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Properties of the set of upper bounds in ordered linear spaces

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§1 Introducrion

Let E be a linear space over $\mathbb{R}$, and P be a convex cone in E satisfying

(P1) $E = P - P$, (P2) $P \cap (-P) = \{0\}$.

An order relation in E can be defined by $x \leq y \Longleftrightarrow y - x \in P$. We call a linear space E equipped with such a positive cone P a (partially) ordered linear space, and denote it by (E, P). For a subset A of E, we denote the set of upper bounds and lower bounds by

$U(A) = \{x \in E \mid y \leq x,\ \forall y \in A\}$, $L(A) = \{x \in E \mid y \geq x,\ \forall y \in A\}$

respectively. These sets have a property of symmetry in the following sence. ([4])

$$U(L(U(A))) = U(A) \qquad (A \subset E). \tag{1}$$

In [4], the method of constructing a completion $(\tilde{E}, \tilde{P})$ of (E, P) by using the set of upper bounds $U(A)$ has been introduced. The relation (1) plays fundamental roles in the construction of $(\tilde{E}, \tilde{P})$. Also, the completion can be represented by the set of the generalized supremum in E which has been introduced in [2]. We will state the summary of those results in the first part of this section.

Let $\mathfrak{B}$ and $\mathfrak{B}'$ be the family of all upper bounded subset and lower bounded subset in E respectively, i.e. $\mathfrak{B} = \{A \subset E | A \neq \emptyset,\ U(A) \neq \emptyset\}$, $\mathfrak{B}' = \{B \subset E | B \neq \emptyset,\ L(B) \neq \emptyset\}$. The relations

$$A \sim B \underset{def}{\Longleftrightarrow} U(A) = U(B) \quad (A, B \in \mathfrak{B}),$$
$$C \sim' D \underset{def}{\Longleftrightarrow} L(C) = L(D) \quad (C, D \in \mathfrak{B}')$$

are clearly equivalence relations. Now we define

$$\tilde{E} = \mathfrak{B}/\sim\ = \{[A] \mid A \in \mathfrak{B}\},$$

where $[A]$ denotes the equivalence class of A. For every $[A] \in \tilde{E}$, two operations

$$u([A]) = U(A), \qquad l([A]) = L(U(A))$$

are well defined. We can see by (1) that $l([A]) \sim A$.

Lemma 1. ([4]) *If $A \sim A'$ and $B \sim B'$ in $\mathfrak{B}$, then for $\lambda > 0$*

$$[A+B] = [A'+B'] = [l([A]) + l([B])]$$
$$[\lambda A] = [\lambda A'] = [\lambda l([A])]$$

hold where $A+B$ and λA denote the set $\{a+b \mid a \in A,\ b \in B\}$ and $\{\lambda a \mid a \in A\}$ respectively.

Definition. *For $[A],\ [B] \in \tilde{E}$ and $\lambda \in \mathbb{R}$,*

(2) $$[A] \leq [B] \underset{def}{\Longleftrightarrow} u([B]) \subset u([A])$$

(3) $$[A] + [B] \underset{def}{=} [A+B]$$

(4) $$\lambda[A] \underset{def}{=} \begin{cases} [\lambda l([A])] & (\lambda > 0) \\ [0^+ l([A])] = [-P] & (\lambda = 0) \\ [\lambda u([A])] & (\lambda < 0), \end{cases}$$

where 0^+C denotes the resession cone of a convex set C defined by $0^+C = \{x \in E \mid C + \lambda x \subset C,\ (\lambda > 0)\}$.

We define two subsets $\tilde{P}$ and $\tilde{E}_1$ of $\tilde{E}$ as follows.

$$\begin{aligned} \tilde{P} &= \{[A] \in \tilde{E} \mid [A] \geq [-P]\} \\ &= \{[A] \in \tilde{E} \mid u([A]) \subset P\} \\ \tilde{E}_1 &= \{[A] \in \tilde{E} \mid u([A]) = a + P \text{ for some } a \in E\}. \end{aligned}$$

We note that the correspondence which assigns $a \in E$ to $[A] \in \tilde{E}_1$ such that $u([A]) = a + P$ is one to one.

Theorem 1. ([4]) *Let E be a Banach space with a closed positive cone. Then $\tilde{E}$ is an order complete vector lattice with the definition* (2),(3),(4), *and*

(*a*) *$\tilde{P}$ is a convex cone in $\tilde{E}$ and satisfies* (P1), (P2), *and* $[A] \leq [B] \Longleftrightarrow [B] - [A] \in \tilde{P}$.

(*b*) *$\tilde{E}_1$ is a subspace which is order isomorphic to (E, P) by the correspondence $E \ni a \longleftrightarrow [A] \in \tilde{E}_1$ where $u([A]) = a + P$.*

Moreover, let $\{A_\sigma\}_{\sigma \in \Sigma} \subset \mathfrak{B}$, and $\{B_\lambda\}_{\lambda \in \Lambda} \subset \mathfrak{B}'$, be arbitrary families such that $\cup_{\sigma \in \Sigma} A_\sigma \in \mathfrak{B}$ and $\cup_{\lambda \in \Lambda} B_\lambda \in \mathfrak{B}'$. Then

(*c*) $\cap_{\sigma \in \Sigma} u([A_\sigma]) = u([\cup_{\sigma \in \Sigma} A_\sigma]), \quad \cap_{\lambda \in \Lambda} l([L(B_\lambda)]) = l([L(\cup_{\lambda \in \Lambda} B_\lambda)])$.

(*d*) $U(L(\cap_{\sigma \in \Sigma} u([A_\sigma]))) = \cap_{\sigma \in \Sigma} u([A_\sigma]), \quad L(U(\cap_{\lambda \in \Lambda} l([L(B_\lambda)]))) = \cap_{\lambda \in \Lambda} l([L(B_\lambda)])$.

Remark. *If (E, P) is order complete, then $(\tilde{E}, \tilde{P})$ is isomorphic to (E, P) as an ordered linear space.*

Let (E, P) be an ordered linear space. For $A \in \mathfrak{B}$, and $A' \in \mathfrak{B}'$ the generalized supremum and the generalized infimum are defined by

$$\begin{aligned} \operatorname{Sup} A &= \{a \in U(A) \mid b \leq a, \ \ b \in U(A) \Longrightarrow a = b\} \qquad (A \in \mathfrak{B}), \\ \operatorname{Inf} A' &= \{a \in L(A') \mid b \geq a, \ \ b \in L(A') \Longrightarrow a = b\} \qquad (A' \in \mathfrak{B}'), \end{aligned}$$

and we denote that $S = \{\operatorname{Sup} A \mid A \in \mathfrak{B}\}$. The basic properties of generalized supremum has been investigated in [2], [3]. In this paper we consider the condition

$$(5) \qquad U(A) = (\operatorname{Sup} A) + P \qquad (\forall A \in \mathfrak{B}),$$

which actually means that for every $x \in U(A)$ there exsits $x_0 \in \operatorname{Sup} A$ such that $x_0 \leq x$. If the space (E, P) satisfies the condition (5), the correspondence

$$\tilde{E} \ni [A] \longleftrightarrow U(A) \longleftrightarrow \operatorname{Sup} A \in S$$

is one to one. In the rest part of this section we will state some results which suggest the importance of the condition (5) in dealing with the generalized supremum. In the case when $dimE < \infty$, some equivalent conditions of (5) are known.([2]) In the infinite dimensional cases, it is not easy to see when the space (E, P) satisfies the condition (5). In this paper we will give some sufficient conditions in §2.

Proposition 1. *Suppose that (E, P) satisfies the condition* (5)*. Then* $\operatorname{Inf} A$ *and* $\operatorname{Sup} A$ *have a symmetric property, that is,*

$$\operatorname{Sup}(\operatorname{Inf}(\operatorname{Sup} A))) = \operatorname{Sup} A \qquad (A \in \mathfrak{B}).$$

proof. Taking the set of minimal points of both sides of (1), we have

$$\operatorname{Sup} A = \operatorname{Sup}(L(U(A))).$$

Moreover by (5),
$\operatorname{Sup}(L(U(A))) = \operatorname{Sup}(L((\operatorname{Sup} A) + P)) = \operatorname{Sup}(L(\operatorname{Sup} A)) = \operatorname{Sup}(\operatorname{Inf}(\operatorname{Sup} A) - P) = \operatorname{Sup}(\operatorname{Inf}(\operatorname{Sup} A))$.

Proposition 2. *Suppose that (E, P) satisfies the condition* (5)*. If* $\operatorname{Sup} A = \{a\}$ *for some $A \in \mathfrak{B}$, then $a = \operatorname{lub} A$ (:the least upper bound of A).*

The proof is trivial. The conclusion of Proposition 2 is not valid when the condition (5) does not hold. The following theorem is the fundamental rules on calculation of the generalized supremum.

Theorem 2. ([4]) *For $A, B \in \mathfrak{B}$,*

(*a*) $U(A+B) \sim' U(A) + U(B)$ *in* $\mathfrak{B}'$,

Moreover, if (E, P) satisfies the condition (5), *then*

(*b*) $\mathrm{Sup}(A+B) + P \supset \mathrm{Sup}\, A + \mathrm{Sup}\, B$,

(*c*) $\mathrm{Sup}(L(\mathrm{Sup}\, A + \mathrm{Sup}\, B)) = \mathrm{Sup}(A+B)$.

Under the condition (5), we define an order relation and a vector operation (the addition $\oplus$ and the scalar multiplication $*$) on S as follows.

Definition. *For $A, B \subset E$ and $\lambda \in \mathbb{R}$,*

$$\mathrm{Sup}\, A \leq \mathrm{Sup}\, B \Leftrightarrow \mathrm{Sup}\, B \subset \mathrm{Sup}\, A + P$$

$$\mathrm{Sup}\, A \oplus \mathrm{Sup}\, B = \mathrm{Sup}(A+B)$$

$$\lambda * \mathrm{Sup}\, A = \begin{cases} \mathrm{Sup}(\lambda l([A])) & (\lambda > 0) \\ \{0\} & (\lambda = 0) \\ \mathrm{Sup}(\lambda u([A])) & (\lambda < 0), \end{cases}$$

for $\mathrm{Sup}\, A$, $\mathrm{Sup}\, B \in S$ *and* $\lambda \in \mathbb{R}$.

Let S_0 be the set of all elements $\mathrm{Sup}\, A \in S$ such that $\mathrm{Sup}\, A = \{a_0\}$ for some $a_0 \in E$. Then by the following theorem, S can be regarded as an order completion of (E, P) which is isomorphic to S_0.

Theorem 3. ([4]) *If (E, P) satisfies* (5), *then S is isomorphic to $\tilde{E}$ as a vector lattice under the one to one correspondence*

$$S \ni \mathrm{Sup}\, A \longleftrightarrow [A] \in \tilde{E},$$

Moreover, S_0 is isomorphic to (E, P) under the same correspondence.

§2 Sufficient conditions for $U(A) = (\mathrm{Sup}\, A) + P$

An ordered linear space (E, P) is said to be **monotone order complete** (m.o.c. for short) if every upper bounded totally ordered subset of E has the least upper bound in E. In the case $E = \mathbb{R}^d$, (E, P) is m.o.c. if and only if P is closed. In the case when E is a Banach space with a closed positive cone P satisfying $P^* - P^* = E^*$, it is known that (E^*, P^*) is m.o.c. where E^* is the topological dual of E and $P^* \underset{def}{=} \{x^* \in E^* \mid x^*(x) \geq 0,\ x \in P\}$.

Proposition 3. *Suppose that an ordered linear space (E, P) is monotone order complete. Then (E, P) satisfies* (5). *In particular,* $\mathrm{Sup}\{a, b\} \neq \emptyset$, $\mathrm{Inf}\{a, b\} \neq \emptyset$ *for every $a, b \in E$, and $U(a, b) = (\mathrm{Sup}\{a, b\}) + P$.*

The proof of this proposition can be seen in [2]. A convex subset C of E is said to be algebraically closed if every straight line of E meets C by a closed interval. A point x of a convex subset $C \subset E$ is called an algebraic interior point of C if for every $z \in E$, there exists $\lambda > 0$ such that $x + \lambda z \in C$. Algebraic exterior points are defined similarly, and we denote the algebraic interior of C by C^i. Moreover, $\partial C = (C^i \cup (C^c)^i)^c$ is called the algebraic boundary of C. Let (E, P) be an ordered linear space and suppose that P is algebraically closed with nonempty algebraic interior. A convex subset F of P is called an exposed face of P if there exists a supporting hyperplane H of P such that $F = P \cap H$. By $\mathfrak{F}(P)$, we denote the set of all exposed faces of P. For $F \in \mathfrak{F}(P)$, $\dim F$ is defined as the dimension of aff$F$ where aff$F$ denotes the affine hull of F. We give another sufficient condition for (5) by using the facial structure of P.

Proposition 4. ([2]) *Suppose that P is algebraically closed and* $\operatorname{int} P \neq \emptyset$. *If* $\dim C < \infty$ *for every* $C \in \mathfrak{F}(P)$, *then* (5) *holds.*

A positive cone P in a topological vector space is said to be **normal** if there exists a neighborhood base of the origin consisting of neighborhoods V satisfying

$$(V + P) \cap (V - P) = V.$$

If P is normal, every order interval $[a, b] = \{x \in E \mid a \leq x \leq b\}$ in E is bounded with respect to the norm. We also recall Bishop-Phelps theorem which asserts that for a bounded closed convex set C in a Banach space E, the set of all bounded linear functional which attains its minimum on C is norm dense in E^*.([6])

Theorem 4. *Let E be a Banach space with a closed positive cone P. If the dual cone P^* has nonempty interior in E^*, then (E, P) has the property* (5).

proof. It is known that P is normal if and only if $P^* - P^* = E^*$ ([1]), and in particular, P is normal in the case that P^* has nonempty interior in E^*. For $x \in U(A)$, we denote $U(A)_x = (x - P) \cap U(A)$. It suffices to show that there exists an minimal point x_0 of $U(A)_x$ such that $x_0 \leq x$. Since P is closed, so is $U(A)_x$. We also have $U(A)_x \subset [a, x] = \{y \in E \mid a \leq y \leq x\}$ for $a \in A$ and hence the normality of P yields that $U(A)_x$ is bounded with respect to the norm in E. Therefore by Bishop-Phelps theorem, we can choose an interior point x_1^* of P^* such that x_1^* attains its minimum on $U(A)_x$ at some point $x_0 \in U(A)_x$. If there exsists $x_1 \in U(A)_x$ such that $x_1 \lneqq x_0$ it follows that $x^*(x_1) < x^*(x_0)$ since x^* is an interior point of P^*. It is a contradiction and x_0 is a minimal point of $U(A)_x$.

Corollary 1. *Let E' be a Banach space and $E = E' \times \mathbb{R}$ and $P = \{(x, t) \in E \mid t \geq \|x\|\}$. Then (E, P) has the property* (5).

Definition. *Let E be a topological vector space with a closed positive cone P. A set $A \subset E$ is said to be* **P-complete** *if it has no covers of the form*

$$\{(x_\alpha - P)^c \mid \alpha \in I\}$$

with $\{x_\alpha\}_{\alpha \in I}$ being a decreasing net in A.

A set A is said to have the **domination property** *if for $x \in A$ there exsists a minimal point x_0 of A such that $x_0 \leq x$.*

In [5], one can see some conditions under which A becomes P-complete or has the domination property.

Proposition 6. ([5]) *Let E be a topological vector space with a closed positive cone P, and let $E \supset A \neq \emptyset$. Then A has a minimal point if and only if there exists $x \in A$ such that $A_x = A \cap (x - P)$ is P-complete. Moreover, A has the domination property if and only if for each $y \in A$ there is some $x \in A_y$ such that A_x is P-complete.*

Theorem 5. *Let E be a reflexive Banach space with a closed positive cone P and suppose that P is normal. Then (E, P) has the property* (5).

proof. Let $x \in U(A)$ and set $U(A)_x = U(A) \cap (x - P)$. We will show that the section $U(A)_x$ has its minimal point. By Proposition 6, it suffices to show that $U(A)_x$ is P-complete. Suppose that there exsits a decreasing net $\{x_\alpha\}_{\alpha \in I}$ in $U(A)_x$ such that

$$U(A)_x \subset \bigcup_{\alpha \in I} (x_\alpha - P)^c.$$

We observe that $U(A)_x \subset [a, x]$ for $a \in A$ and hence the normality of P yields that $U(A)_x$ is bounded with respect to the norm in E. Hence it is weakly compact because the space E is reflexive. Since each $x_\alpha - P$ is weakly closed, we can choose a subcovering $\bigcup_{i=1\cdots n} (x_i - P)^c \supset U(A)_x$ such that $x_1 \geq x_2 \geq \cdots \geq x_n$. It is a contradiction, because $x_n \notin \bigcup_{i=1\cdots n} (x_i - P)^c$ while $x_n \in U(A)_x$.

The hypothesis on the positive cone P in Theorem 5 is weaker than that in Theorem 4. However, (5) does not follow from the condition that E is a Banach space and P is normal. The space $C[0,1]$ with the norm $\|f\| = \sup_{x \in [0,1]} |f(x)|$ and the usual positive cone $P = \{f \mid f(x) \geq 0\ (x \in [0,1])\}$ is a simple example.

Let E be a topological vector space with a closed positive cone P. (E, P) is said to be **boudedly order complete (b.o.c.)** if any bouded decreasing net $\{x_\alpha\}$ has an infimum, where bounded net means that for any neighborhood U of origin $\{x_\alpha\} \subset tU$ for some $t > 0$. P is said to be **Daniell** if any decreasing net $\{x_\alpha\}$ having a lower bound has its infimum to which it converges. If P is Daniell (E, P) is obviously m.o.c., and consequently the condition (5) holds. Moreover, we can easily see the following proposition.

Proposition 7. *Let E be a topological vector space with a positive cone P. If (E, P) is b.o.c. and P is normal, then (E, P) is m.o.c., and it satisfies the condition* (5) *in particular.*

References

[1] T. Ando, *On fundamental properties of a Banach space with cone*, Pacific J. Math. **12** (1962), 1163–1169.

[2] N.Komuro, S.Koshi, *Genaralized supremum in partially ordered linear space*, Proc. of the international conference on nonlinear analysis and convex analysis, World Scientific (1999), 199–204.

[3] S.Koshi, N.Komuro, *Supsets on partially ordered topological linear spaces*, Taiwanese J. of Math. **4-2** (2000), 275–284.

[4] N.Komuro, *The set of upper bounds in ordered linear spaces*, Proc. of the international conference on nonlinear analysis and convex analysis, to appear.

[5] D. T. Luc, *Theory of vector optimization*, Springer-Verlag (1989).

[6] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Springer-Verlag (1993).

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