and (8) we get

$$\|K_1 g_k\|^2 = \|K_2 g_k\|^2 = \frac{1}{2}. \tag{10}$$

From (8) and (9) we get

$$\langle K_2 g_k,\ K_2 g_j \rangle = 0 \quad (k \neq j\,;\ k, j = 1, \dots, N). \tag{11}$$

The state of Alice asked to teleport is of the type

$$\rho = \sum_{s=1}^{N} \lambda_s |\Phi_s\rangle\langle\Phi_s|, \tag{12}$$

$$|\Phi_s\rangle = \sum_{j=1}^{N} c_{sj} |\exp(aK_1 g_j) - \exp(0)\rangle \quad \left(\sum_j |c_{sj}|^2 = 1; s = 1, \dots, N \right) \tag{13}$$

and $|a|^2 = d$. One easily checks that $(|\exp(aK_1 g_j) - \exp(0)\rangle)_{j=1}^N$ and $(|\exp aK_2 g_j) - \exp(0)\rangle)_{j=1}^N$ are ONS in $\mathcal{M}$. That is, the state of Alice asked to teleport lives in an N-dimensional subspace of the Fock space spanned by the ONS.

In order to achieve that $(|\Phi_s\rangle)_{s=1}^N$ is still an ONS in $\mathcal{M}$ we assume

$$\sum_{j=1}^{N} \bar{c}_{sj} c_{kj} = 0 \quad (j \neq k;\ j, k = 1, \dots, N). \tag{14}$$

Denote $c_s = [c_{s1,\dots,}c_{sN}] \in \mathbb{C}^N$, then $(c_s)_{s=1}^N$ is an CONS in $\mathbb{C}^N$.

Now let $(b_n)_{n=1}^N$ be a sequence in $\mathbb{C}^N$,

$$b_n = [b_{n1,\dots,}b_{nN}]$$

with properties

$$|b_{nk}| = 1 \quad (n, k = 1, \dots, N), \tag{15}$$

$$\langle b_n\,,\ b_j\rangle = 0 \quad (n \neq j;\ n, j = 1, \dots, N). \tag{16}$$

Then Alice's measurements are performed with projection

$$F_{nm} = |\xi_{nm}\rangle\langle\xi_{nm}| \quad (n, m = 1, \dots, N) \tag{17}$$

given by

$$|\xi_{nm}\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} b_{nj} |\exp(aK_1 g_j) - \exp(0)\rangle \otimes |\exp(aK_1 g_{j\oplus m}) - \exp(0)\rangle, \tag{18}$$

where $j \oplus m := j + m(\text{mod } N)$. One easily checks that $(|\xi_{nm}\rangle)_{n,m=1}^N$ is an ONS in $\mathcal{M}^{\otimes 2}$. Because $|\xi_{nm}\rangle$ $(n, m = 1, 2, \cdots N)$ does not form a completley orthonormal system of $\mathcal{M} \otimes \mathcal{M}$, we introduce another projection operator $F_0 := \mathbf{1} - \sum_{nm} F_{nm}$. Thus the measurement of the observable F distingishes $\{F_n m\}$'s and F_0, where F_0 corresponds to the case an outcome is zero.

Further, the state vector $|\xi\rangle$ of the entangled state $\sigma = |\xi\rangle\langle\xi|$ is given by

$$|\xi\rangle = \frac{\gamma}{\sqrt{N}} \sum_k |\exp(aK_1 g_k)\rangle \otimes |\exp(aK_2 g_k)\rangle, \tag{19}$$

which is naturally prepared by use of Beam splitting technique, where γ is determined by a normalization condition. However, the physical naturalness requires a sacrifice. That is, the state is not maximally entangled state any longer.

As for unitary operation of Bob, for each $n, m = 1, \cdots, N$ we have U_m, B_n on $\mathcal{M}$ given by

$$\begin{aligned} B_n|\exp(aK_1 g_j) - \exp(0)\rangle &= b_{nj}|\exp(aK_1 g_j) - \exp(0)\rangle \quad (j = 1, \ldots, N) \\ B_n|\exp(0)\rangle &= |\exp(0)\rangle \end{aligned} \tag{20}$$

$$\begin{aligned} U_m|\exp(aK_1 g_j) - \exp(0)\rangle &= |\exp(aK_1 g_{j\oplus m}) - \exp(0)\rangle \quad (j = 1, \ldots, N) \\ U_m|\exp(0)\rangle &= |\exp(0)\rangle \end{aligned} \tag{21}$$

where $j \oplus m := j + m (\mathrm{mod}\ N)$ and define

$$W_{nm} := B_n U_m^* \Gamma(T)^*. \tag{22}$$

In addition we have some arbitrary unitary operator W_0, which we do not specify yet.

4 Fidelity

We need some proper quantity (for e.g., [12]) to measure how close two states are. In this paper we take up *fidelity* [13, 14]. The notion of fidelity is frequently used in the context of quantum information, quantum optics and so on. The fidelity of a state ρ with respect to another state σ is defined by

$$F(\rho, \sigma) := \mathrm{tr}[\sqrt{\sigma^{1/2}\rho\sigma^{1/2}}], \tag{23}$$

which possesses some nice properties.

$$\begin{aligned} 0 \le F(\rho,\sigma) &\le 1 & (24)\\ F(\rho,\sigma)=1 &\Leftrightarrow \rho=\sigma & (25)\\ F(\rho,\sigma) &= F(\sigma,\rho) & (26)\end{aligned}$$

Thus we can say two states ρ and σ are close when the fidelity between them is close to unity. Moreover it satisfies a kind of concavity relation as

$$F(\sum_i p_i\rho_i, \sum_i q_i\sigma_i) \ge \sum_i \sqrt{p_iq_i}F(\rho_i,\sigma_i), \tag{27}$$

where ρ_i's and σ_i's are states and p_i's and q_i's are nonnegative numbers satisfying $\sum_i p_i = \sum_i q_i = 1$. In particular putting $p_j = 1$, one gets

$$F(\rho, \sum_i q_i\sigma) \ge \sqrt{q_j}F(\rho,\sigma_j) \tag{28}$$

for $j = 1, 2, \cdots$.
To estimate $F(\rho, \Xi^*(\rho))$ we begin with a calculation of $\Xi^*(\rho) = \sum_{nm} \Xi^*_{nm}(\rho) + \Xi^*_0(\rho)$.

LEMMA 4.1 *[4] For each $n, m, s\,(= 1, \dots, N)$, it holds*

$$\begin{aligned}(F_{nm}\otimes \mathbf{1})\left(|\Phi_s\rangle \otimes |\tilde{\xi}\rangle\right) &= \frac{\gamma}{N}\left(1-e^{-\frac{d}{2}}\right)|\xi_{nm}\rangle \otimes (\Gamma(T)U_mB_n^*|\Phi_s\rangle)\\ &\quad +\frac{\gamma}{N}\left(\frac{e^{\frac{d}{2}}-1}{e^d}\right)^{\frac{1}{2}}\langle b_n, c_s\rangle_{\mathbf{C}^N}\xi_{nm}\otimes|\exp(0)\rangle\end{aligned}$$

Proof: For all $k, j, r = 1, \dots, N$, we get

$$\begin{aligned}\alpha_{k,j,r} :=\ & \langle |\exp(aK_1g_r) - \exp(0)\rangle \otimes ||\exp(aK_1g_{r\otimes m}) - \exp(0)\rangle\,,\\ & |\exp(aK_1g_j) - \exp(0)\rangle \otimes |\exp(aK_1g_k)\rangle\rangle\\ =\ & \begin{cases}\left(\frac{e^{\frac{a^2}{2}}-1}{e^{\frac{a^2}{2}}}\right) & \text{if } r=j \text{ and } k = r\oplus m\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

and

$$|\exp(aK_2g_{j\oplus m})\rangle = e^{-\frac{a^2}{2}}\left(e^{\frac{a^2}{2}}-1\right)^{\frac{1}{2}}|\exp\ (aK_2g_{j\oplus m}) - \exp(0)\rangle + e^{-\frac{a^2}{2}}|\exp(0$$

On the other hand, we have

$$(F_{nm} \otimes \mathbf{1}) \left(|\Phi_s\rangle \otimes |\tilde{\xi}\rangle \right) = \frac{\gamma}{N} \sum_k \sum_j \sum_r c_{sj} \bar{b}_{nr} \alpha_{k,j,r} \xi_{nm} \otimes |\exp{(aK_2 g_k)}\rangle$$

It follows with $a^2 = d$

$$\begin{aligned}
(F_{nm} \otimes \mathbf{1}) \left(\Phi_s \otimes \tilde{\xi} \right) &= \frac{\gamma}{N} \left(e^{\frac{d}{2}} - 1 \right) e^{-\frac{d}{2}} \xi_{nm} \\
&\quad \otimes \left(\sum_j c_{sj} \bar{b}_{nj} |\exp{(aK_2 g_{j \oplus m})} - \exp{(0)}\rangle \right) \\
&\quad + \frac{\gamma}{N} \left(e^{\frac{d}{2}} - 1 \right)^{\frac{1}{2}} e^{-\frac{d}{2}} \sum_j c_{sj} \bar{b}_{nj} \xi_{nm} \otimes |\exp{(0)}\rangle \\
&= \frac{\gamma}{N} \left(1 - e^{-\frac{d}{2}} \right) \xi_{nm} \otimes (\Gamma(T) U_m B_n^* \Phi_s) \\
&\quad + \frac{\gamma}{N} \left(\frac{e^{\frac{d}{2}} - 1}{e^d} \right)^{\frac{1}{2}} \langle b_n, c_s \rangle_{\mathbb{C}^N} \xi_{nm} \otimes |\exp{(0)}\rangle. \blacksquare
\end{aligned}$$

The following lemma follows immediattiely.

LEMMA 4.2 *For a general state $\rho = \sum_s \lambda_s |\Phi_s\rangle\langle\Phi_s|$, it holds*

$$\begin{aligned}
\Xi_{nm}^*(\rho) &= \frac{\gamma^2}{N^2}(1 - e^{-d/2})^2 \rho + \frac{\gamma^2}{N^2}(\frac{e^{d/2} - 1}{e^d}) \sum_s \lambda_s |\langle b_n, c_s \rangle|^2 |\exp(0)\rangle\langle\exp(0)| \\
&+ \frac{\gamma^2}{N^2}(\frac{e^{d/2} - 1}{e^d})^{1/2}(1 - e^{-d/2})(\sum_s \langle b_n, c_s \rangle^* \lambda_s |\Phi_s\rangle\langle\exp(0)| \\
&+ \sum_s \langle b_n, c_s \rangle \lambda_s |\exp(0)\rangle\langle\Phi_s|). \qquad (29)
\end{aligned}$$

Next we investigate an expression of $\Xi_0^*(\rho)$.

LEMMA 4.3 *It holds*

$$(F_0 \otimes \mathbf{1}) \left(|\Phi_s\rangle \otimes |\tilde{\xi}\rangle \right) = |\Phi_s\rangle \otimes |\exp(0)\rangle \otimes \mathrm{e}^{-|a|^2/4} \frac{\gamma}{\sqrt{N}} \sum_{k=1}^{N} |\exp(aK_2 g_k)\rangle$$

Proof: If we let $\mathcal{L}$ a subsapce spanned by an ONS $\{|\exp(aK_1 g_k) - \exp(0)\rangle\}$ $(k = 1, \cdots, N)$, $\sum_{nm} F_n m$ is a projection onto $\mathcal{L} \otimes \mathcal{L}$.Therefore we obtain

$$(F_0 \otimes \mathbf{1})(\Phi_s \otimes \tilde{\xi}) = (\mathbf{1} \otimes |\exp(0)\rangle\langle\exp)(0)| \otimes \mathbf{1})(\Phi_s \otimes \tilde{\xi}). \qquad (30)$$

Here we used a fact that $|\exp(0)\rangle$ is orthogonal to $\{|\exp(aK_1g_k) - \exp(0)\rangle\}$'s.
■

LEMMA 4.4 *For a general state $\rho = \sum_s \lambda_s |\Phi_s\rangle\langle\Phi_s|$, it holds*

$$\Xi_0^*(\rho) = e^{-|a|^2/2}\frac{\gamma^2}{N}\sum_{k=1}^{N}\sum_{l=1}^{N} W_0|\exp(aK_2g_k)\rangle\langle\exp(aK_2g_l)|W_0^* \tag{31}$$

Now let us estimate the fidelity between $\Xi^*(\rho)$ and ρ. We must first compute

$$\begin{aligned}
\rho^{1/2}\Xi^*(\rho)\rho^{1/2} &= \sum_{nm}\rho^{1/2}\Xi_{nm}^*(\rho)\rho^{1/2} + \rho^{1/2}\Xi_0\rho^{1/2} \\
&= \gamma^2(1-e^{-d/2})^2\rho^{1/2}\rho\rho^{1/2} \\
&+ \rho^{1/2}e^{-|a|^2/2}\frac{\gamma^2}{N}\sum_{k=1}^{N}\sum_{l=1}^{N} W_0|\exp(aK_2g_k)\rangle\langle\exp(aK_2g_l)|W_0^*\rho^{1/2},
\end{aligned}$$

where we used the relation $\langle\exp(0)|\Phi_s\rangle = 0$. Because $\Xi_0^*(\rho)$ is positive operator, $\Xi_0^*(\rho)/\mathrm{tr}[\Xi_0^*(\rho)]$ becomes a state and we can rearrange the expression of fidelity as

$$F(\rho,\Xi^*(\rho)) = F(\rho,\gamma^2(1-e^{-d/2})^2\rho + \mathrm{tr}[\Xi_0^*(\rho)]\Xi_0^*(\rho)/\mathrm{tr}[\Xi_0^*(\rho)]). \tag{32}$$

Thanks to the concavity property (28) of fidelity, we obtain

$$F(\rho,\Xi^*(\rho)) \geq \gamma(1-e^{-d/2})F(\rho,\rho) = \gamma(1-e^{-d/2}). \tag{33}$$

Thus we obtain the following theorem.

THEOREM 4.5 *For any imput state ρ, it holds*

$$F(\rho,\Xi^*(\rho)) \geq \sqrt{\frac{(1-e^{-d/2})^2}{1+(N-1)e^{-d}}}. \tag{34}$$

Therefore the naive teleportation protocol approaches perfect one as the parameter $|a|$ goes to infinity.
With some additional condition, one can strengthen the above estimate to an equality.

PROPOSITION 4.6 *Let $L^2(G) = \mathcal{H}_1 \oplus \mathcal{H}_2$ be the orthogonal sum of the subspaces $\mathcal{H}_1, \mathcal{H}_2$. K_1 and K_2 denote the corresponding projections and $W_0 = \mathbf{1}$.*

$$F(\rho, \Xi^*(\rho)) = \sqrt{\frac{(1-e^{-d/2})^2}{1+(N-1)e^{-d}}}. \tag{35}$$

holds for any imput state ρ.

Proof: Because $\langle \exp(aK_1 g_k) - \exp(0) | \exp(aK_2 g_l) \rangle = 0$ holds, $\rho^{1/2}\Xi_*(\rho)\rho^{1/2} = \gamma^2(1-e^{-d/2})^2\rho^2$ follows. ■

We have cosidered the fidelity of naive teleportation scheme with beam splittings. We showed that as the parameter $|a|$ goes to infinity the fidelity approaches unity and the naive teleporation scheme also approaches a perfect scheme as the teleportation scheme with tests does. In fact the fidelity can be bounded from below by square route of probability to complete successful teleportation with tests.

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