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Centralizers and Monoids in Mathematical Clone Theory

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Abstract

A clone is a set of multi-variable operations which contains all projections and is closed under composition. For a set S of operations the centralizer S^* is the clone of operations which commute with all operations in S.

In this paper we consider some monoids consisting of unary operations and the centralizers of such monoids. We study (1) the centralizers of some monoids which contain the symmetric group, (2) the centralizers of permutation groups and (3) some relatively small monoids whose centralizer is the least clone.

Keywords: Clone; centralizer; monoid

1 Preliminaries

Let $\boldsymbol{k} = \{0, 1, \ldots, k-1\}$ for a fixed $k > 1$. For a positive integer n let $\mathcal{O}_k^{(n)}$ be the set of all n-ary operations on $\boldsymbol{k}$, that is, maps from $\boldsymbol{k}^n$ into $\boldsymbol{k}$, and let $\mathcal{O}_k = \bigcup_{n=1}^{\infty} \mathcal{O}_k^{(n)}$. Denote by $\mathcal{J}_k$ the set of all **projections** e_i^n, $1 \le i \le n$, over $\boldsymbol{k}$ where e_i^n is defined as $e_i^n(x_1, \ldots, x_i, \ldots, x_n) = x_i$ for every $(x_1, \ldots, x_n) \in \boldsymbol{k}^n$.

Definition 1.1 *A subset C of $\mathcal{O}_k$ is a* **clone** *on $\boldsymbol{k}$ if the following conditions are satisfied:*

(i) *C contains $\mathcal{J}_k$.*
(ii) *C is closed under (functional) composition.*

The set of all clones on $\boldsymbol{k}$ is denote by $\mathbf{L}_k$. The set $\mathbf{L}_k$ ordered by inclusion is called the **lattice of clones** *on $\boldsymbol{k}$ and is denoted by $\mathcal{L}_k$.*

The structure of $\mathcal{L}_2$ is completely known ([Po 41]); however, the structure of $\mathcal{L}_k$ for every $k \ge 3$ is extremely complex and still mostly unknown. The cardinality of the lattice of clones is known for each $k \ge 2$: $|\mathcal{L}_2| = \aleph_0$ ([Po 41]) and $|\mathcal{L}_k| = 2^{\aleph_0}$ if $3 \le k < \aleph_0$ ([IM 59]).

To get better understanding of the lattice $\mathcal{L}_k$, we started in [MMR 01] to study the clones corresponding to monoids of unary operations on $\boldsymbol{k}$ by a natural Galois connection.

This paper is an attempt to briefly summarize the work of [MMR 01], [MMR 02] and [MR 03].

Definition 1.2 *A subset M of $\mathcal{O}_k^{(1)}$ is a* **transformation monoid** *(or, simply, a* **monoid***) on $\boldsymbol{k}$ if it satisfies the following conditions:*

(i) *For any $s, t \in M$, the composition $t \circ s$ of s and t belongs to M.*
(ii) *The identity operation $\mathrm{id}_{\boldsymbol{k}}$ $(= e_1^1)$ belongs to M.*

Denote by $\mathcal{M}_k$ the set of all monoids on $\boldsymbol{k}$ ordered by inclusion.

We say that operations $f \in \mathcal{O}_k^{(m)}$ and $g \in \mathcal{O}_k^{(n)}$ **commute** (or **permute**), in symbol $f \perp g$, if for every $m \times n$ matrix $B = (b_{ij})$ over $\boldsymbol{k}$

$$\begin{aligned} & f(g(b_{11}, \ldots, b_{1n}), \ldots, g(b_{m1}, \ldots, b_{mn})) \\ = \; & g(f(b_{11}, \ldots, b_{m1}), \ldots, f(b_{1n}, \ldots, b_{mn})). \end{aligned}$$

Clearly, the relation $\perp$ on $\mathcal{O}_k$ is symmetric.

Definition 1.3 *For a subset F of $\mathcal{O}_k$, define*

$$F^* = \{\, g \in \mathcal{O}_k \mid g \perp f \text{ for all } f \in F \,\}.$$

The set F^* is called the **centralizer** *of F.*

Note: (1) It is easy to see that F^* is a clone on $\boldsymbol{k}$ for any subset F of $\mathcal{O}_k$.
(2) Clearly from the definition, $F \subseteq G$ implies $G^* \subseteq F^*$ for any subsets F, G of $\mathcal{O}_k$.

Definition 1. 4 *For a positive integer h, an h-ary* **relation** *on $\boldsymbol{k}$ is a subset ρ of $\boldsymbol{k}^h$, i.e., a set of h-tuples over $\boldsymbol{k}$. An operation $f \in \mathcal{O}_k^{(m)}$* **preserves** *$\rho$ if, for every $h \times m$ matrix $B = (b_{ij})$ over $\boldsymbol{k}$ whose column vectors all belong to ρ, the h-tuple of values of f on the rows of B also belongs to ρ; i.e.,*

$$(f(b_{11},\dots,b_{1m}),\dots,f(b_{h1},\dots,b_{hm})) \in \rho$$

whenever $(b_{1j},\dots,b_{hj}) \in \rho$ for all $j = 1,\dots,m$.

For an h-ary relation ρ on $\boldsymbol{k}$ set

$$\mathrm{Pol}\,\rho \;=\; \{\, f \in \mathcal{O}_k \mid f \text{ preserves } \rho \,\}.$$

It is easy to see that $\mathrm{Pol}\,\rho$ is a clone on $\boldsymbol{k}$.

In this paper we shall mostly deal with the following special type of relations. For $f \in \mathcal{O}_k^{(m)}$ set

$$f^{\Box} = \{\, (a_1,\dots,a_m,\, f(a_1,\dots,a_m)) \mid a_1,\dots,a_m \in \boldsymbol{k} \,\}.$$

Even more specifically, our main concern is the case $m = 1$, that is, the case of binary relations induced by unary operations. To repeat, for an operation $s \in \mathcal{O}_k^{(1)}$ we set

$$s^{\Box} = \{(x, s(x)) \mid x \in \boldsymbol{k}\}.$$

For $f \in \mathcal{O}_k^{(n)}$ and $s \in \mathcal{O}_k^{(1)}$, $f \in \mathrm{Pol}(s^{\Box})$ is equivalent to

$$f(s(x_1), s(x_2),\dots,s(x_n)) = s(f(x_1,x_2,\dots,x_n))$$

for every $(x_1,x_2,\dots,x_n) \in \boldsymbol{k}^n$. To put it in algebraic terminology, $f \in \mathrm{Pol}(s^{\Box})$ means that s is an endomorphism of the algebra $\langle \boldsymbol{k}; f\rangle$.

From the definition of $f \perp g$ and the symmetry of $\perp$ we obtain for $f, g \in \mathcal{O}_k$

$$f \perp g \iff f \in \mathrm{Pol}\, g^{\Box} \iff g \in \mathrm{Pol}\, f^{\Box}$$

and hence the centralizer F^* of $F \subseteq \mathcal{O}_k$ is

$$F^* \;=\; \bigcap_{f \in F} \mathrm{Pol}\, f^{\Box}.$$

For $F \subseteq \mathcal{O}_k$, $\langle F\rangle$ denotes the clone generated by F, that is, $\langle F\rangle$ is the least clone containing F.

The following definition and theorem are from [Da77, 79].

Definition 1. 5 *For $f \in \mathcal{O}_k^{(n)}$ and $\Sigma \subseteq \mathcal{O}_k$, the operation f is* **parametrically expressible,** *in short,* **p-expressible,** *by Σ if there exist* (i) *integers $m \ge 1$ and $\ell \ge 0$, and* (ii) *$(m+\ell+1)$-ary operations g_i, h_i $(i = 1,\dots,m)$ in $\langle\Sigma\rangle$ such that $f^{\Box}$ consists of all $(x_1,\dots,x_{n+1}) \in \boldsymbol{k}^{n+1}$ satisfying the system of equations*

$$g_i(x_1,\dots,x_{n+\ell+1}) = h_i(x_1,\dots,x_{n+\ell+1}) \qquad (i = 1,\dots,m)$$

for some $x_{n+2},\dots,x_{n+\ell+1} \in \boldsymbol{k}$, i.e.,

$$f^{\Box} = \{(x_1,\dots,x_n,x_{n+1}) \mid \exists x_{n+2},\dots,x_{n+\ell+1} \in \boldsymbol{k} \;\; \forall i \in \{1,\dots,m\} \;\; g_i(x_1,\dots,x_{n+\ell+1}) = h_i(x_1,\dots,x_{n+\ell+1})\}.$$

Theorem 1. 1 (*Kuznetsov criterion*) *Let $f \in \mathcal{O}_k$ and $\Sigma \subseteq \mathcal{O}_k$. Then f is p-expressible by Σ if and only if $\Sigma^* \subseteq \{f\}^*$.*

This property turns out to be quite useful in the study of clones, in particular, of centralizers.

2 Centralizers of Monoids containing the Symmetric Group S_k

2.1 The Sequence $\{N_i\}$ of Monoids

First we consider the centralizers of monoids which sit "above" the symmetric group S_k, i.e., monoids containing S_k.

In [MMR 01] we introduced the operations d_i's as follows: For every $i = 0, 1,\dots,k-1$, a unary operation $d_i : \boldsymbol{k} \longrightarrow \boldsymbol{k}$ is defined by

$$d_i(x) = \begin{cases} 0 & \text{if } 0 \le x \le i, \\ x & \text{if } i < x \le k-1. \end{cases}$$

In particular, $d_0 = e_1^1$ (the identity operation on $\boldsymbol{k}$) and d_{k-2} satisfies the following: $d_{k-2}(x) = 0$ for $x \in \boldsymbol{k} \setminus \{k-1\}$ and $d_{k-2}(k-1) = k-1$.

The monoid N_i for $i \in \boldsymbol{k}$ is defined as the clone generated by the symmetric group S_k on $\boldsymbol{k}$ and the operation d_i, i.e.,

$$N_i = \langle\, S_k \cup \{d_i\}\,\rangle.$$

It is easy to see that $N_0 = S_k$ and $N_1 = \mathcal{O}_k^{(1)}$. Also, N_{k-1} consists of S_k and all constant unary operations.

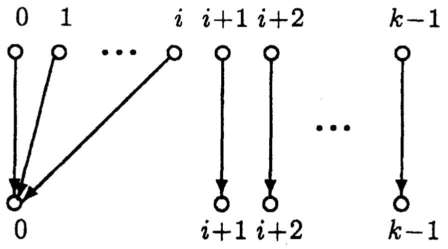

Figure 1. The unary operation d_i

We have a sequence

$$S_k \subset N_{k-1} \subset N_{k-2} \subset \cdots \subset N_3 \subset N_2 \subset \mathcal{O}_k^{(1)}$$

where the symbol $\subset$ denotes proper inclusion.

We also prepare the following unary operations.

Let $\sigma, \tau : \boldsymbol{k} \longrightarrow \boldsymbol{k}$ be the permutations defined as

$$\sigma(x) = \begin{cases} x+1 & \text{if } x \in \boldsymbol{k} \setminus \{k-1\} \\ 0 & \text{if } x = k-1 \end{cases}$$

and

$$\tau(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \\ x & \text{otherwise,} \end{cases}$$

i.e., in the cyclic notation $\sigma = (0\ 1 \cdots k-1)$ and $\tau = (0\ 1)$.

Let $c_0 : \boldsymbol{k} \longrightarrow \boldsymbol{k}$ be a constant operation defined as $c_0(x) = 0$ for all $x \in \boldsymbol{k}$.

For every $i \in \boldsymbol{k}$ let $\chi_i : \boldsymbol{k} \longrightarrow \boldsymbol{k}$ be the unary operation defined as

$$\chi_i(x) = \begin{cases} 1 & \text{if } x = i, \\ 0 & \text{otherwise.} \end{cases}$$

Set $\Gamma_k = \{\chi_i \mid i \in \boldsymbol{k}\}$.

Lemma 2. 1 *For every $i \in \boldsymbol{k}$, χ_i is generated by $\{\sigma, d_{k-2}\}$.*

Proof It is easily verified that

$$\chi_i = \sigma\, d_{k-2}\, \sigma^{-1}\, d_{k-2}\, \sigma^{-(i+1)}. \quad \Box$$

2.2 Application of the Kuznetsov Criterion : $N_i{}^* = N_{k-2}{}^*$

Proposition 2. 2 *For $1 \le i \le k-2$, the operation d_i is p-expressible from N_{k-2}, namely,*

$$d_i{}^{\Box} = \{(x,y) \in \boldsymbol{k}^2 \mid c_0(x) = \chi_1(y) = \cdots = \chi_i(y),\ \chi_{i+1}(x) = \chi_{i+1}(y), \ldots, \chi_{k-1}(x) = \chi_{k-1}(y)\}.$$

Proof Denote by ρ_i the binary relation on the right side of the above equation.

For each $x = 0, 1, \ldots, k-1$, we consider which y satisfies $(x,y) \in \rho_i$ where $y \in \{0, \ldots, k-1\}$.

Case 0: Suppose $x = 0$. Then it is easy to see that $c_0(0) = \chi_1(y)$ implies $y \neq 1$. Similarly, it is implied from $c_0(0) = \chi_j(y)$ that $y \neq j$ for every $j = 1, \ldots, i$. Next, from $\chi_{i+1}(0) = \chi_{i+1}(y)$ it is implied that $y \neq i+1$. Similarly, it is implied from $\chi_\ell(0) = \chi_\ell(y)$ that $y \neq \ell$ for every $\ell = i+1, \ldots, k-1$. So the only possibility is $y = 0$, and clearly $(0,0) \in \rho_i$.

Analogously, we can check, for every $x = 0, \ldots, i$, that $(x, 0)$ is the only pair that satisfies $(x,y) \in \rho_i$ for $y = 0, \ldots, k-1$.

Case $i+1$: Suppose $x = i+1$. In this case $c_0(i+1) = \chi_j(y)$ implies that $y \neq j$ for every $j = 1, \ldots, i$. Now, $\chi_{i+1}(i+1) = \chi_{i+1}(y)$ implies $y = i+1$. And, $\chi_\ell(i+1) = \chi_\ell(y)$ implies $y \neq \ell$ for every $\ell = i+2, \ldots, k-1$. Hence $(i+1, i+1)$ is the only pair that belongs to ρ_i with $i+1$ as the first component.

In the similar way we can check, for every $x = i+1, \ldots, k-1$, that (x,x) is the only pair that satisfies $(x,y) \in \rho_i$ for $y = 0, \ldots, k-1$.

Thus we have proved that $\rho_i = d_i{}^{\Box}$. $\Box$

Corollary 2. 3 *For every i, $1 \le i \le k-2$,*

$$N_i{}^* = N_{k-2}{}^*.$$

Proof Clear from the Kuznetsov criterion, Lemma 2.1 and Proposition 2.2. $\Box$

2.3 The Centralizers of N_i's

In this subsection we assert without proof that $(\mathcal{O}_k{}^{(1)})^* = \mathcal{J}_k$. Then, as a corollary, we have $N_i{}^* = \mathcal{J}_k$ for every $1 \le i \le k-2$.

An operation $f \in \mathcal{O}_k^{(n)}$ is called **conservative** if $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$ for every $(a_1, \ldots, a_n) \in \boldsymbol{k}^n$.

Lemma 2. 4 *For $f \in \mathcal{O}_k^{(n)}$, if $f \in (\mathcal{O}_k^{(1)})^*$ then f is conservative.*

Moreover, we can show the following:

Proposition 2. 5 *For $f \in \mathcal{O}_k^{(n)}$, if $f \in (\mathcal{O}_k^{(1)})^*$ then f is a projection. I.e., $(\mathcal{O}_k^{(1)})^* = \mathcal{J}_k$.*

The proof is tedious and omitted.

Corollary 2. 6 *For every $i = 1, 2, \ldots, k-2$,*

$$N_i{}^* = \mathcal{J}_k.$$

To summarize, we have the following:

$$N_0{}^* (= S_k{}^*) = \{f \in \mathcal{O}_k \mid f \text{ is synchronous }\}$$
$$N_{k-1}{}^* = \{f \in \mathcal{O}_k \mid f \text{ is synchronous and idempotent }\}$$
$$N_{k-2}{}^* = \ldots = N_2{}^* = N_1{}^* (= (\mathcal{O}_k^{(1)})^*) = \mathcal{J}_k$$

(The defintion of a "synchronous" operation appears in the next section.)

3 The Centralizers of the Subgroups of S_k

Next we turn to the centralizers of monoids which sit "below" the symmetric group S_k, i.e., monoids contained in S_k, that is, permutation groups.

3.1 The Centralizers of the Permutation Groups

As we have seen in the previous section, many different monoids have the same clone $\mathcal{J}_k$ as their centralizer. But, surprisingly, the situation is quite different for permutation groups on $\boldsymbol{k}$, i.e., subgroups of the symmetric group S_k. In fact, the $*$-operator is proved to be an injective mapping on the set of permutation groups.

Lemma 3. 1 *For any subgroup H of S_k and any $s \in S_k$, if $s^\square$ is p-expressible by H then $s \in H$.*

Proof If $s^\square$ is p-expressible by H, which is a subgroup of S_k, then $s^\square$ can be expressed as

$$s^\square = \{\, (x, y) \in \boldsymbol{k}^2 \mid t(x) = u(y) \,\}$$

for some $t, u \in H$. This equation can be rephrased as

$$s^\square = \{\, (x, y) \in \boldsymbol{k}^2 \mid (u^{-1}t)(x) = y \,\}.$$

This implies $s = u^{-1}t$ and thus $s \in H$ as desired. □

Theorem 3. 2 *For any subgroups H_1 and H_2 of S_k, if $H_1 \neq H_2$ then $H_1^* \neq H_2^*$.*

Proof Suppose that $H_1 \neq H_2$ and $H_1^* = H_2^*$. Without loss of generality we may assume that there exists a unary operation $s \in H_2 \setminus H_1$. Then it holds that

$$H_1^* \,(= H_2^*) \subseteq (s^\square)^* \,(= \operatorname{Pol} s^\square),$$

since

$$H_2^* = \bigcap_{t \in H_2} \operatorname{Pol} t^\square.$$

It follows from the Kuznetsov criterion that $s^\square$ is p-expressible by H_1. Hence by Lemma 3.1 we have $s \in H_1$, a contradiction. □

3.2 The Centralizer of the Symmetric Group S_k

Marczewski [Ma 64] initiated the study of homogeneous algebras and homogeneous operations. An operation $f \in \mathcal{O}_k$ is *homogeneous* if it belongs to the centralizer of the symmetric group S_k.

Definition 3. 1 *For $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in \boldsymbol{k}^n$, we say that $(x_1, \ldots, x_n)$ is* **similar** *to $(y_1, \ldots, y_n)$ if the following condition is satisfied:*

$$x_i = x_j \iff y_i = y_j \quad \text{for any} \quad 0 \leq i, j < n.$$

Definition 3. 2 *An operation $f \in \mathcal{O}_k^{(n)}$ is* **synchronous** *if the following condition is satisfied for any element $(x_1, \ldots, x_n)$ in $\boldsymbol{k}^n$.*
If $|\{x_1, \ldots, x_n\}| \neq k-1$ then

(1) *$f(x_1, \ldots, x_n) = x_\ell$ for some $1 \leq \ell \leq n$, and*

(2) *$f(y_1, \ldots, y_n) = y_\ell$ for any $(y_1, \ldots, y_n) \in \boldsymbol{k}^n$ which is similar to $(x_1, \ldots, x_n)$,*

and, if $|\{x_1, \ldots, x_n\}| = k-1$ and $f(x_1, \ldots, x_n) = u$ for some $u \in \boldsymbol{k}$ then

(1) *$u = x_\ell$ for some $1 \leq \ell \leq n$ implies $f(y_1, \ldots, y_n) = y_\ell$ for any $(y_1, \ldots, y_n) \in \boldsymbol{k}^n$ which is similar to $(x_1, \ldots, x_n)$, and*

(2) *$u \in \boldsymbol{k} \setminus \{x_1, \ldots, x_n\}$ implies $f(y_1, \ldots, y_n) = v$ where $v \in \boldsymbol{k} \setminus \{y_1, \ldots, y_n\}$ for any $(y_1, \ldots, y_n) \in \boldsymbol{k}^n$ which is similar to $(x_1, \ldots, x_n)$.*

The set of all synchronous operations in $\mathcal{O}_k$ is denoted by $\mathcal{SYN}_k$.

Theorem 3. 3 *For the symmetric group S_k on $\boldsymbol{k}$,*

$$S_k^* = \mathcal{SYN}_k,$$

that is, an operation $f \in \mathcal{O}_k$ is homogeneous if and only if it is synchronous.

Again, we skip the proof. (See [MMR 01].)

3.3 The Centralizer of the Alternating Group A_k

For the alternating group A_k on $\boldsymbol{k}$, Theorem 3.2 implies $S_k^* \neq A_k^*$. In the following we show how the centralizer A_k^* is characterized.

Theorem 3. 4 *Let $n \geq 1$ and $f \in \mathcal{O}_k^{(n)}$. Then $f \in A_k^*$ if and only if the following conditions are satisfied for every h-block equivalence relation θ on $\{1, \ldots, n\}$ with $h \leq k$:*

(1) *If $h \leq k-3$ then f is a projection on $\Pi_n(\theta)$.*

(2) *If $h = k-2$ then on $\Pi_n(\theta)$ the operation f is either a projection or f is uniquely determined by $f(\boldsymbol{a}) \in \{k-2, k-1\}$ where $\boldsymbol{a} \in \Pi_n(\theta)$ satisfies $\operatorname{sgn} \boldsymbol{a} = (0, 1, \ldots, k-3)$.*

(3) *If $h = k-1$ then for $i = 0, 1$ the operation f on $H_{k-1,i}$ is either a projection or it is uniquely determined by $f(\boldsymbol{a}^i) = k-1$.*

(4) *If $h = k$ then for $i = 0, 1$ the operation f on $H_{k,i}$ is a projection.* □

For the proof refer to [MR 03].

4 Monoid whose Centralizer is $\mathcal{O}_k$

Here we look at the largest clone in the lattice $\mathcal{L}_k$, that is, the clone $\mathcal{O}_k$. It is obvious that the least monoid {id} consisting only of the identity operation corresponds to $\mathcal{O}_k$. We show that no other monoid corresponds to $\mathcal{O}_k$.

Proposition 4. 1 *For any monoid $M \subseteq \mathcal{O}_k^{(1)}$,*

$$M^* = \mathcal{O}_k \text{ if and only if } M = \{\mathrm{id}\}.$$

Proof (⇐) Clear. (⇒) Let $M^* = \mathcal{O}_k$. Then by Proposition 2.5 we have

$$\mathrm{id} \in M \subseteq M^{**} = \mathcal{O}_k{}^* \subseteq (\mathcal{O}_k^{(1)})^* = \mathcal{J}_k$$

which shows $M = \{\mathrm{id}\}$. □

5 "Small" Monoids whose Centralizer is $\mathcal{J}_k$

Since $*$-operator induces a Galois connection between monoids and clones, it seems natural to consider that a smaller monoid corresponds to a larger clone and vice versa. However, relatively small monoids were found whose centralizer is $\mathcal{J}_k$, the smallest clone.

5.1 Application of the Kuznetsov Criterion : $\langle\{\sigma, d_{k-2}\}\rangle^* = N_{k-2}{}^*$

Now we look for small monoids whose centralizer is $\mathcal{J}_k$.

Proposition 5. 1 *The transposition τ is p-expressible from $\{\sigma, d_{k-2}\}$, namely,*

$$\tau^{\square} = \{(x, y) \in \boldsymbol{k}^2 \mid \chi_0(x) = \chi_1(y),\ \chi_1(x) = \chi_0(y), \\ \chi_2(x) = \chi_2(y), \ldots, \chi_{k-1}(x) = \chi_{k-1}(y)\}.$$

Proof Similar to the proof of Proposition 2.2. □

Corollary 5. 2 *Let $M_1 = \langle\{\sigma, d_{k-2}\}\rangle$. Then*

$$M_1{}^* = N_{k-2}{}^* \quad (= \mathcal{J}_k).$$

Proof It is well-known that $\{\sigma, \tau\}$ generates the symmetric group S_k. So the assertion is clear from Proposition 5.1. □

Remark: (1) The permutations in M_1 are those generated by σ, that is, $M_1 \cap S_k = \langle\sigma\rangle$. Thus, not all permutations are contained in M_1, i.e., $M_1 \not\supseteq S_k$.
(2) An implication from above is that a monoid M does not need to contain all permutations in order to satisfy $M^* = \mathcal{J}_k$.

5.2 Application of the Kuznetsov Criterion : $\Gamma_k{}^* = N_{k-2}{}^*$

It should be reminded that $\Gamma_k = \{\chi_i \mid i \in \boldsymbol{k}\}$.

Analogous constructions to the one in the preceeding subsection give us the following:

Proposition 5. 3 *The transposition $(i\ j)$ $(i \neq j)$ is p-expressible from Γ_k.*

Proof A construction for $\tau = (0\,1)$ was given in Proposition 5.1. Any transposition $(i\,j)$ can be constructed similarly. □

Corollary 5. 4

$$\Gamma_k{}^* = N_{k-2}{}^* \quad (= \mathcal{J}_k).$$

Proof It is well-known that the set of all transpositions generates the symmetric group S_k. So the assertion is clear. □

Remark: (1) Γ_k consists only of two-valued operations. In particular, Γ_k has no permutations, i.e., $\Gamma_k \cap S_k = \emptyset$.
(2) An implication from above is that transitivity is not necessarily required for a monoid M to satisfy $M^* = \mathcal{J}_k$.

5.3 The Centralizer of $\langle \Gamma_k \rangle \setminus \{\chi_0, \overline{\chi}_1\}$

Even a smaller monoid than Γ_k can be proved to have $\mathcal{J}_k$ as its centralizer.

The monoid $\langle \Gamma_k \rangle$ generated by Γ_k consists of the following operations:

$$\langle \Gamma_k \rangle = \{\chi_0, \chi_1, \ldots, \chi_{k-1}, \overline{\chi}_0, \overline{\chi}_1, \ldots, \overline{\chi}_{k-1}, c_0, c_1, \mathrm{id}_k\}$$

where $\overline{\chi}_i(x) = 1 - \chi_i(x)$ and $c_i(x) = i$ for all $i = 0, 1, \ldots, k-1$.

Set $\Theta_k = \langle \Gamma_k \rangle \setminus \{\chi_0, \overline{\chi}_1\}$, that is,

$$\Theta_k = \{\chi_1, \ldots, \chi_{k-1}, \overline{\chi}_0, \overline{\chi}_2, \ldots, \overline{\chi}_{k-1}, c_0, c_1, \mathrm{id}_k\}.$$

It is readily verified that Θ_k is a submonoid of $\langle \Gamma_k \rangle$.

For this monoid Θ_k we can prove the following:

Proposition 5. 5 *Let $k > 2$. Then $\Theta_k^* = \mathcal{J}_k$.*

Thus the centralizer Θ_k^* of Θ_k is the least clone $\mathcal{J}_k$. The proof is long and omitted here. (See [MR 03])

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