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# Inner and Outer Choquet integral representation

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## 1 Introduction

The Choquet integral is commonly used as the integral for a non-additive set function, because the Choquet integral coincides with expectation in probability theory when the set function is probability measure. The most important properties of the Choquet integral are comonotonic additivity and monotonicity (for short we call c.m.).

Considering the topology, various regularities are proposed and studied [4, 5, 12, 17, 16, 8, 11, 14], especially, [16] propose their regularity in relation to Choquet integral, that is, a c.m. functional on the class of continuous functions with compact support is represented by Choquet integral with respect to an o-regular non-additive measure.

In this paper, we assume the universal set $X$ to be a locally compact Hausdorff space.We say that the representation using o-regular non-additive measure defined by the supremum of the functional is an outer representation. On the other hand, the representation that uses the infinimum of the functional, that is called an inner representation, can be considered. In this paper we study i-regular non-additive measure and inner representation. We make clarify the difference between o-regularity and i-regularity, or between an outer representation and an inner representation. The structure of this paper is as follows: In section 2 we present basic definitions and properties without topological assumption of non-additive measure and Choquet integral with respect to a non-additive measure as a preliminaries. In Section 3, we assume that the universal space $X$ is locally compact space. We define o-regular and i-regular non-additive measure and show some properties.

In section 4, we study the representation of a c.m. functional. We show that the inner representation is possible, although the method is somewhat different from the outer one and the proof is rather complex. The induced i-regular non-additive measure coincides with o-regular one on the class of compact sets.

In section 5, we finish with some concluding remark.

## 2 Non-additive measure and Choquet integral

In this section, we present some basic definition and properties of non-additive measure theory. $X$ denotes the universal set and $\mathcal{B}$ denotes its $\sigma-$ algebra. No topological assumption is needed in this section.

In this paper, we distinguish the term "fuzzy measure" from "non-additive measure". Sugeno's original axioms [15] for a fuzzy measure has some continuity. On the other hand, some authors define a fuzzy measure, that is monotone set function vanishing at $\emptyset$, and is not assumed any continuity. In order to avoid confusion, in this paper, according to Denneberg's monograph [2], we say it a non-additive measure.

**Definition 2.1.** A *non-additive measure* $\mu$ is an extended real valued set function, $\mu : \mathcal{B} \longrightarrow \overline{R^+}$ with the following properties;

$\mu(\emptyset) = 0$, and $\mu(A) \leq \mu(B)$ whenever $A \subset B, A, B \in \mathcal{B}$ where $\overline{R^+} = [0, \infty]$ is the set of extended nonnegative real numbers. In this paper we assume that $\mu$ is finite, that is, $\mu(X) < \infty$.

The class of non-negative measurable functions is denoted by $\mathcal{M}^+$.

**Definition 2.2.** [1, 2] Let $\mu$ be a non-additive measure on $(X, \mathcal{B})$.

1. *The Choquet integral* of $f \in \mathcal{M}^+$ with respect to $\mu$ is defined by

$$(C)\int f d\mu = \int_0^\infty \mu_f(r) dr,$$

where $\mu_f(r) = \mu(\{x|f(x) \geq r\})$.

**Definition 2.3.** [3] Let $f, g \in \mathcal{M}^+$. We say that $f$ and $g$ are comonotonic if $f(x) < f(x') \Rightarrow g(x) \leq g(x')$ for $x, x' \in X$. We denote $f \sim g$, when $f$ and $g$ are comonotonic.

The Choquet integral of $f \in M$ with respect to a non-additive measure has the next basic properties.

**Theorem 2.4.** [2] *Let $f, g \in \mathcal{M}^+$.*

1. *(Monotonicity) If $f \leq g$, then*

$$(C)\int f d\mu \leq (C)\int g d\mu$$

2. *(Comonotonic additivity) If $f \sim g$, then*

$$(C)\int (f+g) d\mu = (C)\int f d\mu + (C)\int g d\mu.$$

# 3 Properties of i-regular non-additive measure

In this section we show some properties of o- regular non-additive measure and i-regular non-additive measure. The properties of o-regular non-additive measure are shown in [6, 7, 9, 10, 8, 11].

In the following we assume that $X$ is a locally compact Hausdorff space, $\mathcal{B}$ the class of Borel subsets of $X$, $O$ the class of open subsets of $X$ and $C$ the class of compact subsets of $X$.

$C^+(X)$ denotes the class of non-negative continuous functions with compact support and$C_1^+(X)$ denotes the subclass of $C^+(X)$ with $0 \leq f \leq 1$ for $f \in C_1^+(X)$.

**Definition 3.1.** Let $\mu$ be a non-additive measure on the measurable space $(X, \mathcal{B})$.

1. $\mu$ is said to be $o$-continuous from below if $O_n \uparrow O \Rightarrow \mu(O_n) \uparrow \mu(O)$ where $n = 1, 2, 3, \cdots$ and both $O_n$ and $O$ are open sets .

2. $\mu$ is said to be $c$-continuous from above if $C_n \downarrow C \Rightarrow \mu(C_n) \downarrow \mu(C)$ where $n = 1, 2, \cdots$ and both $C_n$ and $C$ are compact sets.

First, we define the regular non-additive measures.

**Definition 3.2.** *Let $\mu$ be a non-additive measure on measurable space $(X, \mathcal{B})$. $\mu$ is said to be inner regular if $\mu(B) = \sup\{\mu(C) | C \in \mathcal{C}, C \subset B\}$ for all $B \in \mathcal{B}$.*

*Inner regular non-additive measure is called $i-$ regular if $\mu(C) = \inf\{\mu(O) | O \in \mathcal{O}, C \subset O\}$ for all $C \in \mathcal{C}$.*

*$\mu$ is said to be outer regular if $\mu(B) = \inf\{\mu(O) | O \in \mathcal{O}, O \supset B\}$ for all $B \in \mathcal{B}$.*

*Outer regular non-additive measure is called $o-$regular if $\mu(O) = \sup\{\mu(C) | C \in \mathcal{O}, C \subset O\}$ for all $O \in \mathcal{O}$.*

The next lemmas follow from the definition immediately.

**Lemma 3.3.** *Let $\mu_i$ be an $i-$regular non-additive measure and $\mu_o$ be an $o-$regular non-additive measure. $\mu_i(O) = \mu_o(O)$ for $O \in \mathcal{O}$ if and only if $\mu_i(C) = \mu_o(C)$ for $C \in \mathcal{C}$.*

**Lemma 3.4.** *Let $\mu_i$ be an $i-$regular non-additive measure and $\mu_o$ be an $o-$regular non-additive measure such that $\mu_i(C) = \mu_o(C)$ for $C \in \mathcal{C}$. Then we have $\mu_i(A) \leq \mu_o(A)$ for all $A \in \mathcal{B}$.*

The next two results follow from the definition.

**Proposition 3.5.** *[6] Let $\mu$ be a o- regular non-additive measure.*

1. *$\mu$ is o-continuous from below.*

2. *$\mu$ is c-continuous from above.*

**Remark** Proposition 3.5 (2) is true when non-additive measure are outer regular non-additive measures.

We have the next lemma from the definition and Ulysohn's lemma, although it is still open whether the similar continuity to the lemma above holds in the case of i-regular non-additive measure.

**Proposition 3.6.** *Let $\mu$ be an i-regular non-additive measure. For every compact set $C \in \mathcal{C}$ there exists a sequence $\{O_n\}$ of open sets such that $\mu(O_n) \downarrow \mu(C)$.*

In relation to the Choquet integral, we have the next propositions.

**Proposition 3.7.** *[6] Let $\mu_1$ and $\mu_2$ be c-continuous from above.*

*If for all $f \in K$ $(C)\int f d\mu_1 = (C)\int f d\mu_2$, then $\mu_1(C) = \mu_2(C)$ for all $C \in \mathcal{C}$.*

In the case of i-regular, we have the next similar result.

**Proposition 3.8.** *Let $\mu$ be an i-regular non-additive measure.*

*If for all $f \in K$ $(C)\int f d\mu_1 = (C)\int f d\mu_2$, then $\mu_1(C) = \mu_2(C)$ for all $C \in \mathcal{C}$.*

It follows from Proposition 3.6 that we have

$$\mu_i(\{x|f^n(x) \geq \alpha\}) \downarrow \mu_i(\{x|1_C(x) \geq \alpha\})$$

as $n \to \infty$ for $0 < \alpha \leq 1$ and $i = 1,2$.

The next theorem is the main theorem of this chapter.

**Theorem 3.9.** *Let $\mu_i$ be i-regular non-additive measure and $\mu_o$ be o-regular non-additive measures. If for all $f \in C^+$ $(C)\int f d\mu_i = (C)\int f d\mu_o$, then*

1. *$\mu_i(C) = \mu_o(C)$ for all $C \in \mathcal{C}$,*

2. $\mu_i(O) = \mu_o(O)$ *for all* $C \in \mathcal{O}$,

3. $\mu_i(A) \leq \mu_o(A)$ *for all* $A \in \mathcal{B}$.

# 4 Representation theorems

The Choquet integral is a comonotonically additive and monotone functional. Conversely, we consider a c. m. functional on the class of continuous functions with compact support.

**Definition 4.1.** *Let I be a real valued functional on* $C^+(X)$.

*We say that I is comonotonically additive iff* $f \sim g \Rightarrow I(f+g) = I(f) + I(g)$ *for* $f, g \in C^+(X)$, *and that I is monotone iff* $f \leq g \Rightarrow I(f) \leq I(g)$ *for* $f, g \in C^+(X)$.

*If a functional I is comonotonically additive and monotone, we say that I is a c.m. functional.*

First, we review the outer representation.

**Lemma 4.2.** *[16] Let I be a c.m. functional on* $C^+$.

*We put* $\mu_I^o(O) = \sup\{I(f) | f \in C_1^+, supp(f) \subset O\}$ *for* $O \in \mathcal{O}$

*and* $\mu_I^o(B) = \inf\{\mu_I^o(O) | O \in \mathcal{O}, O \supset B\}$ *for* $B \in \mathcal{B}$. *Then* $\mu_I^o$ *is an outer regular non-additive measure.*

We shall say that this outer regular non-additive measure $\mu_I^o$ is an outer non-additive measure induced by a c.m. functional $I$.

**Proposition 4.3.** *[16] Let* $\mu_I^o$ *be an outer non-additive measure induced by a c.m. functional I.*

1. *If* $f \in C^+, A \subset \{x | f(x) \geq 1\}$, *and* $A \in \mathcal{B}$, *then* $\mu_I(A) \leq I(f)$

2. *If C is a compact set, then* $\mu_I(C) < \infty$.

3. *If O is an open set, then* $\mu_I(O) = \sup\{\mu_I(C) | C \in \mathcal{C}, C \subset O\}$.

*Therefore the non-additive measure induced by a c.m. functional is o-regular.*

From the above lemma, we obtain the following theorem [16].

**Theorem 4.4.** *(Outer representation theorem) [16]*

*Let $I$ be a c.m. functional on $C^+$. If $\mu_I^o$ is an outer regular non-additive measure induced by $I$, then we have $I(f) = (C)\int f d\mu_I^o$ for all $f \in C^+(X)$.*

In the following, we present some preliminary results for inner representation. The proofs are in [16, 6].

Let $O_{n,k} = \{x|f(x) > \frac{k-1}{n}\}, C_{n,k} = \{x|f(x) \geq \frac{k}{n}\}$ where $f \in C^+$, $k$ and $n$ is a positive integer and $1 \leq k \leq n$. Then for all $n$ and $k$, $C_{n,k}$ is a compact set, $O_{n,k}$ is an open set, and $O_{n,k+1} \subset C_{n,k} \subset O_{n,k} \subset supp(f)$. Since $X$ is a locally compact Hausdorff space, for all $n, k$ there exists $f_{n,k} \in C_1^+$ such that $f_{n,k}(x) = 1$ when $x \in C_{n,k}$ and $supp(f_{n,k}) \subset O_{n,k}$.

The functions $f_{n,k}$ have the following properties.

**Lemma 4.5.** *[16]*

1. *For all positive integer $n, k$ and $j$ such that $1 \leq k \leq n$ and $1 \leq j \leq n$, $f_{n,k}$ and $f_{n,j}$ are comonotonic.*

2. *For all positive integer $n$ and $k$ such that $1 \leq k \leq n$,*

   *$f_{n,1} + f_{n,2} + \cdots + f_{n,k}$ and $f_{n,k+1} + f_{n,k+2} + \cdots + f_{n,n}$ are comonotonic.*

**Lemma 4.6.** *[16] Define $f_n \in C_1^+$ for $n = 1, 2, \cdots$ by $f_n = \sum_{k=1}^{n} \frac{1}{n} f_{n,k}$.*

1. *There exists $F \in C_1^+$ which satisfies the following conditions.*

   *(a)* $0 \leq |f - f_n| \leq \frac{1}{n} F$

   *(b)* $x \in supp(f) \Rightarrow F(x) = 1$

(c) $f_n \sim_c F$ *for all* $n$

2. *We have* $\|f - f_n\| \leq \frac{1}{n}$, *where* $\|\cdot\|$ *is the* sup *norm.*

3. *Suppose that* $\dfrac{k}{n} \leq f(x) < \dfrac{k+1}{n}$, *then we have* $\dfrac{k}{n} \leq f_n(x) \leq \dfrac{k+1}{n}$

4. $\lim_{n \to \infty} I(f_n) = I(f)$

Now we can proceed the inner representation. The definition of a non-additive measure is a little bit different from outer representation. The proof is obvious from the definition.

**Lemma 4.7.** *Let I be a c.m. functional on* $C^+$.

*We put* $\mu_I^i(C) = \inf\{I(f) | f \in C_1^+, 1_C \leq f\}$ *for* $C \in \mathcal{C}$ *and* $\mu_I^i(B) = \sup\{\mu_I^i(C) | C \in \mathcal{C}, C \subset B\}$ *for* $B \in \mathcal{B}$. *Then* $\mu_I^i$ *is an inner regular non-additive measure.*

We shall say that the non-additive $\mu_I^i$ is an inner regular non-additive measure induced by a c.m. functional $I$.

**Proposition 4.8.**

*Let* $\mu_I^i$ *be an inner non-additive measure induced by a c.m. functional I. Suppose that* $f \in C_1^+, A \supset supp(f)$, *and* $A \in \mathcal{B}$. *Then we have* $\mu_I^i(A) \geq I(f)$.

**Lemma 4.9.** *Let* $\mu_I^i$ *be an inner non-additive measure induced by a c.m. functional I. Suppose that* $f \in C_1^+, O \subset supp(f)$, *and* $O \in \mathcal{O}$. *Then we have* $\mu_I^i(O) \leq I(f)$.

**Proposition 4.10.** *Let* $\mu_I^i$ *be an inner non-additive measure induced by a c.m. functional I. Then* $\mu_I^i$ *is i-regular, that is,* $\mu_I^i(C) = \inf\{\mu_I^i(O) | O \in \mathcal{O}, O \supset C\}$.

Applying the above mentioned lemmas we can prove the inner representation theorem.

**Theorem 4.11.**

*Let $I$ be a c.m. functional on $C^+$. If $\mu_I^i$ is an i-regular non-additive measure induced by $I$, then we have*

$$I(f) = (C)\int f d\mu_I^i$$

*for all $f \in C_1^+$.*

The next theorem follows from Theorem 3.9, Theorem 4.4 and Theorem 4.11.

**Corollary 4.12.** *Let $I$ be a c.m. functional on $C^+$ and $\mu_I^o$ (resp. $\mu_I^i$) is an o- regular (resp. an i-regular non-additive measure induced by $I$,*

1. $\mu_I^i(C) = \mu_I^o(C)$ *for all* $C \in \mathcal{C}$,
2. $\mu_I^i(O) = \mu_I^o(O)$ *for all* $C \in \mathcal{O}$,
3. $\mu_i(A) \leq \mu_o(A)$ *for all* $A \in \mathcal{B}$.

# 5 Conclusions

We have studied the properties of an i-regular non-additive measure and shown that an inner representation is possible. We showed that an o-regular non-additive measure and an i-regular non-additive measure induced by a c.m. functional takes same value on the class of compact sets and the class of open sets. It will be future works whether they coincides on the class of Borel sets and under what condition they coincides if they are not same on the class of Borel sets.

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